Explorations in Recursion with John Pell and the Pell Sequence

Recurrence Relations and their Explicit Formulas

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John Pell (1611-1685)

John Pell (1611-1685) An "obscure" English Mathematician

- Part of the 17th century intellectual history of England and of Continental Europe.
- Pell was married with eight children, taught math at the Gymnasium in Amsterdam, and was Oliver Cromwell's envoy to Switzerland.
- Pell was well read in classical and contemporary mathematics.
- Pell had correspondence with Descartes, Leibniz, Cavendish, Mersenne, Hartlib, Collins and others.
- His main mathematical focus was on mathematical tables: tables of squares, sums of squares, primes and composites, constant differences, logarithms, antilogarithms, trigonometric functions, etc.

John Pell (1611-1685) An "obscure" English Mathematician

- Many of Pell's booklets of tables and other works do not list himself as the author.
- Did not publish much mathematical work. Is more known for his activities, correspondence and contacts.
- Only one of his tables was ever published (1672), which had tables of the first 10,000 square numbers.
- His best known published work is, "An Introduction to Algebra". It explains how to simplify and solve equations.
- Pell is credited with the modern day division symbol and the double-angle tangent formula.
- Pell is best known, only by name, for the Pell Sequence and the Pell Equation.

John Pell (1611-1685) An "obscure" English Mathematician

- Division Symbol:

$$\div$$

- Double-Angle Tangent Formula:

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

- Pell Sequence:

$$p_n = 2p_{n-1} + p_{n-2} \qquad p_0 = 1, p_1 = 2, n \geq 2$$

- Pell Equation:

$$x^2 + 2y^2 = \pm 1$$

John Pell (1611-1685) An “obscure” English Mathematician

- Both the Pell Sequence and the Pell Equation are erroneously named after him.
- Euler, after reading John Wallis’s “Opera Mathematica”, mistakenly gave credit to Pell for the Pell Equation.
- He had constant financial trouble throughout his life and was twice imprisoned for unpaid debts.
- In summary, Pell seemed easily distracted, had multiple projects going on at once, and many unfinished projects. Not a well known mathematician because of lack of publishing and the desire to remain anonymous.
- Despite all this, he dedicated much of his life to mathematics and therefore is recognized as a minor figure in the history of mathematics.

The Pell Sequence

- Defined by the recurrence relation:

$$p_n = 2p_{n-1} + p_{n-2} \quad p_0 = 1, p_1 = 2, n \geq 2$$

- The first few terms of the Pell Sequence are:

$$1, 2, 5, 12, 29, 70, 168, 408, \ldots..$$

$$p_2 = 2p_{2-1} + p_{2-2} = 2p_1 + p_0 = 2(2) + 1 = 5$$
$$p_3 = 2p_{3-1} + p_{3-2} = 2p_2 + p_1 = 2(5) + 2 = 12$$
$$p_4 = 2p_{4-1} + p_{4-2} = 2p_3 + p_2 = 2(12) + 5 = 29$$
$$etc \ldots\ldots$$

The Pell Sequence

- One solution to the recurrence relation is:

$$p_n = \frac{\sqrt{2}}{4}\left[\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right] \forall n \geq 1$$

- Here is a second solution to the recurrence relation:

$$p_n = \sum_{\substack{i,j,k \geq 0 \\ i+j+2k=n}} \frac{(i+j+k)!}{i!\,j!\,k!}$$

The Pell Sequence

- Here is how to find the first term in the Pell Sequence using the second solution:

$$p_0 = 1$$
$$i + j + 2k = n$$
$$i + j + 2k = 0$$
$$(i, j, k)$$
$$(0,0,0)$$
$$\frac{(0+0+0)!}{0!0!0!} = \frac{1}{1} = 1$$
$$p_0 = 1$$

- Now, it is your turn!

Verification of the Pell Sequence

- Let p_n count the number of ways to fill an $n - foot$ flagpole.
- There are red, white, and blue flags.

$$red = i, blue = j, white = k \qquad p_0 = 1, p_1 = 2, n \geq 2$$

- Red and blue flags are each 1 feet tall and white flags are 2 feet tall.
- If all flags are blue or red or any combination of the 2, then the possibilities are:

$$3^6 = 729$$

Verification of the Pell Sequence

- Consider for all cases which flag is at the top of the flagpole.
- Case 1: If a blue flag is on top then anything underneath is:

$$p_{n-1}$$

- Case 2: If a red flag is on top then anything underneath is:

$$p_{n-1}$$

- Case 3: If a white flag is on top then anything underneath is:

$$p_{n-2}$$

- The cases yield the desired recurrence relation which is the Pell Sequence:

$$p_n = 2p_{n-1} + p_{n-2}$$

Verification of the Pell Sequence

- Here are some examples on a case-by-case basis:
- 1) There is one way to fill a zero-foot flagpole if all flags are zero feet tall.

$$n = 0 \rightarrow p_0 = 1 \rightarrow (i, j, k) \rightarrow (0,0,0) \rightarrow i + j + 2k = 0 + 0 + 2(0) = 0$$

- 2) There are 2 ways to fill a 1-foot flagpole with either a blue or red flag

$$n = 1 \rightarrow p_1 = 2 \rightarrow (i, j, k) \rightarrow (1,0,0) \rightarrow i + j + 2k = 1 \rightarrow 1 + 0 + 2(0) = 1$$

$$or \rightarrow (0,1,0) \rightarrow 0 + 1 + 2(0) = 1$$

- 3) There are 5 ways to fill a 2-foot flagpole:

$$n = 2 \rightarrow p_2 = 5 \rightarrow (i, j, k) \rightarrow (2,0,0), (0,2,0), (0,0,1)(1,1,0), (1,1,0)$$

$$\rightarrow i + j + 2k = 2 \quad red = i, j = blue, k = white$$

Properties of the Pell Sequence

- Here is the Pell Sequence recurrence relation and the first few terms.

$$p_n = 2p_{n-1} + p_{n-2} \qquad p_0 = 1, p_1 = 2, n \geq 2$$

$$1, 2, 5, 12, 29, 70, 169, 408, \ldots$$

- Sometimes the sequence begins with zero.

- Here is one solution to the Pell Sequence.

$$p_n = \frac{\sqrt{2}}{4}\left[\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right], \forall n \geq 1$$

- The only triangular Pell number is 1.

- For a Pell number to be prime, the index needs to be prime.

Properties of the Pell Sequence

- The only Pell numbers that are cubes, squares or any other higher power are:

$$0,1,144$$

- The Pell Numbers can be represented geometrically with the "Silver Rectangle". The ratio of length to width is length "y" and width 1.

- When 2 squares with the side equal to the width are taken out of the rectangle, what remains has the same ratio of length to width as the original rectangle.

- Here is an algebraic representation:

$$\frac{y}{1} = \frac{1}{y-2} \rightarrow y^2 - 2y - 1 = 0 \rightarrow y = \left(1 + \sqrt{2}\right)$$

Properties of the Pell Sequence

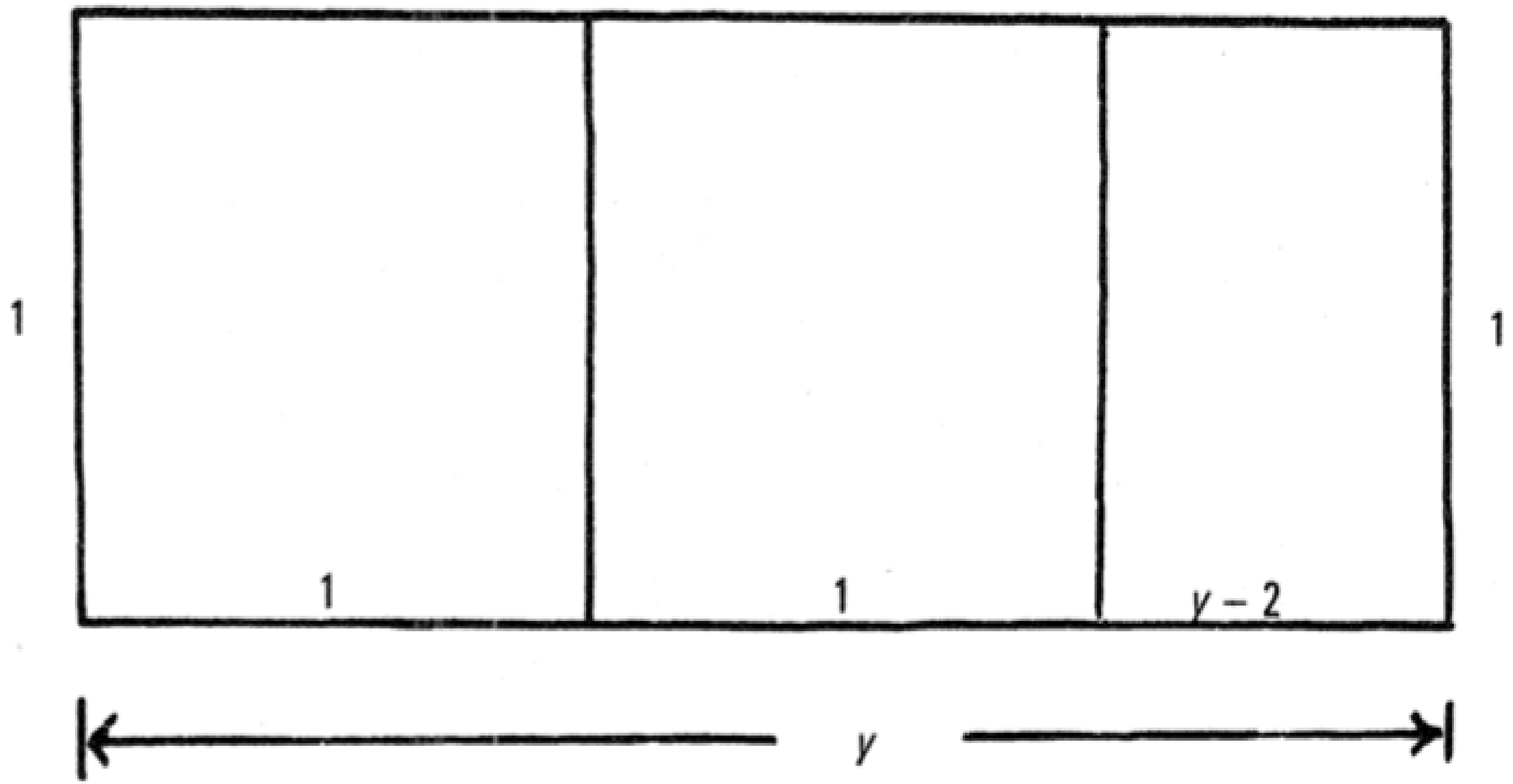
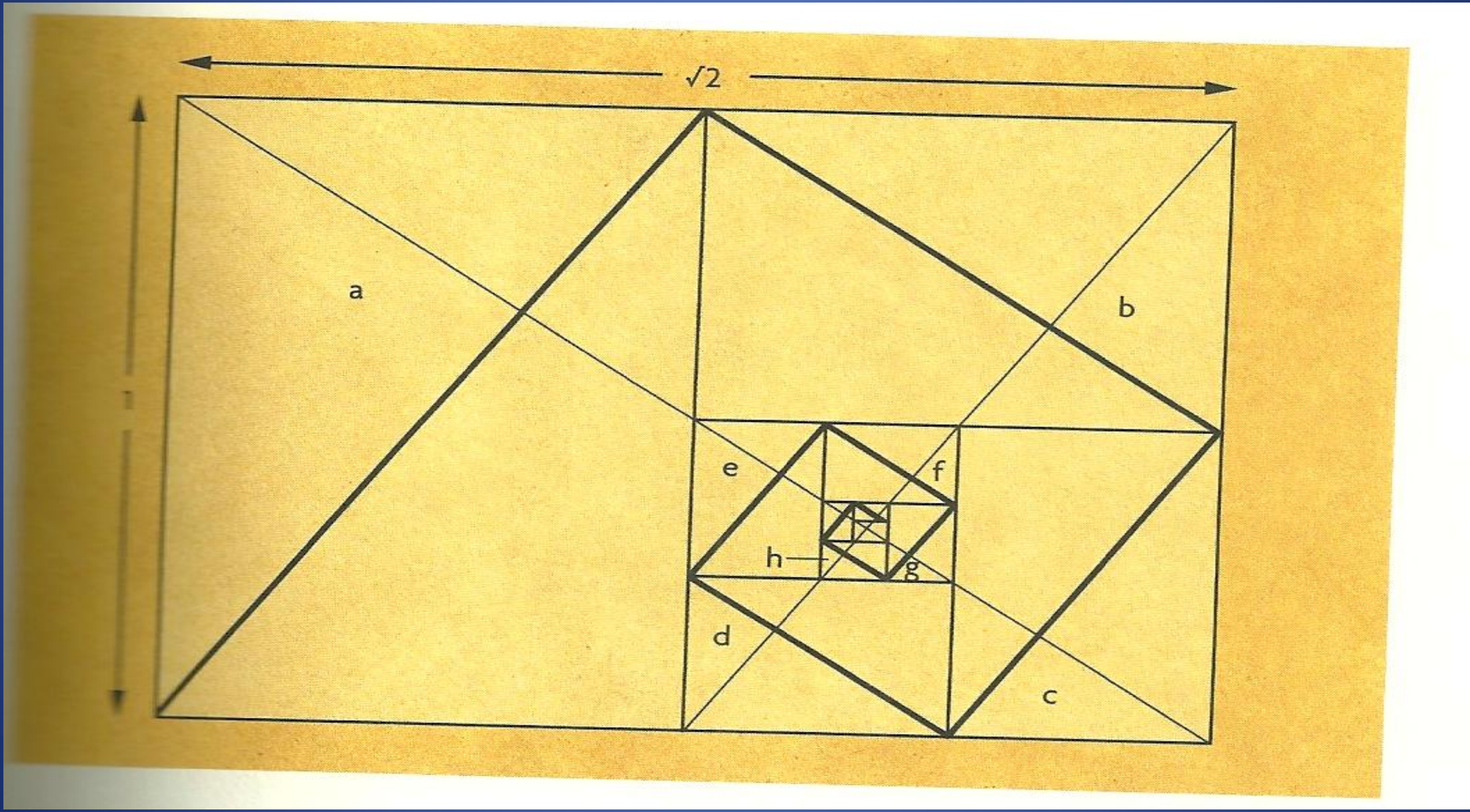


Figure 1

Properties of the Pell Sequence



Properties of the Pell Sequence

- The generating function for the Pell Sequence is:

$$\frac{1}{1-2x-x^2} = \sum_{i=1}^{\infty} P_n x^n$$

- The Pell numbers can be generated by the matrix:

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$$

- Identities of the Pell Sequence can produce Pythagorean Triples and square numbers.

Properties of the Pell Sequence

- The proportion $\sqrt{2}:1$ or $\frac{99}{70}$ is used in paper sizes A3, A4 and others.
- The Pell Numbers are the denominators of the fractions that are the closest rational approximations to the $\sqrt{2}$

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots$$

- The sum of the numerator and the denominator of the previous term is the denominator of the current term.

Properties of the Pell Sequence

- The numerator of the current fraction is the sum of the numerator and 2 times the denominator of the previous fraction.

$$\frac{1}{1},\frac{3}{2},\frac{7}{5},\frac{17}{12},\frac{41}{29},\frac{99}{70},\ldots$$

- Alternating fractions determine approximations closer and closer to the $\sqrt{2}$

$$\frac{1}{1},\frac{7}{5},\frac{41}{29},\ldots<\sqrt{2}\ldots,\frac{99}{70},\frac{17}{12},\frac{3}{2}$$

Properties of the Pell Sequence

- There is a relationship between the Pell Sequence and the Pell Equation.
- The Pell Equation is defined:

$$x^2 + 2y^2 = \pm 1$$

- and, if

$$x = p_{n+1} - p_n \qquad y = p_n$$

- Then x and y will satisfy the Pell Equation.

Properties of the Pell Sequence

- Example:

$$p_2 = 5 \rightarrow x = p_{2+1} - p_2 \rightarrow y = p_2$$

$$\rightarrow x = p_3 - p_2 \rightarrow y = p_2$$

$$\rightarrow x = 12 - 5 = 7 \rightarrow y = 5$$

$$x^2 + 2y^2 = \pm 1$$

$$7^2 + 2(5)^2 \rightarrow 49 - 50 = -1$$

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Introduction to recurrence relations

- A sequence of numbers can be defined recursively by what is known as a recurrence relation.
- The sequence of numbers:

$$1, 2, 5, 12, 29, 70, 169, 408, \ldots..$$

- can be defined with the recurrence relation:

$$p_n = 2p_{n-1} + p_{n-2}$$

- The first few terms are known as the initial conditions of the sequence.

$$p_0 = 1, p_1 = 2, n \geq 2$$

Introduction to Recurrence Relations

- The numbers in the list are the terms of the sequence.

$$p_0 = 1, p_1 = 2, p_2 = 5, etc \ldots$$

- A "solution" to the recurrence relation is:

$$p_n = \frac{\sqrt{2}}{4}\left[\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right] \forall n \geq 1$$

- This is also known as an "explicit" or "closed-form" formula.

4 techniques for solutions to recurrence relations:
Guess and check with the Principle of Mathematical Induction

- Guess and check with the Principle of Mathematical Induction.
- Consider the sequence defined by:

$$a_n = 2a_{n-1} + 1 \quad a_1 = 1 \quad n \geq 2$$

- The first few terms in the sequence can be computed as follows:

$$\begin{aligned}
a_1 &= 1 \\
a_2 &= 2a_{2-1} + 1 = 2a_1 + 1 = 2(1) + 1 = 3 \\
a_3 &= 2a_{3-1} + 1 = 2a_2 + 1 = 2(3) + 1 = 7 \\
a_4 &= 2a_{4-1} + 1 = 2a_3 + 1 = 2(7) + 1 = 15 \\
a_5 &= 2a_{5-1} + 1 = 2a_4 + 1 = 2(15) + 1 = 31 \\
a_6 &= 2a_{6-1} + 1 = 2a_5 + 1 = 2(31) + 1 = 63
\end{aligned}$$

4 techniques for solutions to recurrence relations:
Guess and check with the Principle of Mathematical Induction

- From this data we can notice a pattern and guess a formula:

$$a_1 = 2^1 - 1 = 1$$
$$a_2 = 2^2 - 1 = 3$$
$$a_3 = 2^3 - 1 = 7$$
$$a_4 = 2^4 - 1 = 15$$
$$a_5 = 2^5 - 1 = 31$$
$$a_6 = 2^6 - 1 = 63$$
$$\therefore a_n = 2^n - 1, \forall n \geq 1$$

- Use induction to prove $a_n = 2^n - 1$ holds for all $n \geq 1$

4 techniques for solutions to recurrence relations: Guess and check with the Principle of Mathematical Induction

- Proof: (i) Base cases: For

$$n = 1 \rightarrow a_n = 2^n - 1 \rightarrow a_1 = 2^1 - 1 = 1.$$

- (ii) induction step:

- Assume $a_n = 2^n - 1$ is true, then $a_{n+1} = 2^{n+1} - 1$ is true. Then

$$a_{n+1} = 2a_{(n+1)-1} + 1 \rightarrow 2a_n + 1 \rightarrow 2\left(2^n - 1\right) + 1$$
$$\rightarrow 2^{n+1} - 2 + 1 \rightarrow 2^{n+1} - 1$$

- Therefore by induction $a_n = 2^n - 1$ holds for all $n \geq 1$

4 techniques for solutions to recurrence relations: The Characteristic Polynomial

- Consider the recurrence relation: $a_n = -5a_{n-1} + 6a_{n-2} \quad a_0 = 5, a_1 = 19, n \geq 2$

- Solution $\longrightarrow$

$$a_n = -5a_{n-1} + 6a_{n-2}$$
$$a_n + 5a_{n-1} - 6a_{n-2} = 0$$
$$x^2 + 5x - 6 \rightarrow (x-1)(x+6) = 0$$
$$x_1 = -6, x_2 = 1$$
$$a_n = c_1\left(x_1^n\right) + c_2\left(x_2^n\right)$$
$$a_n = c_1\left(-6^n\right) + c_2\left(1^n\right)$$
$$a_0 = 5 \rightarrow 5 = c_1\left(-6^0\right) + c_2\left(1^0\right)$$
$$5 = c_1 + c_2 \rightarrow equation\,1$$
$$a_1 = 19 \rightarrow 19 = c_1\left(-6^1\right) + c_2\left(1^1\right)$$
$$\rightarrow 19 = -6c_1 + c_2 \rightarrow equation\,2$$

4 techniques for solutions to recurrence relations: The Characteristic Polynomial

- Multiplying equation 1 by 6 and adding equation 1 to equation 2 yields:

$$c_1 = -2, c_2 = 7$$

$$a_n = -2\left(-6^n\right) + 7\left(1^n\right) \rightarrow \therefore a_n = -2\left(-6^n\right) + 7, \forall n \geq 0$$

4 techniques for solutions to recurrence relations: Generating Functions

Consider the recurrence relation: $a_n = 2a_{n-1} \quad a_0 = 1, n \geq 1$

Solution: $\longrightarrow$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$f(x) = a^0 x^0 + \sum_{n=1}^{\infty} (2a_{n-1}) x^n$$

$$f(x) = 1 + 2\sum_{n=1}^{\infty} (a_{n-1}) x^n$$

$$f(x) = 1 + 2x\sum_{n=0}^{\infty} (a_{n-1}) x^{n-1}$$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow f(x) = 1 + 2xf(x)$$

$$f(x) - 2xf(x) = 1 \rightarrow f(x)(1-2x) = 1$$

$$f(x) = \frac{1}{1-2x} \rightarrow f(x) = \sum_{n=0}^{\infty} (2x)^n \rightarrow f(x) = \sum_{n=0}^{\infty} 2^n x^n$$

$$\therefore a_n = 2^n \rightarrow \forall n \geq 0.$$

4 techniques for solutions to recurrence relations: Linear Algebra

- Solve the recurrence relation:

$$a_{n+1} = 3a_n - 2a_{n-1} \qquad a_0 = -4, a_1 = 0, n \geq 1$$

- Solution:

$$v_n = A^n \bullet v_{n-1}$$

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$$

$$A^n v_0 = A^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = A^n \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

4 techniques for solutions to recurrence relations: Linear Algebra

- Next is the characteristic polynomial of A by the diagonalization of A

$$(A-\lambda I)=\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}=\begin{bmatrix}3-\lambda & -2\\ 1 & -\lambda\end{bmatrix}$$

$$\det(A-\lambda I)=(3-\lambda)(-\lambda)-(-2)(1)$$

$$\lambda^2-3\lambda+2=0\rightarrow(\lambda-2)(\lambda-1)\rightarrow\lambda_1=1,\lambda_2=2$$

$$D=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$$

4 techniques for solutions to recurrence relations:
Linear Algebra

- The Eigen vectors of A are: λ_1 and λ_2 The Eigen space for A is:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

- To find the Eigen space for $\lambda_1 = 1$ we have:

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 3-\lambda_1 & -2 \\ 1 & -\lambda_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$2x_1 - 2x_2 = 0$$

$$x_1 - x_2 = 0$$

4 techniques for solutions to recurrence relations: Linear Algebra

- Where $x_2 = t_1$ is free and $x_1 = x_2 = t_1$ and:

$$x = \begin{bmatrix} t_1 \\ t_1 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- To find the Eigen space for $\lambda_2 = 2$ we have:

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 3-\lambda_2 & -2 \\ 1 & -\lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$x_1 - 2x_2 = 0$$

$$x_1 - 2x_2 = 0$$

4 techniques for solutions to recurrence relations: Linear Algebra

- Where $x_2 = t_2$ is free and $x_1 = 2x_2 = 2t_2$ and:

$$x = \begin{bmatrix} 2t_2 \\ t_2 \end{bmatrix} = t_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Then we will write the matrices P, P^{-1}, D to solve for A:

$$P = \begin{bmatrix} t_1 & t_2 \\ t_1 & t_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

4 techniques for solutions to recurrence relations: Linear Algebra

Solution:

$$P^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \to \frac{1}{(1)(1)-(2)(1)}\begin{bmatrix} 1 & -2 \\ -1 & -1 \end{bmatrix}$$

$$P^{-1} = -\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$P^{-1}AP = D \Rightarrow A = PDP^{-1}$$

$$v_n = A^n v_0 = \left(PDP^{-1}\right)v_o = \left(PDP^{-1}\right)\begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

$$P^{-1}v_0 = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

$$PD^n = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix} = \begin{bmatrix} 1 & 2^{n+1} \\ 1 & 2^n \end{bmatrix}$$

$$v_n = PD^n \bullet P^{-1}v_0$$

$$v_n = \begin{bmatrix} 1 & 2^{n+1} \\ 1 & 2^n \end{bmatrix}\begin{bmatrix} -8 \\ 4 \end{bmatrix} = \begin{bmatrix} -8+4\left(2^{n+1}\right) \\ -8+4\left(2^n\right) \end{bmatrix} \begin{matrix} a_{n+1} \\ a_n \end{matrix}$$

$$\therefore a_n = -8+4\left(2^n\right) \to a_n = 4\left[-2+\left(2^n\right)\right] \forall n \geq 0.$$

Curriculum for Instructors and Students

- The curriculum consists of 8 lessons: Introduction to Recurrence Relations, Characteristic Polynomial, Checking Explicit Formulas, Guess and Check with Induction, Pell Sequence, Tower of Hanoi, Generating Functions, Linear Algebra

- Each lesson has a lesson plan, student handout, instructor solutions, and lesson reflection. In the case of the Tower of Hanoi models were made.

- All lessons were done except for Generating Functions and Linear Algebra due to time constraints and students lacking prerequisites.

- The unit was done with my high school Advanced Algebra 2 class with mostly 10^{th} and 11^{th} grade students with a few 12^{th} and 9^{th} grade students. The unit was done January 2011.

Curriculum for Instructors and Students

- A chapter on recursive sequences in their Advanced Algebra 2 book was done before the curriculum. It contained arithmetic and geometric sequences, writing recursive formulas, shifted geometric sequences- (concept of a limit), graphs of sequences, application problems.
- Students had the most success with Introduction to Recurrence Relations, Characteristic Polynomial, Pell Sequence and Tower of Hanoi.
- Students had the least success with Checking the Explicit Formula, and Guess and Check with Induction.
- Here are some examples of student work which are contained within the student handouts.

Characteristic Polynomial – Student Work

5) $a_{n+1} = 7a_n - 10a_{n-1}$ given $a_0 = 10$ and $a_1 = 29$

$x^2 - 7 + 10$ $\quad x^2 - 2x - 5x + 10$

$(x-5)(x-2)$

$a_n = C_1(+5^n) + C_2(+2^n)$

$a_0 = C_1(+5^0) + C_2(+2^0)$

$10 = C_1 + C_2$

$a_1 = C_1(+5^1) + C_2(+2^1)$

$29 = +5C_1 + 2C_2$

$10 = 3 + C_2$

$-3 \quad -3$

$-2 \quad 10 = C_1 + C_2$

$29 = 5C_1 + 2C_2$

$-20 = -2C_1 - 2C_2$

$29 = 5C_1 \;\; 2C_2$

$\frac{9}{3} = \frac{3C_1}{3}$

$C_1 = 3$

$C_2 = 7$

$a_n = 3(5^n) + 7(2^n)$ ✓

Pell Sequence – Student Work

b) Use the characteristic polynomial technique to solve this recurrence relation.

$$P_n - 2P_{n-1} - P_{n-2} = 0$$

$$x^2 - 2x - 1 = 0$$

$$x_1 = 1 + \sqrt{2}$$

$$x_2 = 1 - \sqrt{2}$$

$$x = \frac{2 \pm \sqrt{4 - 4(-1)}}{2}$$

$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$x = \frac{2 \pm 2\sqrt{2}}{2}$$

$$x = 1 \pm \sqrt{2}$$

$$P_n = C_1(1+\sqrt{2})^n + C_2(1-\sqrt{2})^n$$

$$P_0 = C_1(1+\sqrt{2})^0 + C_2(1-\sqrt{2})^0$$

$$1 = C_1 + C_2 \quad \times\ 1\sqrt{2}$$

$$P_1 = C_1(1+\sqrt{2})^1 + C_2(1-\sqrt{2})^1$$

$$2 = (1+\sqrt{2})C_1 + C_2(1-\sqrt{2})$$

$$1+\sqrt{2} = (1+\sqrt{2})C_1 + (1+\sqrt{2})C_2$$

$$(-) \quad 2 = (1+\sqrt{2})C_1 + (1-\sqrt{2})C_2$$

$$\sqrt{2} - 1 = (2\sqrt{2})C_2$$

$$\frac{\sqrt{2}-1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$\frac{2-\sqrt{2}}{4}$$

$$\frac{2-\sqrt{2}}{4} = C_2$$

$$1 = C_1 + \frac{2-\sqrt{2}}{4}$$

$$-\left(\frac{2-\sqrt{2}}{4}\right) \quad -\left(\frac{2-\sqrt{2}}{4}\right)$$

$$\frac{2+\sqrt{2}}{4} = C_1$$

$$P_n = \frac{2+\sqrt{2}}{4}(1+\sqrt{2})^n + \frac{2-\sqrt{2}}{4}(1-\sqrt{2})^n$$

Alternate Pell Formula – Student Work

$P_2 = 5$ true?

$P_2 \rightarrow i + j + 2K = 2$

(1, 1, 0)

(0, 0, 1)

+ (2, 0 0)

(0, 2, 0)

$$\frac{(1+1+0)!}{1!\,1!\,0!} + \frac{(0+0+1)!}{0!\,0!\,1!} + \frac{(2+0+0)!}{2!\,0!\,0!} + \frac{(0+2+0)!}{0!\,2!\,0!}$$

$$\downarrow \quad \frac{2}{1} \qquad \frac{1}{1} \qquad \frac{2}{2} \qquad \frac{2}{2}$$

$$2 + 1 + 1 + 1$$

$P2 = 5$ ✓

Checking the Explicit Formula – Student Work

5) $a_{n+1} = 7a_n - 10a_{n-1}$ given $a_0 = 10$ and $a_1 = 29$

$a_n = \frac{3}{5}(5^n) + \frac{21}{6}(2^n)$

$$\frac{3}{5}(5^n)+\frac{21}{6}(2^n) = 7\left[\frac{3}{5}(5^{n-1})+\frac{21}{6}(2^{n-1})\right] - 10\left[\frac{3}{5}(5^{n-2})+\frac{21}{6}(2^{n-2})\right]$$

$$\frac{3}{5}(5^n)+\frac{21}{6}(2^n) - 7\left[\frac{3}{5}(5^{n-1})+\frac{21}{6}(2^{n-1})\right] + 10\left[\frac{3}{5}(5^{n-2})+\frac{21}{6}(2^{n-2})\right] = 0$$

$$5^{n-2}\cdot 2^{n-2} \quad \frac{3}{5}(5^2)+\frac{21}{6}(2^2) - 7\left[\frac{3}{5}(5^1)+\frac{21}{6}(2^1)\right] + 10\left[\frac{3}{5}(5^0)+\frac{21}{6}(2^0)\right] = 0$$

$$5^{n-2}\, 2^{n-2} \quad 15+14 - 7[3+7] + 10\left[\frac{3}{5}+\frac{21}{6}\right] = 0$$

$$5^{n-2}\cdot 2^{n-2} \quad 29 - 70 + 41 = 0$$

$$5^{n-2}\cdot 2^{n-2} \quad 0 = 0$$

Induction – Student Work

1) Prove that the sum of n consecutive positive odd integers is n^2. In other words prove that $1 + 3 + 5 + ... + (2n - 1) = n^2$

Base Case: $n = 1$

$2 \cdot 1 - 1 = 1^2$

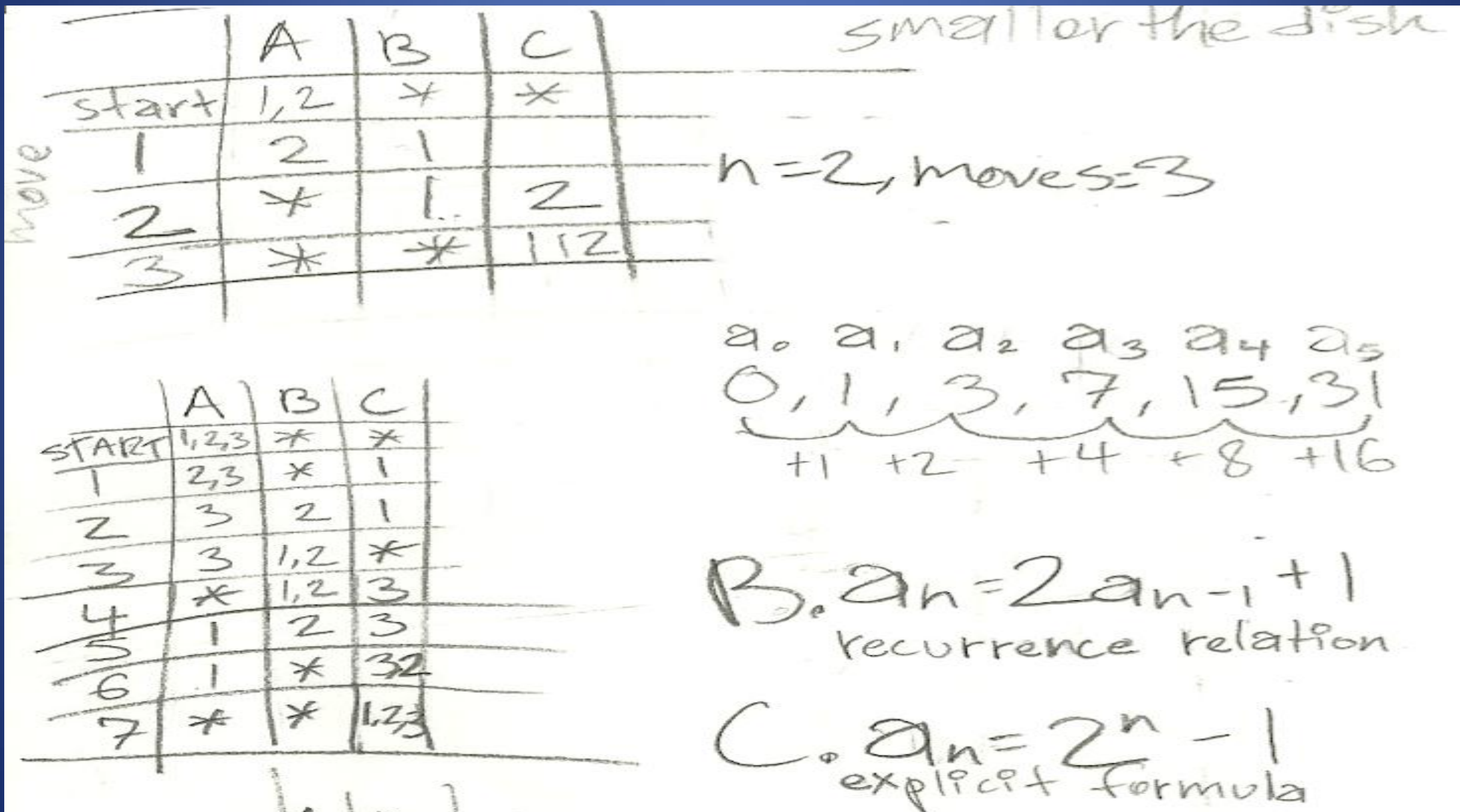
$2 - 1 = 1^2$

$1 = 1$

W.t.S $(2k-1) + 2(k+1) - 1 = k^2 + 2(k+1) - 1$

$$(2k-1) + 2(k+1) - 1 = k^2 + 2(k+1) - 1$$
$$= k^2 + 2k + 2 - 1$$
$$= k^2 + 2k + 1$$
$$= (k+1)(k+1)$$
$$= (k+1)^2$$

Tower of Hanoi – Student Work



Intro to Recurrence Relations – Student Work

15) Write a recurrence relation for the following sequences. Use a_1 for the first term in the sequence

a) 1,1,2,3,5,8,13,... 21, 34, 55, 89

$u_n = u_n \cdot u_{n-1}$

$u_3 = u_2 + u_1$

$u_3 = 1 + 1 = 2$

b) 1,4,9,16, ... 25, 36, 49, 64, 81

$a_n = n^2$

c) 1,2,6,~~16~~ 24, ... 120, 720

$A_n = n!$

d) 4,1,3, -2,5, -7,12, -19,31, ...

$a_{n-2} - a_{n-1} = a_n$

$a_1 = 4$

$a_2 = 1$

$n \geq 3$

Curriculum for Instructors and Students

- Summary of Curriculum:
- Overall it went well, sometimes painful and sometimes beauty
- Small class of 24 students, many smart and motivated students, I have known many of them since 6th grade.
- Summary of M.S.T. 501 project:
- It took about 9-12 months, summer 2010 getting ideas, fall-winter 2010-2011 doing math, winter-spring 2010 paper and power point.