

# Signals and the frequency domain

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A *signal* is a function, in the mathematical sense, normally a function of time. We often refer to functions as signals to convey that the function represents some sort of phenomenon—for example, an audio signal, the electromagnetic signal broadcast in FM radio, a currency exchange rate, or the voltage somewhere in a circuit.

One of the great advances of the 19th century was courtesy of a French mathematician called Joseph Fourier, who showed in his work about heat flow that representing a signal as a *sum of sinusoids* opens the door to more powerful analytical tools. This idea gave rise to what is now known as the *frequency domain*, where we think of signals *as a function of frequency*, as opposed to a function of time. Our goal today is to define, in some sense, what that means.

## Preliminaries

**Periodic signals.** At least to begin, we'll mainly be concerned with signals that are *periodic*. Informally, a periodic signal is one that repeats, over and over, forever. To be more precise:

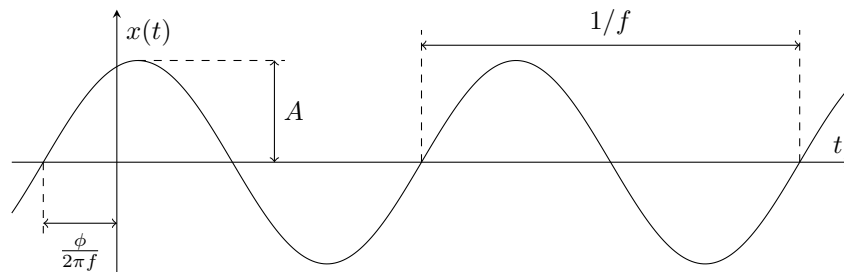
A signal  $x(t)$  is said to be *periodic* if there exists some number  $T$  such that, for all  $t$ ,  $x(t) = x(t+T)$ . The number  $T$  is known as the *period* of the signal.

The *smallest*  $T$  satisfying  $x(t) = x(t+T)$  is known as the *fundamental period*.

**Sinusoids.** Precisely what do we mean by a *sinusoid*? The term “sinusoid” means a sine wave, but we don't just mean the standard  $\sin(t)$ . To enable our analysis, we want to be able to work sine waves of different heights, widths and phases. So, to us, a single sinusoid means a function of the form

$$x(t) = A \sin(2\pi ft + \phi),$$

for some  $A$  (its *amplitude*),  $f$  (its *frequency*) and  $\phi$  (its *phase*).



## The Fourier series

Joseph Fourier's bold idea was to express periodic signals as a sum of sinusoids.

**Theorem.** If  $x(t)$  is a well-behaved periodic signal with period  $T$ , then it can be written in the form

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin(2\pi nft + \phi_n). \quad (1)$$

where  $f = 1/T$ , and for some  $A_0, A_1, A_2, \dots$  (which we call magnitudes) and  $\phi_1, \phi_2, \dots$  (which we call phases). We call this sum the Fourier series of  $x(t)$ .

That “well-behaved” caveat is worth expanding on, briefly. In engineering, every practical signal we ever deal with is “well-behaved” and has a Fourier series. If you talk to a mathematician, they’ll say that we’re sweeping a lot of details under the carpet. They’re of course right, but in real-world applications this tends not to bother us.

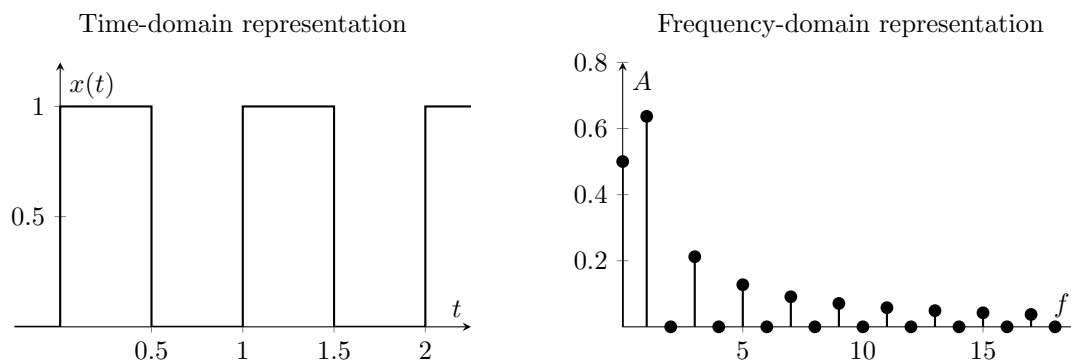
It’s worth reflecting on why Fourier’s claim was so significant. It’s not too hard to believe that *some* periodic signals can be represented by a sum of sinusoids. But included in the “well-behaved” category are signals with sudden jumps (discontinuities) in them. Can a discontinuous signal really be expressed as a sum of smooth sinusoids? Pretty much, yes. The catch is that you might need an infinite number of sinusoids.

## Time-domain and frequency-domain representations

Another fact relating to Fourier series is that the magnitudes  $A_0, A_1, A_2, \dots$  and phases  $\phi_1, \phi_2, \dots$  in (1) uniquely determine  $x(t)$ . That is, if we can find the  $A_0, A_1, A_2, \dots$  and  $\phi_1, \phi_2, \dots$  corresponding to a periodic signal  $x(t)$ , then, in effect, we have another way of describing  $x(t)$ .

When we represent a periodic signal using the magnitudes and phases in its Fourier series, we call that the *frequency-domain representation* of the signal. We often plot the magnitudes in the Fourier series using a stem graph, labeling the frequency axis by frequency.<sup>1</sup> In this sense, this representation is a *function of frequency*.

To emphasize the equivalence between the two, we call plain old  $x(t)$  the *time-domain representation*, since it’s a function of time. For example, here are both representations of a square wave:



Here’s some more terminology:

- In electrical engineering, we call the term  $A_0$  the *DC component*, *DC offset* or simply *offset*. “DC” stands for “direct current”, in contrast to the sinusoids, which “alternate”, though we use this term even if the signal isn’t a current.

In a sense, the DC component is like the “zero frequency component”, since  $\cos(2\pi \cdot 0 \cdot t) = 1$ . We often think of offset in this way, and plot the DC offset at  $f = 0$  in the frequency-domain representation.

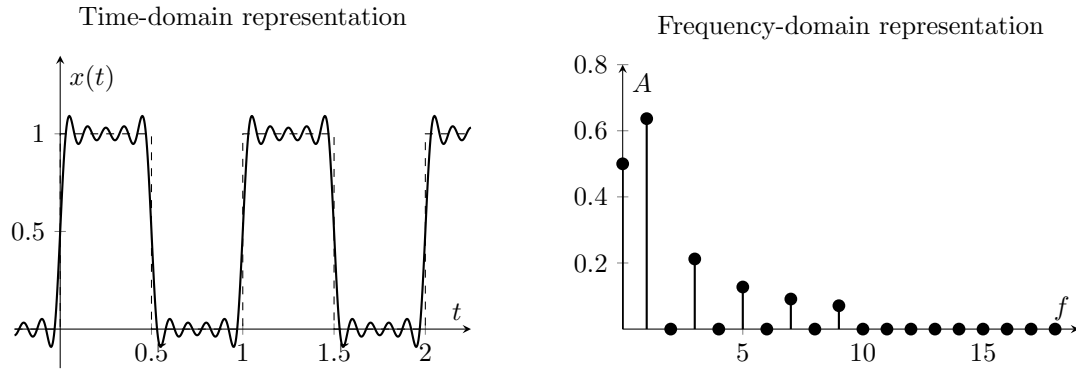
The DC component is often easy to eyeball—it’s equal to the average value of the signal over a period. For example, in the signal above, the DC offset is 0.5.

- The sinusoidal terms are often called *harmonics*, a term borrowed from music. The harmonics will have frequencies  $f, 2f, 3f, 4f$  and so on.
- We also call each harmonic,  $A_n \sin(2\pi nft + \phi_n)$ , the *frequency component* of  $x(t)$  at frequency  $nf$ . For example, if  $f = 10$  Hz, we call the harmonic for which  $n = 3$  the “30 Hz component”, reflecting that in this case  $\sin(2\pi nft + \phi_n)$  is a sinusoid of frequency 30 Hz.

<sup>1</sup>It’s also common to plot the phases on another graph, but we won’t in this course.

## Getting to know the frequency domain

What happens to our humble square wave if we don't pick up on all of the infinitely many frequencies? Say, if everything above 9 Hz disappeared?



We still have an *approximation* to the square wave: it's got the general shape right, it's just not working so well at the corners. In fact, as you might have expected, that discontinuity gives the Fourier series—a sum of continuous functions—a bit of a hard time. Nonetheless, the more frequencies we include, the closer to the true square wave we get. Here's an aphorism that encapsulates this observation: *It takes high frequencies to make jump discontinuities.*

More generally, the following statements provide some intuition for how to think in the frequency domain.

- High frequencies come in where the signal changes *rapidly*.  
At the extreme, when a signal changes *suddenly*, infinitely high frequencies come into play.
- Low frequencies come in where the signal changes *slowly*.  
At the extreme, when it doesn't change at all, we have the DC component (zero frequency).

Now, you might be thinking: This is all very well for periodic signals, but in real life, no signal is *truly* periodic, not least because no signal lasts forever. Then what do we do? Fear not—a lot of the same instincts from periodic signals also apply to signals that are roughly repetitive in the short term, including audio signals and your heartbeat.<sup>2</sup>

## Circuits with capacitors and inductors

The following is a perhaps surprising fact about circuits involving capacitors and inductors.

Any circuit with only sources, resistors, capacitors and inductors acts on frequencies individually. That is, the circuit takes each *frequency component*, multiplies each component by a *gain specific to that frequency*, and then outputs the sum of the new frequency components.

The frequency domain thus gives us a richer way to understand how these circuits work. We'll study this in the coming lectures.

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<sup>2</sup>In fact, there is a close cousin of the Fourier series, known as the Fourier transform, that deals with non-periodic signals much more rigorously.

## Computing Fourier coefficients (*not examinable*)

You might be wondering: It's all very nice that these magnitudes and phases *exist*, but can we compute them? We can, but it turns out that the form in (1) is rather awkward to do these calculations with. Another way to write a Fourier series is

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2\pi nft) + b_n \sin(2\pi nft)]. \quad (2)$$

Showing that (1) and (2) are equivalent is left as an exercise. The coefficients can then be found using the following integrals, where  $T = 1/f$ :

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(2\pi nft) dt, \quad b_n = \frac{2}{T} \int_0^T x(t) \sin(2\pi nft) dt.$$

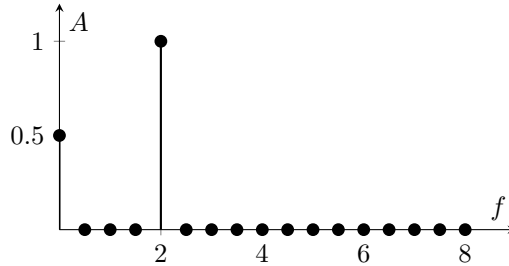
Then  $a_0, a_1, a_2, \dots$  and  $b_1, b_2, \dots$  can be converted to magnitudes and phases to fit the form of (1).

We will not ask you to apply these formulae in this course.

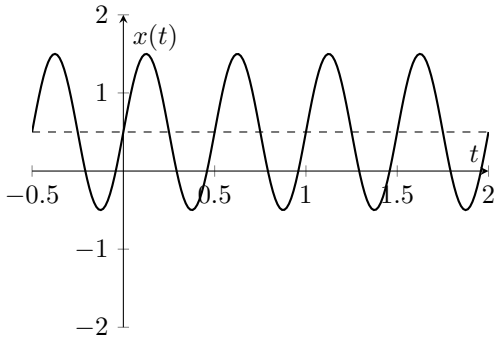
# Exercises

**Exercise 1.** Which of these could be the time-domain representation of the signal whose frequency-domain representation is below?

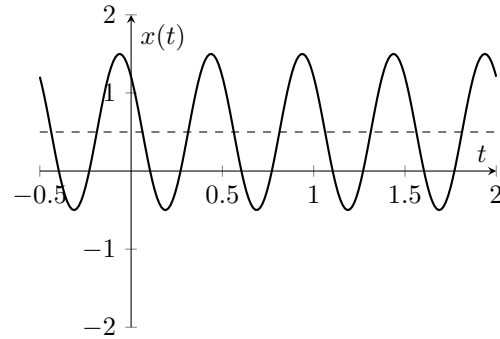
Frequency-domain representation



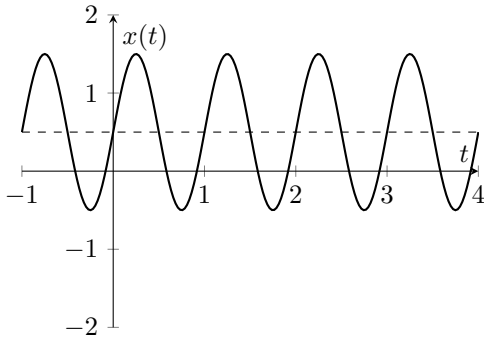
Candidate time-domain representation 1



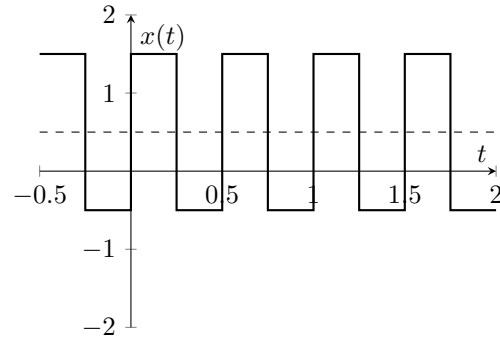
Candidate time-domain representation 2



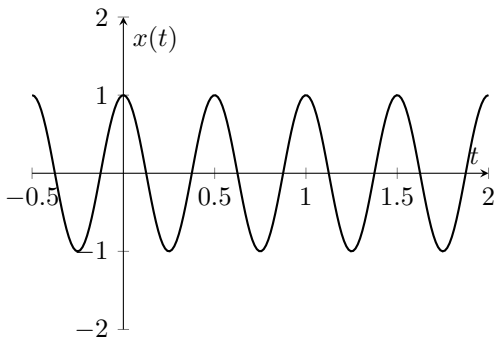
Candidate time-domain representation 3



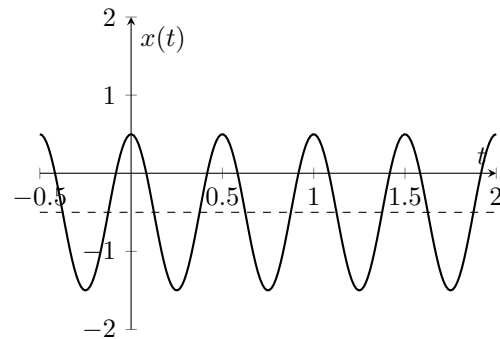
Candidate time-domain representation 4



Candidate time-domain representation 5

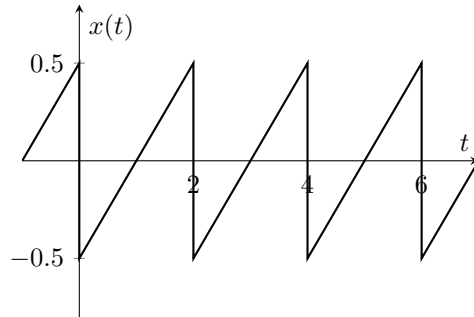


Candidate time-domain representation 6

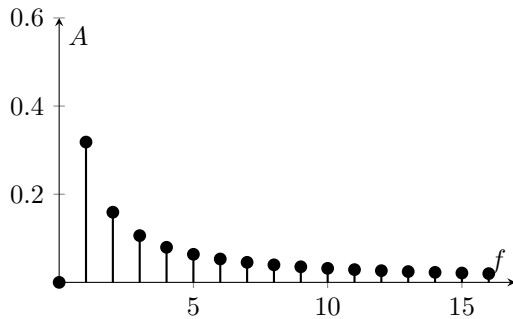


**Exercise 2.** Which do you think is the correct frequency-domain representation of the signal whose time-domain representation is below?

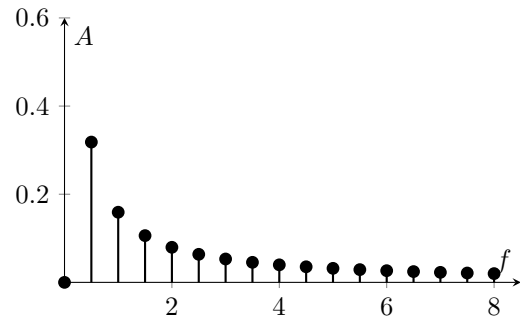
Time-domain representation



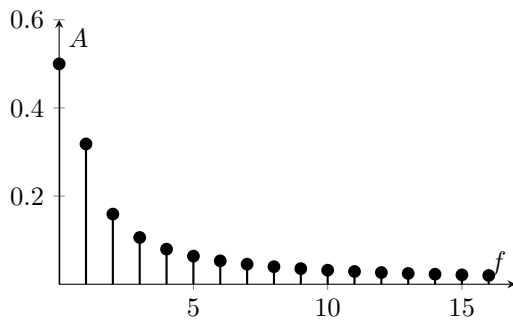
Candidate frequency-domain representation 1



Candidate frequency-domain representation 2



Candidate frequency-domain representation 3



Candidate frequency-domain representation 4

