Competitive Information Design with Asymmetric Senders*

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June 26, 2024

Abstract

We consider a competitive information design game in which a number of *ex-ante asymmetric* senders are competing for a receiver by disclosing information about their respective realizations. Unlike the setting with symmetric senders where a symmetric equilibrium always exists, the equilibrium may not exist under the asymmetric setting. Using the idea of discrete approximation and passing to the limit, we show that if there is no mass point in the senders' priors, then an equilibrium always exists. We next establish the necessary and sufficient conditions for the equilibrium structure. Our characterizations strictly generalize the symmetric equilibrium conditions provided in the symmetric environment studied in previous works. We then use the characterized equilibrium structure to solve the equilibrium for a general two-sender game along with providing a computational method of computing it.

KEYWORDS: Information design; Competition; Ex-ante asymmetry; Nash equilibrium.

*A one-page abstract of this work appeared in the Proceedings of the 25th ACM Conference on Economics and Computation (EC'24). Zhicheng Du, Zihe Wang and Shuo Zhang thank the financial support from National Natural Science Foundation of China (Grant No. 62172422).

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1 Introduction

We consider a competitive information design game in which there are multiple senders vie for the selection of a risk-neutral receiver by disclosing information about their individual state realizations. The receiver aims to select the sender who has the highest expected value of the state. We consider a general setting where the senders may be *ex-ante heterogeneous*, namely, while the senders' state realizations are independently distributed, they do not necessarily follow the same prior distributions. Each sender can only control the disclosure of information regarding his own state realizations, but there is no structural restriction on the set of the feasible information disclosing strategies.

With a risk-neutral receiver who chooses the sender only based on her expected beliefs about the senders' realizations, it is well-known that the sender's full flexibility in choosing any information to be revealed to the receiver can be modeled as each sender being able to choose any mean-preserving contraction (henceforth, MPC) of his own prior distribution. The receiver's expected value of each sender's state realization is then independently drawn according to that sender's designed MPC. Since the receiver's optimal decision is straightforward (as she simply chooses the sender who has the highest expected realized value and breaks ties uniformly at random when she is indifferent with multiple senders), we focus on Nash equilibria played by the senders in their simultaneous-move game. There have been many recent works studying this information design game with competing senders, see, e.g., Brocas et al. (2012); Gul and Pesendorfer (2012); Gentzkow and Kamenica (2016b, 2017); Hwang et al. (2019); Au and Kawai (2020, 2021); Boleslavsky and Cotton (2018) and the related work section for detailed discussions. For tractability, many of these works assume a symmetric environment where senders are all *ex-ante identical*, namely, all senders share a common prior distribution. It is shown that under the symmetric environment, a symmetric equilibrium always exists and its structure can be well-characterized.

Yet in many real-world economic applications, ex-ante heterogeneity arises naturally. For example, a sender could represent a seller who is deciding how much and what product information he wants to provide to customers (often referred to as advertising, see, e.g., Kawai et al., 2022; Sahni and Nair, 2020; Nelson, 1974; Milgrom and Roberts, 1986; Kihlstrom and Riordan, 1984). Consider the following scenario in a duopoly market: the first seller is reputed to produce a product that has a consistent quality that is well known to the customers, while the second seller may produce a product that has variable qualities. To reflect this heterogeneity of the prior quality information, the prior distribution on realizing the product quality for the first seller may need to be a point mass, while the prior distribution for the second seller may have realizations on other quality values. Indeed, this is essentially the example we provide in Section 3 (see Example 3.1), where we show that this simple asymmetric case admits no Nash equilibrium. This observation raises new questions: under which conditions does a Nash equilibrium exist in an asymmetric environment, and what is the equilibrium structure if an equilibrium exists? To answer these questions, in this work, we consider the problem of the existence and the characterization of an equilibrium in the competitive information design game with multiple asymmetric senders.

The first main result of this work is that we provide sufficient conditions under which an equilibrium always exists among senders' competition game. We prove that as long as there is no mass point in senders' prior distributions, an equilibrium always exists (see Theorem 3.1). Intuitively, each sender's action space is constrained by the MPC conditions. A sender with a prior that has mass points would be more likely to result in an MPC also with mass points. Consequently, the other senders may find it always better off spreading his MPC around the mass points, and thus these senders may not have best-response strategies, leading to the non-existence of the equilibrium.

We note that proving the equilibrium existence in our problem is challenging. First, unlike the symmetric setting where the existence of the symmetric equilibrium can be established by a tractable construction method based on solving a fixed point problem, which is the same best response problem faced by all senders. In our problem with asymmetric senders, each sender's best response problem is different as each sender has different action space (the set that includes all MPCs of his prior). Second, like many other economic interactions such as auctions and price competition, the senders' game on information competition is essentially a discontinuous game where the discontinuities may occur when senders choose the MPCs that result in the receiver being indifferent between multiple senders. Indeed, since the first existence conditions given by the seminal papers of Dasgupta and Maskin (1986a,b), there has been affluent literature studying the equilibrium existence in discontinuous games (see, e.g., Reny, 1999; Carmona, 2009; McLennan et al., 2011; Barelli and Meneghel, 2013; Bich and Laraki, 2017; Olszewski and Siegel, 2023). Among these works, a significant breakthrough is the result by Reny (1999) via the *better-reply security* approach, a property of the graph of the mapping from strategy profiles to payoff profiles. Unfortunately, verifying this property is known to be a demanding task, especially given that each sender's action space contains all feasible MPCs (which are infinite-dimensional objects that are subject to infinite-many constraints to satisfy the second-order stochastic dominance). We defer the detail discussions about this line of literature to related work section.

To handle the payoff discontinuities, we use the solution technique that capitalizes on the discrete approximation and the passing to the limit but with careful treatments dedicated to handle the MPC-constrained action space. In particular, we create a sequence of discrete games with a finite, deliberately-designed action space for each sender. Importantly, this space may include distributions that are not initially feasible MPCs of the sender's prior. Nevertheless, we are able to demonstrate that a subsequence of the equilibrium strategies of these discrete games weakly converges to a feasible MPC, which enables us to establish the equilibrium existence. To the best of our knowledge, these constructions have not previously been applied to the determination of equilibrium mixed-strategy profiles with MPC-constrained action spaces.

The second main result of this paper is that we provide structural characterizations of a Nash equilibrium (when it exists). Specifically, we provide necessary and sufficient conditions under which a feasible strategy profile is indeed an equilibrium (see Theorem 4.2). Central to our conditions is the definition of a sender-specific "virtual competitive function" (see Definition 4.1), which can be explicitly constructed based on the sender's prior and the information strategies of other senders. Intuitively, the virtual competitive function of each sender i is closely related to his interim expected utility function induced by the other senders' information strategies and it fully characterizes the virtual competitive environment he faces. And we show that if sender i's strategy is "best-responding" to his virtual competitive function, then he is also best responding to other senders' information strategies. Our equilibrium conditions, developed by leveraging the verification technique developed by Dworczak and Martini (2019), are the verification conditions for the virtual competitive function of each sender, i.e., given a strategy profile, as long as each sender's virtual competitive function satisfies these conditions, then this strategy profile is indeed an equilibrium. In particular, Dworczak and Martini (2019) consider a general programming problem where the sender's (indirect) payoff depends only on the expected value (state) he induces, and they show that to determine the optimality of a feasible MPC, it suffices to show there exists an auxiliary function that satisfies certain conditions. However, they do not provide how to construct such auxiliary function. Instead, we provide an explicit construction for this function, which is our "virtual competitive function". Our equilibrium conditions strictly generalize the conditions identified for the symmetric equilibrium in symmetric settings.

The last contribution of this paper is the applications of our structural characterizations of the Nash equilibrium. In the first application, we show that by utilizing the established verification conditions, we are able to fully characterize the equilibrium of a general two-sender game when the senders' priors are strict uni-modal (see Theorem 5.2). Informally, we show that the equilibrium strategy of each sender, depending on the relative curvature of the priors, either exhibits a synchronous conditional uniformity structure or an asynchronous conditional uniformity structure. In synchronous conditional uniformity structure, both the sender's equilibrium strategy first matches the prior up to a same point, then they both exhibit a conditional uniformity structure after this point. While in asynchronous conditional uniformity structure, both the sender's equilibrium strategy exhibit a conditional uniformity structure in different partitions of the support (and these partitions are not necessarily same for the two senders). Intuitively the uniformity structure in both cases stems from the information competition. When the prior of the other sender is concave over a certain interval, the sender tends to contract his prior as much as possible. However, to prevent the other sender from having profitable deviation, the sender can only contract his prior up to a linear structure. Consequently, a conditional uniformity structure is more likely to make the both sender have no profitable deviation within a certain interval. With this structure in mind, we also provide an algorithmic procedure that can construct the equilibrium strategy profile. The core idea of our algorithm is to iteratively update the boundary points (via solving the local equilibrium structure that are local MPCs to the prior) for each possible conditional uniformity structure that we characterized in Theorem 5.2. In the second application, we revisit the symmetric setting when the senders share the same prior distribution, where it has been shown that a symmetric equilibrium always exists and it is also unique among all symmetric equilibria. We strengthen this result by showing that such symmetric equilibrium is also unique among all *asymmetric* equilibria (see Proposition 5.3). We note that no previous paper that we are aware of has established such equilibrium uniqueness.

1.1 Related Work

Information design and its competition variant. Our work closely relates the line of literature on competitive information design with Bayesian persuasion. Bayesian persuasion (Kamenica and Gentzkow, 2011) studies a game where an informed sender decides how much information she would disclose to a receiver to persuade the receiver to take certain action. As a celebrated model of information design, Bayesian persuasion has led to a plethora of variants studied in recent years. We refer readers to Bergemann and Morris (2019); Kamenica (2019) for a good overview of Bayesian persuasion and recent developments.

Our research contributes to the line of research in Bayesian persuasion with competing senders. Works in this line of research can be categorized into two strands: (1) each sender can only reveal information about their own realized state (Boleslavsky and Cotton, 2015, 2018; Jain and Whitmeyer, 2019; Au and Kawai, 2020, 2021; Hwang et al., 2019; Gradwohl et al., 2022; Ding et al., 2023); (2) all senders share a common state of the world, and each sender can independently disclose information about the common state to the receiver (Gentzkow and Kamenica, 2016a, 2017; Ravindran and Cui, 2020; Hossain et al., 2024).

Our work relates to the first strand where each sender has their own independently realized state and can only reveal information in that state. In this strand, Au and Kawai (2020); Boleslavsky and Cotton (2018); Jain and Whitmeyer (2019) study a setting with two senders competing for a receiver with binary state space. Au and Kawai (2021); Hwang et al. (2019) then focus on a symmetric environment with multiple, continuous state spaces where there is no ex-ante information asymmetry among senders' priors, respectively. Focusing on symmetric equilibrium, they characterize the uniqueness and existence of equilibrium through a tractable construction approach. Our work differs from the above works in that we consider an asymmetric setting with continuous state space. Information asymmetry in our setting poses significant challenges in characterizing the existence and structure of equilibrium (as equilibrium may not even exist). Therefore, their approach does not apply to our setting. In addition to these works, recent works also study competition in a sequential setting (Li and Norman, 2021; Armstrong and Zhou, 2022; He and Li, 2023; Lyu, 2023; Ding et al., 2023). Our work differs from these because we focus on a setting where all senders simultaneously determine their information strategy.

The existence of equilibrium. Our work concerns about the equilibrium existence in the competitive information design game. Due to the presence of the possible ties, our game can be categorized as a discontinuous game in which each sender has a continuous action space but with a discontinuous utility function.

The equilibrium existence for the finite games can be obtained by the seminal Nash's Theorem (Nash Jr, 1950), and for continuous game under specific assumptions, it can be also obtained according to the Glicksberg's Theorem (Glicksberg, 1952). Establishing the equilibrium existence for the discontinuous game is, however, notoriously challenging. Early existence results for games with discontinuous payoffs are obtained by Dasgupta and Maskin (1986a) and Simon (1987) by approximating the original game with a sequence of finite games. Dasgupta and Maskin's result cannot be applied to our setting as they consider an interval action space while ours are MPC-constrained action spaces. Later, Reny (1999) proposes a classical theory for the equilibrium existence in games with discontinuous payoffs via a condition termed as " better reply security", which has been used subsequently by many authors (see, e.g., Carmona, 2009; McLennan et al., 2011; Barelli and Meneghel, 2013; Carmona and Podczeck, 2014; He and Yannelis, 2015; Bich and Laraki, 2017; Olszewski and Siegel, 2023, and Reny 2020 for a survey about equilibrium in discontinuous games literature). Unlike the standard approach on proving the existence of equilibria, which is to approximate the original game by a sequence of games with a finite number of actions, Reny's approach is instead approximating the original game with discontinuous payoffs. It is unclear how to apply Reny's results to establish the equilibrium existence in our setting as it is hard to verify the "better reply security" given the MPC-constrained action spaces. The results in other works are also not applicable to our setting. For example, Olszewski and Siegel (2023) consider the Bayesian game and require "improving deviation", which is also not satisfied in our setting as senders' competition is indeed a constant-sum game. Instead, our approach aligns with the standard approach on approximating the original game by a sequence of games with finite actions (Dasgupta and Maskin, 1986b; Simon, 1987; Maskin and Riley, 2000) but with a subtlety on deliberately constructing these finite actions.

2 Preliminaries

We consider a competitive information design game with $N \geq 2$ *ex-ante* heterogeneous senders, and a single receiver. Each sender $i \in [N]$ is endowed with a proposal, and the value v_i of the proposal of sender i is independently distributed with a prior distribution $F_i \in \Delta([0,1])$ over a common value space $[0,1]$. The receiver only knows the prior distribution of each sender's proposal value, but does not know the realized proposal value. The receiver is an expected-utility-maximizer, and aims to choose the sender who has the highest value of the proposal.

The senders engage in a simultaneous-move game competing for the choice of the receiver. Each sender's objective is to maximize the probability that the receiver chooses his proposal. Without loss of generality, we normalize a sender's payoff to one if the receiver accepts his proposal, and zero otherwise. Each sender i simultaneously chooses a signaling scheme $\{\pi_i(\sigma \mid v), \Sigma_i\}$, where Σ_i is a signal space and $\pi_i(\sigma \mid v) \in [0,1]$ specifies the conditional distribution of signal $\sigma \in \Sigma$ when the proposal with value v is realized. The senders' signaling schemes $\{\pi_i(\sigma \mid v), \Sigma_i\}_{i\in[N]}$ are known to the receiver in advance. Upon observing the realized signal $\sigma_i \sim \Sigma_i$ drawn according to the conditional distribution $\pi_i(\sigma \mid v)$, the receiver can infer a posterior belief about the underlying proposal value v_i of the sender i. Since the receiver is risk neutral, only the conditional expected value $\mathbb{E}[v_i \mid \sigma_i]$ matters for the receiver's decision. In other words, a sender's signaling scheme begets a distribution over posterior distributions of this sender's proposal value. Since the risk-neutral receiver's strategy only depends on her posterior means of senders' proposal qualities, each sender's payoff depends only on the mean of the receiver's posterior induced by the sender's signal and the means of the posterior beliefs induced by other senders' signals (instead of the detailed characteristics of the distributions).

Senders' information strategies. With the above observation, we can represent a sender's information strategy by a distribution over posterior means. A natural next question is which distributions over posterior means can indeed be implemented by some signaling schemes given prior F. This question can be answered using the notion of mean-preserving contraction (MPC), which characterizes feasible distributions to represent senders' information strategies.

Definition 2.1 (Mean-preserving contraction (MPC)). *A distribution $G \in \Delta([0,1])$ is a mean-preserving contraction of a distribution $F \in \Delta([0,1])$ if and only if for all $t \in [0,1]$,*

$$\int_0^t G(x)\ \mathrm{d}x \le \int_0^t F(x)\ \mathrm{d}x \tag{1}$$

where the inequality holds as equality for $t = 1$. We use $\mathsf{MPC}(F)$ to represent the space of all mean-preserving contractions induced from the prior F.

It is well known that a distribution G over posterior means can be induced by some signaling scheme from prior F if and only if the distribution G is an MPC of the prior F (see, e.g., Rothschild and Stiglitz, 1970; Blackwell and Girshick, 1979; Gentzkow and Kamenica, 2016b).

Lemma 2.1. *There exists a signaling scheme that induces the distribution G over posterior means from prior distribution F if and only if $G \in \mathsf{MPC}(F)$.*

With Lemma 2.1, we can without loss of generality assume that each sender i's strategy is to directly choose a distribution $G_i \in \Delta([0,1])$ that satisfies $G_i \in \mathsf{MPC}(F_i)$, without the need to concern the design of the underlying signaling scheme $\{\pi_i(\sigma \mid v), \Sigma_i\}$. In the following discussion, we directly refer to G_i as sender i's action or information strategy.

Solution concept. The timing of our competitive information design game is as follows: First, each sender i simultaneously designs an information strategy (a.k.a., a signaling scheme) $G_i \in \mathsf{MPC}(F_i)$, and each sender i realizes proposal value $x_i \sim G_i$. Second, the receiver observes all realized proposal values $(x_i)_{i\in[N]}$, and chooses the sender that has the maximum proposal value among all senders. When the receiver is indifferent between multiple senders, she chooses one of them uniformly at random.

Notice that each sender i's action space is $\mathsf{MPC}(F_i)$. A mixed strategy specifies a probability measure over $\mathsf{MPC}(F_i)$, which can also be viewed as a convex combination over the action space. Since $\mathsf{MPC}(F_i)$ is also a compact convex set, there exists a pure strategy for each mixed strategy that preserves the expected payoffs of all senders. Therefore, we restrict our attention to the pure-strategy Nash equilibrium (henceforth referred to as equilibrium) of the game described above.

Given a strategy profile $(G_1, \ldots, G_N)$, we denote by $G \triangleq \times_{i=1}^N G_i$, and we use the symbol $-i$ denote "all senders but i", e.g., $G_{-i} \triangleq \times_{j\neq i} G_j$. Given a strategy profile (G_i, G_{-i}), we use $u_i(x) \in [0, 1]$ to denote the sender i's interim expected utility when he realizes the value $x \sim G_i$, i.e., the probability of the receiver choosing the sender i with observing x from sender i. In below discussion, we refer to $u_i(\cdot)$ as the sender i's interim utility function. With this definition, sender i's expected utility can be defined as follows:

$$U_i(G_i, G_{-i}) \triangleq \int_0^1 u_i(x) \, \mathrm{d}G_i(x) \ .$$

An information strategy $\widetilde{G}_i$ is a *best response* to $\widetilde{G}_{-i}$ if it maximizes sender i's expected payoff given that the other senders are using the information strategy $\widetilde{G}_{-i}$. Namely, it satisfies

$$\widetilde{G}_i \in \arg \max_{G_i \in \mathsf{MPC}(F_i)} U_i(G_i, \widetilde{G}_{-i}) \ . \tag{$\mathcal{P}_{\mathsf{BR}}$}$$

A strategy profile $\widetilde{G} = (\widetilde{G}_1, \ldots, \widetilde{G}_N)$ is an equilibrium if and only if for all i, strategy $\widetilde{G}_i$ is a best response to $\widetilde{G}_{-i}$.

3 The Equilibrium Existence

As mentioned in the introduction (Section 1), the *ex-ante* heterogeneity among senders' prior distributions may prevent the existence of equilibrium in the senders' information competition game. We formalize this observation by considering the following simple example:

Example 3.1 (No Equilibrium with asymmetric prior distributions)**.** *Consider a two-sender competitive information design game, the respective priors are defined as below,*

$$F_1(x) = x \ (0 \le x \le 1), \quad F_2(x) = \begin{cases} 0, & \textit{if } 0 \le x < \frac{1}{2} \\ 1, & \textit{if } \frac{1}{2} \le x \le 1 \end{cases} .$$

In the above example, the sender 2's prior F_2 is discontinuous over $[0, 1]$ as it concentrates all the mass at the value $1/2$. Thus, the action space of the sender 2 is $\mathsf{MPC}(F_2) = \{F_2\}$, which only contains the prior distribution itself. In the meantime, the sender 1's prior F_1 is a uniform distribution over the value interval $[0, 1]$ which has the same expected value as the sender 2's prior, and thus, he has a much richer action space $\mathsf{MPC}(F_1)$ that includes all feasible MPCs of the sender 1's prior. We can see that given sender 2's strategy F_2, there exists no best response for sender 1. To see this, we notice that the sender 1 can always be better off deviating to a strategy that puts a bigger mass (less than 1) on the point $1/2 + \epsilon$. The smaller $\epsilon > 0$ is, the higher the utility of sender 1 is. Thus, there exists no equilibrium in the above two-sender game.

Our main result in this section is the below Theorem 3.1 that provides sufficient conditions under which an equilibrium exists for our asymmetric competitive information design game with an arbitrary number of senders.

Theorem 3.1. *There exists a Nash equilibrium in our asymmetric competitive information design game, if each sender i's prior distribution F_i is continuous over $[0, 1]$ and $F_i(0) = 0$.*

The point mass in the sender 2's prior distribution in Example 3.1 essentially constrains his ability to generate a feasible MPC that could spread the distribution around the mean value.The above Theorem 3.1, although it requires $F_i(0) = 0$ for all i, it guarantees that an equilibrium always exists if the prior distributions are all continuous over $[0, 1]$.

Previous works (Hwang et al., 2019; Au and Kawai, 2020) have shown that a symmetric Nash equilibrium always exists when all senders are ex-ante symmetric, namely, they share the same prior distribution F. Their analysis requires that the common prior F has positive, bounded, and continuously differentiable density over $[0, 1]$ (Hwang et al., 2019). Our Theorem 3.1 strictly generalizes the equilibrium existence conditions in two dimensions: (1) we require milder assumptions on senders' prior distributions (even when restricting to symmetric prior distributions); and (2) we allow the asymmetry among senders' prior distributions.

3.1 Proof Challenges and Our Solution Ideas

In this subsection, we provide the proof of Theorem 3.1. We start by highlighting key challenges in proving Theorem 3.1, and then provide a proof sketch.

Challenges in characterizing the equilibrium existence. There are two main challenges in characterizing the equilibrium existence for the studied asymmetric competitive information design game: (1) the considered game is an infinite game, in particular, each player (i.e., the sender) has an infinite number of actions; and (2) each sender's action space is asymmetric due to the asymmetry of priors. If the considered game had a finite number of players with each having a finite action space, then the existence of a Nash equilibrium would follow immediately from Nash's Theorem (Nash, 1950). As we mentioned before, in the studied game, each sender i's action space is the set including all the MPCs of his prior F_i, that is $\mathsf{MPC}(F_i)$, which has infinitely many actions, making it impossible to directly apply Nash's Theorem to show the existence of equilibrium. We would like to note that this challenge is not unique in our problem, it also appears in previous works on characterizing equilibrium existence in the environment with symmetric priors. The focus of the previous works (Hwang et al., 2019; Au and Kawai, 2020) is on the symmetric equilibrium, which can be directly constructed. However, each sender in our setting has different action spaces (due to different priors), which implies that each sender's best response problem is unique, and it is elusive whether there exists a simple and general approach to construct each sender's equilibrium strategy. Meanwhile, as shown in the previous example, equilibrium may even not exist due to the ex-ante asymmetry between senders if considering arbitrary priors.

Key proof ideas. The key idea in our proof is on *discrete approximation* and *passing to the limit*. In particular, we construct a sequence of discrete games in which each sender has a finite action space to approximate the continuous game. We would like to emphasize that, our constructed games are not directly based on discretizing each sender i's original action space $\mathsf{MPC}(F_i)$. Indeed, how to discretize the space $\mathsf{MPC}(F_i)$ is elusive as it is an infinite-dimensional function space. Instead, we carefully construct discrete games by directly specifying a finite action space for every sender and this action space comes as close as possible to the original continuous action space when the discrete game approaches the continuous game. In each of these discrete games, we are able to leverage Nash's Theorem to show the existence of an equilibrium. We then show that by our construction, the sequence of the equilibrium in these discrete finite games converges to a certain strategy profile which is indeed an equilibrium of the considered competitive information design game with asymmetric senders.

We summarize our proof in the following steps:

- In step 1, we carefully construct a sequence of discrete games (see Definition 3.2) where in each game (parameterized by a discretization granularity $m \in \mathbb{Z}^+$), we construct a finite action space for each sender (see Lemma 3.2). This allows us to directly apply Nash's Theorem to show the equilibrium existence for this constructed finite game, which gives us a sequence of discrete equilibrium strategy profile $\{(\widetilde{G}_1^m, \ldots, \widetilde{G}_N^m)\}_{m\in\mathbb{Z}^+}$.

- In step 2, we show that even though each sender i's equilibrium strategy $\widetilde{G}_i^m$ in the discrete game may not be necessarily a feasible MPC of the prior F_i, the sequence of the discrete equilibrium strategy $\{\widetilde{G}_i^m\}_{m\in\mathbb{Z}^+}$, has a subsequence that weakly converges to a certain distribution $\widetilde{G}_i$ which is a feasible distribution of posterior means of sender i, namely $\widetilde{G}_i \in \mathsf{MPC}(F_i)$ (see Lemma 3.3).

- In the last step, we prove that the strategy profile $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$ in limits is indeed an equilibrium of the competitive information design game. We prove this by first showing that there exists no discontinuity in all senders' payoff with respect to realized proposal value (see Lemma 3.4). Thus, we are able to establish the convergence of utility function, and obtain the desired existence of the equilibrium for our competitive information design game (see Lemma 3.5).

We would like to also note that at a high-level, a big chunk of our proof (constructing the m-discrete-approximation game in Definition 3.2 and Lemmas 3.3 to 3.5) are sort of proving the property of "better-reply security" in our game. Thus, Reny's framework offers limited help, as our Theorem 3.1 being straightforward from these lemmas.

3.2 Proof Sketch of Theorem 3.1

In this section, we present detailed steps in proving Theorem 3.1, all missing proofs of other main and technical results are given in Appendix A.

Step 1 – Construct the m-discrete approximation game. According to Kleiner et al. (2021), we know that each sender i's action space $\mathsf{MPC}(F_i)$ is a compact and convex set that contains all feasible distributions of posterior means. By Krein-Milman Theorem, we know that any convex and compact set is the closed and convex hull of its extreme points. Moreover, for any convex set, each point within it can be represented as a convex combination of its extreme points. Thus, each sender i's action space $\mathsf{MPC}(F_i)$ can be equivalently characterized by the extreme points of this convex set $\mathsf{MPC}(F_i)$. In other words, for each sender i, when considering the extreme points of $\mathsf{MPC}(F_i)$ as the action space, any MPC can be seen as a mixed strategy over this set of actions. Recall that from Definition 2.1, the set $\mathsf{MPC}(F_i)$ is derived from an infinite number of linear constraints, implying that the convex space $\mathsf{MPC}(F_i)$ has an infinite number of extreme points, which are also infinite-dimensional objects. Thus, it is difficult to construct discretized action space from $\mathsf{MPC}(F_i)$. In our proof, we circumvent this difficulty by directly constructing a sequence of discrete games in which all senders in each discrete game have a finite action space. We formally define our discretization scheme, parameterized by $m \in \mathbb{Z}^+$, and refer to the constructed discrete game as the m-discrete approximation game.

Definition 3.2 (m-Discrete Approximation Game)**.** *Fix any $m \in \mathbb{Z}^+$, let $\mathsf{P}^m = \{0, \frac{1}{2^m}, \frac{2}{2^m}, \ldots, 1\}$ be a discretized support set. We define the m-discrete approximation game with N senders and a single receiver as follows:*

- *Strategy space (action space): For each sender $i \in [N]$, we define the sender's strategy space (including both pure strategy and mixed strategy) as the set S_i^m that contains all discrete distributions $(p_0, \ldots, p_{2^m})$ supported on the set P^m where p_i denote the probability mass on the point $\frac{i}{2^m}$. and these discrete distributions satisfy the following* *constraints:*

$$\int_0^1 F_i(x)\ \mathrm{d}x \le \sum_{j=0}^{2^m} p_j \cdot (2^m - j) \cdot 2^{-m} + 2^{-m} \tag{2}$$

$$\int_0^{t\cdot 2^{-m}} F_i(x)\ \mathrm{d}x \ge \sum_{j=0}^{t} p_j \cdot (t - j) \cdot 2^{-m} \qquad \forall t \in \{0\} \cup [2^m] \tag{3}$$

It is easy to see that the set S_i^m is a compact and convex set. Each sender i's action space, denoted by A_i^m, is specified by the set that contains all extreme points of S_i^m. [1]

- *Player's payoff: The payoff structure is the same as the payoff structure in the competitive information design game. In particular, each sender aims to maximize the probability that the receiver chooses his proposal. The receiver is an expected-utility-maximizer, and given a strategy profile $(G_1^m, \ldots, G_N^m)$ where each $G_i^m \in \mathsf{A}_i^m$, the receiver selects sender i with probability one if the realization of G_i^m is the unique maximizer among all senders; and with probability $1/k$ if the realization of G_i^m is one of the k maximizers among all senders.*[2]

In the m-discrete approximation game defined above, for each sender i, every strategy $G_i^m \in \mathsf{S}_i^m$ is a discrete distribution supported on $2^m + 1$ discrete points in the set P^m. Constraint (3) relaxes the MPC second-order stochastic dominance condition such that it requires the Inequality (1) holds only on points in P^m. Together with the Constraint (2), strategy G_i^m is ensured that its mean is close to the prior mean.

Remark 3.3. *We note that for any finite $m \in \mathbb{Z}^+$, in the m-discrete approximation game defined above, each sender i's action (resp. strategy) space is explicitly given by A_i^m (resp. S_i^m), which is induced by the above four constraints and the prior F_i. It is worth noting that distributions in the set S_i^m (and also in the set A_i^m) are not necessarily an MPC of the prior F_i.*

In the following, we show that the set S_i^m is non-empty, and moreover the set A_i^m is finite.

Lemma 3.2. *Fix any $m \in \mathbb{Z}^+$, in the m-discrete approximation game, each sender i has a non-empty strategy set S_i^m, and a finite action space A_i^m.*

Step 2 – Proving the limit strategy $\widetilde{G}_i \in \mathsf{MPC}(F_i)$. According to Lemma 3.2, we know that any m-discrete approximation game is a finite game where each sender i has a finite

[1]Again, we shall observe that for each mixed strategy in S_i^m, there exists a pure strategy in A_i^m that preserves the expected payoffs of all players.

[2]Here we adopt a uniformly random tie-breaking rule for convenience of definition.

action space A_i^m. Thus, we can directly apply Nash's Theorem to show that there exists an equilibrium $(\widetilde{G}_1^m, \ldots, \widetilde{G}_N^m)$ for the constructed m-discrete approximation game, where for each sender i, $\widetilde{G}_i^m \in \mathsf{S}_i^m$ is a (mixed) equilibrium strategy. Moreover, all the discrete equilibrium strategies of sender i also form a sequence of CDFs, $\{\widetilde{G}_i^m\}_{m\in\mathbb{Z}^+}$. We then first argue that, by Helly's Selection Theorem (for completeness, see Lemma A.1), this sequence contains a subsequence that weakly converges to a certain CDF. We then show that even though the discrete equilibrium strategy $\widetilde{G}_i^m$ may not be necessarily an MPC of sender i's prior F_i, the limit of the subsequence is indeed an MPC of sender i's prior. We summarize the main results of this step in the following Lemma 3.3.

Lemma 3.3. *For each sender $i \in [N]$, there exists a subsequence $\{m_i(k)\}_{k\in\mathbb{Z}^+} \subseteq \mathbb{Z}^+$ such that $\{\widetilde{G}_i^{m_i(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to a certain CDF $\widetilde{G}_i$, and the CDF $\widetilde{G}_i$ is an MPC of sender i's prior F_i.*

Step 3 – Proving $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$ is indeed a Nash equilibrium. To summarize, we have obtained a sequence of equilibrium strategy profiles $\{(\widetilde{G}_1^m, \ldots, \widetilde{G}_N^m)\}_{m\in\mathbb{Z}^+}$ for a series of constructed finite games by utilizing Nash's Theorem. This sequence contains a subsequence weakly converging to a certain strategy profile $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$, and according to Lemma 3.3, for each sender i, the strategy $\widetilde{G}_i$ is also an MPC of his prior F_i. For simplicity, we abuse the notations and rename the sequence $\{\widetilde{G}_i^{m(k)}\}_{k\in\mathbb{Z}^+}$ as $\{\widetilde{G}_i^m\}_{m\in\mathbb{Z}^+}$ hereafter. We next show that $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$ is indeed one equilibrium of our competitive information design game. We prove this by showing that each sender's utility function, along with the convergence of discrete equilibrium strategies, also converges as the granularity of discretization becomes finer, namely $\lim_{m\to\infty} U_i(\widetilde{G}_i^m, \widetilde{G}_{-i}^m) = U_i(\widetilde{G}_i, \widetilde{G}_{-i})$. However, a caveat here is that if there exist ties in $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$ with strictly positive probability, this would lead to discontinuities in senders' utility functions. Then it is impossible for the utility function to converge along with the convergence of equilibrium strategies. Thus, we need to first prove that, starting from a certain point, there is no tie occurring with strictly positive probability in the strategy profile $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$ (see Lemma 3.4).

We now clarify the definition of "a certain point". Although all feasible distributions of posterior means are all defined on the interval $[0, 1]$, not all the support of the sender's information strategy yield positive utility. Drawing on the concept of the smallest winning bid in the game of first-price auction, we define the *smallest winning value* in our competitive information design game as follows, which will be both useful in the later proof and our equilibrium characterizations in Section 4 and Section 5.1:

Definition 3.4 (Smallest Winning Value)**.** *For any feasible strategy profile $(G_1, \ldots, G_N)$,* *we define the smallest winning value as:*[3] $\underline{x} \triangleq \max_{i\in[N]} \inf \mathsf{supp}(G_i)$.

Note that each sender has zero utility for realizing a proposal value below the value $\underline{x}$. Let smallest winning value of $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$ be $\underline{x}$ and then we have:

Lemma 3.4 (No Ties Occurring in Converging Strategy Profile)**.** *There exists no tie in $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$ at and above $\underline{x}$ if each sender i's prior distribution F_i is continuous over $[0, 1]$ and $F_i(0) = 0$. In other words, there does not exist a point $b \in [\underline{x}, 1]$ and two distinct senders $i, j \in [N]$ with $\widetilde{G}_i, \widetilde{G}_j$ that they simultaneously assign a positive mass at point b.*

We prove Lemma 3.4 by contradiction. The intuition behind the proof is as follows: Without loss of generality, we assume that among N senders, the converging strategies of sender 1 and 2, $\widetilde{G}_1$ and $\widetilde{G}_2$, simultaneously assign the masses p_1 and p_2 respectively, at the point $b > \underline{x}$. In the process of discrete games approximating the continuous game, the probabilities accumulated within the neighborhood of point b in the discrete equilibrium strategies of sender 1 and 2 will converge to p_1 and p_2, respectively. With this observation, we then show that for sufficiently large m, either sender 1 or sender 2 has a profitable deviation with higher utility, thereby contradicting the fact that $\widetilde{G}_1^m$ and $\widetilde{G}_2^m$ are both equilibrium strategies.

With the above Lemma 3.4, we are now ready to show that the limit strategy profile, $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$ indeed forms an equilibrium in our competitive information design game.

Lemma 3.5. *The limit strategy profile $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$ is indeed an equilibrium in the competitive information design game.*

Putting all the pieces together, we can now prove Theorem 3.1.

Proof of Theorem 3.1. Theorem 3.1 follows immediately by combining Lemma 3.2, Lemma 3.3, Lemma 3.4 and Lemma 3.5. ■

4 Characterizing the Equilibrium Structure

In this section, we provide structure characterizations of a Nash equilibrium in our competitive information design game. In particular, we first provide necessary and sufficient conditions for any feasible strategy profile that is indeed an equilibrium (see Theorem 4.2). These conditions can be used to verify whether a given strategy profile is an equilibrium. We then construct an example to show that the equilibrium in general may not be unique and that the expected utility of the same sender may vary under different equilibria (see

[3]Hereafter, we use $\mathsf{supp}(G)$ to represent the support of function G.

Claim 4.6). To better illustrate our idea and approach, in this and next section, we focus on a setting where all senders' prior distributions have full support over $[0,1]$, i.e., the prior F_i for all i is strictly increasing over $[0,1]$. As we show in Theorem 3.1, equilibrium always exists in this case.

Recall that a strategy profile $(G_1, ..., G_N)$ is an equilibrium if and only if, each sender's strategy G_i itself is a solution to his best-response problem (see program $\mathcal{P}_{\mathsf{BR}}$). Understanding each sender's best response strategy is a technically challenging problem as the corresponding program $\mathcal{P}_{\mathsf{BR}}$ is an infinite-dimensional linear program and the sender's action space consists of all MPCs of his prior. Nevertheless, the following recent technical developments obtained by Dworczak and Martini (2019) are helpful to understand each sender's best response problem. In particular, Dworczak and Martini (2019) provides a method to verify the optimality of a feasible MPC for optimizing a certain objective function.

Theorem 4.1 (Dworczak and Martini, 2019)**.** *Suppose that F is the sender's prior distribution with $\mathsf{supp}(F) = [\underline{v}, \overline{v}]$. Consider the following optimization problem*

$$\max_{G \in \mathsf{MPC}(F)} \int_{\underline{v}}^{\overline{v}} u(x)\ \mathrm{d}G(x)\ . \tag{$\mathcal{P}_{\mathsf{OPT}}$}$$

A distribution $G \in \mathsf{MPC}(F)$ is a solution to the above problem if there exists an auxiliary function $\phi : [\underline{v}, \overline{v}] \to \mathbb{R}$ such that (i) $\phi(x)$ is convex over $[\underline{v}, \overline{v}]$; (ii) $\phi(x) \geq u(x)$ for all $x \in [\underline{v}, \overline{v}]$; (iii) $\mathsf{supp}(G) \subseteq \{x \in [\underline{v}, \overline{v}] : u(x) = \phi(x)\}$, and (iv) $\int_{\underline{v}}^{\overline{v}} \phi(x)\ \mathrm{d}G(x) = \int_{\underline{v}}^{\overline{v}} \phi(x)\ \mathrm{d}F(x)$.

Intuitively, the above results say that for the considered optimization problem, given a candidate solution $G \in \mathsf{MPC}(F)$, if one can find an auxiliary function ϕ that satisfies all the listed conditions, then this candidate solution is indeed optimal.

However, it is not straightforward to directly apply Theorem 4.1 to solve the sender's best response in our problem. The main technique challenge to apply Theorem 4.1 lies in the unclear construction of the auxiliary function ϕ. From Theorem 4.1, we can see that given a strategy profile $(G_1, ..., G_N)$, if, for all i, one could construct a function $\phi_i(x)$ that satisfies all listed conditions for each sender i's interim utility function $u_i(x)$, then this strategy profile is indeed an equilibrium. However, Theorem 4.1 does not explicitly provide how to construct function ϕ_i for each sender i. We remark that this challenge arises uniquely in our setting with ex-ante prior heterogeneity. In a symmetric setting, if one considers a symmetric strategy profile, then each sender has the same interim utility function, and thus, the construction of the auxiliary function becomes relatively easier. In our setting, the interim utility function varies if each sender uses different strategies, and we need construct the auxiliary function for each sender.

The main results in this section are as follows: though function $G_{-i}(x)$ may not always equal to sender i's winning probability, we find that we can still be able to explicitly construct the convex function ϕ_i for each sender i based on the given strategy profile $(G_1, \ldots, G_N)$ (see Definition 4.1). Moreover, we provide a set of necessary and sufficient conditions that can verify if a strategy profile $(G_1, \ldots, G_N)$ is indeed an equilibrium by verifying whether the constructed functions $(\phi_i)_{i\in[N]}$ satisfy certain desired conditions (see Theorem 4.2). The core concept of our characterization is the following introduced sender-specific "virtual competitive function":

Definition 4.1 (Virtual Competitive Function)**.** *Given any strategy profile $(G_1, \ldots, G_N)$, we introduce the virtual competitive function $\phi_i : [0, 1] \to \mathbb{R}$ for each sender $i \in [N]$. There exists a partition $\{v_1 = 0, \ldots, v_m = 1\}$[4] such that for every $k \in [m-1]$, v_k is a local MPC boundary point[5] of strategy G_i to prior F_i, and moreover one of the three cases below holds true between v_k and v_{k+1}:*

- ***Case 1:*** *When $\int_0^y G_i(t)\ \mathrm{d}t = \int_0^y F_i(t)\ \mathrm{d}t, \forall y \in [v_k, v_{k+1}]$, we define*

$$\phi_i(x) = G_{-i}(x), \quad \forall x \in [v_k, v_{k+1})\ .$$

- ***Case 2:*** *When $\int_0^y F_i(t)\ \mathrm{d}t > \int_0^y G_i(t)\ \mathrm{d}t, \forall y \in (v_k, v_{k+1})$, and there exist two points $c, d \in \mathsf{supp}(G_i) \cap (v_k, v_{k+1})$ where $c < d$, we define*

$$\phi_i(x) = \frac{G_{-i}(d) - G_{-i}(c)}{d - c}(x - c) + G_{-i}(c), \quad \forall x \in [v_k, v_{k+1})\ .$$

- ***Case 3:*** *When $\int_0^y F_i(t)\ \mathrm{d}t > \int_0^y G_i(t)\ \mathrm{d}t, \forall y \in (v_k, v_{k+1})$, and the set $\mathsf{supp}(G_i) \cap (v_k, v_{k+1})$ contains only one element, denoted as c, we define*

$$\phi_i(x) = G_{-i}(c) \quad \forall x \in [v_k, v_{k+1})\ .$$

For the completeness of definition, we define $\phi_i(1) = \lim_{x\to 1^-} \phi_i(x)$.[6]

With the above definition, we are ready to provide our main results in this section:

[4]The partition may contain infinite points.

[5]We call $a \in [0, 1]$ a local MPC boundary point of distribution G to prior F if (i) $\int_0^a G(x)\ \mathrm{d}x = \int_0^a F(x)\ \mathrm{d}x$, and (ii) $\forall \epsilon > 0$, there exists $\delta \in (0, \epsilon)$ such that $G(a - \delta) \neq F(a - \delta)$ or $G(a + \delta) \neq F(a + \delta)$.

[6]We note that for every (v_k, v_{k+1}), there exists at least one point $t \in \mathsf{supp}(G_i) \cap (v_k, v_{k+1})$, otherwise, we have $\mathsf{supp}(G_i) \cap (v_k, v_{k+1}) = \emptyset$, then $G_i(v_k) = G_i(v_{k+1}^-)$. Since $\int_0^{v_k} F_i(t)\ \mathrm{d}t = \int_0^{v_k} G_i(t)\ \mathrm{d}t$, we have $G_i(v_k) = F_i(v_k)$ by Lemma B.1. Then, we have $\int_0^{v_{k+1}} F_i(t)\ \mathrm{d}t > \int_0^{v_k} G_i(t)\ \mathrm{d}t + G_i(v_k) \cdot (v_{k+1} - v_k) = \int_0^{v_{k+1}} G_i(t)\ \mathrm{d}t = \int_0^{v_{k+1}} F_i(t)\ \mathrm{d}t$, a contradiction. Second, we note that if we choose different points $a, d \in \mathsf{supp}(G_i) \cap [v_k, v_{k+1}]$ for Case 2, then the construction of the function $\phi_i(x)$ will also be different for $x \in [v_k, v_k + 1]$.

Theorem 4.2. *A strategy profile G is an equilibrium if and only if for each sender $i \in [N]$*

(i) the function $\phi_i(x) \geq G_{-i}(x)$ for any $x \in [0,1]$,

(ii) the function ϕ_i is convex over $[0,1]$,

(iii) there exists no $x \in [0,1]$ at which G_i and G_{-i} are both discontinuous.

where function ϕ_i for all i is constructed as in Definition 4.1.

Remark 4.2. *We note that above Theorem 4.2 and Definition 4.1 provide sufficient conditions to verify if a given strategy profile $(G_1, \ldots, G_N)$ is indeed an equilibrium or not. In particular, one can explicitly construct the virtual competitive function for each sender i based on other senders' information strategies and his own prior F_i, and then it suffices to verify if all listed conditions in Theorem 4.2 hold for all senders' virtual competitive function.*

4.1 Intuitions and Implications of Theorem 4.2

In this subsection, we provide intuitions and implications behind the Definition 4.1 and Theorem 4.2. We first provide intuitions on why, given a strategy profile $(G_1, \ldots, G_N)$, the strategy G_i is a best response strategy against all other senders' information strategies if there exists a virtual competitive function ϕ_i satisfying all conditions in Theorem 4.2 for sender i.

G_i "best responding" to virtual competitive function ϕ_i. We first provide intuitions that if there exists a virtual competitive function ϕ_i satisfying all conditions in Theorem 4.2, then sender i's strategy G_i is "best responding" to the virtual competitive function ϕ_i. Here by "best responding" we mean that if one replaces the function $u(x)$ with the function $\phi_i(x)$ in the program $\mathcal{P}_{\mathsf{OPT}}$, then strategy G_i is indeed the optimal solution. To see this, notice that if strategy G_i is the optimal solution w.r.t. the function $\phi_i(x)$ in program $\mathcal{P}_{\mathsf{OPT}}$, then sender i neither benefits from disclosing more information, i.e., spreading the induced posterior; nor disclosing less information, i.e., contracting the induced posterior. Indeed, by defining the constructed function ϕ_i to be either strictly convex or linear over any interval of $\mathsf{supp}(G_i)$, Theorem 4.2 guarantees that, there is no incentive for sender i to contract or disperse either within any interval or between any two intervals of $\mathsf{supp}(G_i)$ from current strategy G_i.

Best responding to function ϕ_i implies best responding to G_{-i}. We now explain why a strategy G_i is best responding to G_{-i} if G_i is "best responding" to his corresponding virtual competitive function ϕ_i. We note that by construction in Definition 4.1, we always have $\phi_i(x) \geq G_{-i}(x)$ for any $x \in [0,1]$ and $\phi_i(x) = G_{-i}(x)$ for any $x \in \mathsf{supp}(G_i)$, which implies that the expected utility when sender i adopts any feasible strategy against ϕ_i, is at least the expected utility when sender i adopts the same strategy against G_{-i}. Due to the fact ϕ_i differs with G_{-i} only within $[0,1] \setminus \mathsf{supp}(G_i)$, in which G_i has zero probability and contributes zero expected utility to sender i. This implies that the expected utility, when sender i adopts G_i against ϕ_i, exactly equals the expected utility when sender i adopts G_i against G_{-i}.

Best responding to function G_{-i} implies best responding to u_i. Recall that function $u_i(x)$ represents the sender i's winning probability when he realizes a reward with interim value $x \sim G_i$, then as we mentioned earlier, $u_i(x)$ does not necessarily equal to $G_{-i}(x)$ at all interim values. However, we can show that functions u_i and G_{-i} are "almost" identical, except at points where G_{-i} is discontinuous. If G_i also has a mass at some point a at which G_{-i} is discontinuous (i.e., there exists a sender $j \neq i$ whose strategy G_j has a mass at a), then the utility induced by the mass at a, that is $u_i(a)$, is lower or equal to $G_{-i}(a)$. Therefore we have $G_{-i}(x) \geq u_i(x)$ for any $x \in [0,1]$ and $G_{-i}(x) = u_i(x)$ for points at which G_{-i} is continuous, which implies that the expected utility when sender i adopts any feasible strategy against G_{-i}, is at least the expected utility when sender i adopts the same strategy against u_i. By Theorem 4.2, we know that the function G_i must be continuous at the points where the function G_{-i} is discontinuous. Therefore, the expected utility when sender i adopts G_i against G_{-i} equals the expected utility when sender i adopts the same strategy against u_i.

Additional useful properties of the equilibrium $(G_1, \ldots, G_N)$. We can also use Theorem 4.2 to obtain the following corollaries which will be helpful for our analysis in a later section to fully characterize the equilibrium for a general two-sender problem. As a feasible strategy of sender i, G_i must be composed of multiple connected local MPCs. From Theorem 4.2, we know that points where the left and right derivatives of the function ϕ_i strictly differ are either local MPC boundary points or belong to an interval where G_i coincides with his prior F_i everywhere. In particular, we have the following implications:

Corollary 4.3. *Given an equilibrium $(G_1, \ldots, G_N)$, for each sender $i \in [N]$, if there exists $a \in [0,1]$ such that $\dot{\phi}_i(a^-) \neq \dot{\phi}_i(a^+)$ or $\ddot{\phi}_i(a) > 0$, then we have $\int_0^a G_i(x)\,\mathrm{d}x = \int_0^a F_i(x)\,\mathrm{d}x$.*[7]

The following result states that the virtual competitive function $\phi_i(x)$ and the function $G_{-i}(x)$ must be equal in support of the equilibrium strategy.

Corollary 4.4. *Given an equilibrium $(G_1, \ldots, G_N)$, for each sender $i \in [N]$, we have $\phi_i(x) = G_{-i}(x)$ for all $x \in \mathsf{supp}(G_i)$.*

We find that each sender's strategy G_i does not have a mass at x when x is large enough. Before providing a detailed statement of this conclusion, we first specify two key definitions.

[7]Hereafter, we use $\dot{F}$ to denote the first derivative of function F, and $\ddot{F}$ to denote the second derivative of function F.

Definition 4.3 (Maximum Winning Value and Continuity Threshold)**.** *For any feasible strategy profile G, we define the maximum wining value as $\overline{x} \triangleq \max_{i\in[N]} \sup \mathsf{supp}(G_i)$. We also define the continuity threshold as*

$$\tau \triangleq \inf\{x \in [\underline{x}, \overline{x}] : \exists i \neq j \ s.t. \ G_i(x) > G_i(\underline{x}) \ and \ G_j(x) > G_j(\underline{x})\} \ .$$

Corollary 4.5. *Given an equilibrium $(G_1, \ldots, G_N)$, each sender i's strategy G_i has no mass in $(\tau, 1]$.*

Non-uniqueness and discontinuous support of Nash equilibrium. We conclude this subsection by showing that in our considered game, there may exist multiple equilibria, and also, the support of the equilibrium strategy may be discontinuous. These two observations stand in contrast to the uniqueness of the symmetric equilibrium and the continuous support in the competitive information design game with symmetric priors.

Claim 4.6. *In the asymmetric setting, there may exist multiple equilibria and the sender's expected utility may vary among different equilibria.*

We prove Claim 4.6 by constructing Example 4.4 (see Figure 1) which admits multiple equilibria and the expected utility of the same sender vary among different equilibria.

Example 4.4 (Multiple equilibria)**.** *Consider the following three-sender competitive information design game, the respective priors being defined as follows:*

$$F_1(x) = x^3 \ (0 \leq x \leq 1), \ F_2(x) = \sqrt{2x - x^2} \ (0 \leq x \leq 1), \ F_3(x) = \sqrt{\frac{x}{2-x}} \ (0 \leq x \leq 1) \ .$$

We can construct two equilibria (G_1, G_2, G_3) and (G'_1, G_2, G_3), where $G_1(x) = x^3$ $(0 \leq x \leq 1)$, $G_2 = F_2$, $G_3 = F_3$ and

$$G'_1(x) = \begin{cases} 0, & \text{if } 0 \leq x < \frac{3}{8} \\ \frac{9}{8}x - \frac{27}{64}, & \text{if } \frac{3}{8} \leq x < \frac{3}{4} \\ x^3, & \text{if } \frac{3}{4} \leq x \leq 1 \end{cases} .$$

The expected utilities of sender 2 and 3 under these two different equilibria are:

$$U_3((G_i)_{i\in[3]}) \approx 0.2118; \ U_2((G_i)_{i\in[3]}) \approx 0.0382; \ U_3(G'_1, G_2, G_3) \approx 0.2127; \ U_2(G'_1, G_2, G_3) \approx 0.0373$$

We can see $U_3(G_1, G_2, G_3) \neq U_3(G'_1, G_2, G_3)$ and $U_2(G_1, G_2, G_3) \neq U_2(G'_1, G_2, G_3)$.

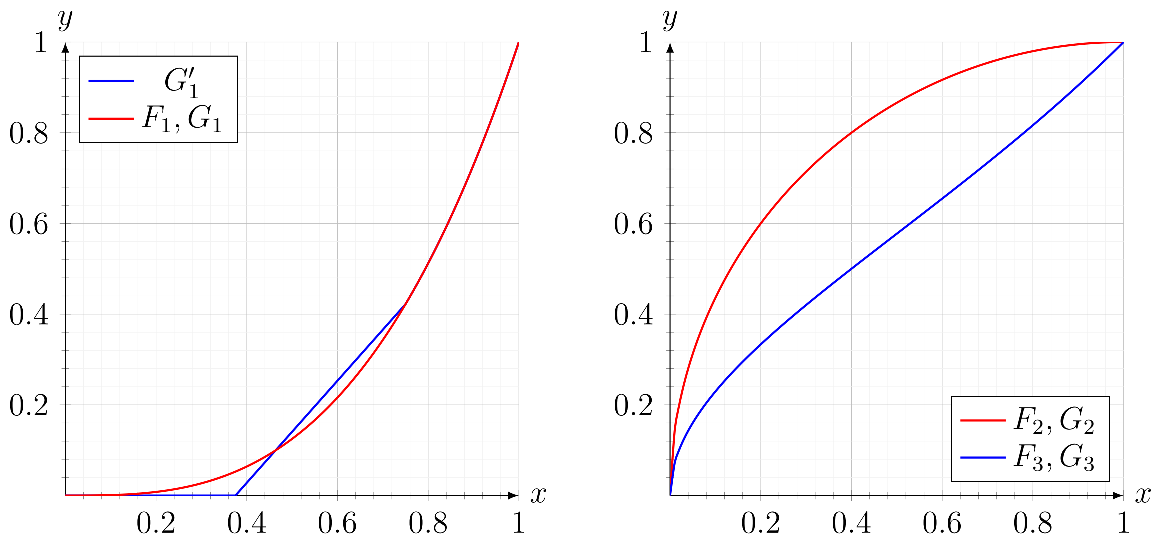

Figure 1: Multiple equilibria: profiles (G_1, G_2, G_3) and (G'_1, G_2, G_3) are two different equilibria in this game and the expected utility of sender 2 and 3 vary among these two equilibria.

Remark 4.5. *We note that in the General Blotto Game, there exists a unique equilibrium in the two-player case (see, e.g., Hart, 2008, Dziubiński, 2013; Ni et al., 2024), and the action space of players is a superset of the action space of a sender with the same prior in the competitive information design game. Therefore, the non-uniqueness of equilibrium arises from MPC constraints in a certain sense.*

We also give an example (see Example 4.6 and Figure 2) showing that in some asymmetric equilibrium, the support of a certain sender's strategy may be discontinuous, even if we assume that each sender's prior has full support over $[0, 1]$.

Example 4.6 (Equilibrium with discontinuous support)**.** *Consider the following three-sender competitive information design game, the respective priors are defined as follows:*

$$F_1(x) = \begin{cases} \sqrt{\frac{x}{3}}, & \text{if } 0 \le x < \frac{1}{3} \\ \frac{1}{3}x + \frac{2}{9}, & \text{if } \frac{1}{3} \le x < \frac{5}{6} \\ 3x - 2, & \text{if } \frac{5}{6} \le x \le 1 \end{cases}, \quad F_2(x) = F_3(x) = \begin{cases} \frac{\sqrt{3}}{2}x, & \text{if } 0 \le x < \frac{1}{3} \\ \frac{\sqrt{78}-2\sqrt{3}}{4}x + \frac{4\sqrt{3}-\sqrt{78}}{12}, & \text{if } \frac{1}{3} \le x < \frac{2}{3} \\ \sqrt{\frac{11}{8}x - \frac{3}{8}}, & \text{if } \frac{2}{3} \le x \le 1 \end{cases}.$$

We can construct an equilibrium (G_1, G_2, G_3) where $G_2 = F_2$, $G_3 = F_3$ and

$$G_1(x) = \begin{cases} \sqrt{\frac{x}{3}}, & \text{if } 0 \le x < \frac{1}{3} \\ \frac{1}{3}, & \text{if } \frac{1}{3} \le x < \frac{2}{3} \\ 2x - 1, & \text{if } \frac{2}{3} \le x \le 1 \end{cases}.$$

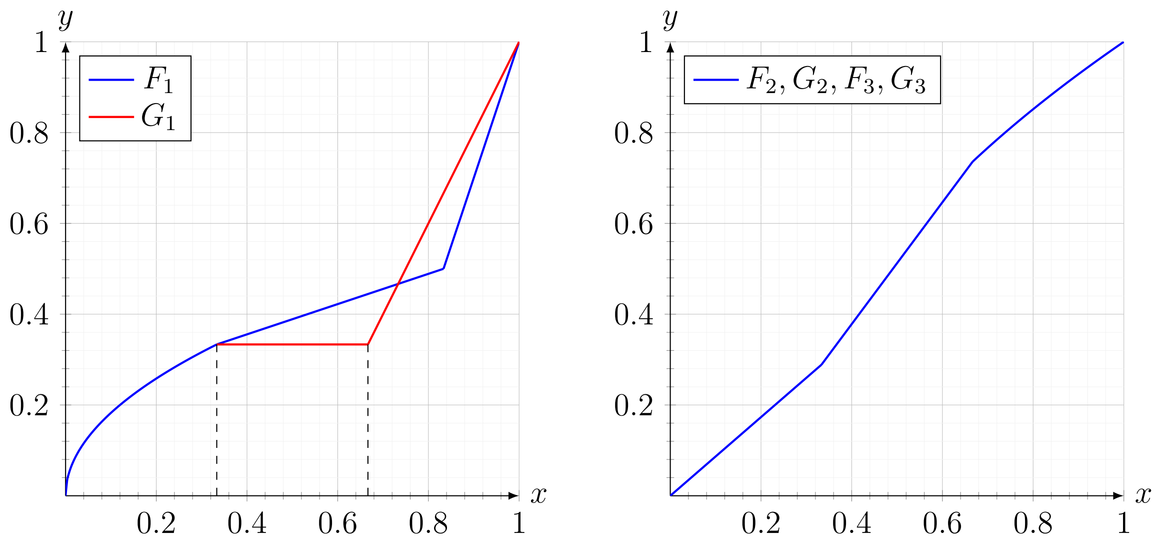

Figure 2: Equilibrium with discontinuous support: profile (G_1, G_2, G_3) is an equilibrium in which strategy G_1 has a discontinuous support $[0, {}^1\!/\!_3] \cup [{}^2\!/\!_3, 1]$.

We can see that G_1's support is discontinuous over $[0, 1]$.

5 Applications

In this section, we leverage the equilibrium structural characterizations presented in Section 4 to pin down the equilibrium in a general two-sender game. In particular, we are able to fully characterize the equilibrium when the both senders' priors exhibit strictly uni-modal structure (see Theorem 5.2). Moreover, we also present an algorithmic procedure to compute the corresponding equilibrium for the two-sender game. We further use this equilibrium structure to show that if both senders have symmetric priors (not necessarily strictly uni-modal distributions), then a unique Nash equilibrium exists, which is symmetric (eliminating the possibility of an asymmetric equilibrium, see Proposition 5.3).

For a general two-sender game with prior distributions satisfying the assumptions in Theorem 3.1, we show that there exist three points $\underline{x}, \tau$ and $\overline{x}$ (see Definition 3.4, Definition 4.3 and Definition 4.3) such that both senders' equilibrium strategies must satisfy: (1) In $[0, \underline{x}]$, either sender 1's strategy or sender 2's strategy allocates a positive probability. In $[\underline{x}, \tau]$, the other sender allocates a positive probability. (2) In $[\tau, \underline{x}]$, both senders' strategies have full support and are continuous within this interval. (3) In $[\overline{x}, 1]$, both senders' strategies allocate zero probability, which also shows that the support of both strategies ends at the same position. We provide an illustration (Figure 3) for a more intuitive understanding of this general characterization in Theorem 5.1.

Theorem 5.1. *For any two-sender game with prior distributions satisfying the assumptions in Theorem 3.1, if (G_1, G_2) is an equilibrium, then we have*

- *either* $\overline{\mathsf{supp}(G_1)} \subseteq [0, \underline{x}] \cup [\tau, \overline{x}]$ *and* $\overline{\mathsf{supp}(G_2)} \subseteq [\underline{x}, \overline{x}]$, *or* $\overline{\mathsf{supp}(G_2)} \subseteq [0, \underline{x}] \cup [\tau, \overline{x}]$ *and* $\overline{\mathsf{supp}(G_1)} \subseteq [\underline{x}, \overline{x}]$,
- $\overline{\mathsf{supp}(G_1)} \cap \overline{\mathsf{supp}(G_2)} - \{\underline{x}\} = [\tau, \overline{x}]$.

where $\underline{x}$ *is defined in Definition 3.4, and* $\overline{x}, \tau$ *are defined in Definition 4.3.*

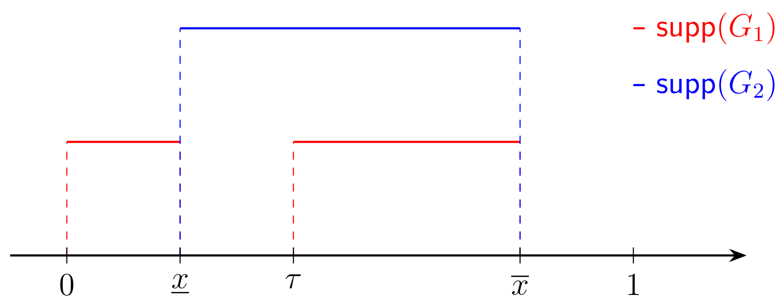

Figure 3: An example of $\mathsf{supp}(G_1), \mathsf{supp}(G_2)$ for equilibrium (G_1, G_2) in a two-sender game.

5.1 Solving the Two-Sender Game

In this subsection, we describe how to leverage Theorem 5.1 to fully characterize the equilibrium when both senders' priors F_1 and F_2 are strictly uni-modal. Before we present our main results, we first define the strictly uni-modal distributions:

Definition 5.1 (Strictly Uni-modal)**.** *A distribution* $F \in \Delta([0, 1])$ *is strictly uni-modal with parameter* $\mu \in [0, 1]$ *if its probability density function* f *is strictly increasing over* $(0, \mu)$ *and strictly decreasing over* $(\mu, 1)$.

Strictly uni-modal distributions capture many canonical distribution functions. For example, all strictly log-concave distributions (have strictly log-concave densities) are strictly uni-modal distributions. We are now ready to present the main results in this section:

Theorem 5.2. *For a two-sender game where both senders have strictly uni-modal priors* F_1, F_2*, the equilibrium strategy profile* G *must fall into one of the following two cases:*

- **Case 1 – Synchronous Conditional Uniformity (SynCU).** *There exists* $a \in [0, 1]$ *such that both the equilibrium strategy* G_1, G_2 *match their prior respectively till to point*

a, and then both exhibit conditional uniformity structure after this point. Namely, for each $i \in [2]$,

$$G_i(x) = \begin{cases} F_i(x), & \text{if } 0 \le x \le a \\ \min\left\{\dfrac{1-F_i(a)}{\overline{x}-a}(x-a)+F_i(a), 1\right\}, & \text{if } a < x \le 1 \end{cases}.$$

- **Case 2 – Asynchronous Conditional Uniformity (AsyCU).** *The profile G forms an asynchronous conditional uniformity structure defined in Definition 5.2.*

Definition 5.2 (Asynchronous Conditional Uniformity)**.** *Given a strategy profile G, let $M_i \triangleq \left\{x \in [\tau, 1] : \int_0^x F_i(t)\ \mathrm{d}t = \int_0^x G_i(t)\ \mathrm{d}t\right\}$ for each sender $i = 1, 2$ where τ is defined as in Definition 4.3, and $M \triangleq M_1 \cup M_2$. An equilibrium (G_1, G_2) exhibits Asynchronous Conditional Uniformity structure if the following conditions hold true: Let $M = \{m_1, \dots, m_w\}$ where $m_1 < \cdots < m_w = 1$, then*

- $0 < \tau = m_1$,
- $M_1 \cap M_2 = \{m_w\}$,
- $m_j \in M_1 \Leftrightarrow m_{j+1} \in M_2$ *and* $m_j \in M_2 \Leftrightarrow m_{j+1} \in M_1$, *for* $\forall j \in [w-2]$,
- *Both strategies G_1 and G_2 are linear over $[m_j, \min\{m_{j+1}, \overline{x}\}]$, for $\forall j \in [w-1]$.*

We provide graph illustrations (see Figure 4 and Figure 5) for both two cases mentioned in Theorem 5.2.

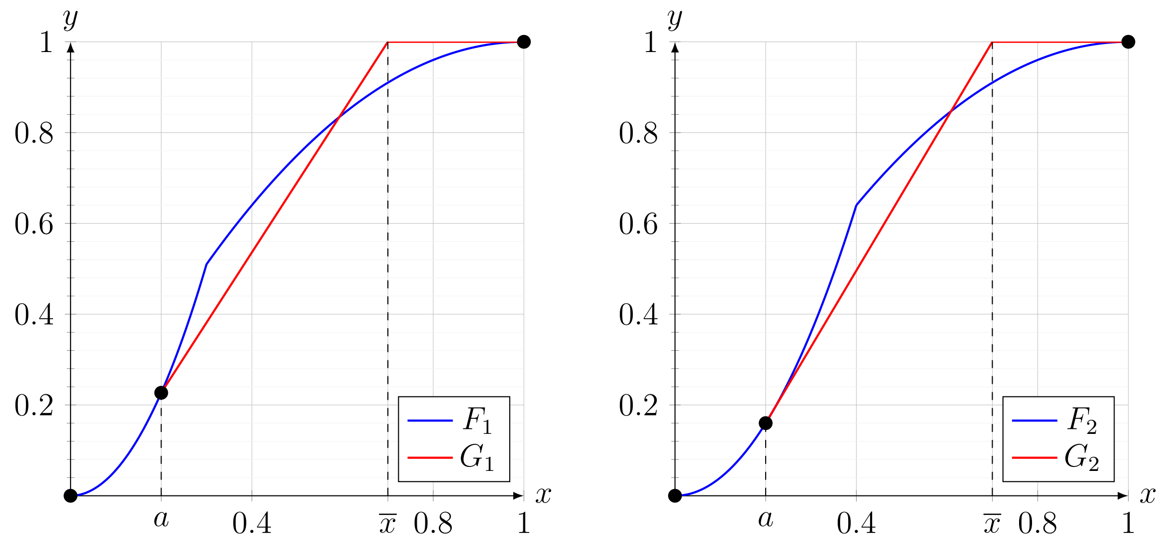

Figure 4: (Case 1 in Theorem 5.2) In this example, both senders' equilibrium strategies coincide with the respective prior up to a same point a, then follow a linear structure over $[a, 1]$. Black points represent the local MPC boundary points.

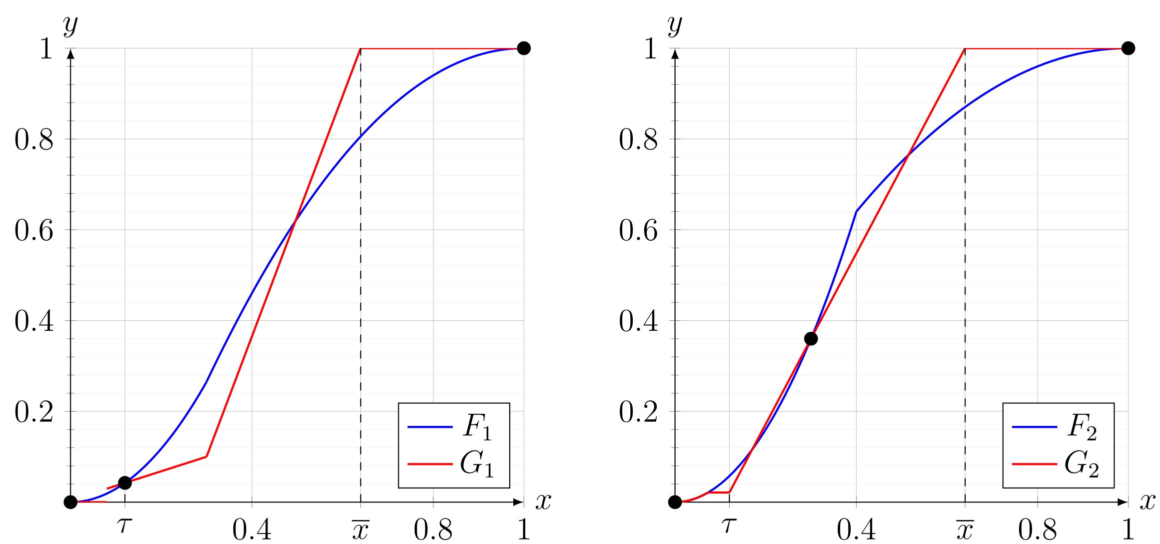

Figure 5: (Case 2 in Theorem 5.2) In this example, sender 1's equilibrium strategy G_1 has a mass point in $[0, \tau]$, and both senders' equilibrium strategies have no mass point over $[0, \tau]$. They exhibit an AsyCU structure we defined in Definition 5.2. Black points represent the local MPC boundary points.

Intuitions behind the SynCU and AsyCU structure. The SynCU and AsyCU structure stems from the competition among the two senders. Informally, at the equilibrium, each sender must ensure himself and the other sender no incentive to spread in every interval. The SynCU structure in Theorem 5.2 says that at the equilibrium, G_1 and G_2 both match their respective prior up to the point a and then they simultaneously exhibit a conditional uniformity structure over $[a, 1]$. We note that from Theorem 4.2, we know when G_1 and G_2 both match their prior, they must be both strictly convex and this can only happen within the interval $[0, \min\{\mu_1, \mu_2\}]$ (as the priors are strictly convex over $[0, \min\{\mu_1, \mu_2\}]$). The intuition behind this convex structure is that, in doing so, neither senders have incentives to contact their information strategies and they are not able to further spread as their strategy already matches the convex portion of their prior. On the other hand, the AsyCU structure says that at the equilibrium, G_1 and G_2 always exhibit a conditional uniformity structure over the interval $[\tau, 1]$, while in $[0, \tau]$, they may either have a point mass or (partially) match the respective prior. As we can see from SynCU and AsyCU structure, both the equilibrium strategy G_1 and G_2 always have a conditional uniformity structure when x is large enough. By having this conditional uniformity structure, both senders have no incentive to spread or contract, as they are both indifferent from disclosing more information and disclosing less information.

An algorithm to compute the equilibrium in Theorem 5.2. We now describe an algorithmic procedure to compute the equilibrium in Theorem 5.2. Let μ_1, μ_2 be the corresponding strictly uni-modal parameters for the prior F_1, F_2, respectively. Our algorithmic procedure utilizes the characterizations obtained in Theorem 4.2 for the general game and Theorem 5.2 for the two-sender game. In particular, Theorem 5.2 has provided a specific characterization of both equilibrium strategies on $[\tau, 1]$. It thus suffices to pin down equilibrium strategies on $[0, \tau]$. Notice that if an equilibrium belongs to Case 1, then we have $\tau = 0$ and so we need not to provide more characterizations of the equilibrium on $[0, \tau]$. If an equilibrium belongs to Case 2 in Theorem 5.2, we have $\tau > 0$ and the equilibrium strategy cannot be fully determined on $[0, \tau]$. Specifically, the algorithm is divided into the following two steps and each step involves enumerating all possibilities for parameters in each case in Theorem 5.2. If the selection of parameters is correct, then our algorithm can accurately calculate the equilibrium. Here are the details of the algorithm.

<u>Step 1. Enumerating all possibilities of Case 1 in Theorem 5.2.</u> We have only one parameter to determine, which is denoted as α. We need to enumerate all possible values of α within the range of $[0, 1]$. Suppose the parameter α is given. For each sender $i = 1, 2$, we find k_i that satisfies

$$\int_\alpha^1 F_i(x)\ \mathrm{d}x = \int_\alpha^1 \min\left\{k_i(x - \alpha) + F_i(\alpha), 1\right\}\ \mathrm{d}x\ .$$

Because both priors F_1 and F_2 have full support over $[0, 1]$, parameters k_1 and k_2 are guaranteed to have a unique solution based on the above equations. For each sender $i = 1, 2$, we construct strategies

$$G_i(x) = \begin{cases} F_i(x), & \text{if } 0 \le x \le \alpha \\ \min\left\{k_i(x-\alpha) + F_i(\alpha), 1\right\}, & \text{if } \alpha < x \le 1 \end{cases} .$$

The strategy profile (G_1, G_2) forms an equilibrium if $G_i \in \mathsf{MPC}(F_i)$ and $k_i \ge \dot{F}_i(\alpha^-)$ hold true for each sender $i = 1, 2$.

Step 2. Enumerating all possibilities of Case 2 in Theorem 5.2. In this case, we have three initial parameters to determine and two sub-cases to consider. The first parameter represents the starting position, denoted by α_1, with the parameter range being $[0, 1]$. We try the first sub-case: $\int_0^{\alpha_1} F_1(x)\ \mathrm{d}x = \int_0^{\alpha_1} G_1(x)\ \mathrm{d}x$ and $\int_0^{\alpha_1} F_2(x)\ \mathrm{d}x > \int_0^{\alpha_1} G_2(x)\ \mathrm{d}x$. Suppose the parameter α_1 is given. The second parameter β_1 represents the right derivative of strategy G_1 at point α_1, that is $\beta_1 = \dot{G}_1(\alpha_1^+)$. The third parameter γ_1 represents the value of function G_2 at point α_1, that is $\gamma_1 = G_2(\alpha_1)$, with the parameter range $[0, F_2(\alpha_1))$. For convenience, we denote the integral of function G_2 from 0 to α_1 as δ_1, that is $\delta_1 = \int_0^{F_2^{-1}(\gamma_1)} F_2(x)\ \mathrm{d}x + \left(\alpha_1 - F_2^{-1}(\gamma_1)\right)\gamma_1$. In the k-th iteration, the inputs are $\alpha_k, \beta_k, \gamma_k$ and δ_k, and the outputs are $\alpha_{k+1}, \beta_{k+1}, \gamma_{k+1}, \delta_{k+1}$ and the expressions of G_1 and G_2 over $[\alpha_k, \alpha_{k+1}]$. Suppose $\alpha_k \in M_i$, we compute

$$\beta_{k+1} = \max\left\{p \ge 0 : \delta_k + \int_{\alpha_k}^{x} \min\left\{p(t-\alpha_k) + \gamma_k, 1\right\}\ \mathrm{d}t \le \int_0^1 F_{-i}(t)\ \mathrm{d}t,\ \forall x \in [\alpha_k, 1]\right\} .$$

Then we obtain the end point of this iteration

$$\alpha_{k+1} = \min\left\{x \in (\alpha_k, 1] : \int_0^x F_{-i}(t)\ \mathrm{d}t = \int_0^x \min\left\{\beta_{k+1}(t-\alpha_k) + \gamma_k, 1\right\}\ \mathrm{d}t\right\} .$$

Note that the set above contains only one single point that is α_{k+1} because both prior distributions are strictly uni-modal. Then we set

$$\begin{aligned} G_i(x) &= \min\left\{\beta_k(x-\alpha_k) + F_i(\alpha_k), 1\right\} \quad \forall x \in [\alpha_k, \alpha_{k+1}] , \\ G_{-i}(x) &= \min\left\{\beta_{k+1}(x-\alpha_k) + \gamma_k, 1\right\} \quad \forall x \in [\alpha_k, \alpha_{k+1}] . \end{aligned}$$

Last, we update the remaining two parameters

$$\gamma_{k+1} = G_i(\alpha_{k+1}) \ ,\ \delta_{k+1} = \int_0^{\alpha_k} F_i(x)\ \mathrm{d}x + \int_{\alpha_k}^{\alpha_{k+1}} G_i(x)\ \mathrm{d}x .$$

If $\alpha_{k+1} = 1$, then we do not need to execute the next $k+1$-th iteration. If the following conditions hold true, then (G_1, G_2) constitutes an Alternating MPC equilibrium.

- $\int_0^1 F_i(x)\ \mathrm{d}x = \int_0^1 G_i(x)\ \mathrm{d}x$ for each sender $\forall i = 1, 2$ and $\sup \mathsf{supp}(G_1) = \sup \mathsf{supp}(G_2)$.
- $\int_x^{\alpha_1} F_i(t)\ \mathrm{d}t \leq \int_x^{\alpha_1} \max\{\beta_1(t - \alpha_1) + F_i(\alpha_1), 0\}\ \mathrm{d}t$ for $\forall x \in [F_{-i}^{-1}(\gamma_1), \alpha_1]$.

If the desired equilibrium has not been found, we then proceed to the second sub-case by reversing the roles of sender 1 and sender 2 (i.e., assume $\int_0^{\alpha_1} F_1(x)\ \mathrm{d}x > \int_0^{\alpha_1} G_1(x)\ \mathrm{d}x$ and $\int_0^{\alpha_1} F_2(x)\ \mathrm{d}x = \int_0^{\alpha_1} G_2(x)\ \mathrm{d}x$), and we repeat the processes mentioned in Step 2.

5.2 Revisit the Symmetric Game

We can also use the equilibrium characterization provided in Theorem 5.1 to fully pin down the uniqueness of the equilibrium in a general two-sender game (where the prior distributions are arbitrary but symmetric).

We recall that Hwang et al. (2019); Au and Kawai (2021) have characterized a unique existence of symmetric equilibrium (among all symmetric strategy profiles) in a symmetric environment, however, this result does not rule out the existence of asymmetric equilibrium. In the two-sender game, we show that the asymmetric equilibrium does not exist when senders have symmetric prior. Combined with the conclusion of Hwang et al. (2019), we can fully determine the uniqueness of equilibrium.

Proposition 5.3. *If two senders' priors are identical, then the equilibrium is unique and symmetric.*

6 Conclusion

In this work, we study a competitive information design game with multiple prior heterogeneous senders. We focus on the Nash equilibrium among senders' game. We first observe that unlike the game with homogeneous senders, the equilibrium, in our game, may not always exist. However, we show that an equilibrium always exists when the senders' prior distributions are all continuous. We establish such equilibrium existence by a careful construction of a sequence of discrete games that are dedicated to handle the complexity of the MPC-constrained action spaces. When the equilibrium exists, we next characterize the necessary and sufficient conditions of the equilibrium strategy profiles. Our conditions operate on the introduced "virtual competitive function", which could be explicitly constructed from the strategy profile, that fully characterizes the virtual competitive environment the sender faces. En route, we provide an example showing that the equilibrium may not be unique when senders are heterogeneous. Lastly, we apply our equilibrium conditions to solve a general two-sender game when their priors are strictly uni-modal and we completely characterize the equilibrium. We also revisit the symmetric setting where the senders are homogeneous, and we use our equilibrium conditions to fully ping down the uniqueness of the equilibrium.

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A Omitted Proofs in Section 3

A.1 Proof of Lemma 3.2

Proof of Lemma 3.2. We first show that to fix any $m \in \mathbb{Z}^+$, the set S_i^m is non-empty. Let $\mu_i \triangleq \int_0^1 x \, \mathrm{d}F_i(x)$ be the mean of the prior F_i. On the one hand, if $\mu_i \in \mathsf{P}^m$, then it is obvious that the degenerate distribution putting all the mass on the point $x = \mu_i$ belongs to S_i^m, showing that S_i^m is non-empty. On the other hand, if $\mu_i \notin \mathsf{P}^m$, then μ_i must lie between two adjacent points in P^m and we can assume that $\mu_i \in (a, b)$ where $a = k \cdot 2^{-m}$ and $b = (k+1) \cdot 2^{-m}$, with $k = \lfloor \mu_i / 2^{-m} \rfloor$. We can construct a discrete distribution B_i^m as follows:

$$B_i^m(x) = \begin{cases} 0, & \text{if } x < a \\ \frac{b - \mu_i}{2^{-m}}, & \text{if } x \in [a, b) \\ 1, & \text{if } x \geq b \end{cases}$$

So the distribution $B_i^m(x)$ has two masses at point a and point b such that the sum of these two masses is one, and the mean of the distribution $B_i^m(x)$ exactly equals to the prior mean μ_i. Clearly, distribution $B_i^m(x)$ satisfies the constraint Equation (2). Below we prove Equation (3) also holds true for $B_i^m(x)$. For $\forall t \in [k]$, we have

$$\int_0^{t \cdot 2^{-m}} F_i(x) \, \mathrm{d}x \geq \int_0^{t \cdot 2^{-m}} B_i^m(x) \, \mathrm{d}x = 0$$

and $\forall t \in \{k+1, ..., 2^m - 1\}$, we have

$$\int_0^{t \cdot 2^{-m}} F_i(x) \, \mathrm{d}x = \int_0^1 F_i(x) \, \mathrm{d}x - \int_{t \cdot 2^{-m}}^1 F_i(x) \, \mathrm{d}x$$
$$\int_0^{t \cdot 2^{-m}} B_i^m(x) \, \mathrm{d}x = \int_0^1 B_i^m(x) \, \mathrm{d}x - \int_{t \cdot 2^{-m}}^1 B_i^m(x) \, \mathrm{d}x$$

while $\int_0^1 F_i(x) \, \mathrm{d}x = \int_0^1 B_i^m(x) \, \mathrm{d}x$, $\int_{t \cdot 2^{-m}}^1 F_i(x) \, \mathrm{d}x \leq \int_{t \cdot 2^{-m}}^1 B_i^m(x) \, \mathrm{d}x$, so we obtain

$$\int_0^{t \cdot 2^{-m}} F_i(x) \, \mathrm{d}x \geq \int_0^{t \cdot 2^{-m}} B_i^m(x) \, \mathrm{d}x \; .$$

Thus, distribution $B_i^m(x)$ satisfies Equation (3), implying that $B_i^m \in \mathsf{S}_i^m$ and the set S_i^m is non-empty.

Notice that the set S_i^m is formed by $O(2^m)$ linear constraints. Thus, there are finitely many extreme points of the set S_i^m, and therefore, the action space A_i^m is finite. ∎

A.2 Proof of Lemma 3.3

To prove the first claim in the above Lemma 3.3, we use the following Helly's Selection Theorem for uniformly bounded monotone CDFs:

Lemma A.1 (Helly's Selection Theorem). *Let $\{G^m\}_{m\in\mathbb{Z}^+}$ be a sequence of CDFs which is tight,*[8] *then there exists a subsequence $\{m(k)\}_{k\in\mathbb{Z}^+} \subseteq \mathbb{Z}^+$ such that $\{G^{m(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to a certain CDF G, namely, we have $\lim_{k\to\infty} G^{m(k)}(x) = G(x)$ for each point x at which G is continuous.*

Proof of Lemma 3.3. We first prove the first claim. To apply Lemma A.1 to the CDFs sequence, first, we need to show that this sequence is tight. Recall that by construction, each i's discrete equilibrium strategy $\widetilde{G}_i^m$ assigns all the probability within the interval $[0,1]$ and obviously, $\forall\epsilon > 0$ we have $\widetilde{G}_i^m(1) - \widetilde{G}_i^m(0^-) = 1 > 1-\epsilon$, which makes the sequence tight. By Lemma A.1, for each $i \in [N]$, there must exist a subsequence $\{m_i(k)\}_{k\in\mathbb{Z}^+} \subseteq [m]$ such that $\{\widetilde{G}_i^{m_i(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to some CDF $\widetilde{G}_i$.

Next, we prove that for each sender i, $\widetilde{G}_i$ is an MPC of his prior F_i. According to the definition of weak convergence, for each sender $i \in [N]$, if the CDFs sequence $\{\widetilde{G}_i^{m_i(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to $\widetilde{G}_i$, then the sequence of real numbers $\{\int_0^1 \widetilde{G}_i^{m_i(k)}(x)\ \mathrm{d}x\}_{k\in\mathbb{Z}^+}$ also converges to $\int_0^1 \widetilde{G}_i(x)\ \mathrm{d}x$, that is

$$\lim_{k\to\infty} \int_0^1 \widetilde{G}_i^{m_i(k)}(x)\ \mathrm{d}x = \int_0^1 \widetilde{G}_i(x)\ \mathrm{d}x.$$

For $\forall k \in \mathbb{Z}^+$, $\widetilde{G}_i^{m_i(k)} \in \mathsf{S}_i^{m_i(k)}$, we have $\int_0^1 F_i(x)\ \mathrm{d}x \le \int_0^1 \widetilde{G}_i^{m_i(k)}(x)\ \mathrm{d}x + 2^{-m_i(k)}$ and $\int_0^1 F_i(x)\ \mathrm{d}x \ge \int_0^1 \widetilde{G}_i^{m_i(k)}(x)\ \mathrm{d}x$ according to the definition of m-discrete approximation game. The properties of the upper and lower bounds of a real number sequence are preserved when passing to the limit, therefore we have

$$\int_0^1 F_i(x)\ \mathrm{d}x \le \lim_{k\to\infty} \int_0^1 \widetilde{G}_i^{m_i(k)}(x)\ \mathrm{d}x + 2^{-m_i(k)} = \int_0^1 \widetilde{G}_i(x)\ \mathrm{d}x$$

and

$$\int_0^1 F_i(x)\ \mathrm{d}x \ge \int_0^1 \widetilde{G}_i^{m_i(k)}(x)\ \mathrm{d}x = \int_0^1 \widetilde{G}_i(x)\ \mathrm{d}x\ .$$

By combining these two inequalities, we obtain $\int_0^1 F_i(x)\ \mathrm{d}x = \int_0^1 \widetilde{G}_i(x)\ \mathrm{d}x$. Then we prove $\int_0^t F_i(x)\ \mathrm{d}x \ge \int_0^t \widetilde{G}_i(x)\ \mathrm{d}x$, $\forall t \in [0,1]$ by contradictions. For convenience, we abuse the notations a little and rename the sequence $\{\widetilde{G}_i^{m(k)}\}_{k\in\mathbb{Z}^+}$ as $\{\widetilde{G}_i^m\}_{m\in\mathbb{Z}^+}$. On the one hand, we

[8] A sequence of functions $\{G^m\}_{m\in\mathbb{Z}^+}$ is tight, if and only if $\forall\epsilon > 0$ there exists an interval $[a,b]$ such that for each $m \in \mathbb{Z}^+$ we have $G^m(b) - G^m(a) > 1-\epsilon$.

assume there exist $t^* \in (0,1)$ and $\epsilon > 0$ such that $\int_0^t F_i(x)\ \mathrm{d}x < \int_0^t \widetilde{G}_i(x)\ \mathrm{d}x$ for $\forall t \in (t^* - \epsilon, t^* + \epsilon)$. When $m > M_1 = \log_2^{\epsilon^{-1}} - 1$ (i.e., $2^{-m} < 2\epsilon$), there exists point $a \in \mathsf{P}^m \cap (t^* - \epsilon, t^* + \epsilon)$ such that $\int_0^a \widetilde{G}_i(x)\ \mathrm{d}x > \int_0^a F_i(x)\ \mathrm{d}x$. On the other hand, due to the fact $\widetilde{G}_i^m \in \mathsf{S}_i^m$, we have $\int_0^a \widetilde{G}_i(x)\ \mathrm{d}x \le \int_0^a F_i(x)\ \mathrm{d}x$.

$$
\begin{aligned}
0 &\ge \lim_{m\to\infty} \int_0^a \widetilde{G}_i^m(x)\ \mathrm{d}x - \int_0^a F_i(x)\ \mathrm{d}x \\
&= \lim_{m\to\infty} \Big(\int_0^a \widetilde{G}_i^m(x)\ \mathrm{d}x - \int_0^a \widetilde{G}_i(x)\ \mathrm{d}x\Big) + \int_0^a \widetilde{G}_i(x)\ \mathrm{d}x - \int_0^a F_i(x)\ \mathrm{d}x
\end{aligned}
$$

Because sequence $\{\widetilde{G}_i^m\}_{m\in\mathbb{Z}^+}$ weakly converges to $\widetilde{G}_i$, we have sequence $\{\int_0^a \widetilde{G}_i^m(x)\ \mathrm{d}x\}_{m\in\mathbb{Z}^+}$ converges to $\int_0^a \widetilde{G}_i(x)\ \mathrm{d}x$. Therefore, we have

$$
0 \ge \int_0^a \widetilde{G}_i(x)\ \mathrm{d}x - \int_0^a F_i(x)\ \mathrm{d}x.
$$

We derive a contradiction. In conclusion, for each sender $i \in [N]$, the converging strategy $\widetilde{G}_i$ satisfies all the constraints in Definition 2.1 and so $\widetilde{G}_i \in \mathsf{MPC}(F_i)$. ■

Recall that we have a finite number of senders in the constructed game in Definition 3.2, it can be shown that, there exists a common subsequence $\{m(k)\}_{k\in\mathbb{Z}^+}$ such that for each sender i, the sequence $\{\widetilde{G}_i^{m(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to a certain CDF $\widetilde{G}_i$. To see this, we first find a subsequence $\{m_1(k)\}_{k\in\mathbb{Z}^+}$ of $\mathbb{Z}^+$ such that $\{\widetilde{G}_1^{m_1(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to $\widetilde{G}_1$. We then find a subsequence $\{m_2(k)\}_{k\in\mathbb{Z}^+}$ of $\{m_1(k)\}_{k\in\mathbb{Z}^+}$ such that $\{\widetilde{G}_2^{m_2(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to $\widetilde{G}_2$. It is known that if a sequence converges, any subsequence of it also converges to the same limit. Hence, $\{\widetilde{G}_1^{m_2(k)}\}_{k\in\mathbb{Z}^+}$ also weakly converges to $\widetilde{G}_1$. In the same manner, we obtain a common subsequence $\{m(k)\}_{k\in\mathbb{Z}^+}$ of $\mathbb{Z}^+$ such that for each sender i, the sequence $\{\widetilde{G}_i^{m(k)}\}_{k\in\mathbb{Z}^+}$ weakly converges to a certain CDF $\widetilde{G}_i$. In other words, there exists a sequence of equilibrium $\{(\widetilde{G}_1^{m(k)}, \dots, \widetilde{G}_N^{m(k)})\}_{k\in\mathbb{Z}^+}$ that weakly converges to a certain strategy profile $(\widetilde{G}_1, \dots, \widetilde{G}_N)$. By Lemma 3.3, each sender's strategy is a feasible distribution of posterior means, so $(\widetilde{G}_1, \dots, \widetilde{G}_N)$ forms a feasible strategy profile and also a candidate equilibrium in the continuous game.

A.3 Proof of Lemma 3.4

Proof of Lemma 3.4. We first observe that, by Definition 2.1 and the assumptions on senders' prior distributions, the mass point of any MPC, if it exists, will exist only within the interval $(0,1)$. We now proceed to the proof by contradiction. Among N senders, we w.l.o.g. suppose sender 1 and 2 with converging strategies $\widetilde{G}_1$ and $\widetilde{G}_2$ simultaneously assigning a positive mass at point $b \in (0,1)$ and $b \geq \underline{x}$, that is

$$\begin{aligned}\widetilde{G}_1(b) - \widetilde{G}_1(b^-) = p_1 > 0 \\ \widetilde{G}_2(b) - \widetilde{G}_2(b^-) = p_2 > 0\ .\end{aligned} \tag{4}$$

Together with the above inequalities and Definition 2.1, we have for each $i \in \{1,2\}$

$$\int_0^b F_i(x)\ \mathrm{d}x > \int_0^b \widetilde{G}_i(x)\ \mathrm{d}x\ . \tag{5}$$

We prove the above inequality using sender 1 as an example. As $\widetilde{G}_1 \in \mathsf{MPC}(F_1)$, we have $\int_0^b F_1(x)\ \mathrm{d}x \geq \int_0^b \widetilde{G}_1(x)\ \mathrm{d}x$. Suppose $\int_0^b F_1(x)\ \mathrm{d}x = \int_0^b \widetilde{G}_1(x)\ \mathrm{d}x$. In order to ensure $\int_0^t F_1(x)\ \mathrm{d}x \geq \int_0^t \widetilde{G}_1(x)\ \mathrm{d}x$ for any $t \in [b,1]$, we must have $F_1(b) \geq \widetilde{G}_1(b) > \widetilde{G}_1(b^-)$. By the fact F_1 is continuous over $[0,1]$, there exists $\epsilon > 0$ such that $F_1(x) > \widetilde{G}_1(x)$ for any $x \in (b-\epsilon, b)$, and $\int_{b-\epsilon}^b F_1(x)\ \mathrm{d}x > \int_{b-\epsilon}^b \widetilde{G}_1(x)\ \mathrm{d}x$. Therefore we have $\int_0^{b-\epsilon} F_1(x)\ \mathrm{d}x = \int_0^b F_1(x)\ \mathrm{d}x - \int_{b-\epsilon}^b F_1(x)\ \mathrm{d}x < \int_0^b \widetilde{G}_1(x)\ \mathrm{d}x - \int_{b-\epsilon}^b \widetilde{G}_1(x)\ \mathrm{d}x = \int_0^{b-\epsilon} \widetilde{G}_1(x)\ \mathrm{d}x$, which contradicts the MPC conditions.

Constructing profitable strategy deviation. With the above Equation (5), we define the following quantity that would be helpful for the analysis

$$\eta \triangleq \min_{i=1,2} \left(\int_0^b F_i(x)\ \mathrm{d}x - \int_0^b \widetilde{G}_i(x)\ \mathrm{d}x \right) > 0\ .$$

Intuitively, we use the Inequality (4), the equilibrium strategy $\widetilde{G}_i^m$ for $i \in \{1,2\}$, and the above defined quantity η to construct a profitable strategy deviation $\bar{G}_i^m$ both for sender 1 and sender 2. We now show how to construct the strategy deviation. Notice that we can reallocate the probabilities of $\widetilde{G}_i^m$ over the interval $[b-\eta_1, b+\eta_2] \subseteq [b-\eta, b+\eta]$ on two masses, one at $b-\eta_1$ and the other at $b+\eta_2$, where η_1 and η_2 will be specified shortly to ensure that the new strategy still satisfies all constraints in Definition 3.2. We now define $k \triangleq 100 \cdot \max\{\frac{\widetilde{G}_1(b)}{p_1}, \frac{\widetilde{G}_2(b)}{p_2}\} \geq 100$. Below we only consider the case where $\widetilde{G}_i(b) < 1$ and $\widetilde{G}_i(b^-) > 0$. The cases where $\widetilde{G}_i(b) = 1$ or $\widetilde{G}_i(b^-) = 0$ are relatively simpler, hence we omit the proofs. To specify η_1 and η_2, we define the following quantities:

$$\begin{aligned}\eta_{2,i} &\triangleq \frac{1}{2} \inf \left\{ t : \widetilde{G}_i(b+t) \geq \min\{1, \widetilde{G}_i(b) + \frac{p_i}{k}\} \right\}, \quad i \in \{1,2\} \\ \eta_{1,i} &\triangleq \frac{1}{2} \inf \left\{ t : \widetilde{G}_i(b-t) \leq \max\{0, \widetilde{G}_i(b^-) - \frac{p_i}{k}\} \right\}, \quad i \in \{1,2\}\end{aligned}$$

With the above definition, we now have for each sender $i \in \{1,2\}$, $\forall x \in [b - \min\{\eta_{1,1}, \eta_{1,2}\}, b+$

$\min\{\eta_{2,1},\eta_{2,2}\}]$,

$$\widetilde{G}_i(b^-) - \frac{p_i}{k} \le \widetilde{G}_i(x) \le \widetilde{G}_i(b) + \frac{p_i}{k} \ .$$

Let $m_1 \triangleq \min\{m' \in \mathbb{Z}^+ \mid \min\{\eta, \eta_{1,1}, \eta_{1,2}\}/2^{-m'} > 2\}$. Then, when $m \ge m_1$, within the interval $(b - \min\{\eta, \eta_{1,1}, \eta_{1,2}\}, b)$, there exist at least two integer multiples of 2^{-m}, and we define

$$\eta_1 \triangleq b - 2^{-m_1} \cdot \max\{t \in \mathbb{Z} | t \cdot 2^{-m_1} < b\} \ .$$

Similarly, let $m_2 \triangleq \min\{m' \in \mathbb{Z}^+ \mid \min\{\frac{\eta_1}{k}, \eta_{2,1}, \eta_{2,2}\}/2^{-m'} > 2\}$. Then, when $m \ge m_2$, within the interval $(b, b + \min\{\frac{\eta_1}{k}, \eta_{2,1}, \eta_{2,2}\})$, there exist at least two integer multiples of 2^{-m}, and we define

$$\eta_2 \triangleq 2^{-m_2} \cdot \min\{t \in \mathbb{Z} | t \cdot 2^{-m_2} > b\} - b \ .$$

As we can see, when $m \ge \max\{m_1, m_2\}$, both $b - \eta_1$ and $b + \eta_2$ are specified and we know $\eta_2 < \frac{\eta_1}{k}$, which implies that the mass on the right of b is much closer than the mass on the left. Now, in any m-discrete approximation game with $m \ge \max\{m_1, m_2\}$, for each sender $i \in \{1, 2\}$, we can construct the strategy deviations $\bar{G}_i^m$ for $\widetilde{G}_i^m$ as follows.

$$\bar{G}_i^m(x) = \begin{cases} \widetilde{G}_i^m(x), & \text{if } x \in [0, b - \eta_1) \cup [b + \eta_2, 1] \\ \frac{\int_{b-\eta_1}^{b+\eta_2} \widetilde{G}_i^m(x) \ \mathrm{d}x}{\eta_2 + \eta_1}, & \text{if } x \in [b - \eta_1, b + \eta_2) \end{cases}$$

Clearly, we have $\int_0^1 \bar{G}_i^m(x) \ \mathrm{d}x = \int_0^1 \widetilde{G}_i^m(x) \ \mathrm{d}x$. The intuition behind the construction is based on discrete equilibrium strategies, dividing the probabilities in the range $[b - \eta_1, b + \eta_2]$ into two parts, and allocating them to $b - \eta_1$ and $b + \eta_2$ to form two new masses, while maintaining the expectation unchanged. As a consequence, the new mass at $b + \eta_2$ may bring higher utility to either sender 1 or 2.

Proving the constructed strategy is feasible. We next show for sufficiently large m, for each $i \in \{1, 2\}$, we have $\bar{G}_i^m \in \mathsf{S}_i^m$, that is $\bar{G}_i^m$ is a feasible strategy satisfying all constraints in the m-discrete approximation game.

We take the proof of sender 1 as an example. It is easy to see that, when $m \ge \max\{m_1, m_2\}$, strategy $\bar{G}_1^m$ satisfies the constraint Equation (2). For Constraint (3), since $\bar{G}_1^m$ and $\widetilde{G}_1^m$ differ only within the interval $[b-\eta_1, b+\eta_2)$, it suffices to prove that $\int_0^t F_1(x) \ \mathrm{d}x \ge \int_0^t \bar{G}_1^m(x) \ \mathrm{d}x$, for any $t \in \mathsf{P}^m \cap [b - \eta_1, b + \eta_2)$. According to the definition of weak convergence, there exists $m_3 > 0$ such that when $m \ge m_3$, we have $|\int_0^b \widetilde{G}_i^m(x) \ \mathrm{d}x - \int_0^b \widetilde{G}_i(x) \ \mathrm{d}x| < \eta - \eta_1 < \eta - \eta_2, i \in \{1, 2\}$. Now notice that if there exists a point $c \in [b - \eta_1, b]$ and c is an integer multiples of 2^{-m}, we have

$$
\begin{aligned}
\int_0^c F_1(x)\ \mathrm{d}x - \int_0^c \bar{G}_1^m(x)\ \mathrm{d}x &= \int_0^c F_1(x)\ \mathrm{d}x - \int_0^c \widetilde{G}_1^m(x)\ \mathrm{d}x + \int_0^c \widetilde{G}_1^m(x)\ \mathrm{d}x - \int_0^c \bar{G}_1^m(x)\ \mathrm{d}x \\
&= \int_0^b F_1(x)\ \mathrm{d}x - \int_0^b \widetilde{G}_1^m(x)\ \mathrm{d}x - \left(\int_c^b F_1(x)\ \mathrm{d}x - \int_c^b \widetilde{G}_1^m(x)\ \mathrm{d}x\right) \\
&\quad - \left(\int_{b-\eta_1}^c \bar{G}_1^m(x)\ \mathrm{d}x - \int_{b-\eta_1}^c \widetilde{G}_1^m(x)\ \mathrm{d}x\right) \\
&\geq \int_0^b F_1(x)\ \mathrm{d}x - \int_0^b \widetilde{G}_1^m(x)\ \mathrm{d}x - (b-c) - (c-b+\eta_1) \\
&> \int_0^b F_1(x)\ \mathrm{d}x - \int_0^b \widetilde{G}_1(x)\ \mathrm{d}x - (\eta-\eta_1) - (b-c) - (c-b+\eta_1) \\
&\geq \eta - (\eta-\eta_1) - (b-c) - (c-b+\eta_1) = 0
\end{aligned}
$$

If there exists a point $d \in [b, b+\eta_2)$ and d is an integer multiples of 2^{-m}, we have $\int_0^d F_1(x)\ \mathrm{d}x - \int_0^d \bar{G}_1^m(x)\ \mathrm{d}x \geq 0$ using a similar proof method. In summary, we have proved that $\int_0^t F_1(x)\ \mathrm{d}x \geq \int_0^t \bar{G}_1^m(x)\ \mathrm{d}x$, for any $t \in \mathsf{P}^m \cap [b-\eta_1, b+\eta_2)$, and therefore, $\forall m \geq \max\{m_1, m_2, m_3\}$, we have $\bar{G}_i^m \in \mathsf{S}_i^m$ for each $i \in \{1, 2\}$.

Organizing all necessary inequalities. We now detail all the inequalities that are necessary to prove that the constructed strategy is profitable. Taking sender 1 as an example, if $\widetilde{G}_1$ assigns a mass p_1 at b, then in the convergence of the discrete equilibrium strategy sequence, the probability aggregated in the neighborhood near b will gradually converge to p_1. This means that for any interval $(b-\xi_1, b+\xi_2) \subseteq [0,1]$ and for $\forall \alpha \in (0,1)$, it holds that $\widetilde{G}_1^m(b+\xi_2) - \widetilde{G}_1^m(b-\xi_1) \geq \alpha p_1$ for sufficiently large m. Let $m_4 \triangleq \min\{m' \in \mathbb{Z}^+ | \eta_2/2^{-m'} \geq 3\}$ and when $m \geq \max\{m_1, m_2, m_3, m_4\}$, the definitions of $b-\xi_1, b+\xi_2$ are as follows

$$
\begin{aligned}
\xi_1 &\triangleq b - 2^{-m_4} \cdot \max\{t \in \mathbb{Z} | t \cdot 2^{-m_4} < b\} \\
\xi_1 &\triangleq b - 2^{-m_4} \cdot \max\{t \in \mathbb{Z} | t \cdot 2^{-m_4} < b\} \quad .
\end{aligned}
$$

Till now, the relationship between these parameters defined above can be summarized as $b - \min\{\eta, \eta_{1,1}, \eta_{1,2}\} < b - \eta_1 < b - \xi_1 < b < b + \xi_2 < b + \eta_2 < b + \min\{\frac{\eta_1}{k}, \eta_{2,1}, \eta_{2,2}\}$. And it follows that, for $i \in \{1, 2\}$, we have $\widetilde{G}_i(b^-) - \frac{p_i}{k} \leq \widetilde{G}_i(b-\eta_1) \leq \widetilde{G}_i(b-\xi_1) \leq \widetilde{G}_i(b^-) < \widetilde{G}_i(b) \leq \widetilde{G}_i(b+\xi_2) \leq \widetilde{G}_i(b+\eta_2) \leq \widetilde{G}_i(b) + \frac{p_i}{k}$.

The above inequalities describe the relationship of function values between different positions in the converging strategy. Regarding the relationship of values at the same positions between discrete equilibrium strategy functions and converging strategy functions, we have the following conclusion. There exists $m_5 > 0$ such that, when $m \geq$

$\max\{m_1, m_2, m_3, m_4, m_5\}$, for each sender $i \in \{1, 2\}$, we have

$$
\begin{aligned}
|\widetilde{G}_i^m(b-\eta_1) - \widetilde{G}_i(b-\eta_1)| &\leq \frac{2p_i}{k} \\
|\widetilde{G}_i^m(b-\xi_1) - \widetilde{G}_i(b-\xi_1)| &\leq \frac{2p_i}{k} \\
|\widetilde{G}_i^m(b+\xi_2) - \widetilde{G}_i(b+\xi_2)| &\leq \frac{2p_i}{k} \\
|\widetilde{G}_i^m(b+\eta_2) - \widetilde{G}_i(b+\eta_2)| &\leq \frac{2p_i}{k}.
\end{aligned}
$$

We use the proof of $|\widetilde{G}_1^m(b-\eta_1) - \widetilde{G}_1(b-\eta_1)| \leq \frac{2p_1}{k}$ as an example, and the proofs of the other conclusions all employ similar approaches and methods. Although we have $\widetilde{G}_1^m$ weakly converges to $\widetilde{G}_1$, according to the definition of weak convergence, only the continuous points in $\widetilde{G}_1$ converge pointwise, and $b-\eta_1$ may not necessarily be a continuous point. Our solution is to find two continuous points on the left and right sides of $b-\eta_1$ that converge pointwise, and then use this relationship to obtain the target conclusion. Specifically, we can find $b_l \in (b - \min\{\eta, \eta_{1,1}, \eta_{1,2}\}, b-\eta_1)$ and $b_r \in (b-\eta_1, b-\xi_1)$ such that $\widetilde{G}_1$ is continuous both on point b_l and point b_r. Let $\epsilon = \min\{\frac{p_1}{k}, \frac{p_2}{k}\}$ and $M > 0$ such that when $m \geq M$, we have

$$
|\widetilde{G}_1^m(b_l) - \widetilde{G}_1(b_l)| < \epsilon \leq \frac{p_1}{k},\ |\widetilde{G}_1^m(b_r) - \widetilde{G}_1(b_r)| < \epsilon \leq \frac{p_1}{k}\ .
$$

When $\widetilde{G}_1^m(b-\eta_1) - \widetilde{G}_1(b-\eta_1) \geq 0$, we have

$$
\begin{aligned}
|\widetilde{G}_1^m(b-\eta_1) - \widetilde{G}_1(b-\eta_1)| &= \widetilde{G}_1^m(b-\eta_1) - \widetilde{G}_1(b-\eta_1) \\
&\leq \widetilde{G}_1^m(b_r) - \widetilde{G}_1(b-\eta_1) \\
&< \widetilde{G}_1(b_r) + \frac{p_1}{k} - \widetilde{G}_1(b-\eta_1) \\
&\leq \widetilde{G}_1(b^-) + \frac{p_1}{k} - \widetilde{G}_1(b^-) + \frac{p_1}{k} = \frac{2p_1}{k},
\end{aligned}
$$

and when $\widetilde{G}_1^m(b-\eta_1) - \widetilde{G}_1(b-\eta_1) < 0$, we have

$$
\begin{aligned}
|\widetilde{G}_1^m(b-\eta_1) - \widetilde{G}_1(b-\eta_1)| &= \widetilde{G}_1(b-\eta_1) - \widetilde{G}_1^m(b-\eta_1) \\
&\leq \widetilde{G}_1(b-\eta_1) - \widetilde{G}_1^m(b_l) \\
&< \widetilde{G}_1(b-\eta_1) - \widetilde{G}_1(b_l) + \frac{p_1}{k} \\
&\leq \widetilde{G}_1(b^-) - \widetilde{G}_1(b^-) + \frac{p_1}{k} + \frac{p_1}{k} = \frac{2p_1}{k}.
\end{aligned}
$$

To sum up, we have proved that $|\widetilde{G}_1^m(b-\eta_1) - \widetilde{G}_1(b-\eta_1)| \leq \frac{2p_1}{k}$.

Showing the constructed strategy deviation is profitable. With the above inequalities, we are now ready to show that either sender 1 or sender 2 is profitable to deviate to the constructed strategy.

For sufficiently large m, we first compute the lower bound of the mass at $b+\eta_2$ for $\bar{G}_1$ and $\bar{G}_2$. Taking sender 1 as an example, for all $m \geq \max\{m_1, m_2, m_3, m_4, m_5\}$, the increase of mass at $b-\eta_1$ is as follows.

$$
\begin{aligned}
&\bar{G}_1^m(b-\eta_1) - \widetilde{G}_1^m(b-\eta_1) = \frac{\int_{b-\eta_1}^{b+\eta_2} [\widetilde{G}_1^m(x) - \widetilde{G}_1^m(b-\eta_1)]\ \mathrm{d}x}{\eta_2+\eta_1} \\
=\ & \frac{\int_{b-\eta_1}^{b-\xi_1} [\widetilde{G}_1^m(x) - \widetilde{G}_1^m(b-\eta_1)]\ \mathrm{d}x + \int_{b-\xi_1}^{b+\eta_2} [\widetilde{G}_1^m(x) - \widetilde{G}_1^m(b-\eta_1)]\ \mathrm{d}x}{\eta_2+\eta_1} \\
\leq\ & \frac{(\eta_1-\xi_1)[\widetilde{G}_1^m(b-\xi_1) - \widetilde{G}_1^m(b-\eta_1)] + (\eta_2+\xi_1)[\widetilde{G}_1^m(b+\eta_2) - \widetilde{G}_1^m(b-\eta_1)]}{\eta_2+\eta_1} \\
\leq\ & \frac{(\eta_1-\xi_1)[\widetilde{G}_1(b-\xi_1) - \widetilde{G}_1(b-\eta_1) + \frac{4p_1}{k}] + (\eta_2+\xi_1)[\widetilde{G}_1(b+\eta_2) - \widetilde{G}_1(b-\eta_1) + \frac{4p_1}{k}]}{\eta_2+\eta_1} \\
\leq\ & \frac{(\eta_1-\xi_1)(\widetilde{G}_1(b^-) - \widetilde{G}_1(b^-) + \frac{p_1}{k} + \frac{4p_1}{k}) + (\eta_2+\xi_1)(\widetilde{G}_1(b) + \frac{p_1}{k} - \widetilde{G}_1(b^-) + \frac{p_1}{k} + \frac{4p_1}{k})}{\eta_2+\eta_1} \\
\leq\ & \frac{(\eta_1-\xi_1)\frac{5p_1}{k} + (\eta_2+\xi_1)(p_1 + \frac{6p_1}{k})}{\eta_2+\eta_1} \\
=\ & \frac{5p_1}{k} + \frac{(\eta_2+\xi_1)(p_1+\frac{p_1}{k})}{\eta_1+\eta_2}\ .
\end{aligned}
$$

The relationships between parameters defined above can be summarized as $\xi_1 \leq 2^{-m_4} \leq \frac{\eta_2}{3} \leq \eta_2 \leq \frac{\eta_1}{k} \leq \frac{\eta_1}{100}$, and we have

$$
\bar{G}_1^m(b-\eta_1) - \widetilde{G}_1^m(b-\eta_1) \leq \frac{5p_1}{k} + \frac{(\eta_2+\xi_1)(p_1+\frac{p_1}{k})}{\eta_1+\eta_2} \leq \frac{7p_1}{k}\ .
$$

Thus, the mass of $\bar{G}_1$ at $b+\eta_2$ can be calculated as below.

$$
\begin{aligned}
\bar{G}_1^m(b+\eta_2) - \bar{G}_1^m(b-\eta_1) &\geq \widetilde{G}_1^m(b+\eta_2) - [\widetilde{G}_1^m(b-\eta_1) + \frac{7p_1}{k}] \\
&\geq \widetilde{G}_1(b+\eta_2) - \frac{2p_1}{k} - [\widetilde{G}_1(b-\eta_1) + \frac{2p_1}{k} + \frac{7p_1}{k}] \\
&\geq \widetilde{G}_1(b) - \frac{2p_1}{k} - [\widetilde{G}_1(b) - p_1 + \frac{9p_1}{k}] \\
&= p_1 - \frac{11p_1}{k}\ .
\end{aligned}
$$

Thus, for all $m \geq \max\{m_1, m_2, m_3, m_4, m_5\}$, the mass at $b+\eta_2$ for $\bar{G}_1^m$ is at least $p_1 - \frac{11p_1}{k}$, and similarly, we can establish that the mass at $b+\eta_2$ for $\bar{G}_2^m$ is at least $p_2 - \frac{11p_2}{k}$.

With slightly abusing notations, we use $U_1(\bar{G}_1^m, \widetilde{G}_2^m)$ to represent the utility of sender 1 when sender 1 adopts strategy $\bar{G}_1^m$, sender 2 adopts strategy $\widetilde{G}_2^m$, and the rest of the senders ($i \neq 1, 2$) adopt strategy $\widetilde{G}_i^m$. The meanings of other definitions can be inferred in a similar manner. Given other senders' strategies unchanged, we do know that at least one of the senders 1 and 2 has a profitable strategy deviation, but we do not know specifically which sender it is. Therefore, we aim to prove that, the sum of the utilities of sender 1 and 2 when each independently adopts the strategy deviation given others' strategies unchanged, denoted as $U_1(\bar{G}_1^m, \widetilde{G}_2^m) + U_2(\widetilde{G}_1^m, \bar{G}_2^m)$, is strictly greater than the sum of their utilities when each i adopts the strategy $\widetilde{G}_i$, denoted as $U_1(\widetilde{G}_1^m, \widetilde{G}_2^m) + U_2(\widetilde{G}_1^m, \widetilde{G}_2^m)$. We have

$$
\begin{aligned}
& U_1(\bar{G}_1^m, \widetilde{G}_2^m) + U_2(\widetilde{G}_1^m, \bar{G}_2^m) - \Big(U_1(\widetilde{G}_1^m, \widetilde{G}_2^m) + U_2(\widetilde{G}_1^m, \widetilde{G}_2^m)\Big) \\
\geq\ & \left(\bar{G}_1^m(b+\eta_2) - \bar{G}_1^m(b-\eta_1)\right) \widetilde{G}_2^m(b+\xi_2) \mathbb{E}_{x_{-1,2} \sim \widetilde{G}_{-1,2}^m} \left[\frac{\mathbf{1}\big\{b+\eta_2 \geq \max_{q \in [n]/\{1,2\}} x_q\big\}}{1 + \sum_{q=3}^n \mathbf{1}\{x_q = b+\eta_2\}} \right] \\
+\ & \left(\bar{G}_2^m(b+\eta_2) - \bar{G}_2^m(b-\eta_1)\right) \widetilde{G}_1^m(b+\xi_2) \mathbb{E}_{x_{-1,2} \sim \widetilde{G}_{-1,2}^m} \left[\frac{\mathbf{1}\big\{b+\eta_2 \geq \max_{q \in [n]/\{1,2\}} x_q\big\}}{1 + \sum_{q=3}^n \mathbf{1}\{x_q = b+\eta_2\}} \right] \\
-\ & \Big(\widetilde{G}_1^m(b+\eta_2)\widetilde{G}_2^m(b+\eta_2) - \widetilde{G}_1^m(b-\eta_1)\widetilde{G}_1^m(b-\eta_1)\Big) \mathbb{E}_{x_{-1,2} \sim \widetilde{G}_{-1,2}^m} \left[\frac{\mathbf{1}\big\{b+\eta_2 \geq \max_{q \in [n]/\{1,2\}} x_q\big\}}{1 + \sum_{q=3}^n \mathbf{1}\{x_q = b+\eta_2\}} \right] \\
-\ & \Big(\widetilde{G}_1^m(b+\eta_2) - \widetilde{G}_1^m(b+\eta_2^-)\Big) \Big(\widetilde{G}_2^m(b+\eta_2) - \widetilde{G}_2^m(b+\eta_2^-)\Big) \mathbb{E}_{x_{-1,2} \sim \widetilde{G}_{-1,2}^m} \left[\frac{\mathbf{1}\big\{b+\eta_2 \geq \max_{q \in [n]/\{1,2\}} x_q\big\}}{1 + \sum_{q=3}^n \mathbf{1}\{x_q = b+\eta_2\}} \right] \\
\geq\ & \mathbb{E}_{x_{-1,2} \sim \widetilde{G}_{-1,2}^m} \left[\frac{\mathbf{1}\big\{b+\eta_2 \geq \max_{q \in [n]/\{1,2\}} x_q\big\}}{1 + \sum_{q=3}^n \mathbf{1}\{x_q = b+\eta_2\}} \right] \\
\times\ & \Bigg(\left(1 - \frac{11}{k}\right) p_1 \left(\widetilde{G}_2(b) - \frac{2p_2}{k}\right) + \left(1 - \frac{11}{k}\right) p_2 \left(\widetilde{G}_1(b) - \frac{2p_1}{k}\right) \\
-\ & \left(\widetilde{G}_1(b) + \frac{3p_1}{k}\right)\left(\widetilde{G}_2(b) + \frac{3p_2}{k}\right) + \left(\widetilde{G}_1(b) - \frac{3p_1}{k} - p_1\right)\left(\widetilde{G}_2(b) - \frac{3p_2}{k} - p_2\right) \\
-\ & \left(\widetilde{G}_1(b) + \frac{3p_1}{k} - \widetilde{G}_1(b) + \frac{2p_1}{k}\right)\left(\widetilde{G}_2(b) + \frac{3p_2}{k} - \widetilde{G}_2(b) + \frac{2p_2}{k}\right)\Bigg) \\
=\ & \mathbb{E}_{x_{-1,2} \sim \widetilde{G}_{-1,2}^m} \left[\frac{\mathbf{1}\big\{b+\eta_2 \geq \max_{q \in [n]/\{1,2\}} x_q\big\}}{1 + \sum_{q=3}^n \mathbf{1}\{x_q = b+\eta_2\}} \right] \left(p_1 p_2 \left(1 + \frac{2}{k} + \frac{19}{k^2}\right) - \left(p_1 \widetilde{G}_2(b) + p_2 \widetilde{G}_1(b)\right) \frac{17}{k} \right),
\end{aligned}
$$

in which $\widetilde{G}_{-1,2}^m = (\widetilde{G}_3^m, \ldots, \widetilde{G}_N)$ and $x_{-1,2} = (x_3, \ldots, x_N)$ represents the realizations of $\widetilde{G}_{-1,2}^m$.

The first inequality is divided into four lines, which means four parts. Since $\bar{G}_1^m$ differs with $\widetilde{G}_1^m$ only within $[b-\eta_1, b+\eta_2]$ and so do $\bar{G}_2^m$ and $\widetilde{G}_2^m$, this inequality only considers the expected utility within $[b-\eta_1, b+\eta_2]$. When computing sender 1's expected utility $U_1(\bar{G}_1^m, \widetilde{G}_2^m)$, we only consider the utility from the mass at $b+\eta_2$ and leave out the utility from the mass at $b-\eta_1$. Since $\widetilde{G}_2^m$ aggregates a large probability near point b, forming a mass at point c slightly greater than b can just defeat sender 2 when his posterior mean is around b. This expectation actually calculates the probability of forming a mass at point $b+\eta_2$ to defeat all senders except for sender 1 and 2, under the rule of uniformly random tie-breaking. The meaning of the second line of this inequality is similar to that of the first line, except that it swaps the roles of sender 1 and sender 2. When each sender i adopts strategy $\widetilde{G}_i^m$, the sum of utilities for sender 1 and sender 2 within the interval $[b-\eta_1, b+\eta_2]$ is divided into two parts, as shown in the third and fourth lines of this inequality. The reason for dividing it into two parts is that, when both sender 1 and sender 2 have a realization of $b+\eta_2$, under uniformly random tie-breaking, the sum of their probabilities of winning a tie together is less than twice the probability of each person winning a tie individually.

Note that here we use uniformly random tie-breaking rule just for convenience. Because the point $b+\eta_2$ is strictly bigger than $\underline{x}$, the smallest winning value of $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$, it is obvious that there exists $m_6 > 0$ such that when $m \geq \max\{m_1.m_2, m_3, m_4, m_5, m_6\}$, we have

$$\mathbb{E}_{x_{-1,2}\sim\widetilde{G}^m_{-1,2}}\left[\frac{\mathbf{1}\left\{b+\eta_2 \geq \max_{q\in[n]/\{1,2\}} x_q\right\}}{1+\sum_{q=3}^n \mathbf{1}\{x_q = b+\eta_2\}}\right] > 0 \ .$$

Combined with the inequalities above, we obtain

$$\begin{aligned}
&U_1(\bar{G}_1^m, \widetilde{G}_2^m) + U_2(\widetilde{G}_1^m, \bar{G}_2^m) - [U_1(\widetilde{G}_1^m, \widetilde{G}_2^m) + U_2(\widetilde{G}_1^m, \widetilde{G}_2^m)] \\
&\geq \mathbb{E}_{x_{-1,2}\sim\widetilde{G}^m_{-1,2}}\left[\frac{\mathbf{1}\left\{b+\eta_2 \geq \max_{q\in[n]/\{1,2\}} x_q\right\}}{1+\sum_{q=3}^n \mathbf{1}\{x_q = b+\eta_2\}}\right]\left(p_1p_2\left(1+\frac{2}{k}+\frac{19}{k^2}\right) - \left(p_1\widetilde{G}_2(b) + p_2\widetilde{G}_1(b)\right)\frac{17}{k}\right) \\
&\geq \mathbb{E}_{x_{-1,2}\sim\widetilde{G}^m_{-1,2}}\left[\frac{\mathbf{1}\left\{b+\eta_2 \geq \max_{q\in[n]/\{1,2\}} x_q\right\}}{1+\sum_{q=3}^n \mathbf{1}\{x_q = b+\eta_2\}}\right] p_1p_2\left(1+\frac{2}{k}+\frac{19}{k^2}-\frac{17}{50}\right) \\
&= \mathbb{E}_{x_{-1,2}\sim\widetilde{G}^m_{-1,2}}\left[\frac{\mathbf{1}\left\{b+\eta_2 \geq \max_{q\in[n]/\{1,2\}} x_q\right\}}{1+\sum_{q=3}^n \mathbf{1}\{x_q = b+\eta_2\}}\right] p_1p_2\left(\frac{33}{50}+\frac{2}{k}+\frac{19}{k^2}\right) > 0 \ ,
\end{aligned}$$

which implies that at least one of the following two inequalities holds true

$$U_1(\bar{G}_1^m, \widetilde{G}_2^m) > U_1(\widetilde{G}_1^m, \widetilde{G}_2^m) \ , \quad U_2(\widetilde{G}_1^m, \bar{G}_2^m) > U_2(\widetilde{G}_1^m, \widetilde{G}_2^m) \ .$$

To sum up, when $m > \max\{m_1, m_2, m_3, m_4, m_5, m_6\}$, we can construct a strategy deviation $\bar{G}_1^m \in \mathsf{S}_1^m$ for sender 1, and $\bar{G}_2^m \in \mathsf{S}_2^m$ for sender 2, such that at least one of $\bar{G}_1^m$ and $\bar{G}_2^m$ is a profitable strategy deviation. This contradicts the fact that $\widetilde{G}_1^m$ and $\widetilde{G}_2^m$ both are equilibrium strategies in the m-discrete approximation game. Therefore, we conclude that the initial assumption is invalid, indicating the non-existence of ties within the interval $[\underline{x}, 1]$ in their limit strategy profile $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$. ■

A.4 Proof of Lemma 3.5

Proof of Lemma 3.5. We define sender i's utility when other senders adopt strategy $\widetilde{G}_{-i}$ and sender i reports the value x as $u_i(x)$. So the expected utility when sender i plays any strategy G_i and others adopt strategy $\widetilde{G}_{-i}$ can be writen as $\int_0^1 u_i(x)\ \mathrm{d}G_i(x)$. By Lemma 3.4, we know there is no tie within the interval $[\underline{x}, 1]$ in the profile $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$. By the definition of weak convergence, it follows that the sequence of real numbers $\{\int_0^1 u_i(x)\ \mathrm{d}\widetilde{G}_i^m(x)\}_{m\in\mathbb{Z}^+}$ converges to $\int_0^1 u_i(x)\ \mathrm{d}\widetilde{G}_i(x)$, that is

$$\lim_{m\to\infty} \int_0^1 u_i(x)\ \mathrm{d}\widetilde{G}_i^m(x) = \int_0^1 u_i(x)\ \mathrm{d}\widetilde{G}_i(x)\ . \tag{6}$$

For each sender $i \in [N]$ and any feasible strategy $G_i \in \mathsf{MPC}(F_i)$, we can construct a CDFs sequence $\{G_i^m\}_{m\in\mathbb{Z}^+}$ where each $G_i^m \in \mathsf{S}_i^m$, and the sequence weakly converges to strategy G_i. The construction method is as follows. For $\forall m \in \mathbb{Z}^+$, we define G_i^m as below.

$$G_i^m(x) = \begin{cases} G_i(x), & \text{if } x \in \mathsf{P}^m \\ G_i(\min\{t : t \in \mathsf{P}^m,\ t \geq x\}), & \text{o.w.} \end{cases}$$

It is obvious that $G_i^m \in \mathsf{S}_i^m$. Next, we show that the sequence $\{G_i^m\}_{m\in\mathbb{Z}^+}$ weakly converges to G_i, namely $\{G_i^m\}_{m\in\mathbb{Z}^+}$ converges pointwise at all continuous points of G_i. All continuous points of G_i can be divided into two sets A and B. For $\forall x \in A$, there exists $M > 0$ such that when $m > M$, we have $x \bmod 2^{-m} = 0$. Set B contains all the remaining points. For $\forall x \in A$, by the definition of G_i^m, it follows that there exists $M > 0$ such that when $m > M$, we have $G_i^m(x) = G_i(x)$. Therefore, every point in A pointwise converges towards G_i. For $\forall x \in B$ and $\forall m \in \mathbb{Z}^+$, x lies between two adjacent points of P^m, $\lfloor \frac{x}{2^{-m}} \rfloor \cdot 2^{-m}$ and $\lceil \frac{x}{2^{-m}} \rceil \cdot 2^{-m}$, and we have $G_i^m(x) = G_i^m(\lfloor \frac{x}{2^{-m}} \rfloor \cdot 2^{-m}) = G_i(\lfloor \frac{x}{2^{-m}} \rfloor \cdot 2^{-m})$. Since $\{G_i(\lfloor \frac{x}{2^{-m}} \rfloor \cdot 2^{-m})\}$ converges to $G_i(x^-)$ and G_i is continuous at point x, we have $\{G_i^m(x)\}$ also converges to $G_i(x)$. Therefore, every point in A pointwise converges towards G_i. In summary, we prove that $\{G_i^m\}_{m\in\mathbb{Z}^+}$ weakly converges to G_i.

By Equation (6), we have

$$
\begin{aligned}
&\int_0^1 u_i(x)\ \mathrm{d}G_i(x) - \int_0^1 u_i(x)\ \mathrm{d}\widetilde{G}_i(x) \\
&= \lim_{m\to\infty}\left(\int_0^1 u_i(x)\ \mathrm{d}G_i(x) - \int_0^1 u_i(x)\ \mathrm{d}G_i^m(x)\right) + \lim_{m\to\infty}\left(\int_0^1 u_i(x)\ \mathrm{d}G_i^m(x) - \int_0^1 u_i(x)\ \mathrm{d}\widetilde{G}_i^m(x)\right) \\
&= \lim_{m\to\infty}\left(u_i(x)G_i(x)|_{x=0}^1 - \int_0^1 G_i(x)\ \mathrm{d}u_i(x) - u_i(x)G_i^m(x)|_{x=0}^1 + \int_0^1 G_i^m(x)\ \mathrm{d}u_i(x)\right) \\
&\quad + \lim_{m\to\infty}\left(\int_0^1 u_i(x)\ \mathrm{d}G_i^m(x) - \int_0^1 u_i(x)\ \mathrm{d}\widetilde{G}_i^m(x)\right) \\
&= \lim_{m\to\infty}\left(\int_0^1 \left(G_i^m(x) - G_i(x)\right)\ \mathrm{d}u_i(x)\right) + \lim_{m\to\infty}\left(\int_0^1 u_i(x)\ \mathrm{d}G_i^m(x) - \int_0^1 u_i(x)\ \mathrm{d}\widetilde{G}_i^m(x)\right)\ ,
\end{aligned}
$$

By the construction of sequence $\{G_i^m\}_{m\in\mathbb{Z}^+}$, we have for $\forall m \in \mathbb{Z}^+$, $G_i(x) \geq G_i^m(x)$ for $\forall x \in [0,1]$, which implies that

$$
\lim_{m\to\infty}\left(\int_0^1 \left(G_i^m(x) - G_i(x)\right)\ \mathrm{d}u_i(x)\right) \leq 0\ . \tag{7}
$$

In addition to the fact $(\widetilde{G}_1^m, \ldots, \widetilde{G}_N^m)$ is an equilibrium in the m-th discrete approximation game, we have

$$
\int_0^1 u_i(x)\ \mathrm{d}\widetilde{G}_i^m(x) \geq \int_0^1 u_i(x)\ \mathrm{d}G_i^m(x)\ . \tag{8}
$$

Combining Inequalities (7) and (8), we have

$$
\begin{aligned}
&\int_0^1 u_i(x)\ \mathrm{d}G_i(x) - \int_0^1 u_i(x)\ \mathrm{d}\widetilde{G}_i(x) \\
&= \lim_{m\to\infty}\left(\int_0^1 \left(G_i^m(x) - G_i(x)\right)\ \mathrm{d}u_i(x)\right) + \lim_{m\to\infty}\left(\int_0^1 u_i(x)\ \mathrm{d}G_i^m(x) - \int_0^1 u_i(x)\ \mathrm{d}\widetilde{G}_i^m(x)\right) \leq 0\ ,
\end{aligned}
$$

which shows that for sender i, $\widetilde{G}_i$ is a best response strategy to function u_i, and $(\widetilde{G}_1, \ldots, \widetilde{G}_N)$ is indeed an equilibrium in our competitive information design game. ■

B Omitted Proofs in Section 4

B.1 Lemma B.1 and Proof

Lemma B.1. *A distribution G is an MPC of prior F. If there exists $a \in [0,1]$ such that $\int_0^a F(x)\ \mathrm{d}x = \int_0^a G(x)\ \mathrm{d}x$, then it holds that $F(a) = G(a)$.*

Proof of Lemma B.1. We prove it by contradiction. Consider each $a \in [0,1]$ such that $\int_0^a F(x)\ \mathrm{d}x = \int_0^a G(x)\ \mathrm{d}x$. First we assume $G(a) > F(a)$. Obviously, we have $a < 1$. Then $\exists \epsilon > 0$ such that for all $x \in (a, a+\epsilon)$ we have $G(x) > F(x)$. This implies that $\int_a^{a+\epsilon} G(x)\ \mathrm{d}x > \int_a^{a+\epsilon} F(x)\ \mathrm{d}x$ and $\int_0^{a+\epsilon} G(x)\ \mathrm{d}x > \int_0^{a+\epsilon} F(x)\ \mathrm{d}x$, which contradicts the MPC constraints. Therefore, the assumption is invalid and $G(a) \le F(a)$. Then we assume $G(a) < F(a)$. Obviously, we have $a > 0$. Then $\exists \epsilon > 0$ such that for all $x \in (a-\epsilon, a)$ we have $G(x) < F(x)$. This implies that $\int_{a-\epsilon}^{a} G(x)\ \mathrm{d}x < \int_{a-\epsilon}^{a} F(x)\ \mathrm{d}x$ and $\int_0^{a-\epsilon} G(x)\ \mathrm{d}x > \int_0^{a-\epsilon} F(x)\ \mathrm{d}x$, which contradicts the MPC constraints. Therefore, the assumption is invalid and $G(a) = F(a)$. ■

B.2 Proof of Theorem 4.2

To facilitate the analysis, we focus on a strategy profile $(G_1, \ldots, G_N)$ where for each sender $i \in [N]$, there exists $\epsilon > 0$ such that $G_i(x+\epsilon) - G_i(x-\epsilon) > 0$ for $\forall x \in \mathsf{supp}(G_i)$. We note that this assumption is without loss of generality as if there exists an interval $[a,b]$ in which G_i is constant, then any point belonging to $[a,b] \cap \mathsf{supp}(G_i)$ does not affect the calculation of any sender's utility or the equilibrium structure. To prove Theorem 4.2, first, we present Lemma B.2 and its proof as follows.

Lemma B.2. *Suppose $(G_1, \ldots, G_N)$ is an equilibrium. For each sender $i \in [N]$, if strategy G_i is discontinuous at $a \in (0,1)$, then there exists $\epsilon > 0$ such that the function G_{-i} is constant over $(a-\epsilon, a)$; if the function G_{-i} is discontinuous at $a \in (0,1)$, then strategy G_i must be continuous at a.*

Proof of Lemma B.2. We prove the first claim. We assume sender i's equilibrium strategy, G_i has a mass at $a \in (0,1)$. If $G_{-i}(a) = 0$, since G_{-i} is increasing over $[0,a]$, we directly have G_{-i} is constant over $[0,a]$. If $G_{-i}(a) > 0$, we prove it by contradiction and assume there exists some sender $k \neq i$ such that $\mathsf{supp}(G_k) \cap (a-\epsilon, a) \neq \emptyset$ for $\forall \epsilon > 0$. If $G_k(a) < 1$, then there exists $p \in (a-\epsilon, a)$ and $q \in (a,1) \cap \mathsf{supp}(G_k)$ such that sender k can contract the probabilities around p and q onto $a+\epsilon_1$ for sufficiently small $\epsilon_1 > 0$ to achieve a profitable deviation, which shows the assumption is invalid. If $G_k(a) = 1$, then we can also show the assumption is invalid using the similar idea. Therefore, we have that there exists $\epsilon > 0$ such that G_{-i} is constant over $(a-\epsilon, a)$.

Then it suffices to prove that for each sender i, there exists no $a \in (0,1)$ at which G_i and G_{-i} are both discontinuous. We prove it by contradiction and assume there exists sender i and a $a \in (0,1)$ at which G_i and G_{-i} are both discontinuous. We define sender i's utility when other senders adopt strategy G_{-i} and sender i reports the value x as $u_i(x)$. Because

G_{-i} has a mass at a, we have $u_i(a) < \lim_{x\to a^+} u_i(x)$ under the uniformly random tie-breaking rule. We have already proved $\int_0^a F_i(x)\ \mathrm{d}x > \int_0^a G_i(x)\ \mathrm{d}x$ and $a < 1$. We define

$$\sigma = \frac{1}{2}\min\{\int_0^a F_i(x)\ \mathrm{d}x - \int_0^a G_i(x)\ \mathrm{d}x, 1-a\}\ ,$$

and $G_i(a) - G_i(a^-) = p_i$. Based on strategy G_i, we can construct a deviation as below

$$G_i'(x) = \begin{cases} G_i(x), & \text{if } 0 \le x < a-\sigma \\ G_i(x) + rp_i, & \text{if } a-\sigma \le x < a \\ G_i(x) - (1-r)\,p_i, & \text{if } a \le x < a + \frac{\sigma r}{1-r} \\ G_i(x), & \text{if } a + \frac{\sigma r}{1-r} \le x \le 1 \end{cases}\ ,$$

in which $r \in (0, \frac{1-a}{\sigma+1-a})$ to ensure $a + \frac{\sigma r}{1-r} < 1$. It is obvious that $\int_0^1 F_i(x)\ \mathrm{d}x = \int_0^1 G_i'(x)\ \mathrm{d}x$. For any $x \in (a-\sigma, a+\frac{\sigma r}{1-r})$, we have

$$\int_0^x F_i(t)\ \mathrm{d}t - \int_0^x G_i'(t)\ \mathrm{d}t \ge \int_0^a F_i(t)\ \mathrm{d}t - \int_0^a G_i(t)\ \mathrm{d}t - \sigma \ge 0\ ,$$

which shows that the deviation $G_i' \in \mathsf{MPC}(F_i)$. Now we can calculate the utility increase brought by deviation G_i' when $r < \min\{\frac{1-a}{\sigma+1-a}, 1 - \frac{u_i(a)}{\lim_{x\to a^+} u_i(x)}\}$ as below

$$\begin{aligned} &\int_0^1 u_i(x)\ \mathrm{d}G_i'(x) - \int_0^1 u_i(x)\ \mathrm{d}G_i(x) \\ &= \int_{a-\sigma}^{a+\frac{\sigma r}{1-r}} u_i(x)\ \mathrm{d}G_i'(x) - \int_{a-\sigma}^{a+\frac{\sigma r}{1-r}} u_i(x)\ \mathrm{d}G_i(x) \\ &\ge (p_i - rp_i)\lim_{x\to a^+} u_i(x) - p_i u_i(a) > 0\ . \end{aligned}$$

Therefore, when $r < \min\{\frac{1-a}{\sigma+1-a}, 1-\frac{u_i(a)}{\lim_{x\to a^+} u_i(x)}\}$, the deviation G_i' can bring utility increase to sender i, which violates the equilibrium conditions and shows the assumption is invalid. Therefore we have proved that for each sender i, there exists no $a \in (0,1)$ at which G_i and G_{-i} are both discontinuous. ■ Below, we divide the proof of Theorem 4.2 into two parts: the first part proves the necessity of Theorem 4.2, and the second part proves the sufficiency of Theorem 4.2.

Proof of Necessity in Theorem 4.2.

Given $(G_1, \dots, G_N)$ is an equilibrium, we show $(G_1, \dots, G_N)$ satisfies all the conditions in Theorem 4.2. Directly by Lemma B.2, we have there exists no $x \in [0,1]$ at which G_i and G_{-i} are both discontinuous. That is (iv) in Theorem 4.2 holds true. We divide the remaining proof into the following three steps.

Step 1. We prove $\phi_i(x) \geq G_{-i}(x)$ for $\forall x \in [0, 1]$. It suffices to prove, for $\forall k \in [m-1]$, it always holds that $\phi_i(x) \geq G_{-i}(x)$ for $\forall x \in [v_k, v_{k+1})$ and $\phi_i(1) \geq G_{-i}(1)$. For convenience, we denote v_k as a and v_{k+1} as b.

(1) If Case 1 holds true over $[a, b]$, obviously we have $\phi_i(x) \geq G_{-i}(x)$ for each $x \in [a, b)$.

(2) If Case 2 holds true over $[a, b]$, we have $\phi_i(c) = G_{-i}(c)$ and $\phi_i(d) = G_{-i}(d)$ according to Definition 4.1. If we assume that there exists $e \in (c, d)$ such that $G_{-i}(e) > \phi_i(e)$, then sender i can contract the probabilities around c and d onto $e + \epsilon$ to achieve a profitable deviation for sufficiently small $\epsilon > 0$. The reason for transferring probabilities to $e + \epsilon$ instead of e is to prevent the occurrence of a tie at e which may make this contraction not profitable. If there exists $e \in [a, c)$ such that $G_{-i}(e) > \phi_i(e)$, then sender i can spread a probability λ around c onto $e+\epsilon$ and $d+\epsilon$ for sufficiently small $\epsilon > 0$ to achieve a profitable deviation (as long as λ is small enough, the new distribution is still an MPC of F_i). If there exists $e \in [d, b)$ such that $G_{-i}(e) > \phi_i(f)$, then sender i can spread the a probability λ around d onto $c + \epsilon$ and $e + \epsilon$ for sufficiently small $\epsilon > 0$ to achieve a profitable deviation. Therefore, we show $\phi_i(x) \geq G_{-i}(x)$ for any $x \in (a, b)$. In addition to ϕ_i is right-continuous at a, so we have $\phi_i(x) \geq G_{-i}(x)$ for any $x \in [a, b)$.

(3) If Case 3 holds true over $[a, b]$, we have there exists $\epsilon > 0$ such that G_{-i} is constant over $(c - \epsilon, c)$, and G_{-i} is continuous at point c according to Lemma B.2. Since G_{-i} is increasing over $[a, b]$ and $\phi_i(c) = G_{-i}(c)$, we have $\phi_i(x) \geq G_{-i}(x)$ for any $x \in (a, c]$. Since ϕ_i and G_{-i} are both right continuous, we have $\phi_i(a) \geq G_{-i}(a)$. If we assume that there exists $e \in (c, b)$ such that $G_{-i}(e) > \phi_i(e)$ and $f \in (c - \epsilon, c)$, then sender i can spread the probability around c onto $f + \epsilon_2$ and point $e + \epsilon_2$ for sufficiently small $\epsilon_2 > 0$ to achieve a profitable deviation. The deviation is feasible as long as the probability spread is small enough. Therefore, we have $\phi_i(x) \geq G_{-i}(x)$ for any $x \in (a, b)$. In addition to ϕ_i is right-continuous at a, so we have $\phi_i(x) \geq G_{-i}(x)$ for any $x \in [a, b)$.

Since each sender i has no mass at one constrained by the MPC conditions, function G_{-i} is left-continuous at one. In addition to $\phi_i(x) \geq G_{-i}(x)$ for $\forall x \in [v_{m-1}, v_m)$ and $v_m = 1$, we have $\phi_i(1) \geq G_{-i}(1)$. Till now, we have finished the first step of the proof. That is (ii) in Theorem 4.2 holds true.

Step 2. We prove for each $k \in [m-1]$, it always holds that $\phi_i(x)$ is convex over $[v_k, v_{k+1})$. For convenience, we denote v_k as a and v_{k+1} as b. Obviously, by Definition 4.1, ϕ_i is linear and also convex over $[a, b)$, if Case 2 or Case 3 holds true over $[a, b]$. If Case 1 holds true over $[a, b]$, we have $[a, b] \in \mathsf{supp}(G_i)$ and $\phi_i(x) = G_{-i}(x)$ for $\forall x \in [a, b)$. If there exist $c, d, e \in [a, b)$ such that $c < d < e$ and

$$G_{-i}(d) > \frac{e-d}{e-c} G_{-i}(c) + \frac{d-c}{e-c} G_{-i}(e) \ ,$$

then sender i can contract the probabilities around c and e onto $d+\epsilon$ to achieve a profitable deviation for sufficiently small $\epsilon > 0$. Therefore, to ensure there is no incentive for sender i to make any contraction inside $[a, b]$, ϕ_i must be convex over $[a, b)$. Till now, we have finished the second step of the proof.

Step 3. We prove for each $k \in [m-2]$, it always holds that $\phi_i(v_{k+1}^-) = \phi_i(v_{k+1}^+)$ and $\dot{\phi}_i(v_{k+1}^-) \le \dot{\phi}_i(v_{k+1}^+)$. For convenience, we denote v_k as a, v_{k+1} as b, and v_{k+2} as c. For each interval $[v_k, v_{k+1}]$, ϕ_i has three possible cases, resulting in nine different adjacency scenarios between every two adjacent segments. We will analyze each scenario separately in what follows.

(1) Suppose Case 1 holds true over $[a, b]$ and Case 1 also holds true over $[b, c]$. This scenario is obvious, as if G_i coincides with the prior F_i everywhere within two adjacent intervals, these two intervals can be merged into one.

(2) Suppose Case 1 holds true over $[a, b]$ and Case 2 holds true over $[b, c]$. There exist two distinct points $d, e \in [b, c] \cap \mathsf{supp}(G_i)$ and $d < e$. If we assume $\phi_i(b^-) < \phi_i(b^+)$, then there exists sufficiently small $\epsilon > 0$ such that sender i can contract the probabilities around e and $b - \epsilon$ onto $d + \epsilon_1$ to achieve a profitable deviation (for sufficiently small $\epsilon_1 > 0$), which shows the assumption is invalid. If we assume $\phi_i(b^-) > \phi_i(b^+)$, since Case 1 holds true over $[a, b]$, we have $\phi_i(b^-) = G_{-i}(b^-) > \phi_i(b^+) \ge G_{-i}(b^+)$, which contradicts the fact that G_{-i} is non-decreasing. Therefore, this assumption is also invalid. If we assume $\phi_i(b^-) = \phi_i(b^+)$ but $\dot{\phi}_i(b^-) > \dot{\phi}_i(b^+)$, then there exists point $f \in (a, b)$ such that sender i can contract the probabilities around e and f onto the $d + \epsilon$ to achieve a profitable deviation (for sufficiently small $\epsilon > 0$), which shows the assumption is invalid. Therefore, we prove that $\phi_i(b^-) = \phi_i(b^+)$ and $\dot{\phi}_i(b^-) \le \dot{\phi}_i(b^+)$.

(3) Suppose Case 1 holds true over $[a, b]$ and Case 3 holds true over $[b, c]$. There exists $d \in (b, c) \cap \mathsf{supp}(G_i)$. According to Definition 4.1, G_i has a mass at d. According to Lemma B.2, there exists $\epsilon > 0$ such that G_{-i} is constant over $(d - \epsilon, d]$. Therefore, we have $\phi_i(x) = G_{-i}(x)$ for $\forall x \in (d - \epsilon, d)$. If we assume $\phi_i(b^-) \le \phi_i(b^+)$, there exists $f \in (a, b)$ such that sender i can contract the probabilities around f and d onto point $d - \epsilon_1$ to achieve a profitable deviation(for sufficiently small $\epsilon_1 \in (0, \epsilon)$). If we assume $\phi_i(b^-) > \phi_i(b^+)$, since Case 1 holds true over $[a, b]$, we have $\phi_i(b^-) = G_{-i}(b^-) > \phi_i(b^+) \ge G_{-i}(b^+)$, which contradicts the fact G_{-i} is non-decreasing. Therefore, this scenario cannot occur even if we have $\phi_i(b^-) = \phi_i(b^+)$.

(4) Suppose Case 2 holds true over $[a, b]$ and Case 1 holds true over $[b, c]$. There exist two distinct points $d, e \in \mathsf{supp}(G_i) \cap [a, b]$ and $d < e$. If we assume $\phi_i(b^-) < \phi_i(b^+)$, there exists sufficiently small $\epsilon > 0$ such that we can find another MPC of prior, G_i', that is identical to G_i everywhere outside $[a, b]$, but has more probabilities than G_i in $(b - \epsilon, b)$. Then there exists $d' \in (b - \epsilon, b)$ such that sender i, based on strategy G_i', can contract the probabilities around d' and $b + \epsilon_1 + \epsilon_2$ onto point $b + \epsilon_1$ to achieve a profitable deviation (for sufficiently small $\epsilon_1 > 0,\ \epsilon_2 > 0$), which shows the assumption is invalid. If we assume $\phi_i(b^-) > \phi_i(b^+)$, then sender i can contract the probabilities around d and $b + \epsilon_1$ onto the $e + \epsilon_2$ to achieve a profitable deviation (for sufficiently small $\epsilon_1 > 0,\ \epsilon_2 > 0$), which shows the assumption is invalid. If we assume $\phi_i(b^-) = \phi_i(b^+)$ but $\dot{\phi}_i(b^-) > \dot{\phi}_i(b^+)$, then there exists point $f \in (b, c)$ such that sender i can contract the probabilities around d and f onto $e + \epsilon$ to achieve a profitable deviation (for sufficiently small $\epsilon > 0$), which shows the assumption is invalid. Therefore, we prove that $\phi_i(b^-) = \phi_i(b^+)$ and $\dot{\phi}_i(b^-) \leq \dot{\phi}_i(b^+)$.

(5) Suppose Case 2 holds true over $[a, b]$ and Case 2 holds true over $[b, c]$. There exist $d, e \in [a, b] \cap \mathsf{supp}(G_i)$ and $d < e$. There exist $f, g \in [b, c] \cap \mathsf{supp}(G_i)$ and $f < g$. If we assume $\phi_i(b^-) < \phi_i(b^+)$, there exists sufficiently small $\epsilon > 0$ such that we can find another MPC of prior, G_i', that is identical to G_i everywhere outside $[a, b]$, but has more probabilities than G_i in $(b - \epsilon, b)$. Then there exists point $d' \in (b - \epsilon, b)$ such that sender i, based on strategy G_i', can contract the probabilities around d' and g onto $f + \epsilon_1$ to achieve a profitable deviation(for sufficiently small $\epsilon_1 > 0$), which shows the assumption is invalid. If we assume $\phi_i(b^-) > \phi_i(b^+)$, there exists sufficiently small $\epsilon > 0$ such that we can find another MPC of prior, G_i', that is identical to G_i everywhere outside $[b, c]$, but has more probabilities than G_i in $(b, b + \epsilon)$. Then there exists $f' \in (b, b + \epsilon)$ such that sender i, based on strategy G_i', can contract the probabilities around d and f' onto $e + \epsilon_1$ to achieve a profitable deviation(for sufficiently small $\epsilon_1 > 0$), which shows the assumption is invalid. If we assume $\phi_i(b^-) = \phi_i(b^+)$ but $\dot{\phi}_i(b^-) > \dot{\phi}_i(b^+)$, then sender i can contract the probabilities around d and f onto $e + \epsilon$ to achieve a profitable deviation (for sufficiently small $\epsilon > 0$), which shows the assumption is invalid. Therefore, we prove that $\phi_i(b^-) = \phi_i(b^+)$ and $\dot{\phi}_i(b^-) \leq \dot{\phi}_i(b^+)$.

(6) Suppose Case 2 holds true over $[a, b]$ and Case 3 holds true over $[b, c]$. There exist $d, e \in [a, b] \cap \mathsf{supp}(G_i)$ and $d < e$. There exists $f \in (b, c) \cap \mathsf{supp}(G_i)$ at which G_i has a mass. According to Lemma B.2, there exists $\epsilon > 0$ such that G_{-i} is constant over $(f - \epsilon, f]$. Therefore, we have $\phi_i(x) = G_{-i}(x)$ for $\forall x \in (f - \epsilon, f]$. If we assume $\phi_i(b^-) \leq \phi_i(b^+)$, then sender i can contract the probabilities around e and f onto $f - \epsilon_1$ to achieve a profitable deviation(for sufficiently small $\epsilon_1 \in (0, \epsilon)$). If we assume $\phi_i(b^-) > \phi_i(b^+)$, then sender i can contract the probabilities around d and f onto $e - \epsilon_2$ to achieve a profitable deviation(for sufficiently small $\epsilon_2 > 0$). Therefore, this scenario cannot occur even if we have $\phi_i(b^-) = \phi_i(b^+)$.

(7) Suppose Case 3 holds true over $[a, b]$ and Case 1 holds true over $[b, c]$. There exists $d \in (a, b) \cap \mathsf{supp}(G_i)$ at which G_i has a mass. We can find another MPC of prior, G_i', that is identical to G_i except spreading the mass in G_i to $d - \epsilon_1$ and $b - \epsilon_1$. G_i' will achieve better utility than strategy G_i. Then, based on G_i', sender i can contract the probabilities at $b - \epsilon_1$ and around $b + \epsilon_1 + \epsilon_2$ onto $b + \epsilon_1$ to achieve a profitable deviation, which shows the assumption is invalid. If we assume $\phi_i(b^-) > \phi_i(b^+)$, since Case 3 holds true over $[a, b]$, we have $\phi_i(b^-) = \phi_i(d) = G_{-i}(d)$. Therefore, it holds that $G_{-i}(b^+) \leq \phi_i(b^+) < \phi_i(b^-) = G_{-i}(d)$, which contradicts the fact G_{-i} is non-decreasing and shows the assumption invalid. Since $\dot{\phi}_i(b^-) = 0$, it holds that $\dot{\phi}_i(b^-) \leq \dot{\phi}_i(b^+)$ automatically. Therefore, we prove that $\phi_i(b^-) = \phi_i(b^+)$ and $\dot{\phi}_i(b^-) \leq \dot{\phi}_i(b^+)$.

(8) Suppose Case 3 holds true over $[a, b]$ and Case 2 holds true over $[b, c]$. There exists $d \in (a, b) \cap \mathsf{supp}(G_i)$ at which G_i has a mass. There exist $e, f \in [b, c] \cap \mathsf{supp}(G_i)$ and $e < f$. According to Lemma B.2, there exists $\epsilon > 0$ such that G_{-i} is constant over $(d - \epsilon, d]$ and we have $\phi_i(x) = G_{-i}(x)$ for $\forall x \in (d - \epsilon, d]$. Since $\phi_i(x) \geq G_{-i}(x)$ for $\forall x \in [d, b)$, $G_{-i}(d) = \phi_i(d)$ and G_{-i} is non-decreasing, we have $G_{-i}(x) = \phi_i(d)$ for $\forall x \in [d, b)$. If we assume $\phi_i(b^-) < \phi_i(b^+)$, since G_{-i} is constant over $[d - \epsilon, b)$, there exists $\epsilon_1 > 0$ such that sender i can spread probabilities at d to $d - \epsilon_1$ and $b - \epsilon_1$ with no utility decrease. Then, sender i can contract the probabilities around $b - \epsilon_1$ and f onto $e + \epsilon_2$ to achieve a profitable deviation (ϵ_2 is sufficiently small), which shows the assumption is invalid. If we assume $\phi_i(b^-) > \phi_i(b^+)$. Since we have $\phi_i(b^-) = \phi_i(d) = G_{-i}(d)$, then we get $G_{-i}(b^+) \leq \phi_i(b^+) < \phi_i(b^-) = G_{-i}(d)$, which contradicts the fact G_{-i} is non-decreasing. If we assume $\phi_i(b^-) = \phi_i(b^+)$ but $\dot{\phi}_i(b^-) > \dot{\phi}_i(b^+)$, since $\dot{\phi}_i(b^-) = 0$, the assumption is also invalid. Therefore, we prove that $\phi_i(b^-) = \phi_i(b^+)$ and $\dot{\phi}_i(b^-) \leq \dot{\phi}_i(b^+)$.

(9) Suppose Case 3 holds true over $[a, b]$ and Case 3 holds true over $[b, c]$. There exists $d \in (a, b) \cap \mathsf{supp}(G_i)$ at which G_i has a mass. There exists $e \in (b, c) \cap \mathsf{supp}(G_i)$ at which G_i has a mass too. According to Lemma B.2, there exists $\epsilon > 0$ such that G_{-i} is constant over $(e - \epsilon, e]$ and we have $\phi_i(x) = G_{-i}(x)$ for $\forall x \in (e - \epsilon, e]$. If we assume $\phi_i(b^-) < \phi_i(b^+)$, sender i can contract the probabilities around d and e onto $e - \epsilon_1$ to achieve a profitable deviation(for sufficiently small $\epsilon_1 \in (0, \epsilon)$). If we assume $\phi_i(b^-) > \phi_i(b^+)$, since ϕ_i Case 3 holds true over $[a, b]$, we have $\phi_i(b^-) = \phi_i(d) = G_{-i}(d)$. Therefore, we have $G_{-i}(b^+) \leq \phi_i(b^+) < \phi_i(b^-) = G_{-i}(d)$, which contradicts the fact G_{-i} is non-decreasing. Therefore, the assumption is invalid. Since $\dot{\phi}_i(b^-) = \dot{\phi}_i(b^+) = 0$, we have $\phi_i(b^-) = \phi_i(b^+)$ and $\dot{\phi}_i(b^-) \leq \dot{\phi}_i(b^+)$. Till now, we have proved that for each $k \in [m-2]$, it always holds that $\phi_i(v_{k+1}^-) = \phi_i(v_{k+1}^+)$ and $\dot{\phi}_i(v_{k+1}^-) \leq \dot{\phi}_i(v_{k+1}^+)$. Combining all the above proofs and Lemma B.2, we conclude that, if a strategy profile $(G_1, \ldots, G_N)$ is an equilibrium, then it satisfies all the conditions in Theorem 4.2. ■

Proof of Sufficiency in Theorem 4.2. We divide this part of proof into three steps.

Step 1. We show for each sender $i \in [N]$, if his virtual competitive function ϕ_i satisfies Conditions (i), (ii) and (iii) in Theorem 4.2, then strategy G_i is a best response to the function ϕ_i. By Theorem 4.2, the virtual competitive function ϕ_i directly satisfies Conditions (i) and (ii) in Theorem 4.1. For Condition (iii) in Theorem 4.1, we consider each $k \in [m-1]$ and denote v_k as a, v_{k+1} as b for convenience. In Case 1, $\phi_i(x) = G_{-i}(x)$ holds true for any $x \in [a, b]$ by Definition 4.1. In Case 2, first we have $c, d \in \mathsf{supp}(G_i)$ and $\phi_i(c) = G_{-i}(c)$, $\phi_i(d) = G_{-i}(d)$. If there exists $e \in (c, d) \cap \mathsf{supp}(G_i)$, we have $\phi_i(e) \geq G_{-i}(e)$ by (i) in Theorem 4.2. If $\phi_i(e) > G_{-i}(e)$, then sender i can spread probabilities around e to $c + \epsilon$ and $d + \epsilon$ to achieve a profitable deviation (for sufficiently small $\epsilon > 0$). So $\phi_i(e) = G_{-i}(e)$. If there exists $e \in (a, c) \cap \mathsf{supp}(G_i)$, or there exists $f \in (d, b) \cap \mathsf{supp}(G_i)$, we can prove $\phi_i(e) = G_{-i}(e)$ and $\phi_i(f) = G_{-i}(f)$ using the same idea. In Case 3, the equation holds automatically. In summary, we prove that $\phi_i(x) = G_{-i}(x)$ for any $x \in \mathsf{supp}(G_i)$. For Condition (iv) in Theorem 4.1, we have either $G_i(x) = F_i(x)$ for any $x \in [v_k, v_{k+1}]$, or ϕ_i is linear over $x \in [v_k, v_{k+1}]$ and G_i forms a MPC of F_i over the same interval. For either type, we have

$$\int_{v_k}^{v_{k+1}} \phi_i(x) \, \mathrm{d}G_i(x) = \int_{v_k}^{v_{k+1}} \phi_i(x) \, \mathrm{d}F_i(x) \ ,$$

which leads to

$$\int_0^1 \phi_i(x) \, \mathrm{d}G_i(x) = \int_0^1 \phi_i(x) \, \mathrm{d}F_i(x) \ .$$

To sum up, if the virtual competitive function ϕ_i satisfies (i), (ii), and (iii) in Theorem 4.2, then G_i forms a best response strategy to the function ϕ_i according to Theorem 4.1.

Step 2. Based on the fact G_i is a best response to the function ϕ_i, we show G_i is also a best response to the function G_{-i}. By Theorem 4.2, we have $\phi_i(x) \geq G_{-i}(x)$ for $\forall x \in [0, 1]$ and $\phi_i(x) = G_{-i}(x)$ for $\forall x \in \mathsf{supp}(G_i)$, which implies that

$$\int_0^1 \phi_i(x) \, \mathrm{d}G'_i(x) \geq \int_0^1 G_{-i}(x) \, \mathrm{d}G'_i(x), \quad \forall G'_i \in \mathsf{MPC}(F_i) \ ,$$

and

$$\int_0^1 \phi_i(x) \, \mathrm{d}G_i(x) = \int_0^1 G_{-i}(x) \, \mathrm{d}G_i(x) \ .$$

Therefore, we have G_i is also a best response to the function G_{-i}.

Step 3. Based on the fact G_i is a best response to the function G_{-i} and combining Condition (iv) in Theorem 4.2, we show G_i is also a best response to the function u_i. We define sender i's utility when other senders adopt strategy G_{-i} and sender i reports the value $x \sim G_i$ as $u_i(x)$. Obviously we have $u_i(x) \le G_{-i}(x)$ for $\forall x \in [0,1]$ under any tie-breaking rule, so we have

$$\int_0^1 u_i(x)\ \mathrm{d}G_i'(x) \le \int_0^1 G_{-i}(x)\ \mathrm{d}G_i'(x), \quad \forall G_i' \in \mathsf{MPC}(F_i)\ .$$

Function u_i and function G_{-i} are unequal only at points where G_{-i} is discontinuous. We define set Z as follows

$$Z = \{x \in \mathsf{supp}(G_i) : G_{-i}(x^-) \neq G_{-i}(x^+)\}\ .$$

By (iv) in Theorem 4.2, G_i is continuous at any single point of set Z. Therefore, the utility brought to sender i by every single point of set Z is zero and it holds that

$$\int_0^1 u_i(x)\ \mathrm{d}G_i(x) = \int_0^1 G_{-i}(x)\ \mathrm{d}G_i(x)\ .$$

Therefore, we have G_i is also a best response strategy to the function $u_i(x)$.

In summary, if a feasible strategy profile $(G_1, \ldots, G_N)$ satisfies all the conditions in Theorem 4.2, then it forms a Nash equilibrium in our competitive information design game. ■

B.3 Proof of Corollary 4.4

Proof of Corollary 4.4. We consider each interval $[v_k, v_{k+1}]$. For convenience, we denote v_k as a and v_{k+1} as b. For Case 1, $\phi_i(x) = G_{-i}(x)$ holds true for any $x \in [a,b]$ by Definition 4.1. For Case 2, first we have $c, d \in \mathsf{supp}(G_i)$ and $\phi_i(c) = G_{-i}(c)$, $\phi_i(d) = G_{-i}(d)$. If there exists a point $e \in (c,d) \cap \mathsf{supp}(G_i)$, we have $\phi_i(e) \ge G_{-i}(e)$ by (ii) in Theorem 4.2. If $\phi_i(e) > G_{-i}(e)$, then sender i can spread probabilities around the point e to point $c+\epsilon$ and point $d+\epsilon$ to achieve a profitable deviation (for sufficiently small $\epsilon > 0$). So $\phi_i(e) = G_{-i}(e)$. If there exists a point $e \in (a,c) \cap \mathsf{supp}(G_i)$, or there exists a point $f \in (d,b) \cap \mathsf{supp}(G_i)$, we can prove $\phi_i(e) = G_{-i}(e)$ and $\phi_i(f) = G_{-i}(f)$ using the same idea. For Case 3, the equation holds automatically. In summary, we prove that $\phi_i(x) = u_i(x)$ for any $x \in \mathsf{supp}(G_i)$. ■

B.4 Proof of Corollary 4.5

Proof of Corollary 4.5. Assume there exists $y > \tau$ and $k \in [N]$ such that $G_k(x)$ has mass at point y.

Firstly, let's prove that $\exists \epsilon > 0$ such that $\forall i \neq k, (y-\epsilon, y) \cap \mathsf{supp}(G_i) = \emptyset$. Assume

$\exists i \in [N], \forall \epsilon > 0, (y-\epsilon, y) \cap \mathsf{supp}(G_i) \neq \emptyset$. Consequently, $\lim_{x \to y^-} \phi_i(x) = \prod_{j \neq i} G_j(y^-)$. Since $G_k(x)$ has mass at y,

$$\prod_{j \neq i} G_j(y^-) < \prod_{j \neq i} G_j(y) = \phi_i(y)$$

This implies that $\phi_i(x)$ is discontinuous at y, which contradicts Theorem 4.2. Therefore, $\exists \epsilon \in (0,1)$ such that $\forall i \neq k, (y-\epsilon, y) \cap \mathsf{supp}(G_i) = \emptyset$. Similarly, $\forall i \neq k$, G_i has no mass at y.

Since $y \in \mathsf{supp}(G_k)$, by Theorem 4.1, $\phi_k(y) = \prod_{i \neq k} G_i(y)$. Due to the existence of $\epsilon > 0$ such that for all $i \neq k$, $(y-\epsilon, y) \cap \mathsf{supp}(G_i) = \emptyset$, it follows that $\prod_{i \neq k} G_i(x)$ is constant in $(y-\epsilon, y)$. According to Theorem 4.1, $\phi_k(x) \geq \prod_{i \neq k} G_i(x)$, and $\phi_k(x)$ is a convex function. Therefore, $\forall x < y$, $\phi_k(x) = \prod_{i \neq k} G_i(y) > 0$. By the definition of $\underline{x}$, there exists $t \in [0, \underline{x} + \epsilon]$ such that $t \in \mathsf{supp}(G_k)(\epsilon > 0$ and ϵ is sufficiently small). Therefore, $\phi_k(t) = \prod_{i \neq k} G_i(t) = \prod_{i \neq k} G_i(y)$. Since $\prod_{i \neq k} G_i(x)$ is non-decreasing and $\phi_k(x) \geq \prod_{i \neq k} G_i(x)$, when $x \in (\underline{x}, y)$, $\prod_{i \neq k} G_i(x) = \phi_k(x) = \prod_{i \neq k} G_i(y)$. Consequently, for any $x \in [\underline{x}, y)$, $\forall i \neq k, G_i(x) = G_i(\underline{x})$. This contradicts the assumption that $y > \tau$. Therefore, for $x > \tau$, there is no mass in $G_i(x)$ for any i. ■

C Omitted Proofs in Section 5

C.1 Proof of Theorem 5.2

To prove Theorem 5.2, we need to utilize the following conclusions: Lemma C.1, Lemma C.2, Theorem 5.1 and Lemma C.3.

Lemma C.1. *Given (G_1, G_2) is an equilibrium, then G_1 and G_2 both have no mass at any $x \geq \tau$.*

Proof of Lemma C.1. According to Corollary 4.5, for $\forall x > \tau$, there is no mass in G_1 and G_2. To begin with, we prove that G_1 and G_2 do not simultaneously have mass at point τ. Since $G_{-1} = G_2$ and $G_{-2} = G_1$, if G_1, G_2 simultaneously have mass at point τ, we have G_{-1} and G_1 are both discontinuous at τ, which contradicts Lemma B.2. Therefore, G_1 and G_2 do not simultaneously have mass at point τ. Next, we prove that either G_1 or G_2 has no mass at the point τ.

Case 1: When $\underline{x_1} = \underline{x_2}$, it is evident that $\underline{x} = \tau = \underline{x_1} = \underline{x_2}$. We assume G_1 has mass at τ, and G_2 has no mass at τ. There exists $\epsilon, \lambda > 0$ such that G_1' increases the probability compared to G_1 by $\frac{\lambda}{2}$ at $\tau - \frac{\epsilon}{2}$ and $\tau + \frac{\epsilon}{2}$, and decreases the probability by λ at τ. As long as ϵ

and λ are sufficiently small, $G_1' \in \mathsf{MPC}(F_1)$. At this point, strategy G_1' achieves $\frac{\lambda}{2} \cdot G_2(\tau + \frac{\epsilon}{2})$ higher utility than strategy G_1. Therefore, G_1 can achieve a profitable deviation, leading to a contradiction with the equilibrium, which demonstrates the invalidity of the assumption.

Case 2: When $\underline{x_1} \neq \underline{x_2}$, without loss of generality, assume $\underline{x_1} < \underline{x_2}$, and in this case, $\underline{x} = \underline{x_2}$.

Case 2.1: When G_2 has mass at τ, and G_1 has no mass at τ. According to Lemma B.2, it is known that $\exists \epsilon > 0, (\tau - \epsilon, \tau) \cap \mathsf{supp}(G_1) = \emptyset$. There exists $\epsilon, \lambda > 0$ such that G_2' increases the probability compared to G_2 by $\frac{\lambda}{2}$ at $\tau - \frac{\epsilon}{2}$ and $\tau + \frac{\epsilon}{2}$, and decreases the probability by λ at τ. As long as ϵ and λ are sufficiently small, $G_2' \in \mathsf{MPC}(F_2)$. At this point, the utility difference between G_2' and G_2 is $\frac{\lambda}{2} \cdot [G_1(\tau + \frac{\epsilon}{2}) - G_1(\tau)]$. Since G_1 has no mass at τ and, by the definition of τ, $G_1(\tau + \frac{\epsilon}{2}) > G_1(\tau)$, the utility of G_2' is greater than that of G_2. Therefore, G_2 can achieve a profitable deviation, leading to a contradiction with the equilibrium, demonstrating the invalidity of the assumption.

Case 2.2: when G_1 has a mass at τ, a deviation would occur using the same argument.

■

Lemma C.2. *Given (G_1, G_2) is an equilibrium, then we have $\tau = \sup\{x : \exists i \in \{1, 2\}, \dot{\phi}_i(x) = 0\}$.*

Proof of Lemma C.2. Case 1: When $\underline{x_1} \neq \underline{x_2}$, without loss of generality, we assume $\underline{x_1} < \underline{x_2}$.

Case 1.1: When $\tau > \underline{x_2}$, since $\mathsf{supp}(G_2) \cap [\underline{x_2}, \tau) \neq \emptyset$, there exists $\widetilde{x} \in \mathsf{supp}(G_2) \cap [\underline{x_2}, \tau)$. According to Corollary 4.4, $\phi_2(\widetilde{x}) = G_1(\widetilde{x})$. By the definition of τ, $\forall x > \tau, G_1(x) > G_1(\tau)$, so $\forall x > \tau, \phi_2(x) \geq G_1(x) > G_1(\tau) \geq G_1(\widetilde{x}) = \phi_2(\widetilde{x})$. Therefore, $\sup\{x : \dot{\phi}_2(x) = 0\} = \tau$(otherwise, $\exists x > \tau, \phi_2(x) = \phi_2(\widetilde{x})$, which contradicts the fact that $\forall x > \tau, \phi_2(x) > \phi_2(\widetilde{x})$). Because $\mathsf{supp}(G_1) \cap [\underline{x_1}, \underline{x_2}) \neq \emptyset$, there exists $\hat{x} \in \mathsf{supp}(G_1) \cap [\underline{x_1}, \underline{x_2})$, $\phi_1(\hat{x}) = G_2(\hat{x}) = 0$. According to the definition of $\underline{x_2}$, $\exists \hat{x}' \in [\underline{x_2}, \tau)$ such that $G_2(\hat{x}') > 0$. According to Lemma C.1, G_1, G_2 have no mass at τ, so by the definition of τ, $\forall x > \tau, \exists x' \in \mathsf{supp}(G_1) \cap (\tau, x)$. Then, $\phi_1(x) \geq \phi_1(x') = G_2(x') \geq G_2(\hat{x}') > 0$. Therefore, $\sup\{x : \dot{\phi}_1(x) = 0\} \leq \tau$ (otherwise, $\exists x > \tau, \phi_1(x) = 0$). Therefore, we have $\sup\{x : \exists i \in \{1, 2\}, \dot{\phi}_i(x) = 0\} = \tau$.

Case 1.2: When $\tau = \underline{x_2}$, in this case, $\tau = \underline{x}$. According to Lemma C.1, G_1 and G_2 have no mass at τ. By the definitions of τ and $\underline{x}$, for $\forall x > \tau$, there exists $x' \in \mathsf{supp}(G_1) \cap (\tau, x)$ such that $\phi_1(x) \geq \phi_1(x') = G_2(x') > 0$. Therefore, $\sup\{x : \dot{\phi}_1(x) = 0\} = \tau$ (otherwise, $\exists x > \tau, \phi_1(x) = 0$). Similarly, $\sup\{x : \dot{\phi}_2(x) = 0\} \leq \tau$. Therefore, we have $\sup\{x : \exists i \in \{1, 2\}, \dot{\phi}_i(x) = 0\} = \tau$.

Case 2: When $\underline{x_1} = \underline{x_2}$, the analysis is the same as in Case 1.2. Therefore, we conclude that $\sup\{x : \exists i \in \{1, 2\}, \dot{\phi}_i(x) = 0\} = \tau$. ■

Proof of Theorem 5.1. First, we prove $\overline{\mathsf{supp}(G_1)} \cap [\tau, 1] = \overline{\mathsf{supp}(G_2)} \cap [\tau, 1]$ by contradiction. Suppose $(\overline{\mathsf{supp}(G_1)} - \overline{\mathsf{supp}(G_2)}) \cap [\tau, 1] \neq \emptyset$. We w.l.o.g. assume $\exists a, b \in [\tau, 1]$, $\overline{\mathsf{supp}(G_1)} \cap (a, b) = \emptyset$, and $G_2(b) - G_2(a) \neq 0$. According to Lemma C.1, G_2 has no mass over $[a, b]$, so there exist $c, d \in (a, b)$ such that $(c, d) \subseteq \overline{\mathsf{supp}(G_2)}$. According to Corollary 4.4, we have $\phi_2(c) = G_1(c)$ and $\phi_2(d) = G_1(d)$. Since $\overline{\mathsf{supp}(G_1)} \cap [a, b] = \emptyset$, $\phi_2(c) = G_1(c) = G_1(d) = \phi_2(d)$, so ϕ_2 is constant over (c, d), $\dot{\phi}_2(d^-) = 0$. According to Lemma C.2, $\sup\{x : \dot{\phi}_2(x) = 0\} \leq \tau$ which forms a contradiction. Therefore, $\overline{\mathsf{supp}(G_1)} \cap [\tau, 1] = \overline{\mathsf{supp}(G_2)} \cap [\tau, 1]$ which implies that $\sup \mathsf{supp}(G_1) = \sup \mathsf{supp}(G_2)$, $\overline{x} \in \overline{\mathsf{supp}(G_1) \cap \mathsf{supp}(G_2)}$. Since G_2 has no mass in $[\tau, 1]$, $\overline{\mathsf{supp}(G_2)} \cap [\tau, 1]$ is a union of intervals.

Next, we prove that $\overline{\mathsf{supp}(G_1)} \cap [\tau, 1] = \overline{\mathsf{supp}(G_2)} \cap [\tau, 1] = [\tau, \overline{x}]$. Suppose there exist $a, b, c, d \in [\tau, 1]$ $(a < b < c < d)$, such that $[a, b] \subseteq \overline{\mathsf{supp}(G_1)} \cap [\tau, 1]$ and $[c, d] \subseteq \overline{\mathsf{supp}(G_1)} \cap [\tau, 1]$ and $(b, c) \cap \overline{\mathsf{supp}(G_1)} = \emptyset$. Since $\overline{\mathsf{supp}(G_1)} \cap [\tau, 1] = \overline{\mathsf{supp}(G_2)} \cap [\tau, 1]$, $G_1(b) = G_1(c)$, $G_2(b) = G_2(c)$. According to Corollary 4.4, we have $\phi_2(c) = G_1(c) = G_1(b) = \phi_2(b)$, so ϕ_2 is constant in (b, c), $\dot{\phi}_2(c^-) = 0$. According to Lemma C.2, $\sup\{x : \dot{\phi}_2(x) = 0\} \leq \tau$. This shows the assumption is invalid. Since G_1 and G_2 has no mass in $[\tau, 1]$, $\overline{\mathsf{supp}(G_1)} \cap [\tau, 1]$ and $\overline{\mathsf{supp}(G_2)} \cap [\tau, 1]$ would be unions of intervals, $\tau \in \overline{\mathsf{supp}(G_1) \cap \mathsf{supp}(G_2)}$ and $\overline{x} \in \overline{\mathsf{supp}(G_1) \cap \mathsf{supp}(G_2)}$, we have $\overline{\mathsf{supp}(G_1)} \cap [\tau, 1] = \overline{\mathsf{supp}(G_2)} \cap [\tau, 1] = [\tau, \overline{x}]$.

According to the definition of τ and $\underline{x}$, we have either $\overline{\mathsf{supp}(G_1)} \subseteq [0, \underline{x}] \cup [\tau, \overline{x}]$, $\overline{\mathsf{supp}(G_2)} \subseteq [\underline{x}, \overline{x}]$, or vice versa, and $\overline{\mathsf{supp}(G_1) \cap \mathsf{supp}(G_2)} - \{\underline{\mathrm{x}}\} = [\tau, \overline{x}]$. ■

Lemma C.3. *We w.l.o.g. assume $\underline{x_1} \leq \underline{x_2}$.*

- *If $0 < \underline{x} < \tau < \overline{x}$, then we have $\int_0^{\underline{x}} F_1(x)\ \mathrm{d}x = \int_0^{\underline{x}} G_1(x)\ \mathrm{d}x$, $\int_0^{\underline{x}} F_2(x)\ \mathrm{d}x > \int_0^{\underline{x}} G_2(x)\ \mathrm{d}x$, $\int_0^{\tau} F_1(x)\ \mathrm{d}x > \int_0^{\tau} G_1(x)\ \mathrm{d}x$ and $\int_0^{\tau} F_2(x)\ \mathrm{d}x = \int_0^{\tau} G_2(x)\ \mathrm{d}x$.*

- *If $0 < \underline{x} = \tau < \overline{x}$, then we have $\int_0^{\tau} F_1(x)\ \mathrm{d}x = \int_0^{\tau} G_1(x)\ \mathrm{d}x$ and $\int_0^{\tau} F_2(x)\ \mathrm{d}x > \int_0^{\tau} G_2(x)\ \mathrm{d}x$.*

- *If $0 = \underline{x} = \tau < \overline{x}$, then we have $\int_0^{\tau} F_1(x)\ \mathrm{d}x = \int_0^{\tau} G_1(x)\ \mathrm{d}x$ and $\int_0^{\tau} F_2(x)\ \mathrm{d}x = \int_0^{\tau} G_2(x)\ \mathrm{d}x$.*

Proof of Lemma C.3. First, we prove $\tau < \overline{x}$ by contradiction. We assume $\tau = \overline{x}$. By Theorem 5.1, we have sender 2's expected utility equals one, while sender 1's expected utility equals zero. Since both priors are full-support over $[0, 1]$, sender 1 can switch his current strategy to his prior and gain a strictly positive expected utility, which violates the equilibrium conditions and makes the assumption invalid. By Theorem 5.1, we have $0 \leq \underline{x} \leq \tau$. The case where $0 = \underline{x} < \tau$ is impossible, as it contradicts the assumption that $x_1 \leq x_2$. Consider Case 1 where $0 < \underline{x} < \tau < \overline{x}$. By Theorem 4.2, we have $\dot{\phi}_1(\underline{x}^-) < \dot{\phi}_1(\underline{x}^+)$, which implies that $\int_0^{\underline{x}} F_1(x)\ \mathrm{d}x = \int_0^{\underline{x}} G_1(x)\ \mathrm{d}x$ according to Corollary 4.3. Because $G_2(x) = 0$

for $\forall x \in [0, \underline{x}]$ and F_2 is full-support over $[0,1]$, it holds that $\int_0^{\underline{x}} F_2(x)\ \mathrm{d}x > \int_0^{\underline{x}} G_2(x)\ \mathrm{d}x$. In the same manner, we have $\int_0^{\tau} F_1(x)\ \mathrm{d}x > \int_0^{\tau} G_1(x)\ \mathrm{d}x$ and $\int_0^{\tau} F_2(x)\ \mathrm{d}x = \int_0^{\tau} G_2(x)\ \mathrm{d}x$. Consider Case 2 where $0 < \underline{x} = \tau < \overline{x}$. By Theorem 4.2, we have $\dot{\phi}_1(\tau^-) < \dot{\phi}_1(\tau^+)$, which implies that $\int_0^{\tau} F_1(x)\ \mathrm{d}x = \int_0^{\tau} G_1(x)\ \mathrm{d}x$ according to Corollary 4.3. Because $G_2(x) = 0$ for $\forall x \in [0, \tau]$ and F_2 is full-support over $[0,1]$, it holds that $\int_0^{\tau} F_2(x)\ \mathrm{d}x > \int_0^{\tau} G_2(x)\ \mathrm{d}x$. Consider Case 3 where $0 = \underline{x} = \tau < \overline{x}$. We directly have $\int_0^{\tau} F_1(x)\ \mathrm{d}x = \int_0^{\tau} G_1(x)\ \mathrm{d}x$ and $\int_0^{\tau} F_2(x)\ \mathrm{d}x = \int_0^{\tau} G_2(x)\ \mathrm{d}x$. ■

Lemma C.4. *Given an equilibrium $(G_1, \ldots, G_N)$ is an Alternating MPC equilibrium, both strategies G_1 and G_2 are linear over each interval $[m_j, \min\{m_{j+1}, \overline{x}\}]$ for $\forall j \in [w-1]$. Specifically, for $\forall j \in [w-1]$ and for each sender $i = 1, 2$, if $m_j \in M_i$, then we have*

$$G_i(x) = \min\left\{k_i(x - m_j) + F_i(m_j), 1\right\} \quad \forall x \in [m_j, m_{j+1}]\ ,$$

and

$$G_{-i}(x) = \min\left\{k_{-i}(x - m_j) + G_{-i}(m_j), 1\right\} \quad \forall x \in [m_j, m_{j+1}]\ .$$

If $j = 1$ then $k_i = G_i(m_j^+)$, otherwise $k_i = G_i(m_j^-)$. And we have

$$k_{-i} = \max\{k : \int_0^{m_j} G_{-i}(t)\ \mathrm{d}t + \int_{m_j}^{x} \min\left\{k(t - m_j) + G_{-i}(m_j), 1\right\} \mathrm{d}t \le \int_0^{x} F_{-i}(t)\ \mathrm{d}t,\ \forall x \in [m_j, 1]\}.$$

Proof of Lemma C.4.

By the definition of set M, for each $j \in [w-1]$, we know $\int_0^y F_1(x)\ \mathrm{d}x > \int_0^y G_1(x)\ \mathrm{d}x$ and $\int_0^y F_2(x)\ \mathrm{d}x > \int_0^y G_2(x)\ \mathrm{d}x$ for $\forall y \in (m_j, m_{j+1})$. By Theorem 4.2 and Corollary 4.4, we have G_1 and G_2 are both linear over $[m_j, \min\{m_{j+1}, 1\}]$. By Corollary 4.5, we have G_1 and G_2 are both continuous over $[\tau, 1]$. By Definition 5.2, we have $m_j \notin M_1 \cap M_2$ and for each $j \in [w-1]$, either $m_j \in M_1$ or $m_j \in M_2$. Combining all the facts above, we have already proved the characterizations of G_1 and G_2 in the case of Alternating MPC equilibrium.

For $\forall j \in [w-1]$ and for each sender $i = 1, 2$, if $m_j \in M_i$, then we have $G_i(m_j) = F_i(m_j)$. If $j = 1$ then $\dot{G}_i(m_j^-)$ not necessarily equals to $\dot{G}_i(m_j^+)$, so we let $k_i = \dot{G}_i(m_j^+)$. If $j > 1$ then we have $\dot{G}_i(m_j^-) = \dot{G}_i(m_j^+)$ by Theorem 4.2, and so we let $k_i = \dot{G}_i(m_j^-)$.

Next we explain why we specify the parameter k_{-i} like above. We know that G_{-i} shoots out a straight line from point m_j to point m_{j+1} to form a local MPC, which means that the slope of the line must remain the same between (m_j, m_{j+1}). By Theorem 4.2 and Corollary 4.4, we have strategy G_{-i} is convex over $[\tau, \overline{x}]$. On the one hand, if there exists $a \in [m_j, 1]$ such that $\int_0^{m_j} G_{-i}(x)\ \mathrm{d}x + \int_{m_j}^{a} \min\left\{k_{-i}(t - m_j) + G_{-i}(m_j), 1\right\}\ \mathrm{d}t > \int_0^1 F_{-i}(t)\ \mathrm{d}t$, then G_{-i} definitely cannot form an MPC of his prior F_{-i}. On the other hand, if $\int_0^{m_j} G_{-i}(x)\ \mathrm{d}x + \int_{m_j}^{x} \min\{k(t-m_j)+G_{-i}(m_j), 1\}\ \mathrm{d}t < \int_0^1 F_{-i}(t)\ \mathrm{d}t$ for $\forall x \in [m_j, 1]$, then G_{-i} also cannot form an MPC of his prior F_{-i} at m_{j+1}. ■

Now we can provide the complete proof of Theorem 5.2.

We first provide a proof sketch. The proof essentially utilize Theorem 5.1. When each sender has a strictly uni-modal prior, the classification of equilibrium is based on whether there is a continuous interval in the set M_1 or M_2. If there is one, then we prove that $\tau = 0$ and both strategies must coincide with their respective prior from zero to a certain point, and then behave as a straight line to form a local MPC from that boundary point to one. This describes Case 1 in Theorem 5.2 (see example in Figure 4). On the other hand, if there does not exist a continuous interval both in M_1 and M_2, then M_1 and M_2 must contain only discrete points. We prove that, starting from τ, strategies G_1 and G_2 alternately form a local MPC of their respective prior, until reaching one, where they once again form the local MPC simultaneously. Moreover, due to both priors are strictly uni-modal over $[0, 1]$, the strategies G_1 and G_2 are both linear over any two adjacent local MPC boundary points in set M.

Proof of Theorem 5.2. We consider whether there exists a_1, a_2 such that $\tau \le a_1 < a_2 \le \overline{x}$ and $\int_0^y F_1(t)\ \mathrm{d}t = \int_0^y G_1(t)\ \mathrm{d}t$ or $\int_0^y F_2(t)\ \mathrm{d}t = \int_0^y G_2(t)\ \mathrm{d}t$ for $\forall y \in [a_1, a_2]$.

(1) We first consider the situation when a_1 and a_2 exist. There are two possibilities to consider.

(1.1) When $\tau \le a_1 < a_2 \le \overline{x}$ and $\int_0^y F_1(t)\ \mathrm{d}t = \int_0^y G_1(t)\ \mathrm{d}t$ for $\forall y \in [a_1, a_2]$.

First, we prove that $[a_1, a_2] \subseteq [\tau, \min\{\mu_1, \mu_2\}]$ **and** $\int_0^y F_2(t)\ \mathrm{d}t = \int_0^y G_2(t)\ \mathrm{d}t$ **for** $\forall x \in [a_1, a_2]$**.** By $\int_0^y F_1(t)\ \mathrm{d}t = \int_0^y G_1(t)\ \mathrm{d}t$ for $\forall y \in [a_1, a_2]$, we have $G_1(x) = F_1(x)$ for $\forall x \in [a_1, a_2]$. By Theorem 5.1 and Corollary 4.4, we have G_1 is convex over $[\tau, \overline{x}]$. Combining these two facts, we have F_1 is strictly convex over $[a_1, a_2]$ which implies that $[a_1, a_2] \subseteq [0, \mu_1]$. Since the interval $[a_1, a_2] \subseteq \mathsf{supp}(G_2)$, so we have $\phi_2(x) = G_1(x)$ for $\forall x \in [a_1, a_2]$ according to Corollary 4.4, which implies that ϕ_2 is strictly convex over $[a_1, a_2]$. By Corollary 4.3, we have $\int_0^y F_2(t)\ \mathrm{d}t = \int_0^y G_2(t)\ \mathrm{d}t$ and $G_2(y) = F_2(y)$ for $\forall y \in [a_1, a_2]$. In the same manner, we have F_2 is also strictly convex over $[a_1, a_2]$ which implies that $a_2 \le \mu_2$. Therefore, we have $[a_1, a_2] \subseteq [\tau, \min\{\mu_1, \mu_2\}]$.

Second, we prove that there don't exist b_1, b_2, b_3, b_4 **such that** $\tau \le b_1 < b_2 < b_3 < b_4 \le \overline{x}$**,** $\int_0^y F_1(t)\ \mathrm{d}t = \int_0^y G_1(t)\ \mathrm{d}t$ **for** $\forall y \in [b_1, b_2] \cup [b_3, b_4]$, and $\forall \epsilon > 0$, $\exists y \in (b_2, b_2+\epsilon] \cup [b_3-\epsilon, b_3]$ such that $\int_0^y F_1(t)\ \mathrm{d}t > \int_0^y G_1(t)\ \mathrm{d}t$. Otherwise, We define $z_1 \triangleq \min\{t \in (b_2, b_3] : \int_0^t F_1(x)\ \mathrm{d}x = \int_0^t G_1(x)\ \mathrm{d}x\}$ and $z_2 \triangleq \min\{t \in (b_2, b_3] : \int_0^t F_2(x)\ \mathrm{d}x = \int_0^t G_2(x)\ \mathrm{d}x\}$. W.l.o.g. we assume $z_1 \le z_2$. By Theorem 4.2 and Corollary 4.4, we have G_1 and G_2 are both linear over $[a_1, z_1]$. This contradicts the fact that F_1 is strictly convex over $[a_1, z_1]$ and $\int_0^{z_1} F_1(t)\ \mathrm{d}t = \int_0^{z_1} G_1(t)\ \mathrm{d}t$, which shows the assumption is invalid.

Therefore, there do not exist two discontinuous intervals in which all points are MPC points.

Third, we prove $\int_0^t F_1(x)\mathrm{d}x = \int_0^t G_1(x)\mathrm{d}x$, for any $t \in [0, a_2]$. We define $c_i = \min\{t : \int_0^x F_i(y)\mathrm{d}y = \int_0^x G_i(y)\mathrm{d}y, \forall x \in [t, a_2]\}$ for $i = 1, 2$. Using the argument in the first step, we can show $c_1 \le c_2$ and $c_2 \le c_1$. Thus $c_1 = c_2$.

Actually, it holds that $c_1 = c_2 = \tau$. Otherwise, we let $c_3 = \max\{t \in [0, c_1) | \int_0^t F_1(x)\mathrm{d}x = \int_0^t G_1(x)\mathrm{d}x\}$ and $c_4 = \max\{t \in [0, c_1) | \int_0^t F_2(x)\mathrm{d}x = \int_0^t G_2(x)\mathrm{d}x\}$. Since τ is a local MPC boundary point of either G_1 or G_2, we have $\tau \le \max\{c_3, c_4\}$. W.l.o.g., we assume $c_3 \ge c_4$. we have ϕ_1 is linear over $[c_3, c_1]$ and G_2 is linear over $[c_3, c_1]$ by Theorem 4.2. This implies that G_2 is linear over $[c_3, c_1]$, which contradicts the fact that F_2 is strictly convex over $[c_3, c_1]$ and $\int_0^{c_3} F_2(x)\ \mathrm{d}x = \int_0^{c_3} G_2(x)\ \mathrm{d}x$.

If $c_1 = c_2 = \tau > 0$, then one of senders 1 and 2 can switch his strategy to his prior and gain a utility increase which violates the equilibrium conditions. Therefore, we have $c_1 = c_2 = \tau = 0$.

W.l.o.g., we assume $[0, a_2]$ is the longest interval, i.e., $a_2 = \max\{t : \int_0^y F_1(x)\mathrm{d}x = \int_0^y F_1(x)\mathrm{d}x, \forall y \in [0, t]\}$.

Last we prove $\int_0^y F_1(t)\ \mathrm{d}t > \int_0^y G_1(t)\ \mathrm{d}t$, $\int_0^y F_2(t)\ \mathrm{d}t > \int_0^y G_2(t)\ \mathrm{d}t$ for $\forall y \in (a_2, 1)$, and G_1, G_2 are both linear over $[a_2, \overline{x}]$. We define $d_1 \triangleq \min\{t \in (a_2, 1] : \int_0^t F_1(x)\ \mathrm{d}x = \int_0^t G_1(x)\ \mathrm{d}x\}$ and $d_2 \triangleq \min\{t \in (a_2, 1] : \int_0^t F_2(x)\ \mathrm{d}x = \int_0^t G_2(x)\ \mathrm{d}x\}$. W.l.o.g., we assume $d_1 \le d_2$. By Definition 4.1, we have ϕ_1, ϕ_2 are both linear over $[a_2, d_1]$. Combining the fact $[a_2, d_1] \cap \mathsf{supp}(G_1) = [a_2, d_1] \cap \mathsf{supp}(G_2) = [a_2, \min\{d_1, \overline{x}\}]$ with Corollary 4.4, we have G_1 and G_2 are also linear over $[a_2, \min\{d_1, \overline{x}\}]$. If $d_1 < \overline{x}$, then we have G_1 is linear over $[a_2, d_1]$, which contradicts the fact $\int_0^{d_1} F_1(x)\ \mathrm{d}x = \int_0^{d_1} G_1(x)\ \mathrm{d}x$ and F_1 is S-shape over $[0, 1]$. So we have $d_1 \ge \overline{x}$ and $d_1 = 1$. That is to say, G_1, G_2 are both linear over $[a_2, \overline{x}]$. Hence, the Nash equilibrium corresponds to the equilibrium in Case 1 where $a = a_2$.

(1.2) When $\tau \le a_1 < a_2 \le \overline{x}$ and $\int_0^y F_2(t)\ \mathrm{d}t = \int_0^y G_2(t)\ \mathrm{d}t$ for $\forall y \in [a_1, a_2]$. The proof is exactly the same as in (1.1).

(2) When such a_1 and a_2 do not exist, then there exists no interval in the MPC set M, and we let $M = \{m_1, \ldots, m_w\}$. By Lemma C.3, we have $m_1 = \tau$ and in addition, if $m_1 > 0$, we have $m_1 \notin M_1 \cap M_2$. For each $j \in \{1, \ldots, w-2\}$, we have G_1 and G_2 are both linear over $[m_j, m_{j+1}]$, moreover G_1 and G_2 are both linear over $[m_{w-1}, \overline{x}]$ by the definition of virtual competitive function and Theorem 4.2. There are two possibilities to consider depending on the value of τ.

(2.1) When $\tau = 0$. Then we have $m_1 \in M_1 \cap M_2$ and $m_2 \in M_1 \cup M_2$. Since prior F is in S-shape, we have $G_1(m_2) = 1$ if $m_2 \in M_1$, and $G_2(m_2) = 1$ if $m_1 \in M_2$. Thus, we always have $m_2 = 1$. Then G_1 and G_2 are both linear in $[0, \overline{x}]$. This can be considered a special

Nash equilibrium in Case 1 where $a = 0$.

(2.2) When $\tau > 0$. We assume $m_j \in M_1 \cap M_2$, in other words, either G_1 or G_2 forms a local MPC over $[m_j, m_{j+1}]$, which contradicts the fact both priors are strictly uni-modal and shows the assumption is invalid. When $j < w-1$, we assume $m_j \in M_1$ and $m_{j+1} \in M_1$. Then we have $\int_0^y F_1(x)\ \mathrm{d}x > \int_0^y G_1(x)\ \mathrm{d}x$ for $\forall y \in [m_j, m_{j+1}]$. We know G_1 is linear over $[m_j, m_{j+1}]$, which contradicts the fact G_1 forms an MPC over $[m_j, m_{j+1}]$ and F_1 is strictly uni-modal. And if we assume $m_j \in M_2$ and $m_{j+1} \in M_2$, we will also find the contradiction in the same manner. So we have either $m_j \in M_1,\ m_{j+1} \in M_2$ or $m_j \in M_2,\ m_{j+1} \in M_1$.

Till now, we have proved that when $\tau > 0$, the equilibrium satisfies all the conditions in Definition 5.2 and thus it is an Alternating MPC equilibrium. ■

C.2 Proof of Proposition 5.3

Proof of Proposition 5.3. The prior distributions of sender 1 and sender 2 are both denoted by F, and let (G_1, G_2) be the strategy profile. According to Theorem 5.1,

$$\overline{\mathsf{supp}(G_1) \cap \mathsf{supp}(G_2)} - \{\underline{x}\} = [\tau, \overline{x}]\ .$$

Case 1: When $\forall x \in [\tau, \overline{x}]$, $G_1(x) = G_2(x)$, we have $\mathsf{supp}(G_1) = \mathsf{supp}(G_2)$. We prove this by contradiction. Suppose $\mathsf{supp}(G_1) \neq \mathsf{supp}(G_2)$. According to Theorem 5.1, w.l.o.g, we have $\overline{\mathsf{supp}(G_1)} \subseteq [0, \underline{x}] \cup [\tau, \overline{x}]$, $\overline{\mathsf{supp}(G_2)} \subseteq [\underline{x}, \overline{x}]$. Since $\mathsf{supp}(G_1) \cap (\underline{x}, \tau) = \emptyset$, and according to Lemma C.1, G_1 has no mass at τ, we have $G_1(\underline{x}) = G_1(\tau)$. Since $\forall x \in [0, \underline{x})$, $G_2(x) = 0$, $\forall x \in [\underline{x}, \tau]$, $G_2(x) \leq G_2(\tau) = G_1(\tau) = G_1(x)$. Since $\mathsf{supp}(G_1) \neq \mathsf{supp}(G_2)$, we have $\int_0^\tau G_1(x)dx > \int_0^\tau G_2(x)dx$. Since $\forall x \in [\tau, 1]$, $G_1(x) = G_2(x)$, we have $\int_0^1 G_1(x)dx > \int_0^1 G_2(x)dx$, which contradicts $G_1, G_2 \in \mathsf{MPC}(F)$. Therefore, if $\forall x \in [\tau, \overline{x}]$, $G_1(x) = G_2(x)$. We have $\mathsf{supp}(G_1) = \mathsf{supp}(G_2) = [\tau, \overline{x}]$. Therefore, $G_1 = G_2$.
Case 2: $\exists x \in [\tau, \overline{x}], G_1(x) \neq G_2(x)$. Let

$$y = \sup\{x : \forall x \in [\tau, \overline{x}], G_1(x) \neq G_2(x)\}.$$

Because G_1, G_2 are both right-continuous, we have $y > \tau$ (otherwise, it is the same as Case 1). We consider about the set $S = \{x : x \in [\tau, y), G_1(x) = G_2(x)\}$, which may not be empty.

- **Case 2.1**: If $S \neq \emptyset$. Let $\widetilde{x} = \sup S$. We have $G_1(\widetilde{x}) = G_2(\widetilde{x}), G_1(y) = G_2(y)$ and $\forall x \in (\widetilde{x}, y), G_1(x) \neq G_2(x)$. We consider the case $\forall x \in (\widetilde{x}, y), G_1(x) > G_2(x)$ and the analysis for the other case is exactly the same. Because $[\widetilde{x}, y] \subseteq \mathsf{supp}(G_1), [\widetilde{x}, y] \subseteq \mathsf{supp}(G_2)$, according to Corollary 4.4, we have $\forall x \in [\widetilde{x}, y], \phi_2(x) = G_1(x) > G_2(x) = \phi_1(x)$. According to Theorem 4.2, ϕ_1, ϕ_2 are convex functions, therefore, there exists a point

$a \in [0,1]$ such that $\dot{\phi}_1(a^-) \neq \dot{\phi}_i(a^+)$ or $\ddot{\phi}_1(a) > 0$ (otherwise, ϕ_1 would be linear on $[\widetilde{x}, y]$, which contradicts $\phi_2(x) > \phi_1(x)$ and ϕ_2 being convex). According to Corollary 4.3, we have $\int_0^a G_1(x)dx = \int_0^a F(x)dx$. Because $G_2 \in \mathsf{MPC}(F)$, then $\int_0^a G_2(x)dx \leq \int_0^a F(x)dx = \int_0^a G_1(x)dx$. Since $x \in (a,y), G_1(x) > G_2(x)$, $x \in [y,1], G_1(x) = G_2(x)$. Therefore, $\int_a^1 G_1(x)dx > \int_a^1 G_2(x)dx$. Hence, $\int_0^1 G_1(x)dx > \int_0^1 G_2(x)dx$, contradicting $G_1, G_2 \in \mathsf{MPC}(F)$. Thus, Case 2.1 does not occur.

- **Case 2.2**: If $S = \emptyset$, we have $\forall x \in [\tau, y)$, $G_1(x) \neq G_2(x)$.

  Case 2.2.1: When $\underline{x} < \tau$, according to Lemma C.3, we have $\int_0^\tau F(x)\,\mathrm{d}x = \int_0^\tau G_2(x)\,\mathrm{d}x$ and $\int_0^{\underline{x}} F(x)\,\mathrm{d}x = \int_0^{\underline{x}} G_1(x)\,\mathrm{d}x$. Therefore, we have $G_2(\tau) = F(\tau)$, $G_1(\underline{x}) = F(\underline{x})$(otherwise, it contradicts $G_1, G_2 \in \mathsf{MPC}(F)$). Since $\mathsf{supp}(G_1) \cap (\underline{x}, \tau) = \emptyset$ and G_1 has no mass at τ, we have $G_1(\tau) = G_1(\underline{x}) = F(\underline{x})$. Therefore, we have $G_2(\tau) = F(\tau) \geq F(\underline{x}) = G_1(\tau)$. Since $S = \emptyset$, we have $G_2(\tau) > G_1(\tau)$. Since $G_1(y) = G_2(y)$, and according to Lemma C.1, G_1, G_2 have no mass in $[\tau, \overline{x}]$. Therefore, $\forall x \in [\tau, y)$, $G_2(x) > G_1(x)$, and $\int_\tau^1 G_2(x)dx > \int_\tau^1 G_1(x)dx$. Since $G_1 \in \mathsf{MPC}(F)$, we have $\int_0^\tau G_1(x)dx \leq \int_0^\tau F(x)dx = \int_0^\tau G_2(x)dx$, therefore, $\int_0^1 G_2(x)dx > \int_0^1 G_1(x)dx$, which contradicts $G_1, G_2 \in \mathsf{MPC}(F)$.

  Case 2.2.2: When $\underline{x} = \tau$, since G_1, G_2 have no mass at τ, we have $G_1(\tau) > G_2(\tau) = 0$. Since $G_1(y) = G_2(y)$, we have $\int_\tau^y G_1(x)dx > \int_\tau^y G_2(x)dx$ and $\int_y^1 G_1(x)dx = \int_y^1 G_2(x)dx$. Therefore, $\int_\tau^1 G_1(x)dx > \int_\tau^1 G_2(x)dx$. Since $[0, \underline{x}] \cap \mathsf{supp}(G_1) \neq \emptyset$, therefore $\int_0^\tau G_1(x)dx > 0$. Since $\int_0^\tau G_2(x)dx = 0$. We have $\int_0^1 G_1(x)dx > \int_0^1 G_2(x)dx$, which contradicts $G_1, G_2 \in \mathsf{MPC}(F)$. Therefore, Case 2.2 does not occur. Thus, $G_1 = G_2$.

According to Proposition 5.3, we have if $F_1 = F_2$, then $G_1 = G_2$. According to (Hwang et al., 2019), a symmetric Nash equilibrium always exists and is unique when all senders have identical prior distribution. ■