APMEP Régionale de CAEN

Transformation « à la Curtz »

Curtz like Transformation

March 2008

Richard Choulet
richardchoulet@yahoo.fr

Résumé : L'idée de cette transformation m'a été donnée en cherchant la fonction génératrice de la suite A135364 du fameux site Encyclopedia On Line of Integer Sequences de Neil J. A. Sloane ; elle propose d'associer à toute suite d'entiers une autre suite par l'intermédiaire d'un triangle comme le définit Paul Curtz.

ABSTRACT *: The idea of this transformation which associates to a given (u_n) sequence another sequence, was suggested by observing the array which leads to sequence A135364 of Paul Curtz, in the famous OEIS of N. J. A. Sloane.*

1 Usual and unusual notation

To cancel some first terms of a sequence -e.g., here one term- :

- For a sequence $(u_n)_{n\geqslant 0}$, I note $(\check{u}_0,\ u_0,\ u_1,\ u_2,\ \ldots)$ the sequence $(u_n)_{n\geqslant 1}$

To add some first new terms at a given sequence -e.g., here two terms- :

- For a sequence A of the OEIS for example, I note $(a,\ b,\ \mathrm{A})$ the new sequence for which $u_0 = a$, $u_1 = b$ and $u_n = \mathrm{A}(n-2)$ $(n \geqslant 2)$. Eventually, if there is no risk of confusion, I cancel the brackets.

I identify sequence and ordinary generating fonction. I shall write for example :

$$\frac{1-z-z^2+z^3}{1-3z+2z^2-z^3} = \mathrm{A135364},$$

and I shall find also :

$\frac{1-2z+2z^2}{1-3z+2z^2-z^3} = (1,\ 1,\ 3,\ 8,\ 19,\ 44,\ 102,\ 237,\ 551,\ 1281,\ 2978,\ \ldots) = (1,\ \mathrm{A097550})$

$\frac{1+z^2}{1-3z+2z^2-z^3} = (1,\ 3,\ 8,\ 19,\ 44,\ 102,\ 237,\ 551,\ 1281,\ 2978,\ \ldots)=\mathrm{A097550}$

$\frac{1-z+z^2}{1-3z+2z^2-z^3} = (1,\ 2,\ 5,\ 12,\ 28,\ 65,\ 151,\ 351,\ 816,\ 1897,\ \ldots) = (\check{1},\ \check{1},\ \mathrm{A034943})$

$\frac{2-4z+z^2}{1-3z+2z^2-z^3} = (2,\ 3,\ 7,\ 17,\ 40,\ 93,\ \ldots) = (\check{1},\ \mathrm{A135364})$.

2 The transformation

2.1 Definition

Let be a, b given in $\mathbb{C}$.

The $\mathcal{T}_{a,b}$ bijection transforms the (u_n) sequence (of complex numbers) onto the (v_n) sequence such as :

I consider the infinite array, whose terms are noted $a_{n,p}$ (n row indice et p column indice),

- The zeroes of the array are not written (top triangular half array)
- I note $a_{1,1} = a$, $a_{2,2} = b$ and $a_{n,1} = u_{n-2}$ (for every $n \geqslant 2$)
- The number $a_{n,n}$ of this array (first non zero of the n-th column), gives the reason of the arithmetic sequence of the $(n+1)$-th column which follows (for every $n \geqslant 1$)
- The first number (non zero) of this $(n+1)$-th column is the sum $\sum_{p=1}^{n} a_{p,n}$ (for every $n \geqslant 3$)
- And finally, $\boxed{\mathcal{T}_{a,b}((u_n)) \text{ is the sequence } (a_{n,n})_{n\geqslant 3}}$.

$$\begin{vmatrix} a & & & & & \\ u_0 & b & & & & \\ u_1 & a+b & u_0+b & & & \\ u_2 & 2a+b & u_0+2b & a+u_0+u_1+2b & & \\ u_3 & 3a+b & u_0+3b & a+2u_0+u_1+3b & 3a+2u_0+u_1+u_2+5b & \\ u_4 & 4a+b & u_0+4b & a+3u_0+u_1+4b & 4a+3u_0+2u_1+u_2+5b & 7a+5u_0+2u_1+u_2+u_3+12b \end{vmatrix}$$

-Beginning of the announced array-

2.2 Theorem

When Φ is the o.g.f of the (u_n) sequence, then the o.g.f $\widehat{\Phi}$ of the (v_n) sequence is given by the following formula :

$$\boxed{\widehat{\Phi}(z) = \frac{(1-z)^2}{1-3z+2z^2-z^3}\Phi(z) + a\frac{z}{1-3z+2z^2-z^3} + b\frac{1-z+z^2}{1-3z+2z^2-z^3}}.$$

The proof

At first, I have clearly : $a_{n+1,n+1} = \sum_{p=1}^{n} a_{p,n}$ ($n \geqslant 2$) and $a_{n,p} = a_{p,p} + (n-p)a_{p-1,p-1}$ ($n \geqslant p \geqslant 2$).

With the definition of arithmetic sequence, I obtain the recurrence relation :

$$v_n = v_{n-1} + \sum_{p=0}^{n-2}(n-p-1)v_p + na + nb + u_n$$

for every $n \geqslant 3$ and the first values $v_0 = b + u_0$, $v_1 = u_0 + u_1 + 2b + a$.

Therefore by sommation comes with $\Phi(z) = \sum_{n=0} u_n z^n$ and $\widehat{\Phi}(z) = \sum_{n=0} v_n z^n$:

$$\widehat{\Phi}(z) - v_0 = \frac{az}{(1-z)^2} + \frac{bz}{(1-z)^2} + \Phi(z) - u_0 + v_0 z + \frac{z^2}{(1-z)^2}\widehat{\Phi}(z) + z(\widehat{\Phi}(z) - v_0)$$

and thus finally the announced result.

3 « Old » and new sequences

NOTA BENE : In the subsections which are following, « **NEW** » means that this sequence is not yet in OEIS.

3.1 Examples of sequences which egal the third difference sequence

$\mathcal{T}_{1,1}(0) = \frac{1+z^2}{1-3z+2z^2-z^3} = (1,\ 3,\ 8,\ 19,\ 44,\ 102,\ 237,\ 551,\ 1281,\ 2978,\ \ldots)$=(A097550)

$\mathcal{T}_{1,1}(\frac{1}{1-z}) = \mathcal{T}_{1,1}$(A000012) $= \frac{2-z+z^2}{1-3z+2z^2-z^3} = (2,\ 5,\ 12,\ 28,\ 65,\ 151,\ 351,\ 816,\ 1897,\ \ldots) =$ ($\check{1}$, $\check{1}$, $\check{1}$, A034943)

$\mathcal{T}_{1,1}(\frac{1}{(1-z)^2}) = \mathcal{T}_{1,1}$(A000027) $= \frac{2+z^2}{1-3z+2z^2-z^3} = (2,\ 6,\ 15,\ 35,\ 81,\ 188,\ 437,\ 1016,\ 2362,\ \ldots)$ **NEW**

$\mathcal{T}_{1,1}(\lambda) = \frac{\lambda(1-z)^2+1+z^2}{1-3z+2z^2-z^3} = (\lambda+1,\ \lambda+3,\ 2\lambda+8,\ 5\lambda+19,\ 12\lambda+44,\ \ldots) = \lambda$($\check{1}$, A034943)+(A097550)

$\mathcal{T}_{1,1}(1) = \frac{2-2z+2z^2}{1-3z+2z^2-z^3} = (2,\ 4,\ 10,\ 24,\ 56,\ 130,\ 302,\ 702,\ 1632,\ 3794,\ 8820,\ 20504,\ \ldots) =$ $2*$($\check{1}$, $\check{1}$, A034943)

$\mathcal{T}_{1,2}(\frac{1}{1-z}) = \frac{3-2z+2z^2}{1-3z+2z^2-z^3} = (3,\ 17,\ 40,\ 93,\ 216,\ 502,\ 1167,\ 2713,\ 6307,\ 14662,\ 34085,\ \ldots) =$ ($\check{1}$, $\check{2}$, A135364)

$\mathcal{T}_{1,2}(\frac{1}{(1-z)^2}) = \frac{3-z+2z^2}{1-3z+2z^2-z^3} = (3,\ 8,\ 20,\ 47,\ 109,\ 253,\ 588,\ 1367,\ 3178,\ 7388,\ 17175,\ \ldots)$ **NEW**

$\mathcal{T}_{0,1}(1) = \frac{2-3z+2z^2}{1-3z+2z^2-z^3} = (2,\ 3,\ 17,\ 40,\ 93,\ 216,\ 502,\ 1167,\ 2713,\ 6307,\ 14662,\ 34085,\ \ldots) =$ ($\check{1}$, A135364)

$\mathcal{T}_{1,0}(1,\ 0,\ -2,\ -3,\ -4,\ \ldots) = \mathcal{T}_{1,0}(\frac{1-2z-z^2+z^3}{1-3z+2z^2-z^3}) = (1,\ 2,\ 3,\ 17,\ \ldots) =$ A135364 $= \frac{1-z-z^2+z^3}{1-3z+2z^2-z^3}$.

3.2 Beginning of a collection

$\mathcal{T}_{0,0}(\frac{1-\sqrt{1-4z}}{2z}) = \mathcal{T}_{0,0}(1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ldots) = \mathcal{T}_{0,0}$(A000108) $= \frac{(1-z)^2(1-\sqrt{1-4z})}{2z(1-3z+2z^2-z^3)}$ $= (1,\ 2,\ 5,\ 14,\ 40,\ 116,\ 344,\ 1047,\ 3273,\ 10500,\ 34503,\ \ldots)$ **NEW**

$\mathcal{T}_{0,0}(\frac{1}{(1-z)^3}) = \mathcal{T}_{0,0}(1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ldots) = \mathcal{T}_{0,0}$($\check{0}$, A000217) $= \frac{1}{(1-z)(1-3z+2z^2-z^3)} =$ $(1,\ 4,\ 11,\ 27,\ 64,\ 150,\ 350,\ 815,\ 1896,\ 4409,\ 10251,\ \ldots)$ **NEW**

$\mathcal{T}_{0,0}(\frac{1}{(1-z)^4}) = \mathcal{T}_{0,0}(1,\ 4,\ 10,\ 20,\ 35,\ 56,\ 84,\ldots) = \mathcal{T}_{0,0}$($\check{0}$, A000292) $= \frac{1}{(1-z)^2(1-3z+2z^2-z^3)} =$ $(1,\ 5,\ 16,\ 43,\ 107,\ 257,\ 607,\ 1422,\ 3318,\ 7727,\ 17978,\ \ldots)$ **NEW**

$\mathcal{T}_{0,1}(\frac{1}{1+z}) = \mathcal{T}_{0,1}(1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ldots) = \mathcal{T}_{0,1}(\text{A033999}) = \frac{2-2z+z^2+z^3}{(1+z)(1-3z+2z^2-z^3)} =$ $(2,\ 2,\ 7,\ 15,\ 37,\ 84,\ 197,\ 456,\ 1062,\ 2467,\ 5737,\ 13335,\ \ldots)$ **NEW**.
The recurrence relation is $v_{n+4} = 2v_{n+3} + v_{n+2} - v_{n+1} + v_n$.

4 Various results

I consider a and b in $\mathbb{N}$ or $\mathbb{Z}$, and (u_n) sequence of integer numbers associated with Φ.

4.1 Some links

I notice that :

$$\mathcal{T}_{a,b}(0) = \frac{az}{1-3z+2z^2-z^3} + \frac{b(1-z+z^2)}{1-3z+2z^2-z^3}$$

while :

$$\mathcal{T}_{a,b}(1) = \frac{(a-1)z}{1-3z+2z^2-z^3} + \frac{(b+1)(1-z+z^2)}{1-3z+2z^2-z^3}.$$

This gives the result :

$$\mathcal{T}_{a,b}(1) = \mathcal{T}_{a-1,b+1}(0).$$

More generally, in the same vein :

$$\mathcal{T}_{a,b}(\Phi + \alpha) = \mathcal{T}_{a-\alpha,b+\alpha}(\Phi),$$

for every α. The proof is clear with the definition of $\mathcal{T}_{a,b}$.

4.2 Invariant

The equation which gives the invariant fonction Φ is :

$$-z(1-z+z^2)\Phi(z) = az + b(1-z+z^2),$$

and we conclude that :

♣ for $b \neq 0$, there is no invariant (which leads to an integer sequence!)

♢ for $b = 0$, the only invariants are given by the formula : $\Phi_a(z) = \frac{-a}{1-z+z^2}$ (with $a \in \mathbb{Z}$). The basic sequence $\frac{1}{1-z+z^2}$ is in fact $(1,\ 1,\ 0,\ -1,\ -1,\ 0,\ 1,\ldots)$ i.e., A010892. In other words, the only invariants are the $-a$A010892 (with $a \in \mathbb{Z}$).

4.3 By iteration

If we consider the iteration n times of $\mathcal{T}_{a,b}$ and Φ_a an invariant of $\mathcal{T}_{a,b}$, a classical result gives

$$\mathcal{T}_{a,b}^{(n)}(\Phi)(z) = \Phi_a(z) + \Big(\frac{(1-z)^2}{1-3z+2z^2-z^3}\Big)^n (\Phi(z) - \Phi_a(z)).$$

4.4 Reciprocal bijection

The o.g.f Φ which is transformed in $\widehat{\Phi}$ by $\mathcal{T}_{a,b}$ is given above :

$$\Phi(z) = \frac{(1-3z+2z^2-z^3)\widehat{\Phi}(z) - az - b(1-z+z^2)}{(1-z)^2}.$$

An example :
$\mathcal{T}_{1,1}^{-1}(\frac{1}{1-z}) = \frac{-2z+z^2}{(1-z)^3} = (0,\ -2,\ -5,\ -9,\ -14,\ -20,\ -27,\ -35,\ -44,\ -54,\ -65,\ \ldots)$
$= -$(A000096). Here $v_n = -(0,5n^2+1,5n)$.

5 References

`http://www.research.att.com/~njas/sequences/`

and the following sequences of OEIS

A000012, A000027, A000096, A000108, A000217, A000292 whose author is N. J. A. Sloane himself,
A010892 (S. Plouffe),
A033999 (V. Danilov),
A034943 (N. J. A. Sloane),
A097550 (D. N. Verma) and the winner is ...
A135364 (Paul Curtz).