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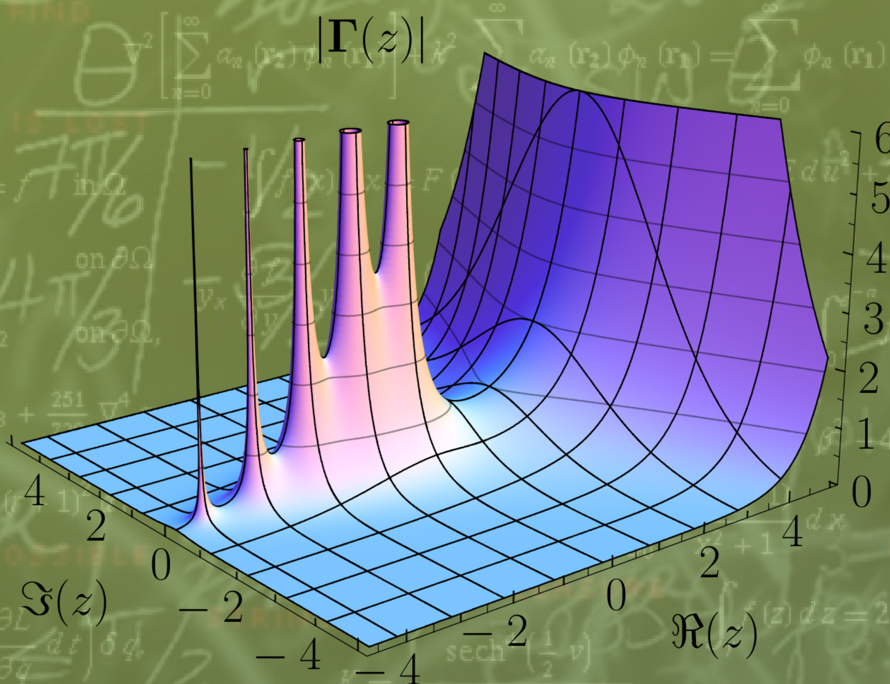

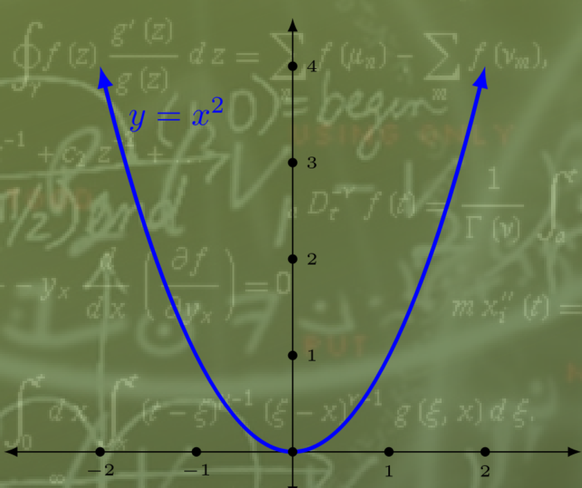

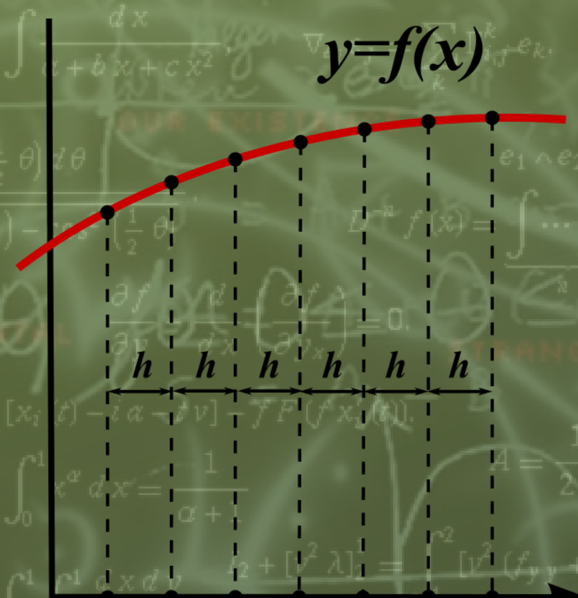

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Editorial

Advances in Science, Technology and Engineering Systems Journal (ASTESJ) is an online-only journal dedicated to publishing significant advances covering all aspects of technology relevant to the physical science and engineering communities. The journal regularly publishes articles covering specific topics of interest.

Current Issue features key papers related to multidisciplinary domains involving complex system stemming from numerous disciplines; this is exactly how this journal differs from other interdisciplinary and multidisciplinary engineering journals. This issue contains 26 accepted papers in Mathematics domain.

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Three-Pass Protocol Implementation in Vigenere Cipher Classic Cryptography Algorithm with Keystream Generator Modification

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A R T I C L E I N F O

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A B S T R A C T

Vigenere Cipher is one of the classic cryptographic algorithms and included into symmetric key cryptography algorithm, where to encryption and decryption process use the same key. Vigenere Cipher has the disadvantage that if key length is not equal to the length of the plaintext, then the key will be repeated until equal to the plaintext length, it course allows cryptanalysts to make the process of cryptanalysis. And weaknesses of the symmetric key cryptographic algorithm is the safety of key distribution factor, if the key is known by others, then the function of cryptography itself become useless. Based on two such weaknesses, in this study, we modify the key on Vigenere Cipher, so when the key length smaller than the length of plaintext entered, the key will be generated by a process, so the next key character will be different from the previous key character. In This study also applied the technique of Three-pass protocol, a technique which message sender does not need to send the key, because each using its own key for the message encryption and decryption process, so the security of a message would be more difficult to solved.

1. Introduction

A security issue is one of the most important aspects of an information system. A message that contains important information can be misused by irresponsible people, therefore a message contains important information, in order to secure an important message it is necessary a technique to secure it, cryptography is the science and art to maintain the security of a message [11].

The development of cryptography itself has already begun a long time ago, there are two types of cryptography, classical and modern cryptography, classical cryptography works based on character mode, and modern cryptography works based on bit mode. And if viewed from the key, cryptographic algorithm using symmetric and asymmetric key do not revise any of the current designations.

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Vigenere Cipher is a classic cryptographic algorithm, classical cryptography is generally included into the symmetric key algorithm, where to do the encryption and decryption process use the same key. In this case, key security and key distribution become the main factor, when the key and the ciphertext is known, then, of course, plaintext will be known also. This is one of the drawbacks of symmetric key algorithms.

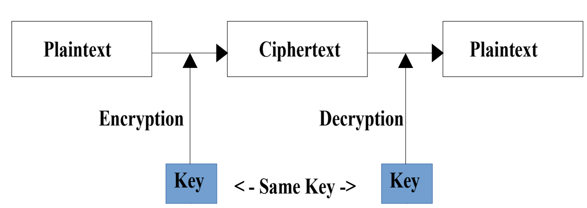

Figure 1: Symmetric key algorithm scheme

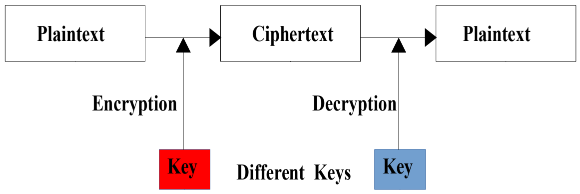

Figure 2: Asymmetric key algorithm scheme

Vigenere Cipher itself also has drawbacks which have been solved by a kasiski method, The drawback is if the key length is not the equal to the length of plaintext, then the key will be repeated continuously until the same as the plaintext length, it can cause the occurrence of what is called the histogram, which is the same ciphertext or repetitive content, kasiski method collect all the repeated histogram to calculated the distance between the histogram to find the length of the initial key.

With the existence of these flaws, we interested to try and examine how to minimize weaknesses. And we try to modify Vigenere Cipher algorithm with the Key Stream Generator method, which if key length is not the same length of the plaintext, then the next key will be generated by a process, this will cause key to getting the same length as the plaintext length becomes unrepetitive and it is expected to thus it would be more difficult to solve by kasiski method.

Furthermore, this paper also will apply what the so-called Three-Pass protocol, a method for securing messages without having to distribute keys, because neither the sender nor the recipient can use owned key each to encryption and decryption process. It is expected also that by applying the layered method (Vigenere Cipher key modification and use Three-Pass protocol methods), can increase the security level of classical cryptography algorithms Vigenere Cipher.

2. Theories

2.1. Three-Pass Protocol

The Three-Pass protocol is a framework that allows a party may send a message encrypted securely to the other party without having provided the key [2], this is possible because between the sender and receiver using a key belonging to the respective to perform the encryption and decryption process. Called the Three-pass protocol as do three exchange time before a message decrypted into meaningful messages. Three-Pass protocol invented by Adi Shamer about 1980 [2]. in applying Three-Pass protocol does not always have to use a cryptographic algorithm, because basically, this technique has its own function, namely to use the function exclusive-OR (XOR) [1], but in practice, to improve the reliability of this technique combined with cryptographic algorithms.

The implementation of this technique is still less attention [1], In the previous study, This technique can be a solution to the classic problem of the symmetric key algorithm which should send a key to the recipient. A message sender only needs to send a message to the recipient, and the important one (the key) does not need to distribute.

Here is a schematic representation Three-Pass protocol:

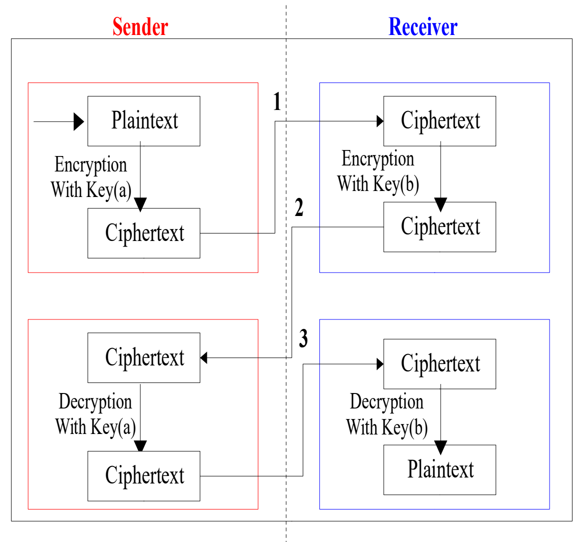

Figure 3: Three-Pass Protocol Process Scheme

2.2. Vigenere Cipher

Vigenere Cipher is one of the classic cryptographic algorithms that included into the category of polyalphabetic substitution [3] and a symmetric key cryptographic algorithm, whereby for encryption and decryption process used same keys. In the process of encryption and decryption, Vigenere Cipher using a table called tabula recta [11], it is a 26 x 26 matrix containing alphabet letters. This Algorithm was discovered by Blaise de Vigenere of France in the century to sixteen, in 1586 and this algorithm cannot be solved until 1917 [3] [11] Friedman and Kasiski solved it [3].

-- PLAINTEXT --

KEY	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
A	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
B	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A
C	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
D	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
E	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
F	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E
G	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F
H	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
I	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
J	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I
K	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J
L	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K
M	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
N	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M
O	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
P	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Q	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
R	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
S	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
T	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
U	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
V	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
W	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V

Figure 4: Vigenere Cipher table

Mathematically, the process of encryption and decryption Vigenere Cipher can be seen in the following equation:

$$C_i = E\ (\ P_i + K_i\ )\ mod\ 26 \qquad (1)$$

$$P_i = D\ (\ C_i - K_i\ )\ mod\ 26 \qquad (2)$$

Which C is the ciphertext, P is the plaintext, K is the Key, E is Encryption than D is Decryption. For the encryption process, every plaintext alphabet combined with the alphabet keys, the alphabet which intersected (by table) between the plaintext and ciphertext is the alphabet ciphertext. If the key alphabet is smaller than the plaintext, then the key will be repeated until equal to the length of the plaintext.

For example, supposing that the plaintext "THIS IS MY PAPER" with the keyword "UP", then, based on the table square vigenere, illustrations encryption can be seen as follows:

Plaintext	:	THIS IS MY PAPER
Keyword	:	UPUP UP UP UPUPU
Ciphertext	:	NWCH CH GN JPJTL

For the decryption process, with vigenere cipher table and used the same key, then the resulting plaintext from the ciphertext comparison with the keys, the alphabet corresponding to the key and the ciphertext, then the alphabet is the plaintexts.

Ciphertext	:	NWCH CH GN JPJTL
Keyword	:	UPUP UP UP UPUPU
Plaintext	:	THIS IS MY PAPER

The drawback of Algorithm Vigenere Cipher is if the key length is smaller than the plaintext length, then the key will be repeated, because it most likely will produce the same ciphertext as long as the same plaintext, in the example above, the character "IS" in the encryption into ciphertext the same as "CH" , this can be exploited by cryptanalysts to break off the ciphertext. Kasiski method is a method of cryptanalysis that broke Vigenere Cipher algorithms, methods kasiski collect the same characters of ciphertext to calculate the distance to ultimately find a number of the key length. After a long key is found, the next step is determining what the keywords to use exhaustive key search [11].

2.3. Keystream Generator

KeyStream Generator is a process for generating the key by using a function, so that be randomly generated the key. With the random key, it will further increase the reliability of a cryptographic algorithm because it would be difficult to solve.

The function used may be any function by use of bait as an input that is not the same for each process, so it will produce different output depending on that input.

Process keystream generator receives the initial fill of U (as a user key), and then processed to generate keys K_i, the next insert is from the previous process, and so on until the process reached K_n.

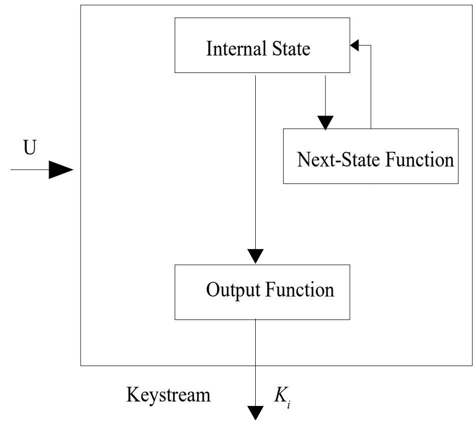

Figure 5: Key Stream Generator Process

3. Methodology

In this paper, it will be implemented how to implement cryptographic algorithms classic Vigenere Cipher by the key modification and how it is applied in the Three-pass protocol that the sender and the receiver do not have to exchange keys.

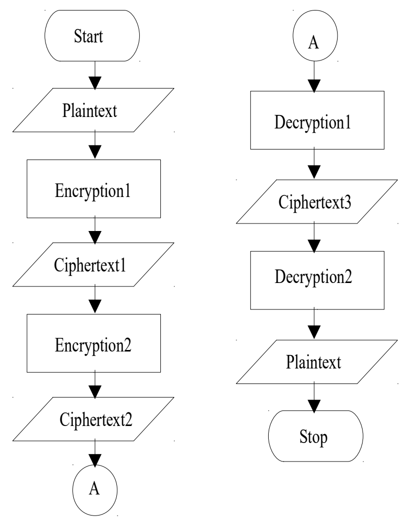

Figure 6: Flowchart of Three-Pass Protocol scheme

The key modification of Vigenere Cipher in this study, we use the key stream generator, the key will be generated when the key length is not equal to the length of the plaintext, when the key length equal to the length of the plaintext, the key generating process is not necessary.

In generating the keys, we use the following equation:

$$K_i = (K_{i-1} + n)\ mod\ 26 \qquad (3)$$

where K_i is a character key that will be generated, K_{i-1} is the index of the previous key character and n is the length of the keys to the K_{i-1}.

Alphabet in the alphabet *A to Z* can be described as an index of a row of numbers from *0 to 25* with *0 = A* and *25 = Z*, an illustration of the process of generating the key is illustrated as follows: suppose the length of plaintext is 10, and the length of the is key 6, suppose the key text "MYCODE", when the standard key Vigenere Cipher, key will be "MYCODEMYCO", then by using the above functions, the whole key generate to "MYCODEKRZI", where *"K"* is generated from the summation of index *"E"= 4* plus the length of the key of "M" to "E"= 6 equals 10 then in modulo with 26 generating a character to *10 ="K"*, and so on.

For encryption and decryption is then performed as usual. Flowchart for encryption and decryption process vigenere cipher can be seen as follows:

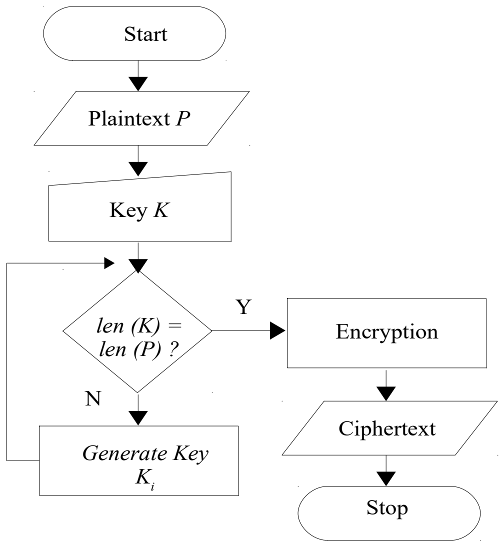

Figure 7: Vigenere Cipher Encryption Process

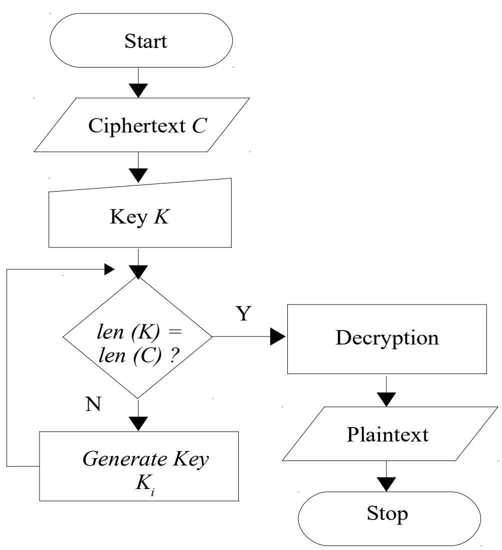

Figure 8: Vigenere Cipher Decryption processed

In this paper, to prove each process, we use python, we made it to both encryption and decryption process in plaintext manual input or plaintext taken from a file with .txt extension.

4. Testings And Implementations

As a test, we will do the encryption and decryption of a message with a cryptographic algorithm classic Vigenere Cipher that reads "THE FAMILY AND THE FAV" uses two keys are assumed as a key owned by the sender "KEY" and keys owned by the recipient "BUNG".

First, the sender will encrypt the plaintext with his own key and produce the first ciphertext, then the first ciphertext sent to the recipient, the recipient encrypts back the first ciphertext with his own key and produce the second ciphertext, and then the second ciphertext is sent back by the recipient to the sender, then sender decrypt the second ciphertext and generates the third ciphertext, the third ciphertext is sent back to the recipient to decrypt into plaintext.

For the illustrations can be seen as follows:

First Encryption by Sender:

Plaintext	:	THE FAMILY AND THE FAV
Sender Key	:	KEY BFKQXF OYJ VIW LBS
First Ciphertext	:	DLC GFWYID OLM OPA QBN

Second Encryption by Recipient:

First Ciphertext	:	DLC GFWYID OLM OPA QBN
Recipient Key	:	BUN GKPVCK TDO ANB QGX
Second Ciphertext	:	EFP MPLTKN HOA OCB GHK

First Decryption by Sender:

Second Ciphertext	:	EFP MPLTKN HOA OCB GHK
Sender Key	:	KEY BFKQXF OYJ VIW LBS
Third Ciphertext	:	UBR LKBDNI TQR TUF VGS

Second Decryption by Recipient:

Third Ciphertext	:	UBR LKBDNI TQR TUF VGS
Recipient Key	:	BUN GKPVCK TDO ANB QGX
Plaintext	:	THE FAMILY AND THE FAV

In the above process, we can see that there have been three exchanges between the sender and the recipient, the first exchange is the sender sends the first ciphertext, the second exchange is the recipient sends back the second ciphertext to the sender, the last exchange is the sender sends the third ciphertext, and finally the recipient decrypts the message to be plaintext.

For comparison, let's see how if the message decrypted by standard Vigenere cipher algorithm method. Where the sender sends the key to the recipient for decrypt the ciphertext simultaneously through different paths and assuming that the key delivered safely.

Encryption by Sender:

Plaintext	:	THE FAMILY AND THE FAV
Key	:	KEY KEYKEY KEY KEY KEY
Ciphertext	:	DLC PEKSPW KRB DLC PET

Decryption by Recipient:

Ciphertext	:	DLC PEKSPW KRB DLC PET
Key	:	KEY KEYKEY KEY KEY KEY
Plaintext	:	THE FAMILY AND THE FAV

For encryption process using the key that has been modified, it appears that the ciphertext generated more random, the word "THE" produce different cipher word, it will be different if we decrypt without key modification, the word "THE" will be decrypted the same word "DLC", then "FA" equal to "PE", this will be loopholes for cryptanalyst to make the cryptanalysis process using kasiski methods. By modifying the key, kasiski method will be more difficult to do, moreover, the key also has to distributed, this, of course, requires us to ensure that the distribution of the key should be completely safe. But when we applying the method of Three-Pass protocol, the key does not need to be distributed.

5. Conclusions

From the research, it seemed that by modifying keys, classical algorithm Vigenere Cipher actually has better reliability compared with standard Vigenere Cipher, this is due to the modification of the keys that are generated from a process that is done, so that when the key length is not equal to the length of the plaintext, then the key will not be repeated, but will be generated by a function, this has resulted in a more random keys rather than having to repeat the key as in the standard Vigenere Cipher algorithm.

And Vigenere Cipher can also be applied to the method of Three-Pass protocol, so although Vigenere Cipher included into the algorithm symmetric key, the sender of the message does not have to send the key used to encrypt the message, because each can use its own key both to encrypt and decrypt, of course, this is very useful when the key distribution security more vulnerable to tapping, and moreover, if the distribution of key is secure, the message sent does not have to be encrypted, isn’t it ?.

In this study, we simply apply safeguards messages using the standard alphabet consists of 26 characters, for further research, may be applied to a more complex character.

References

[1] Robbi Rahim, Ali Ikhwan, “Study of Three Pass Protocol on data Security” International Journal of Science and Research (IJRS) 5 (11), November 2016.

[2] Boni Oktaviana, Andysah Putera Utama Siahaan, “Three-Pass Protocol Implementation in Caesar Cipher Classic Cryptography” IOSR Journal of Computer Engineering (IOSR-JCE) 18(4) III (Jul-Aug, 2016): 26-29.

[3] Al-Amin Mohammed Aliyu, Abdulrahman Olaniyan “Vigenere Cipher: Trends, Review and Possible Modifications” International Journal of Computer Applications (0975-8887) 135(11), February 2016.

[4] Saloni Garg, Sonam Khera, Archana Aggarwal, “Extended Vigenere Cipher with Stream Cipher” International Journal of Engineering Science and Computing (IJESC) 6 (5), May 2016.

[5] Nimai Agarwal, Poonam Choudhari, Manivannan S S, “A Cryptosystem based on Vigenere Cipher with Enhanced Key Mechanism for data Security a Crypto System Based on Vigenere Cipher to Enhance Data Security”, International Journal of Pharmacy & Technology 8 (4), Dec 2016.

[6] Andysah Putera Utama Siahaan “Three-Pass Protocol Concept in Hill Cipher Encryption Technique”, Seminar Nasional Aplikasi Teknologi Informasi (SNATi), Yogyakarta, 6 Agustus 2016.

[7] Hendro Eko Prabowo, Arimaz, “Hangga Method of Vigenere Cipher Double Columnar Transposition as Modified Cryptographic Technique in Key Establishment ", National Seminar of the 6th SNST of 2015.

[8] Syaiful Anwar, Indra Nugroho, Asep Ahmadi "Cryptography Implementation With Shift Vigenere Cipher Encryption And Checksum Using CRC32 on Text", Journal of Information Systems 2, 2015.

[9] Benni Purnama, "Securing a Secret Message Through Vigenere Cipher Cryptography with a Layered Key", Journal of Media Processor 9 (3), October 2014.

[10] Imam Saputra, Mesran, Nelly Astuti Hasibuan, Robbi Rahim, “Vigenere Cipher Algorithm with Grayscale Image Key Generator for Secure Text File” International Journal of Engineering Research & Technology (IJERT) 6 (01), January 2017.

[11] Dony Ariyus "Introduction to Cryptography, Theory, Analysis and Implementation", Andi Publisher, Yogyakarta, 2008.

[12] Saifuldun Mostafa, Hayder Saad, Mustafa Musa Jaber, Mohammed Hasan Ali, Karam Dhafer: The Design Trends of Keystream Generator For Stream Cipher For High Immunity Attacks”, Researchgate, January 2016.

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Modification of Symmetric Cryptography with Combining Affine Chiper and Caesar Chiper which Dynamic Nature in Matrix of Chiper Transposition by Applying Flow Pattern in the Planting Rice

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ABSTRACT

Classical cryptography is a way of disguising the news done by the people when there was no computer. The goal is to protect information by way of encoding. This paper describes a modification of classical algorithms to make cryptanalis difficult to steal undisclosed messages. There are three types of classical algorithms that are combined affine chiper, Caesar chiper and chiper transposition. Where for chiperteks affine chiper and Caesar chiper can be looped as much as the initial key, because the result can be varied as much as key value, then affine chiper and Caesar chiper in this case is dynamic. Then the results of the affine and Caesar will be combined in the transposition chiper matrix by applying the pattern of rice cultivation path and for chipertext retrieval by finally applying the pattern of rice planting path. And the final digit of the digit shown in the form of binary digits so that 5 characters can be changed to 80 digit bits are scrambled. Thus the cryptanalyst will be more difficult and takes a very long time to hack information that has been kept secret.

1. Introduction

A security issue is one of the most important aspects of an Advancements in the field of computer technology allows thousands of people and computers around the world connected in a virtual world known as *cyberspace* or Internet. But unfortunately, technological advances are always followed by a downside to the technology itself. One is the susceptibility of data security, giving rise to the challenges and demands the availability of a data security system is as sophisticated as the technology advances of the computer itself. This is the background of the development of a data security system to protect data transmitted through a communications network [1]. There are several ways to do security data through one channel, one of which is cryptography. In cryptography, data transmitted via the network will be disguised in a way that even if the data can be read then it cannot be understood by unauthorized parties. Data to be transmitted and not experienced isitilah encoding known as plaintext, and after camouflaged with a way of encoding, then this will turn plaintext into ciphertext. The functions that are fundamental in cryptography is encryption and decryption [2].

Many cryptographic techniques are implemented to safeguard information, but the present condition is much too way or the work done by cryptanalysis to break it. Though an important thing in the delivery of the message is to maintain the security of the information that are not easily known or manipulated by other parties. One solution that can be done is to modify the cryptography solved or create a new cryptography so that it can be an alternative for securing messages [3].

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This study design a symmetric key cryptography by using 3 classical cryptographic algorithm, of which two are affine and Caesar chiperteksnya results will be dynamic (changing). And the matrix transposition cipher is a medium to combine chiperteks affine and Caesar on the technique of laying of the bits in the matrix using the rice planting furrow pattern and sampling technique of bits to be chiperteks end using the groove pattern-making transplanting rice. And in this study chiperteks be displayed in the form of binary numbers are 1 and 0. This research could be useful as a modification that serve as an alternative cryptographic security of information, so that the cryptanalyst would have difficulty or even require a very long time in the hacking of information [4-10].

2. Theoretical Basic

Encryption is the process of converting a plain text to be a message in the ciphertext.

$C = E\ (M)$

Where:
M = original message
E = encryption process
C = message in code (for brevity called the password), while

Decryption is the process of changing the password message in one language into the original message back.

$M = D\ (C)$

Where:
D = the decryption process

Generally, apart from using certain functions in the encryption and decryption, it is often a function was given an additional parameter called the key terms. For example, the original text: "experience is the best teacher". Once encrypted password algorithm with key xyz and PQR into cipher text: V583ehao8 @ $%.

2.1. Algorithm of Affine Chiper

Affine cipher on the method of affine is an extension of the method Caesar Cipher, which divert the plaintext with a value and add it with a shift of P produce ciphertext C is expressed by the function of congruent:

$$\mathbf{C \equiv m\ P + b\ (mod\ n)}$$

In which n is alphabet size, m is an integer which must be relatively prime to n (if not relatively prime, then the decryption can not be done) and b is the number of shifts (Caesar cipher is the specialty of affine cipher with m = 1). To carry out the description, equation (4) herus solved to obtain a solution P. congruence exists only if inver m (mod n), expressed in m^{-1}.Jikam $^{-1}$ exists then decryption is done by the following equation:

$$\mathbf{P \equiv m^{-1}(C - b)\ (mod\ n)}$$

2.2. Algorithm of Caesar Chiper

Cryptographic one of the oldest and simplest is Caesar cryptography. Historically, this is how Julius Caesar send important letters to the governors. Caesar cryptographic formula, generally can be written as follows:

$C = E\ (P) = (P + k) \bmod 26$
$P = D\ (C) = (C\ k) \bmod 26$, where
P is the plaintext,
C is the ciphertext,
K is shifting letter corresponding to the desired key.

2.3. Algorithm of Chiper Transposition

Cryptographic in column (column cipher), plaintext letters are arranged in groups consisting of several letters. Then the letters in this group write a column by column, the order of columns can be fickle. Cryptography column is one example of cryptographic methods of transposition. Example Cryptography Column:

The phrase 'FATHER HAS ARRIVED YESTERDAY AFTERNOON', if arranged in columns 7 letter, it will be the following columns:

AYAHSUD
AHTIBAK
EMARINS
OREAAAA

Figure 1. Structure of Character Chiper Transposition

To complete the last column that contains 7 letters, then the rest is filled with the letters' A 'can be any letter or as a complementary letter. Tesebut sentence after 7 columns encrypted with a key sequence of letters and 6,725,431, the result of the encryption:

DKSAATAEUANASBIAHIRAAAEOYHMR

Figure 2. Column Encryption of Chiper Transposition

3. Research Methods

3.1. Research Design

In the development of classical cryptography is included 3 classic algorithm that is affine cipher, Caesar cipher and a transposition cipher. Wherein the plaintext to be executed by using affine cipher and the Caesar cipher in advance separately, and in the encryption process is dynamic, ie chiperteks generated by affine and caesarean may change according to the wishes, of which a maximum of change as the keys that are used at the beginning of the process affine and Caesar.

Changes that occur chiperteks generated by looping the same process. Below we can see the design of stages of research:

And to process returns to the plaintext message from chiperteks called process descriptions, can be seen in the design below:

3.2. 3.2. Limitations of Problem

To not extend the scope of the discussion of the given limitations in this study, namely;

1. The process of encryption and decryption is done on the text.
2. Key b at the same affine with key k at Caesar specified by the author of the message and should be a number, whereas m in the affine key is a prime number.

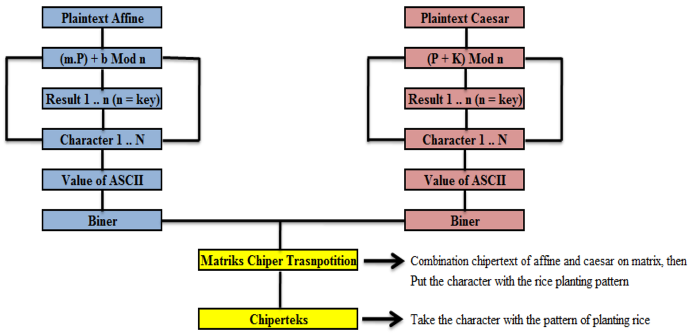

Figure 3. Research Design Process Encryption

3. Affine cipher algorithm and a Caesar cipher generates a dynamic chiperteks, where the results are changed just as much as the initial key specified by the author of the message.
4. Recurrence happened not more than the limit, and the limit on the initial key = used affine key (b) and Caesar (key k).
5. Laying chiperteks affine and Caesar on Marik transposition cipher using rice cropping patterns and making the character to chiperteksnya is to use pattern-making transplanting rice.

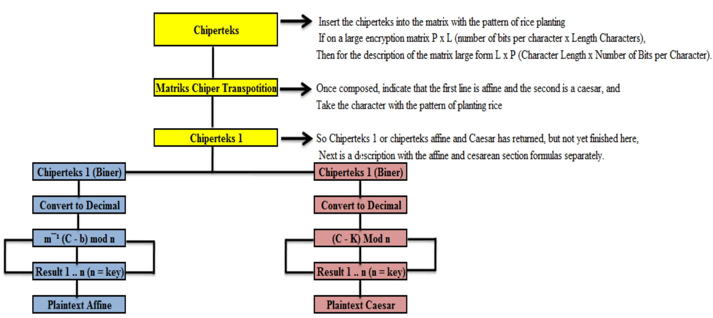

Figure 4. Process Research Design Description

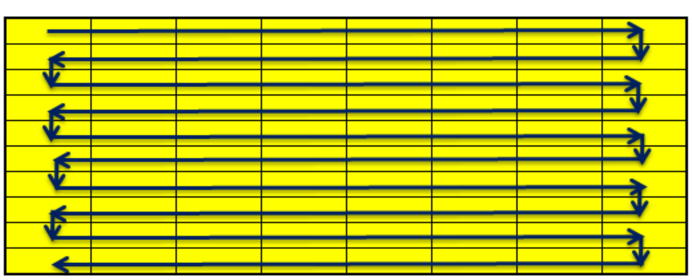

Figure 5. Pattern of Rice Planting

4. Results and Discussions

This study to design the development of classical cryptography involving three classic algorithms are affine cipher, Caesar cipher and a transposition cipher, where the encryption result of the affine cipher and the Caesar cipher is dynamic. And matrices on transposition cipher contain binary blend chiperteks of affine cipher and the Caesar cipher.

4.1. Rice Planting Pattern Flow

Method rice planting is usually done in a long horizontal continuous with mapped fields. This study used the same way as the planting of rice, using a matrix transposition ciphe

4.2. Flow Pattern Making Rice Planting

Rice planting take groove pattern is used as a pattern for the process description, which is to get back the encrypted plaintext before the rice planting groove pattern on the matrix transposition cipher.

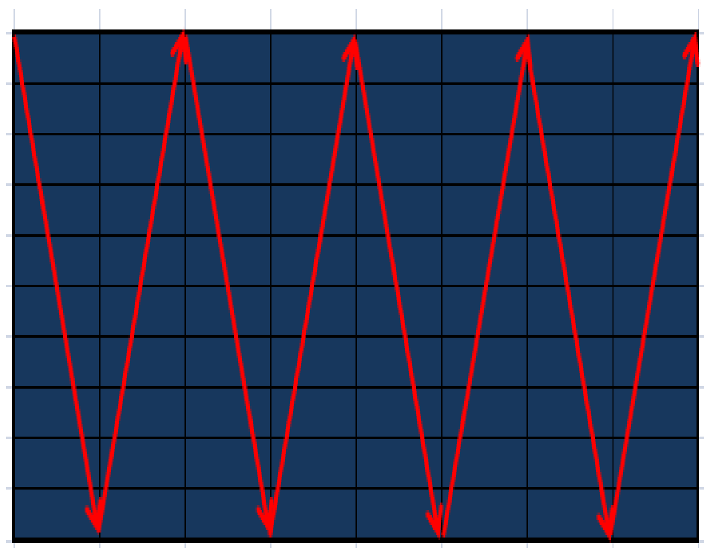

Figure 6. Flow Patterns Rice Planting Decision

4.3. Flowchart of Encryption

In cryptography this modification, first in plaintext encryption process we execute with affine cipher and the Caesar cipher separately. Where the results of the affine cipher encryption and later Caesar cipher is a dynamic, namely chiperteks generated by each algorithm can be fickle. It is very helpful to add to the confusion of the cryptanalyst. And to create dynamic results, the authors do looping on the affine cipher or Caesar cipher formula denagn respectively. But in this study, we limit the repetition can be done, namely a maximum of keys are initialized at the beginning. For example, key in the beginning is 15, then looping may occur up to 15 times.

a.Affine Chipper Encryption

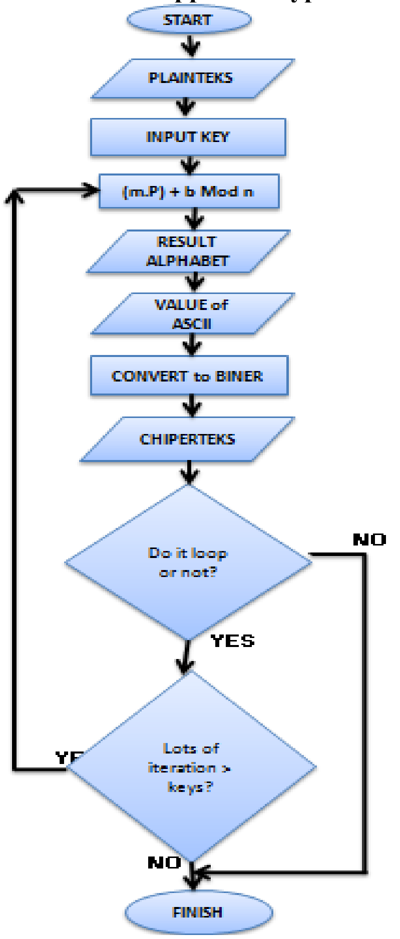

Figure 7. Flowchart of Affine Chiper

b. Caesar Chiper Encryption

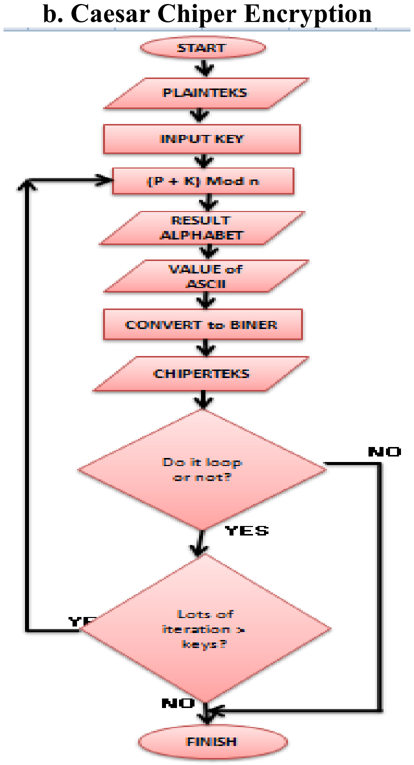

Figure 8. Flowchart of Caesar Chiper

c. Encryption of Chiper Transposition

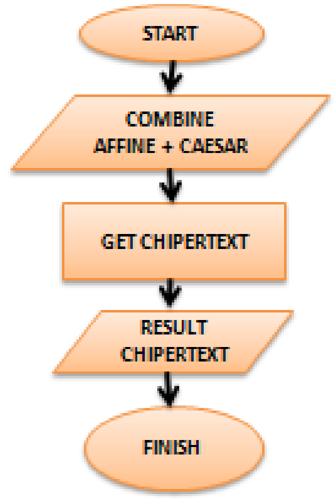

Figure 9. Flowchart encryption of Chiper Trasnposition

In transposition ciphers will be included chiperteks of affine cipher and the Caesar cipher to the matrix. Chiperteks of affine and Caesar in the form of binary digits, and the digits of the binary is placed into the matrix to follow the pattern of grooves cropping with the provisions of the first line starts from the affine cipher followed by a Caesar cipher and so on so forth until all the binary digits chiperteks on affine and Caesar composed both in the matrix transposition cipher.

4.4. Flowchart of Description

For the description of the process of restoring the messages that have been encrypted, or in other words to restore the plaintext of chiperteks, starting from a transposition cipher.

a. Description of Transposition Chiper

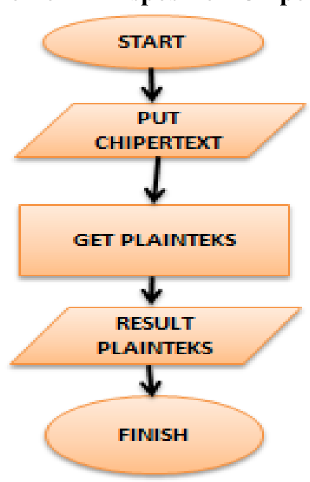

Figure 10. Flowchart Description of Chiper Trasnposisi

After receiving the plaintext of a transposition cipher, the plaintext is chiperteks results affine and Caesar cipher encryption. It is still necessary for the next stage of the description that the message be the same again. And the next process is mendeksripsikan the description of a transposition cipher using a formula descriptions affine cipher and the Caesar cipher.

b. Affine Chiper Description

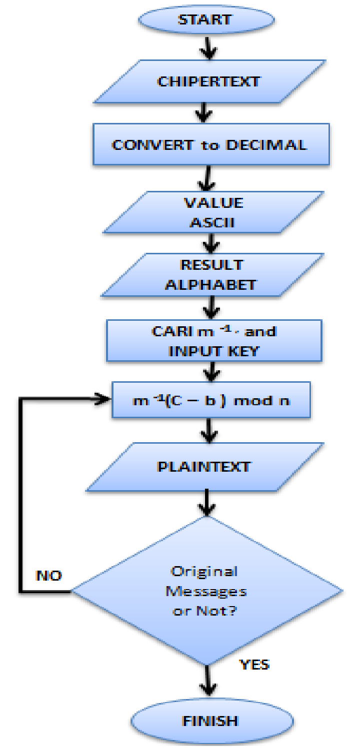

Figure 11. Flowchart of Affine Chiper

c. Chipper Description

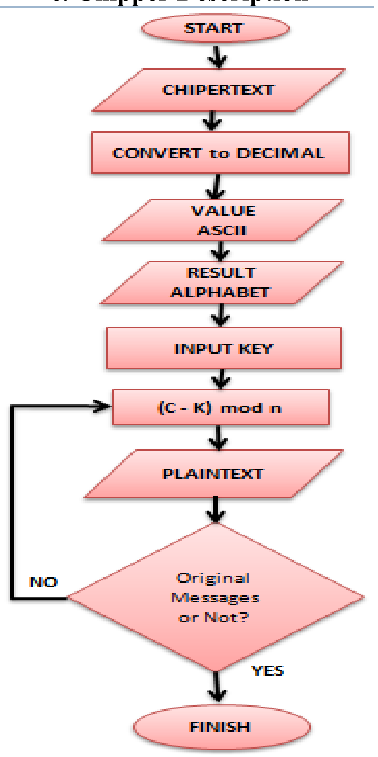

Figure 12. Flowchart of Caesar Chiper

4.5. Discussion of combined Affine chiper and Caesar chiper in the matrix of Chiper Transposition

For processes in detail, if we presuppose the plaintext is Y and chiperteks is C were converted into bits, then:

Y = {Y1, Y2, Y3, Y4, Y5, ..., Yn}

Y1 = {C1, C2, C3, ..., C8}
Y2 = {C9, C10, C11, ..., C16}
Y3 = { C17, C18, C19, ..., C24}… and so on up to Yn.

To process the cipher transposition, then we take chiperteks on affine cipher and the Caesar cipher and place in the matrix with the provisions of the algorithm is:

The first line is the first character or that has been converted into 8-digit bit chiperteks of affine, which started with a pattern of grooves rice planting ie from left to right on the first row (odd) and so on.

Then for the second row, followed by the first character of the Caesar cipher chiperteks or 8-digit bit after conversion is done according to the rules also cropping the groove pattern from left to right of the second row (even) and so on.

So can we symbolize results occur:

C1.A	C2.A	C3.A	C4.A	C5.A	C6.A	C7.A	C8.A
C8.B	C7.B	C6.B	C5.B	C4.B	C3.B	C2.B	C1.B
C9.A	C10.A	C11.A	C12.A	C13.A	C14.A	C15.A	C16.A
C16.B	C15.B	C14.B	C13.B	C12.B	C11.B	C10.B	C9.B
C17.A	C18.A	C19.A	C20.A	C21.A	C22.A	C23.A	C24.A
C24.B	C23.B	C22.B	C21.B	C20.B	C19.B	C18.B	C17.B
C25.A	C26.A	C27.A	C28.A	C29.A	C30.A	C31.A	C32.A
C32.B	C31.B	C30.B	C29.B	C28.B	C27.B	C26.B	C25.B
C33.A	C34.A	C35.A	C36.A	C37.A	C38.A	C39.A	C40.A
C40.B	C39.B	C38.B	C37.B	C36.B	C35.B	C34.B	C33.B

Description : A symbol is Affine Chiper, and B Symbol is Caesar Chiper.

Figure 13. Matrix Encryption of Chiper Transposition (Put plaintext)

See matrix above, the number of bits is 8 x 10 = 80 bits, which consists of 40 bits and 40 bits affine Caesar. And the last step in the process of encryption to obtain chiperteks the above matrix is to apply the rice cropping making workflow, from left column top to bottom and so on.

The matrix can be seen in the figure below:

C1.A	C2.A	C3.A	C4.A	C5.A	C6.A	C7.A	C8.A
C8.B	C7.B	C6.B	C5.B	C4.B	C3.B	C2.B	C1.B
C9.A	C10.A	C11.A	C12.A	C13.A	C14.A	C15.A	C16.A
C16.B	C15.B	C14.B	C13.B	C12.B	C11.B	C10.B	C9.B
C17.A	C18.A	C19.A	C20.A	C21.A	C22.A	C23.A	C24.A
C24.B	C23.B	C22.B	C21.B	C20.B	C19.B	C18.B	C17.B
C25.A	C26.A	C27.A	C28.A	C29.A	C30.A	C31.A	C32.A
C32.B	C31.B	C30.B	C29.B	C28.B	C27.B	C26.B	C25.B
C33.A	C34.A	C35.A	C36.A	C37.A	C38.A	C39.A	C40.A
C40.B	C39.B	C38.B	C37.B	C36.B	C35.B	C34.B	C33.B

Figure 14. Matrix Encryption of Chiper Transposition (Decision Chiperteks)

From the matrix above, we can take chiperteks by following the rice planting groove pattern as above, then chipertek is:

C ={C1A, C8B, C9A, C16B, C17A, C24B, C25A, C32B, C33A, C40B, C39B, C34A, C31B, C26A, C23B, C18A, C15B, C10A , C7B, C2A, ..., C33B, C40A, C25B, C32A, C27B, C24A, C9B, C16A, C1B, C8A}

Chiperteks that have been obtained in the above then we descriptions to get the plaintext back. This is what will be done recipient messages or information. The algorithm description on transposition cipher chiperteks above is:

1. Put chiperteks transposition cipher encryption on the results into the matrix, where the matrix is a comprehensive description width x length (number of characters x number of bits per character) or the inverse of the encryption process. Peletakannya same way with encryption that is by adopting a rice planting furrow.

C1A	C8B	C9A	C16B	C17A	C24B	C25A	C32B	C33A	C40B
C2A	C7B	C10A	C15B	C18A	C23B	C26A	C31B	C34A	C39B
C3A	C6B	C11A	C14B	C19A	C22B	C27A	C30B	C35A	C38B
C4A	C5B	C12A	C13B	C20A	C21B	C28A	C29B	C36A	C37B
C5A	C4B	C13A	C12B	C21A	C20B	C29A	C28B	C37A	C36B
C6A	C3B	C14A	C11B	C22A	C19B	C30A	C27B	C38A	C35B
C7A	C2B	C15A	C10B	C23A	C18B	C31A	C26B	C39A	C34B
C8A	C1B	C16A	C9B	C24A	C17B	C32A	C25B	C40A	C33B

Figure 15. Matrix Description of Chiper Transposition (Put Chiperteks)

2. After putting chiperteks into the matrix, then the next step is to take the initial plaintext or chiperteks of affine and Caesar by following the flow of the rice-planting decision.

P1A	P8B	P9A	P16B	P17A	P24B	P25A	P32B	P33A	P40B
P2A	P7B	P10A	P15B	P18A	P23B	P26A	P31B	P34A	P39B
P3A	P6B	P11A	P14B	P19A	P22B	P27A	P30B	P35A	P38B
P4A	P5B	P12A	P13B	P20A	P21B	P28A	P29B	P36A	P37B
P5A	P4B	P13A	P12B	P21A	P20B	P29A	P28B	P37A	P36B
P6A	P3B	P14A	P11B	P22A	P19B	P30A	P27B	P38A	P35B
P7A	P2B	P15A	P10B	P23A	P18B	P31A	P26B	P39A	P34B
P8A	P1B	P16A	P9B	P24A	P17B	P32A	P25B	P40A	P33B

Figure 16. Matrix Description of Chiper Transposition (Plaintext Intake / Chiperteks1)

3. And can be seen in the matrix with the pattern of grooves taking over the rice planting, that plaintext or chiperteks 1 which means chiperteks of affine and Caesar already apparent. If we write the tracks arrow then these bits are arranged well and return to the previous message. And it seems clear also that the symbol is a bit affine cipher A and B are Caesar cipher. The results of the description if written is:

P = {P1A, P2A, P3A, P4A, P5A, P6A, P7A, P8A, C1B, C2B, C3B, C4b, C5b, C6b, C7B, C8B, C9A, C10A, C11A, C12A, C13A, C14A, C15A, C16A ... ..., C33A, C34A, C35A, C36A, C37A, C38A, C39A, C40A, C33B, C34B, C35B, C36B, C37B, C38B, C39B, C40B}

4.6. Test Result

There are some that can can of this algorithm, the invention includes some strengths and weaknesses in this algorithm. Namely:

a. Advantages :

1. The results of the dynamic at chiperteks early stage in the process of the affine cipher and the Caesar cipher, becomes a strength because the results can vary and this will make the cryptanalyst becomes more difficult to hack.
2. Chiperteks end that is sent to the message recipient is the number of binary digits, so if 5 characters or one word shall we send, will generate 80 digit bit. And even this 80-digit randomly arranged by rice cropping pattern groove, so that in testing it becomes an advantage in this algorithm.

b. Weaknesses :

However, there are things that still looks weak in the algorithm, namely a recurrence that may be performed in affine and Caesar does not exceed the value of the key early, it can be said that the larger the key will be the better, but if the keys are initialized at the beginning of the small, then the less recurrence happens that a hacker does not require a long time to try to hijack the message.

5. Conclusions

a. Chiperteks dynamic process and Caesar affine cipher is also an important point on the modification of this algorithm, as plaintext consists of one message alone can generate an arbitrary chiperteks. And this technique can also trickthe cryptanalyst.

b. The use of algorithms rice planting groove pattern on the matrix transposition cipher can prove that this technique generates a symmetric cryptographic methodology stronger.

c. The combination of several methods on classical cryptography such as affine cipher, Caesar cipher and a transposition cipher increase the level of difficulty of this cryptography. The encryption process is layered with berebeda method will make hackers need at a very long time to steal the information or message.

d. Chiperteks display in the form of binary digits that doubled the number of bits in each character, will help add to the confusion cryptanalyst for one word alone can produce tens or hundreds of binary bits consist of 1 and 0.

References

[1] Widodo Achmad, "Block Cipher Cryptography Design Based on Planting Paddy and Rice Field Techniques". Bandung, April 2015.

[2] Dunkom. Binary Numbers, (Binary), http: //www.dunovtek. Wordpress. Com / 2011/08/05 / number-binary-binary, 2011.

[3] Ridwan, "Designing The Key Symmetry Cryptography Algorithm By Expanding the Vigenere Algorithm and Analysis of Kasiski's Methods

Against the Algorithm", http: //informatika.stei.itb.ac.id/~inalinal.munir Cryptography. 2006-2007 / Makalah1 / Makalah1-040.pdf.

[4] Munir, R., 2006, Cryptography, Informatics Publishers, Bandung.

[5] Hadiwibowo.Pengamanan Information and Cryptography - Adding khasanah reading cryptology and information security for the people of Indonesia. Http://hadiwibowo.wordpress.com., December 23, 2006.

[6] http://download.portalgaruda.org/article.php?article=423826&val=7596&title=Implementasi%20Combination%20Algorithms%20Encryption%20Aes%20128 [12]% 20And% 20Algorithm% 20Compression% 20Shannon-Fano.

[7] Piper, Fred and Sean Murphy.Cryptography A Very Short Introduction Oxford; 2002.

[8] Munir, Rinaldi. 2004. Diktat Lecture IF2153 Discrete Mathematics - Fourth Edition. Bandung: Tenik Informatika Study Program, STEI ITB. 2004. Classic Chiper System. http: //kur2003.if.itb.ac.id/file/System%20Chiper%20Klasik.doc.

[9] Ardiansyah Rifky, "Development of Viginere Chiper Algorithm Using Binary-Based Lock Method", http://digilib.unmuhjember.ac.id/files/disk1/37/umj-1x-rifkyardia-1813-1-jurnal.pdf.

[10] Mischanz. 2009. Character and Table ASCII, http://mischanz.wordpress.com/2009/12/12/character-and-table-ascii.

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Collaborative Encryption Algorithm Between Vigenere Cipher, Rotation of Matrix (ROM), and One Time Pad (OTP) Algoritma

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A B S T R A C T

Cryptography is still developing today. Classical cryptography is still in great demand for research and development. Some of them are Vigenere Cipher and One Time Pad (OTP) Algorithm. Vigenere Cipher is known as the alphabet table used to encrypt messages. While OTP is often used because it is still difficult to solve. Currently there are many ways to solve the Vigenere Cipher algorithm. OTP itself has constraints with the distribution of keys that are too long. The key length in the OTP algorithm is the same as the plaintext length. Key random creation also increases the intensity of key distribution. This requires a secure network at a high cost. The key repeater also lowers the message security level. EM2B Key algorithm is able to overcome key problem in OTP. EM2B and Increment of Key (L_k) collaborations produce key lengths equal to plaintext. The Rotation of Matrix (ROM) algorithm contributes to manipulating the length of plaintext characters. ROM works with Square Matrix tables that also scramble the contents of plaintext. The addition of Vigenere Cipher further enhances plaintext security. The algorithm was developed with the EnC1 function, Subrange (S_i), and Length of Plaintext (L_p). This research produces a very strong ciphertext. Because plaintext undergoes four stages of encryption to become cipherteks. Then the length of the plaintext changes with the number of cells in the matrix. The value of Si and L_p is added to the plaintext to be a flag for the decryption process.

1. Introduction

In this increasingly sophisticated era almost all the good circles of government, industry, business to personal companies do the work using computer. The capabilities possessed by computer devices is no doubt, this is proved by the level of accuracy to a high speed in completing a job. Besides the bias advantage obtained from the use of computer, the most important thing to be considered is part of its security which if the information/data stored in the computer suffered damage or loss then it could lead to huge losses. The condition of a computer that is not secured properly, will be a great opportunity to the hackers to enter the computer access and steal all the data he wants. Some examples of hacking cases in 2016, among others "Ransomware emerges as a top cyber threat to business, UK second only to US in DDoS attacks, 412 million user accounts exposed in Friend Finder Networks hack, Financial Conduct Authority concerned about cyber security of banks, and other cases caused by the weakness of the security system. For that we need a computer security system. Security of data in a computer is very important to protect the data from other parties that do not have the authority to determine the content of the data [1]. Security concerns relate to risk areas such as external data storage, dependency on the public internet, lack of control, multi-tenancy and integration with internal security [2].

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2. Literature Review

The NIST *Computer Security Handbook* [NIST95] defines the term *computer security* as follows: The protection afforded to an automated information system in order to attain the applicable objectives of preserving the integrity, availability, and confidentiality of information system resources (includes hardware, software, firmware, information/data, and telecommunications) [3]. Cryptography is one of the areas of science that studies about information security / data to avoid adverse effects due to misuse of information by irresponsible parties. Cryptography has an important role in maintaining the confidentiality of information both in the computer and at the time of transaction data.

So a more hardheaded goal of cryptography is to make it too work intensive for attacker [11]. The basic terms used in cryptography are discussed bellow:

Plaintext

In crytography, plaintext is a simple readable text before being encrypted into ciphertext [12]. The data can be read and understood without any special measure is called plaintext [13].

Ciphertext

In Cryptography, the transformation of original message into non-readable message before the transmission is known as cipher text [16]. It is amessage obtained by some kind of encryption operation on plain text.

Encryption

Encryption is a process of converting plain text into cipher text. Encryption process requires encryption algorithm and key to convert the plain text into cipher [17]. In cryptography encryption performed at sender end.

Decryption

Decryption is the reverse process of encryption. It converts the cipher text into plain text. In cryptography decryption performed at receiver end [11].

Key

The key is the numeric or alphanumeric text used for the encryption of plain text and decryption of cipher text [16]

Currently the scientists in the field of cryptography has been a lot of research about the science of cryptography by creating a variety of new algorithms developed from previous algorithms. The level of security offered varies, there is a security priority by applying the stages of a very complex algorithm, and by offering the speed of the execution process of the algorithm they created.

Vigenere Cipher is one of the classic cryptographic algorithms that until now is still widely developed by researchers. In the cryptography, it contains different methods among them the cryptography with the Vigenère matrix. Cipher was proposed by BLAISEDE VIGENÈRE in 1583 and has reigned about 03centuries.

Another classic cryptographic algorithm is *Vernam Cipher*, known as the One *Time Pad* algorithm. In its development *One Time Pad* algorithm is still much in demand by researchers, because the key length generated randomly equal to the length of the message. But the length of the key to be distributed becomes an interesting thing to develop, because when sending a key must require a secure network where the secure network is expensive. The key looping is also a weak point in the algorithm, which means that if the message is encrypted and decrypted then to encrypt the same message must use a new key, it means that network security is a top priority due to frequent distribution of keys.

The key looping is also a weak point in the algorithm, which means that if the message is encrypted and decrypted then to encrypt the same message must use a new key, it means that network security is a top priority due to frequent distribution of keys. This is the case with Vigenere ciphers though considered safe for centuries but then its weaknesses have been identified. Friedrich Kasiki invented a method to identify the period and therefore the key and plaintext [11]

This research provides a solution to the length of keys to be distributed that have lengths to be equal to the message. Doubts over repetition of key biases that result in cryptanalysis can easily know the contents of the message can be overcome by the collaboration with the algorithm *Vigenere Cipher* and *Rotation of Matrix (ROM)*. These three algorithms support each other to ensure the security level of the message and make it difficult for the irresponsible party to guess the contents of the original message.

The message will be sent encryption process four stages where each stage will produce *1st* ciphertext, *2nd* ciphertext, *3rd* ciphertext, and *final ciphertext* [10]. The first stage, the message is encrypted using the *Vigenere Cipher* algorithm by first determining the index of each message character and key. Both indexes are summed and modulated to the specified character length of the index table. The result of this stage is called *1st ciphertext*. . The next step is to determine the value of subrange and Length of plaintext. The subrange value is derived from the sum of each message index and the Length of Plaintext is determined by the number of characters of the message itself. After the subrange and length of plaintext values are obtained then the index value for each key character multiplied by the subrange value and modulated by length of plaintext, this result is called Encryption *1st ciphertext (EnC1)*. *2nd ciphertext* is obtained from multiplication between *1st ciphertext and EnC1* value. The third stage is using the *Rotation of Matrix (ROM)* algorithm where in this stage the message that has been converted into *2nd ciphertext* is inserted into a matrix table. This algorithm aims to scramble messages that have been entered into the matrix table using the rotation method and manipulate the number of characters from the message.

The workings of this algorithm is very easy but requires precision because the rotation value obtained from a key algorithm called *E2mbs Key* Algorithm and the length of the key characters will determine the number of matrix tables to rotate. After all the process at this stage is done then obtained *3rd ciphertext*, and the final stage is using *One Time Pad* algorithm, where as we know this algorithm is run by converting all characters of the message and key into binary numbers which then operated by exclusive or (XOR) method. The *One Time Pad* algorithm in this research has been modified in the key generation section. Previous research has always used a random machine to generate a key. The need for modifications to the key generation process aims to avoid making such long keys when restoring the original message from ciperthext. The key used to encrypt messages on the *One Time Pad* algorithm is the primary key that has been altered using the *E2mbs* Key algorithm, then to increase the length of the key so that it is the same as the message display, the increment key method is used. The encryption results in this final stage produce *Final Ciphertext* which will be sent to the recipient of the message along with the main key where the length of the character does not have to be the same length of the message. In this classic algorithm has been found a way to find out the contents of the original message without having to know the key, This research is explained that the decryption process will be faster if the recipient of the message to obtain the main key and follow the correct procedure. However, if decrypted by guessing the key then kriptanalis must go through a long route and the weight of the possibility to find out the original message content.

3. Materials and Methods

According to (William, 2010) Cryptography is a technique applied for encryption and decryption. In the field of cryptography there are several techniques available for encryption/decryption. These techniques can be generally classified into two major groups, i.e. Conventional and public key Cryptography [4].

To understand the workings of cryptographic algorithms in this research, the following will be presented a block diagram explaining the flow of the encryption process.

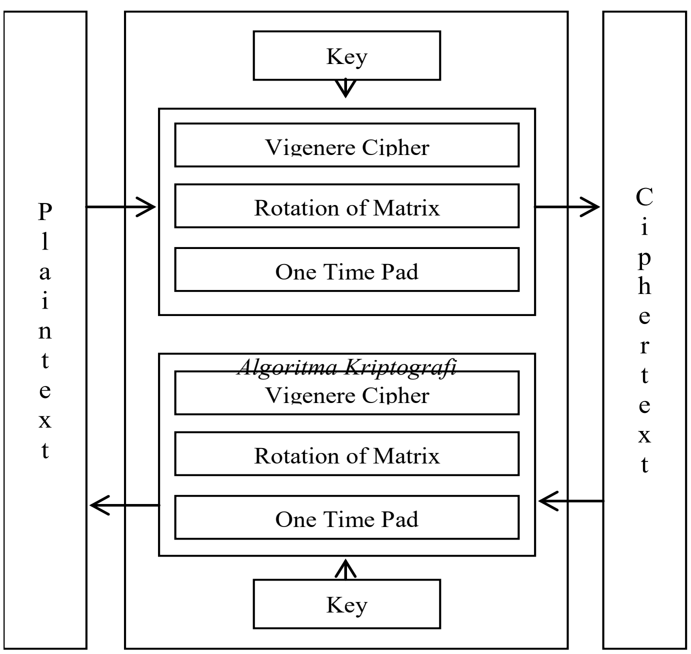


Figure 1. Block Diagram of Cryptography Algorithm Encryption and Decryption.

From the picture above can be known in general the process of encryption of a plaintext into ciphertext and the decryption of ciphertext to plaintext.

In this cryptographic algorithm used two tables as parameter to determine character index among others, Plaintext Index Table (0, 1, 2, …, 36) of 37 characters to identify characters (A, B, ..., Z, spaces, 0, 1 , ..., 9) and ASCII tables as character parameters changed into ASCII characters.

Paintext	Index	Paintext	Index	Paintext	Index	Paintext	Index
A	0	K	10	U	20	3	30
B	1	L	11	V	21	4	31
C	2	M	12	W	22	5	32
D	3	N	13	X	23	6	33
E	4	O	14	Y	24	7	34
F	5	P	15	Z	25	8	35
G	6	Q	16	sp	26	9	36
H	7	R	17	0	27		
I	8	S	18	1	28		
J	9	T	19	2	29		

Figure 2. Plaintext Index Table

Dec	Hx	Oct	Char	Dec	Hx	Oct	Html	Chr	Dec	Hx	Oct	Html	Chr	Dec	Hx	Oct	Html	Chr
0	0	000	NUL (null)	32	20	040		Space	64	40	100	@	@	96	60	140	`	`
1	1	001	SOH (start of heading)	33	21	041	!	!	65	41	101	A	A	97	61	141	a	a
2	2	002	STX (start of text)	34	22	042	"	"	66	42	102	B	B	98	62	142	b	b
3	3	003	ETX (end of text)	35	23	043	#	#	67	43	103	C	C	99	63	143	c	c
4	4	004	EOT (end of transmission)	36	24	044	$	$	68	44	104	D	D	100	64	144	d	d
5	5	005	ENQ (enquiry)	37	25	045	%	%	69	45	105	E	E	101	65	145	e	e
6	6	006	ACK (acknowledge)	38	26	046	&	&	70	46	106	F	F	102	66	146	f	f
7	7	007	BEL (bell)	39	27	047	'	'	71	47	107	G	G	103	67	147	g	g
8	8	010	BS (backspace)	40	28	050	(	(	72	48	110	H	H	104	68	150	h	h
9	9	011	TAB (horizontal tab)	41	29	051	)	)	73	49	111	I	I	105	69	151	i	i
10	A	012	LF (NL line feed, new line)	42	2A	052	*	*	74	4A	112	J	J	106	6A	152	j	j
11	B	013	VT (vertical tab)	43	2B	053	+	+	75	4B	113	K	K	107	6B	153	k	k
12	C	014	FF (NP form feed, new page)	44	2C	054	,	,	76	4C	114	L	L	108	6C	154	l	l
13	D	015	CR (carriage return)	45	2D	055	-	-	77	4D	115	M	M	109	6D	155	m	m
14	E	016	SO (shift out)	46	2E	056	.	.	78	4E	116	N	N	110	6E	156	n	n
15	F	017	SI (shift in)	47	2F	057	/	/	79	4F	117	O	O	111	6F	157	o	o
16	10	020	DLE (data link escape)	48	30	060	0	0	80	50	120	P	P	112	70	160	p	p
17	11	021	DC1 (device control 1)	49	31	061	1	1	81	51	121	Q	Q	113	71	161	q	q
18	12	022	DC2 (device control 2)	50	32	062	2	2	82	52	122	R	R	114	72	162	r	r
19	13	023	DC3 (device control 3)	51	33	063	3	3	83	53	123	S	S	115	73	163	s	s
20	14	024	DC4 (device control 4)	52	34	064	4	4	84	54	124	T	T	116	74	164	t	t
21	15	025	NAK (negative acknowledge)	53	35	065	5	5	85	55	125	U	U	117	75	165	u	u
22	16	026	SYN (synchronous idle)	54	36	066	6	6	86	56	126	V	V	118	76	166	v	v
23	17	027	ETB (end of trans. block)	55	37	067	7	7	87	57	127	W	W	119	77	167	w	w
24	18	030	CAN (cancel)	56	38	070	8	8	88	58	130	X	X	120	78	170	x	x
25	19	031	EM (end of medium)	57	39	071	9	9	89	59	131	Y	Y	121	79	171	y	y
26	1A	032	SUB (substitute)	58	3A	072	:	:	90	5A	132	Z	Z	122	7A	172	z	z
27	1B	033	ESC (escape)	59	3B	073	;	;	91	5B	133	[	[	123	7B	173	{	{
28	1C	034	FS (file separator)	60	3C	074	<	<	92	5C	134	\	\	124	7C	174			|
29	1D	035	GS (group separator)	61	3D	075	=	=	93	5D	135	]	]	125	7D	175	}	}
30	1E	036	RS (record separator)	62	3E	076	>	>	94	5E	136	^	^	126	7E	176	~	~
31	1F	037	US (unit separator)	63	3F	077	?	?	95	5F	137	_	_	127	7F	177	&#127;	DEL

Source: www.LookupTables.com

128	Ç	144	É	160	á	176	░	192	└	208	╨	224	α	240	≡
129	ü	145	æ	161	í	177	▒	193	┴	209	╤	225	ß	241	±
130	é	146	Æ	162	ó	178	▓	194	┬	210	╥	226	Γ	242	≥
131	â	147	ô	163	ú	179	│	195	├	211	╙	227	π	243	≤
132	ä	148	ö	164	ñ	180	┤	196	─	212	╘	228	Σ	244	⌠
133	à	149	ò	165	Ñ	181	╡	197	┼	213	╒	229	σ	245	⌡
134	å	150	û	166	ª	182	╢	198	╞	214	╓	230	µ	246	÷
135	ç	151	ù	167	º	183	╖	199	╟	215	╫	231	τ	247	≈
136	ê	152	ÿ	168	¿	184	╕	200	╚	216	╪	232	Φ	248	°
137	ë	153	Ö	169	⌐	185	╣	201	╔	217	┘	233	Θ	249	∙
138	è	154	Ü	170	¬	186	║	202	╩	218	┌	234	Ω	250	·
139	ï	155	¢	171	½	187	╗	203	╦	219	█	235	δ	251	√
140	î	156	£	172	¼	188	╝	204	╠	220	▄	236	∞	252	^{n}
141	ì	157	¥	173	¡	189	╜	205	═	221	▌	237	φ	253	^{2}
142	Ä	158	₧	174	«	190	╛	206	╬	222	▐	238	ε	254	■
143	Å	159	ƒ	175	»	191	┐	207	╧	223	▀	239	∩	255	

Source: www.LookupTables.com

Figure 3. Three-Pass Protocol Process Scheme based on ASCII table
Source: http://www.asciitable.com

The *Vigenere cipher* is a method of encrypting alphabetic text by using a series of different Caesar ciphers based on the letters of a keyword. It is a simple form of polyalphabetic substitution in which each alphabet can replace with several cipher alphabets [5]. The encoding by the *Vigenère matrix* is the type by substitution. It consists to employ a key composed by a word or by an expression. It utilizes a square composed by the alphabet 25 times in such way it signifies a square matrix where each cell contains a letter of the alphabet (it exists several variants of the matrix) [6]. Message is divided in blocks. Length's block is equal length's key K. Maximal Length of key **is** 208 bits (26*8=208). Key length then could be 128,192,256... for one matrix. To perform the encryption using *Vigenere Cipher* algorithm development required the following functions:

$$C_i = P_i + K_i \bmod 37 \qquad (1)$$

This function is then developed by multiplying each index to the value of 1st ciphertext (Ci1) with Encryption 1st ciphertext. Encryption 1st ciphertext (EnC1) is the sum of subrange with length of plaintext.

3.1. Cryptography

The word Cryptography is derived from the Greek, namely from word Cryptos meaning of the word hidden and graphein means writing. Cryptography can be interpreted as an art or a science which researched how the data is converted into a certain shape that is difficult to understand [1]. Cryptography aims to maintain the confidentiality of information or data that can not be known by unauthorized parties (unauthorized person)

Cryptography components

Basically, the cryptographic component consists of several components, such as:

1. Encryption: is very important in cryptography, is a way of securing the transmitted data that are kept confidential. The original message is called plaintext, which is converted into code that is not understood. Encryption can be interpreted with a cypher or code. Similarly, do not understand a word then we will see it in a dictionary or glossary. Unlike the case with encryption, to convert plain text into text-code we use algorithms to encode the data that we want.
2. Decryption: is the opposite of encryption. The encrypted message is returned to the original form (original text), called the message encryption algorithm used for encryption is different from the algorithm used for encryption.
3. Keywords: that means here is the key used for encryption and description.

Security of cryptographic algorithms depending on how the algorithm works, therefore this kind of algorithm is called finite algorithm. Limited algorithm is an algorithm used a group of people to keep the messages they send.

3.2. Vigenere Chiper

The Vigenère cipher is a method of encrypting alphabetic text by using a series of different Caesar ciphers based on the letters of a keyword. It is a simple form of polyalphabetic substitution. [2]. The characters used in the Cipher Vigenere that is A, B, C, ..., Z and united with the numbers 0, 1, 2, ..., 25. The encryption process is done by writing the key repeatedly. Writing the key repeatedly performed until each character in messages have a couple of key characters. Furthermore, the characters in the message is encrypted using the Caesar Cipher key values that have been paired with numbers. The Confederacy's messages were far from secret and the Union regularly cracked their messages. Throughout the war, the Confederate leadership primarily relied upon three key phrases, "Manchester Bluff", "Complete Victory" and, as the war came to a close, "Come Retribution".[3] The table below show the example of encription using Vigenere Chiper consist of Plaintext, Key and Chipertextin Figure 4.

Plaintext	E	L	W	I	N		P	U	R	B	A		M	A	N	O	R	S	A
Key	M	A	N	O	R	S	A	M	A	N	O	R	S	A	M	A	N	O	R
Chipertext	R	M	K	X	F	S	Q	H	S	P	P	R	F	B	A	P	F	H	S

Figure 4. Examples Using Encryption Vigenere Cipher (Bruen, 2005)

Examples of encryption in Figure 1, the message character "**E**" is encrypted with a key "**M**" and generate cipher text "**R**". The results obtained from the code encrypting the message "**E**" is worth **5** and a key character "**M**" which is worth **13**. Each character value added **5 + 13 = 18**. Because **18** is less than the **26** which is the number of characters used, then **18** divided by **26**. The rest of the division is **18** which is a character "**R**". The encryption process can be calculated by the following equation (Stalling, 2011):

$$Ei=(Pi+Ki) \bmod 26 \qquad (2.0)$$

where Ei, Pi and Ki an encrypted character, the character of the message and the character key. While the decryption process can use the following equation:

$$Di=(Ci-Ki) \bmod 26 \qquad (2.1)$$

with Di is the result of the decryption code, Ci is character cipher text or cipher, Ki is a key character. While other methods to perform the encryption process with Vigenere Cipher method that uses tabula recta (also called Vigenere square; Figure 5).

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
a	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
b	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A
c	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
d	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
e	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
f	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E
g	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F
h	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
i	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
j	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I
k	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J
l	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K
m	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
n	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M
o	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
p	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
q	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
r	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
s	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
t	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
u	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
v	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
w	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
x	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
y	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
z	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y

Figure 5. Tabula recta Vigenere algorithm.

The leftmost column of squares states key letters, while the top line states **plaintext letters**. Each line in the rectangle states the **letters ciphertext** obtained by Caesar Cipher, in which the number of shifts letter plaintext specified numerical values of letters that key (**ie, a = 0, b = 1, c = 2, ..., z = 25**) , Vigenère square is used to obtain the ciphertext by using a key that has been determined. If the key length is shorter than the length of the plaintext, then the key are repeated it's use (the periodic system). When the key length is **m**, then the period is said to be **m**.

3.3. Vernam Chiper

Cryptography for most people is something that is very difficult and we as beginners tend to be lazy to learn it. However there is a cryptographic method that is rather easy to learn and the experts have stated that this method is a cryptographic method that is safe enough to use. The method is commonly known by the name of One Time Pad (OTP) or better known as the Vernam Cipher [4]. Vernam Cipher invented by Major J. Maugborne and G. Vernam in 1917. Algorithms One Time Pad (OTP) is a diversified symetric key algorithm, which means that the key used to encrypt and decrypt the same key. In the process of encryption, algorithm it uses the stream cipher derived from the XOR between bits of plaintext and key bits. In this method, the plain text is converted into ASCII code and then subjected to an **XOR** operation on the key that has been converted into ASCII code.

3.4. One Time Pad

One-time pad (OTP) is a stream cipher to encrypt and decrypt one character each time [5]. This algorithm was found in 1917 by Major Joseph Mauborgne as improvement of Vernam Cipher to produce a perfect security. Mauborgne proposes the use of One-Time Pad containing a row of characters randomly generated key. One pad is used only once (one-time) only to encrypt a message, after the pad has been used demolished so as not be reused for other encrypting messages. Encryption can be expressed as the sum modulo 26 of the plaintext character with one key character one-time pad [6]. This is the equation of one-time pad encryption 26 characters shown in Equation 2.2 below:

$$Ci = (Pi + Ki) \bmod 26 \qquad (2.2)$$

If the character that is used is a member of the set of 256 characters (such as characters with ASCII encoding), then the encryption equation shown in equation 2.3 below.

$$Pi = (Ci - Ki) \bmod 26 \qquad (2.3)$$

After the sender encrypts the message with the key, he destroyed the key. Recipient of the message using the same pad to decrypt the ciphertext characters into characters plaintext with equation 2.4 below.

$$Pi = (Ci - Ki) \bmod 26 \qquad (2.4)$$

for the 26-letter alphabet, or for the 256-character alphabet with equation 2.5 below.

$$Pi = (Ci - Ki) \bmod 256 \qquad (2.5)$$

The ways of working one time pad method:

$$C = \mathrm{P\ XOR\ K} \qquad (2.6)$$
$$\mathrm{P} = \mathrm{C\ XOR\ K} \qquad (2.7)$$

Note that the key length should be equal to the length of the plaintext, so there is no need to repeat the use of the key during the encryption process (as in vernam cipher).

3.5. Rotation Matrix

Rotation matrix is shifting ciphertext character that has been incorporated into the matrix column clockwise, along the defined distance of the key. How to determine the length of shifts and the number of shifts can be seen in the following Table 1 below:

Table 1. Calculation of Long Shifts in Matrix

Plaintext	E	L	W	I	N		P	U	R	B	A		M	A	N	O	R	S	A
Key	M	A	N	O	R	S	A	M	A	N	O	R	S	A	M	A	N	O	R
Chipertext	R	M	K	X	F	S	Q	H	S	P	P	R	F	B	A	P	F	H	S

Ki	M	A	N	O	R	S	A	
A = Dec(Ki)	77	65	78	79	82	83	65	(2.8)
B = A+C_{n-1}	76	90	91	79	83	87	70	(2.9)
C = A mod 26	25	13	0	1	4	5	13	(2.10)
D = B+C mod 26	23	25	13	2	9	14	5	(2.11)
Char	W	Y	M	B	I	N	E	
G = Dec(Char)	87	89	77	66	73	78	69	(2.12)
Rg = G mod 26	9	11	25	14	21	0	17	(2.13)

Explanation:

Ki : Key

A : Decimal ASCII value of the key characters

B : Results Summation with a decimal value key with the previous result (C).

C : The decimal value of a key character mod 26

D : Number of B + C mod 26

Char : The result of the shift in the index table alphabetic characters as much as the value of D
G : Decimal ASCII value of the key characters
Rg : Long shifts in the character matrix table

Whereas for the amount of shift is determined by the character key.

Forms of research conducted by the authors in this paper is a review of literature. The literature review is a framework, concepts, or orientation to perform the analysis and classification of facts collected in a study. Referral sources from books and journals, which are referred to in this paper are directly related to the object under researched, that is the plaintext encryption. The method of research that used in this paper is a flowchart. This method includes determining a model of encryption, the completion of the encryption algorithm, encryption simulation manufacture and analysis of simulation results encryption. Flowchart design simulations on complete this research can be seen in Figure 6.

Plain Text	E	L	W	I	N		P	U	R	B	A		M	A	N	O	R	S	A
Index Plaintext	5	12	23	9	14	0	16	21	18	2	1	0	13	1	14	15	18	19	1
Index Key	13	1	14	15	18	19	1	13	1	14	15	18	19	1	13	1	14	15	18
P + K mod 26	18	13	11	24	6	19	17	8	19	16	16	18	6	2	1	16	6	8	19
Cipertext_1	R	M	K	X	F	S	Q	H	S	P	P	R	F	B	A	P	F	H	S

Figure 6. Table Testing of Encryption Vigenere Cipher

Rotation Key Algorithm

Rotation key generation algorithm aims to improve the security of the plaintext by changing the keys into a new character. Furthermore, the new character is converted to decimal and then look for the value of the modulation to determine the length of a shift towards the characters in the matrix table. Consider the following Figure 7.

Key	M	A	N	O	R	S	A
A = ASCII Code	77	65	78	79	82	83	65
B = A + C(n-1)	76	90	91	79	83	87	70
C = Mod(ASCII Code)	25	13	0	1	4	5	13
D = Mod (B + C)	23	25	13	2	9	14	5
Chiper_Key	W	Y	M	B	I	N	E
X = ASCII Code	87	89	77	66	73	78	69
Y = Mod(X)	9	11	25	14	21	0	17
	RT 9	RT11	RT25	RT14	RT21	RT0	RT17

Figure 7. Process Key Algorithm determining Long Shifts on Matrix rotation

Rotation algorithm

In this process, all the characters ciphertext that has been generated on Vigenere Cipher algorithm written into the matrix table where condition matrix which must consist of a matrix of squares, where the number of rows equals the number of columns (Figure 8). To run this algorithm, first consider the following steps.

1. Write down all of the ciphertext into the matrix
2. The matrix will be formed must consist of the same number of rows and columns.
3. To define a matrix of rows and columns calculate the amount of n ciphertext; n = length of ciphertext.
4. If **0 < n <= 9**, it will form a **3 x 3** matrix.

   If **9** < **n** <= **16**, it will form a matrix of **4 x 4**

   If **16** < **n** <= **25**, it will form a matrix of **5 x 5,** etc.
5. If the matrix column empty, fill it with the alphabet from A until the empty column full of character.
6. Copy the first column then make a new ending column
7. After that copy the first line and then make a new ending line.
8. Make the first shift in which the shift length is determined by the rotation of key algorithms. Rotation matrix will end after reaching n MAX of keys.

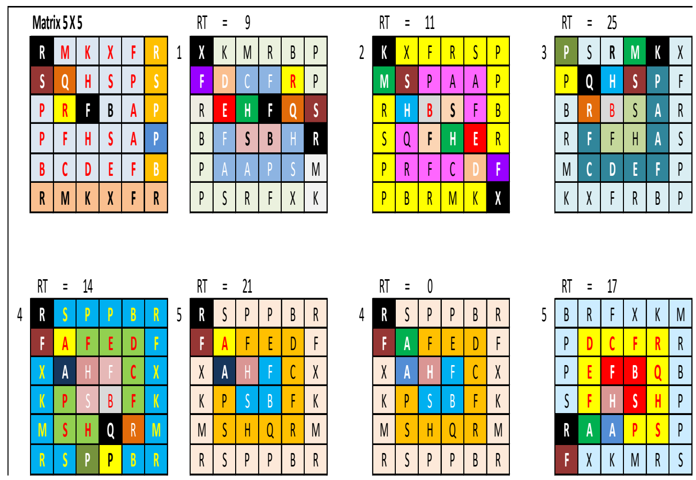


Figure 8. Process of Rotation Matrix

Retrieved end Cipertext: **BRFXKMPDCFRRPEFBQBSFHSHPRAAPSPFXKMS**

Vernam Cipher

Encryption can be expressed as the result of the Exclusive OR (XOR) of the plaintext character with an OTP key characters. This algorithm acts to encrypt the plaintext where the key is generated randomly. This key is valid only disposable where if you want to decrypt the same message then the generated key will be changed.

One Time Pad

In this method, there are two things that need to be encrypted is the key and the ciphertext obtained from vigenere chipper. This method works in advance is change the main key with One Time Pad methode to produce the Ciphertext of the key then this key will be the **second key** or a **new key** (Figure 9).

Ei : Pi XOR Ki
The main key : MANORSA
Random key :
Key Ciphertext : "€ +« "n%

Plaintext	Binary Bit	Random Bit	XOR	Key
M	01001101	11100101	10101000	“
A	01000001	11000001	10000000	€
N	01001110	01100101	00101011	+
O	01001111	11100100	10101011	«
R	01010010	11111010	10101000	“
S	01010011	00111101	01101110	n
A	01000001	01100100	00100101	%

Figure 9. Encryption Results of Main Key

The next new key that has been generated is used to encrypt the ciphertext that has been obtained from vigenere cipher.

From the results of the main key encryption in the previous process we obtain a new key and then the key is repeated until the key length equal to the length of the plaintext. The result of the encryption of the plaintext and the key will be the end of the process to produce a strong ciphertext.

Plaintext :
BRFXKMPDCFRRPEFBQBSFHSHPRAAPSPFXKMRS
Key :
“€+«“n%“€+«“n%“€+«“n%“€+«“n%“€+«“n%“

Ciphertext :
ÛÊm¾Ò#uý├m¨·>`¯┬zÚ¹(m¹╚{¨Ú/u¹ðm¾Ò#w¹

The encryption process is represented in binary form :

Pi : 01000010 01010010 01000110
01011000 01001011
01001101
Ki : 10101000 10000000 00101011
10101011 10101000
01101110
Ci : 11101010 11010010 01101101
11110011 11100011
00100011

Pi : 01010000 01000100 01000011
01000110 01010010
01010010
Ki : 00100101 10101000 10000000
00101011 10101011
10101000
Ci : 01110101 11101100 11000011
01101101 11111001
11111010

Pi : 01010000 01000101 01000110
01000010 01010001
01000010
Ki : 01101110 00100101 10101000
10000000 00101011
10101011
Ci : 00111110 01100000 11101110
11000010 01111010
11101001

Pi : 01010011 01000110 01001000
01010011 01001000
01010000
Ki : 10101000 01101110 00100101
10101000 10000000
00101011
Ci : 11111011 00101000 01101101
11111011 11001000
01111011

Pi : 01010010 01000001 01000001
01010000 01010011
01010000
Ki : 10101011 10101000 01101110
00100101 10101000
10000000
Ci : 11111001 11101001 00101111
01110101 11111011
11010000

Pi : 01000110 01011000 01001011
01001101 01010010
01010011
Ki : 00101011 10101011 10101000
01101110 00100101
10101000
Ci : 01101101 11110011 11100011
00100011 01110111
11111011

Fig. 8 The encryption process with the one-time pad method

4. Results and Discussions

The results of encryption on this method obtain ciphertext to be sent to the recipient where there are two keys that are sent, among others, the key to decrypt the plaintext and the key to decrypt the key.

4.1. Description

One Time Pad

To know the plaintext from the ciphertext received by the rightful recipient of the message, that is by first using the one-time pad method. The message recipients require a new key as a reference to recover the plaintext.

Ciphertext :
ÛÊm¾Ò#uý├m¨·>`¯┬zÚ¹(m¹╚{¨Ú/u¹ðm¾Ò#w¹
New Key : “€+«“n%

Ciphertext and the key then is XOR ed using the formula:

Di : Ci XOR Ki

The process of the message recipient to get a new message that is still in the form of Ciphertext are:

BRFXKMPDCFRRPEFBQBSFHSHPRAAPSPFXKMRS.

After successful decryption of the message recipient to decrypt back to find out the main key is still the one-time pad method.

Ciperteks key : “€+«“n%

Key : åÁeäú=d

Main Key = Ci XOR Ki

“ : 10101000 XOR å : 11100101 = 01001101 : M
€ : 10000000 XOR Á : 11000001 =01000001 : A
+ : 00101011 XOR e : 01100101 =01001110 : N
« : 10101011 XOR ä : 11100100 =01001111 : O
“ : 10101000 XOR ú : 11111010 =01010010 : R
N : 01101110 XOR =: 00111101 =01010011 : S
% : 00100101 XOR d : 01100100 =01000001 : A

Main Key : MANORSA

Rotation algorithm

The next decryption algorithm is an algorithm in which the rotation that has been encrypted ciphertext of the previous methods to be put into a matrix table. See the Figure 10 below.

Ciphertext: BRFXKMPDCFRRPEFBQBSFHSHPRAAPSPFXKMRS

B	**R**	F	X	K	M
P	**D**	C	F	R	R
P	E	F	B	Q	B
S	F	**H**	S	H	P
R	**A**	**A**	P	S	P
F	X	K	M	R	S

Figure 10. Matrix Ciphertext

In the previous encryption process, ciphertext character that is in the column of the matrix are moved forward along key value, but on the decryption algorithm ciphertext character that is in the matrix table is sliding backwards or revers along key value and repeated as much as the key length. The key value is obtained from the following process in the Figure 11 and Figure 12.

Key	M	A	N	O	R	S	A
A = ASCII Code	77	65	78	79	82	83	65
B = A + C(n-1)	76	90	91	79	83	87	70
C = Mod(ASCII Code)	25	13	0	1	4	5	13
D = Mod (B + C)	23	25	13	2	9	14	5
Chiper_Key	W	Y	M	B	I	N	E
X = ASCII Code	87	89	77	66	73	78	69
Y = Mod(X)	9	11	25	14	21	0	17
	RT 9	RT11	RT25	RT14	RT21	RT0	RT17

Figure 11. The process of determining the value of the key.

After a successful key value is determined next process is to do a rotation matrix algorithm.

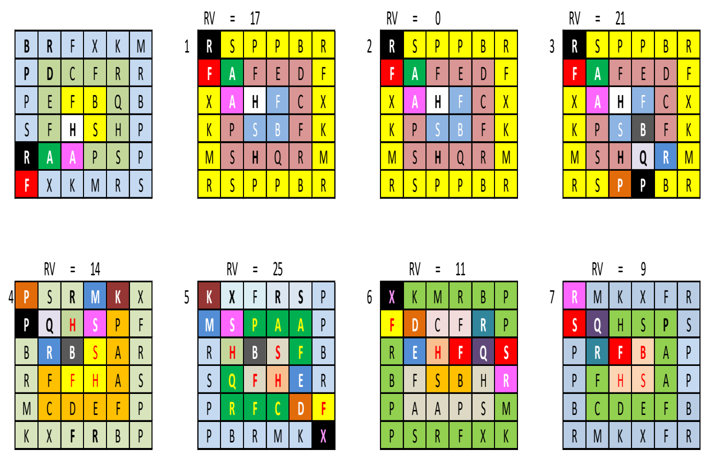


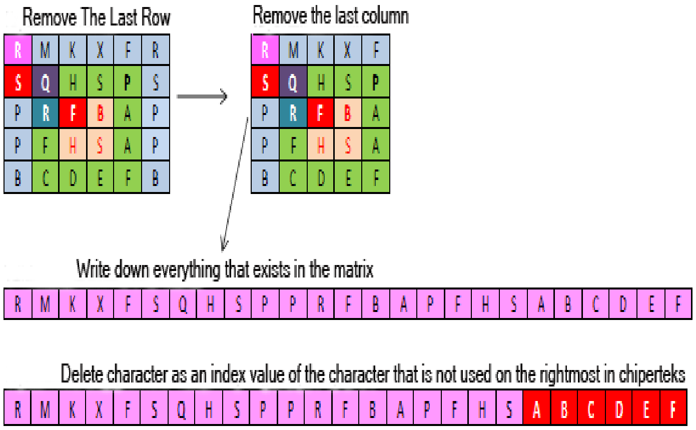
Figure 12. Process Description ciphertext with Matrix Ciphertext Algorithm

The results of the first rotation indicated at number one in the previous figure 10 wherein **R** is on the line five-column one. If we look at previous matrix characters that are in row one column one is **B**, then calculated spin clockwise as key values obtained from the previous process. The key value is taken values that are at the end of the index is **17**, the next process is the value at the previous index up until the beginning of the index, this is certainly the opposite of encryption algorithms. Having calculated the characters are on the order of **17** is **R** then **R** is placed on row one column one on the following matrix and is followed by the next character, then repeated continuously until the last process (Figure 13). In the process of this algorithm obtained the final matrix below:

R	M	K	X	F	R
S	**Q**	H	S	**P**	S
P	**R**	**F**	**B**	A	P
P	F	H	S	A	P
B	C	D	E	F	B
R	M	K	X	F	R

Figure 13. The Results of Algorithm Rotation Matrix

The next stage removes the last row and the last column of the matrix.



The Ciphertext become:

R	M	K	X	F	S	Q	H	S	P	P	R	F	B	A	P	F	H	S

Algoritma Vigenere

Last decryption algorithm using the algorithm vigenere cipher by using the key with formula:

$$Di = (Ci - Ki) \bmod 26$$

So the result we can see below:

Ciphertext : RMKXFSQHSPPRFBAPFHS
Key : MANORSAMANORSAMANOR
Plaintext : ELWIN PURBA MANORSA

R	M	K	X	F	S	Q	H	S	P	P	R	F	B	A	P	F	H	S
18	13	11	24	6	19	17	8	19	16	16	18	6	2	1	16	6	8	19
M	A	N	O	R	S	A	M	A	N	O	R	S	A	M	A	N	O	R
13	1	14	15	18	19	1	13	1	14	15	18	19	1	13	1	14	15	18
5	12	23	9	14	0	16	21	18	2	1	0	13	1	14	15	18	19	1
E	L	W	I	N		P	U	R	B	A		M	A	N	O	R	S	A

5. Conclusions

The encryption mechanism used in the present research is still modest or simple, but is expected to be useful as a first step to enter into the world of cryptography, particularly in the implementation of message security algorithms using other combinations. To the future, this research is expected to be developed, used and applied in the areas of life that is more complex.

References

[1] Zaeniah, Bambang Eka Purnama, "An Analysis of Encryption and Decryption Application by using One Time Pad Algorithm", *(IJACSA)* International Journal of Advanced Computer Science and Applications, 6(9), 2015.

[2] K Hashizume , "An analysis of security issues for cloud computing", Journal of Internet Services and Applications, a Springer open Journal: 1-13, 2013.

[3] Michael Nieles, Kelley Dempsey, Victoria Yan Pillitteri, "An Introduction to Information Security", National Institute of Standards and Technology Special Publication 800-12 Revision 1, 2017.

[4] William Stallings, "Cryptography and Network Security: Principles & Practices", Fifth edition, Prentice Hall, ISBN-13: 978-0136097044, 2010.

[5] C. Bhardwaj, "Modification of Vigenère Cipher by Random Numbers, Punctuations & Mathematical Symbols," Journal of Computer Engineering (IOSRJCE) ISSN No: 2278-0661, 2012.

[6] Anuja Priyam, "Extended Vigenère using double Transposition Cipher with One Time Pad Cipher", Intl J Engg Sci Adv Research; 1(2):62-65, ISSN No: 2395-0730, 2015.

[7] Sundram Prabhadevi, Rahul De, Pratik Shah, "Cost Effective Poly Vernam Cipher With Cache Optimization", Journal of Theoretical and Applied Information Technology, ISSN: 1992-8645, 2013.

[8] Shukla, R. and Prakash, H.O., "Sampurna Suraksha: Unconditionally Secure and Authenticated One Time Pad Cryptosystem", *IEEE Conference:* International Conference on Machine Intelligence and Research Advancement 174-178, DOI: 10.1109/ICMIRA.2013.40, 2013.

[9] Borowski, M., Lesniewicz, M.,"Modern usage of old one-time pad", IEEE Conference :Communications and Information Systems: 1-5, ISBN:978-***1-***4673-1422-0, 2012

[10] Elwinus Mendrofa, Elwin Yunith Purba, Boy Yako Siahaan, Rahmad W Sembiring, "Manipulation Vigenere Cipher Algorithm With Vernam Cipher Trough Matrix Table Rotation", 2nd International Conference of Computer, Environment, Social Science, Health Science, Agriculture & Technology (ICEST), ISBN: 979-458-964-0, 2017.

[11] Aized Amin Soofi, Irfan Riaz, Umair Rasheed, "An Enhanced Vigenere Cipher For Data Security", International Journal Of Scientific & Technology Research 5(03), March 2016, ISSN:22778616.

[12] M. Rouse. (2007, Plain text. Available: http://searchsecurity.techtarget.com/definition/plaintext, 2017.

[13] T. Gunasundari and K. Elangovan, "A Comparative Survey on Symmetric Key Encryption Algorithms," International Journal of Computer Science and Mobile Applications, ISSN no.2321-8363, 2014.

[14] Digital Economy Promotion Agency, under the Administrative Supervision of the Minister of Digital Economy and Society, 2016.

[15] Avialable at: —http://www.sipa.or.th", 2017.

[16] M. Agrawal and P. Mishra, "A comparative survey on symmetric key encryption techniques," International Journal on Computer Science and Engineering (IJCSE), vol. 4, pp. 877-882, 2012.

[17] V. Beal. (2009, Encryption. Available: http://www.webopedia.com/TERM/E/encryption.html, 2017.

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Combination of Caesar Cipher Modification with Transposition Cipher

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A R T I C L E I N F O

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The caesar cipher modification will be combine with the transposition cipher, it would be three times encryption on this experiment that is caesar modification at first then the generated ciphertext will be encrypted with transposition, and last, the result from transposition will be encrypted again with the second caesar modification, similarly at the decryption but the process is reversed. In the modification of caesar cipher, what would be done is the shift of letters are not based on the alphabet but based on ASCII table, plaintext will get the addition of characters before encryption and then the new plaintext with the addition of characters will be divided into two, they are plaintext to be encrypted and plaintext are left constantly (no encryption), The third modification is the key that is used dynamically follows the ASCII plaintext value.

1. Introduction

Cryptography is one way to conceal a message so that it is not easy to read or understood by people who are no right to access it. Cryptographic algorithm is divided into two that is classic and modern cryptographic algorithm. Classical cryptographic algorithm is an algorithm that used in antiquity that generally this algorithm only uses substitution and permutation method. Along with the development of algorithm technology it is considered less secure so then the new algorithms were created, then the modern cryptographic algorithm was born that use various method and in the process it used binary, hexadecimal, etc.

Classical cryptographic algorithm is starting to be abandoned by some people because it is considered less secure or considered it has expired, therefore it is possible to make modifications to the one of the classical cryptographic algorithm that is caesar cipher and combine it to another classical algorithm that is transposition cipher, so the results of ciphertext will be more complex and difficult to solve.

Modifications of caesar cipher have been done by other research, manipulating ciphertext by making readable ciphertext, so cryptanalis or unauthorized parties will not be suspicious if the message they are reading is a ciphertext [1]. In another research, caesar cipher was manipulated by changing the order of plaintext characters then performed substitution according to the key [2].

The modification of Caesar cipher will be executed by shifting the character based on ASCII table, plaintext will get the addition of characters before encryption and then the new plaintext with the addition of characters will be divided into two, plaintext that will be in encryption and plaintext which is left fixed (without encryption), and the last modification is to use a dynamic key following the value of plaintext ASCII.

2. Materials and Methods

In secure communication field, there are many studies which involves cryptography. Cryptography would be a well-known method for secure communication from the present of intruder. It is the algorithm method in which security goals are preferred. There are the processes to transcribe information into different form so that only authorized parties can access it. [3]

A cryptographic algorithm would be quite efficient when there is a guarantee for the data security. However, time of execution is more important, since it does not have to spend a lot of time to execute. [4]

There are two techniques of encryption: Substitution Technique and Transposition Technique. In substitution

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technique, the letters of plaintext are replaced by other letters or any number or by symbols. Example Caesarcipher, hill cipher, monoalphabetic cipher etc. In transposition technique, some sort of permutation is performed on plaintext. Example: rail fence method, columnar method etc[5].

Cryptography is divided into two types, Symmetric key and Asymmetric key cryptography. In Symmetric key cryptography a single key is shared between sender and receiver. The sender uses the shared key and encryption algorithm to encrypt the message. The receiver uses the shared key and decryption algorithm to decrypt the message.

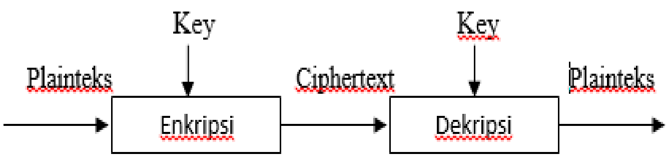

Figure 1. The encryption and decryption process of symmetric keys

In Asymmetric key cryptography each user is assigned a pairof keys, a public key and a private key. The public key isannounced to all members while the private key is kept secretby the user. The sender uses the public key which was announced by the receiver to encrypt the message. Thereceiver uses his own private key to decrypt the message[6].

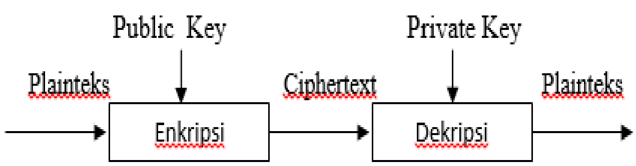

Figure 2. The encryption and decryption process of asymmetric keys.

2.1. Caesar Cipher

At the encryption of caesar cipher, each letter in the plaintext replaced with another letter with a fixed position apart by a numerical value, it is used as a secret key. [3]

Caesar cipher would be the simplest method of encryption because it is very easy to calculate but rarely used because of lack of robustness. The novelty of the introduced encryption technique is Simple in implementation but difficult in intercepting. And with caesar cipher data transmission can be successfully delivered using LASER. [7]

In cryptography, Caesar cipher is one of the simplest and most famous encryption techniques. The password includes a substitution password in which each letter in the plaintext is replaced by another letter that has a certain position difference in the alphabet. For example, if using shear 3, B will be E, U becomes X, and K becomes N so the plaintext of "buku" will become "EXNX" in encoded text. Caesar's name was taken from Julius Caesar a Roman general, consul, and dictator who used this password to communicate with his commander.

The encoding process (encryption) can mathematically use modulus operation by converting the letters into numbers, A = 0, B = 1, ..., Z = 25. The password (En) of "letter" x with shear n is mathematically written with:

$$E_n(x) = (x + n) \bmod 26$$

While in the process of solving the code (decryption), the decryption (Dn) is:

$$D_n(x) = (x - n) \bmod 26 \qquad [8]$$

Caesar Cipher can be combined with the vigenere cipher [9]. But, it would be possible to combined Caesar cipher with transposition cipher for secure encryption. Because the combination of this two techniques provides more secure and strong cipher. The final cipher text is so strong that is very difficult to solve. The transposition method only change position of characters and substitution method only replaces the letter with any other letter. The above described method provides much more secure cipher with combination of both the transposition and substitution method. [10]

2.2. Transposition Cipher

Ciphertext is obtained by changing the position of the letters inside the plaintext. In other words, this algorithm transfers the sequence of letters in plaintext. Another name for this method is the permutation, because transpose each character in the text is the same as permutating the characters [11].

Examples of transposition cipher are as follows:

Plaintext: "ABCDEFGHI”

A	B	C
D	E	F
G	H	I

The result of transposition cipher is: ADGBEHCFI

Transposition is often combined with other techniques. With the power of computers, substitution and transposition encryption can be easily performed. The combination of these two classic techniques provides a more secure and strong cipher. The key is like a password for cipher text which is so strong that no one can break it. Transposition method only provides the rearrangement of characters of the plaintext. The attacker may attack on the cipher text to known plaintext. Above described method contain transposition as well as substitution method which makes secure and strong ciphertext. [12]

3. Results and Discussions

Encryption and decryption process through 3 processes. In the encryption process, the plaintext will be encrypted twice with the modified caesar cipher and once with the transposition cipher algorithm. Modifying the caesar cipher algorithm which the caesar not only uses the alphabet letters but also uses the characters in the ASCII table (32-126 characters), Key is used dynamically because the key is one of the ASCII value of one plaintext character. So, different plaintext, different key. Mathematically the process of encryption and decryption are as follows:

$$Ex_= ((A-32)+K) \bmod 127)$$

$$Dx = ((A-32)-K) \bmod 127$$

3.1. Description

A=ASCII character inthe plaintext K= Key(one of the ASCII characters from plaintext mod 32).

The last modification is by adding certain words to the plaintext, in the encryption process the system will add a certain word on the back of the plaintext then plaintext divided into 2 parts: the unencrypted part, and the encrypted part. Conversely in the process of decryption, system will automatically delete the additional words so plaintext will be back to normal. The process of encryption and decryption is as follows:

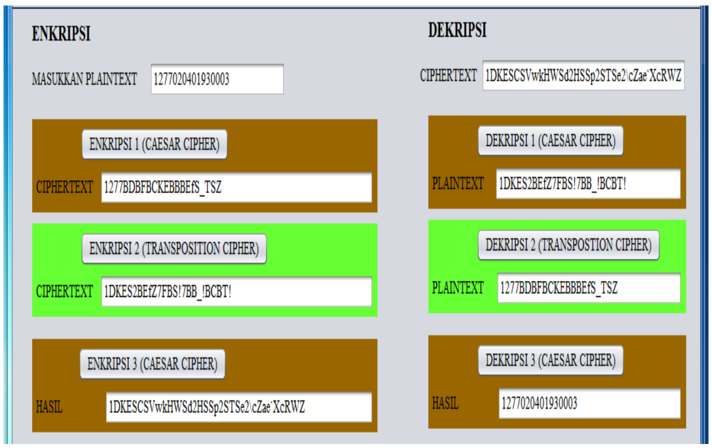

Figure 3. The process of encryption and decryption system

The explanation of the first caesar modification is as follows: The plaintext used is the NIK Number of KTP (identity Card) consisting of 16 characters, So the plaintext is 1277020401930003, then added with the word 'TAMBAH', so the plaintext on the system become 22 characters, they are 127702040193TAMBAH. Furthermore, the plaintext is divided into two, they are the unencrypted characters and the character that will be encrypted. The unencrypted charactersare the the four characters at the beginning of the plaintext, they are '1277'and the character that will be encrypted is '02040193TAMBAH'. The key in this encryption is the ASCII value of the second character ('2') which is 50. Then put into the formula where the conversion is executedat every character on the plaintext that will be encrypted.

$$Ex = ((A-32)+K) \bmod 127)$$

The ASCII Value of character ‘0’, which is 48.

$$Ex = ((48-32)+50) \bmod 127)$$
$$= ((16+50) \bmod 127)$$
$$= 66 \bmod 127$$
$$= 66$$

The ASCII value of 66 is character 'B'. And so on until the last character. So from the above data '127702040193TAMBAH' become '1277BDBFBCKEBBBEfS_TSZ'

Since the encryption/decryption key depend on the ASCII value of the message so if the key character changed then all of ciphertext will be changed. As in the picture below:

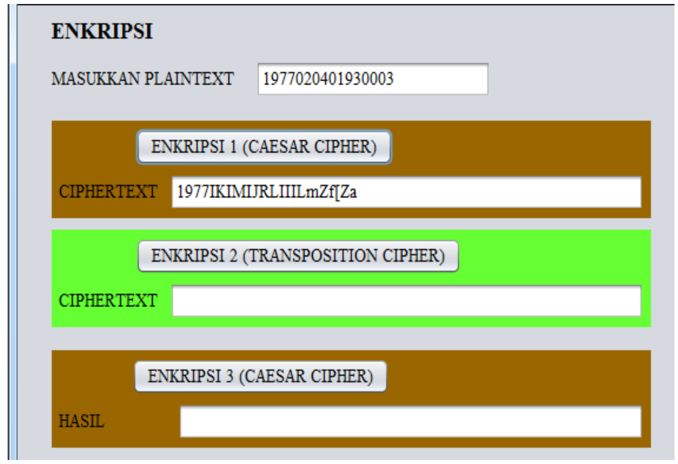

Figure 4. The process of encryption system

Then, the result of the first Caesar modification is encrypted with columnar transposition with 5 as key, since the result of the first encryption just 22 characters then the plaintext will be added 3 characters of ‘!’. There is not any modification at this encryption step, so it will execute as usual.

Plaintext is ‘1277BDBFBCKEBBBEfS_TSZ’

1	2	7	7	B
D	B	F	B	C
K	E	B	B	B
E	F	S	_	T
S	Z	X	X	X

So, the ciphertext is :

“1DKES2BEfZ7FBS!7BB_!BCBT!”

Furthermore, the result of columner transposition will be encrypted again with the second Caesar modification. The second modification is similar to the first modification. The difference is the number of unencrypted characters is 5. The word added into this step is “KRIPTOGRAFI” and the key used is the first ASCII value.

The plaintext for the second Caesar modification is ‘1DKES2BEfZ7FBS!7BB_!BCBT!KRIPTOGRAFI”. The unencrypted characters is ‘1DKES’. And the character that will be encrypted is ‘2BEfZ7FBS!7BB_!BCBT!KRIPTOGRAFI’ with The key in from the ASCII value of the first character ('1') which is 49. After all of the part has been known, then put it into the formula.

$$Ex = ((A-32)+K) \bmod 127)$$

The ASCII Value of character ‘2’, which is 50

$$Ex = ((50-32)+49) \bmod 127) = ((18+49) \bmod 127)$$
$$= 67 \bmod 127$$
$$= 67$$

The ASCII value of 67 is character 'C'. And so on until the last character. So from the above data ‘1DKES2BEfZ7FBS!7BB_!BCBT!KRIPTOGRAFI’ become ‘1DKESCSVwkHWSd2HSSp2STSe2\cZae`XcRWZ’.

Decryption is the inversion of the encryption process, if there is character addition at the encryption process then there is character reduction at the decryption process. The decryption process is refer to the following Figure 5:

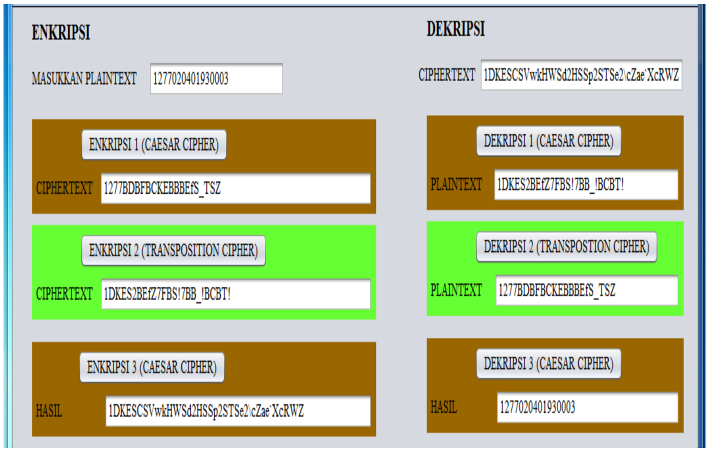

Figure 5. The process of encryption and decryption system

4. Conclusions

From the above explanation, it is known that the original plaintext and final ciphertext have different number of characters and in the process occurs 3 times encryption where each process occurs characters addition. So, the result of ciphertext is very complex and to solve it there are so many possibilities that should be tried cryptanalis such as: must find the sequence of algorithms used, determine the added character part, determine the unencrypted character part and search for the key used, this is not simple, because different plaintext, different keys will be used.

References

[1] Purnama, B. and Rohayati, H, A New Modified Caesar Chiper Cryptography Method With Legible Ciphertext From A Message To Be Encrypted, Procedia Computer Science 59: 195 – 204, 2015.

[2] Abraham, O. and Shefiu, G.,O, An Improved Caesar Cipher (Icc) Algorithm, [IJESAT], International Journal of Engineering Science & Advanced Technology 2(5):1198-1202, 2012.

[3] Han, L. C. and Mahyuddin, N. M, An Implementation of Caesar Cipher and XOR Encryption Technique in a Secure Wireless Communication, International Conference on Electronic Design (ICED):111 –116, 2014.

[4] Gowda, S, N, Innovative Enhancement Of The Caesar Cipher Algorithm For Cryptography, IEEE, 2016.

[5] Mishra, A, Enhancing Security Of Caesar Cipher Using Different, IJRET, International Journal of Research in Engineering and Technology 2(9): 327-332, 2013

[6] Jain, A., Dedhia, R., & Patil, A, Enhancing the Security of Caesar Cipher Substitution Method using a Randomized Approach for More Secure Communication. International Journal of Aplications 129(13): 6-11, 2015.

[7] Abedin, S., Tasbin, T. and Hira, A. Optical Wireless Data Transmission with Enhanced Substitution Caesar Cipher Wheel Encryption, International Conference on Electrical, Computer and Communication Engineering (ECCE): 552-556, 2017.

[8] Pradipta, A, Implementation Caesar Chiper Alphabet Plural Method in Criptografi for Information Security. Indonesian Journal on Networking and Security 5(3):16-19, 2016.

[9] Senthil, K., Prasanthi, K, and Rajaram, R. IEEE, 2013.

[10] Shrivastava, G., Sharma, R, and Chouhan, M, International Journal of Engineering Sciences & Research Technology 2(6): 1475-1478, 2013.

[11] Sasongko, J, Protction of Information Data Using Clasic Criptografi, Jurnal Teknologi Informasi DINAMIK 10(3):160-167, 2005.

[12] Padiya, S, D, and Dakhane, D, N, Plaintext Based Transposition Method, International Journal of Advance Research in Computer Science and Software Engineering. 2(7): 234-236, 2012.

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Analysis of Learning Development With Sugeno Fuzzy Logic And Clustering

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A B S T R A C T

In the first journal, I made this attempt to analyze things that affect the achievement of students in each school of course vary. Because students are one of the goals of achieving the goals of successful educational organizations. The mental influence of students' emotions and behaviors themselves in relation to learning performance. Fuzzy logic can be used in various fields as well as Clustering for grouping, as in Learning Development analyzes. The process will be performed on students based on the symptoms that exist. In this research will use fuzzy logic and clustering. Fuzzy is an uncertain logic but its excess is capable in the process of language reasoning so that in its design is not required complicated mathematical equations. However Clustering method is K-Means method is method where data analysis is broken down by group k (k = 1,2,3, .. k). To know the optimal number of Performance group. The results of the research is with a questionnaire entered into matlab will produce a value that means in generating the graph. And simplify the school in seeing Student performance in the learning process by using certain criteria. So from the system that obtained the results for a decision-making required by the school.

1. Introduction

The quality of education is largely determined by the ability of schools to manage the learning process and more specifically the learning process that takes place in the classroom. In particular students who are still in junior high school who have adolescence, adolescence is a critical period in the development cycle of a person, where at this time there are many changes, both biological changes, psychological and social changes. This phase of change often leads to conflict between adolescents with themselves and conflicts with the surrounding environment. If the conflict cannot be resolved properly then in its development can bring negative impact, especially on adolescent character maturation and not infrequently trigger the occurrence of mental disorders.

Mentally and emotional disorders such as depression, behavior problems and substance abuse among children and adolescents cause a heavy burden for families, nations and themselves. For that required an expert to diagnose developmental learning disorders in children by knowing the graph of learning disorders of the learners themselves. By knowing the graph of the level of learning disorders of students, teachers and parents can find out how to educate and more to control the level of patience in educating children.

The data used for the diagnosis of developmental learning disorders categorized mental emotional problems and learning behavior is from the problem below:

1. Behavior
2. Feelings
3. Relationships
4. Individuals who never mature

From the above variables, the authors analyze by applying the fuzzy logic method, because it is very flexible and has a tolerance to data that is not appropriate and based on natural language.

And also the author tries to add the processing in the analysis of Student Learning Performing Grouping by applying Clustering method by assessing:

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1. Material
2. Discipline
3. Attitude

2. Research Methods

In order to obtain the data that can be in this study, then held the factors that affect student learning disorders is a factor in everyday life. Learning disorder factors are acceptable in a variable that is:

Table 1. Types of Bounce Emotions and Behavior Problems in Learning

No	Category	Type Problems	Value	Information
1.	Learning Problems Borderline	- Behavior	0 - 17	- Conduct Pull Away (Overambitious)
		- feelings	0 – 16,5	- Excessive worries about problems
		- Relationships	0 – 20	- Difficulty establishing relationships with peers
		–Individuals who never grew	0 – 18	- Lack of consistency
2.	Abnormal Learning Problems	- Behavior	13 - 25	- Conduct contrary (not like being reprimanded / given positive feedback, do not want to follow the rules)
		- feelings	14 - 25	- Not able to solve problems
		- Relationships	16 - 25	- Aggressive against peers
		- Individuals who never grew	15 - 25	- Very inconsistent

Information:
Here the author previously explained in the table above that categorized:

1. Borderline learning problems describe the behavior and disorders of children who are changing.
2. And if abnormal learning problems are excessive child's behavior and disorder.

3. Discussion

Illustrations to clarify the notion of a fuzzy rule system. Suppose we want to build a system to control the storage tool, then the process is needed:

3.1. Fuzzification

The input of truth value is certain (crisp input) is converted to fuzzy input form, which is linguistic value whose semantic is determined based on membership function.

3.2. Inference

To distinguish from Firs-Order logic, syntax, a fuzzy rule is written as IF antecendent THEN consequent.

3.3. Defuzzification

At this stage of defuzification we calculate the average (Weight Average / WA) of each predicate on each variable using the following equation:

$$y^* = \frac{\sum y\mu_R(y)}{\sum \mu_R(y)}$$

Information:

α_n = the value of the predicate rules to-n

Z_n = index to the output value -n

4. General Needs

Emotional and mentality problems that are not resolved properly, it will have a negative impact on the development of adolescents in the future, especially on the maturation of his character and not infrequently trigger the occurrence of emotional and behavioral mental disorders that can be high-risk behavior. This shows that 80% of adolescents aged 11-15 years are said to have exhibited risk behaviors such as misbehavior in school, substance abuse, and anti-social behavior (stealing, fighting, pulling, or ditching) and 50% of them also show behavior Other high risk factors such as drunk driving, sexual intercourse without contraception, and minor criminal behavior.

Motivation of teachers to work will encourage to achieve the success of teaching and learning activities. One measure of the success of teaching and learning can be seen from the value obtained from students didikannya.

From the results of questioning based on the questionnaire can be retrieved important data available to be re-managed into information that will be used as source based on the criteria or variables contained in the questionnaire.

Start the fuzzy logic toolbox on matlab by typing fuzzy on the commad window. Next will appear FIS Editor window view with sugeno type. To start the FIS with the usual sugeno type start by adding and adjusting the input and output.

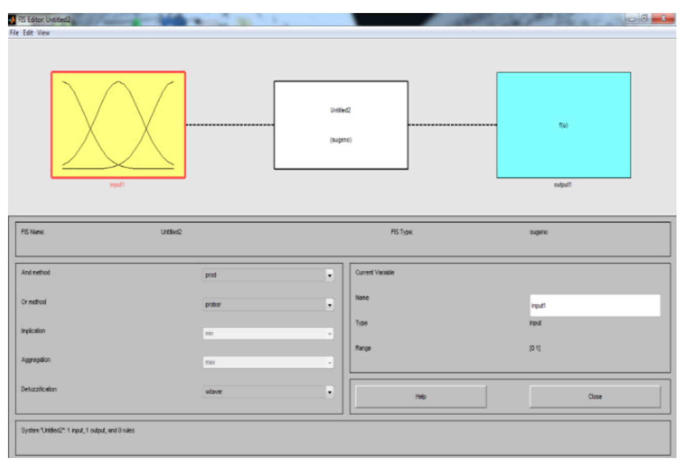
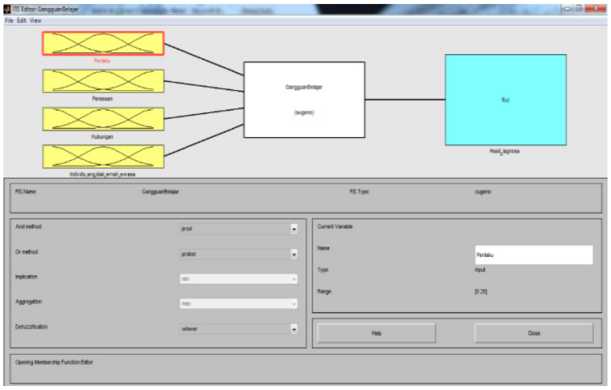


Figure 1. FIS Editor

The next step is to add the desired number of parameters of behavior, feelings, relationships, individuals who never mature.



igure 2. FIS Editor Tipe *Sugeno*

Description, picture above is Variable in Sugeno type. Each input parameter has a certain value, each being a membership function (MF) that has a different type.

For category I using type zmf, category II using type zmf, category III using type zmf and category IV using type zmf. The output variables in the form of results have two categories of MF, namely Borderline (changing nature) and Abnormal (hard and excessive bars). Both categories use tippet zmf.

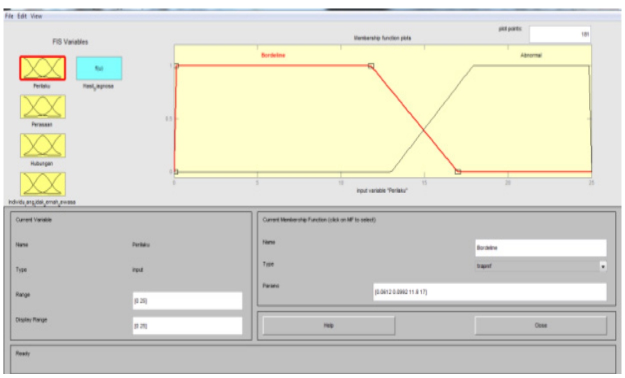
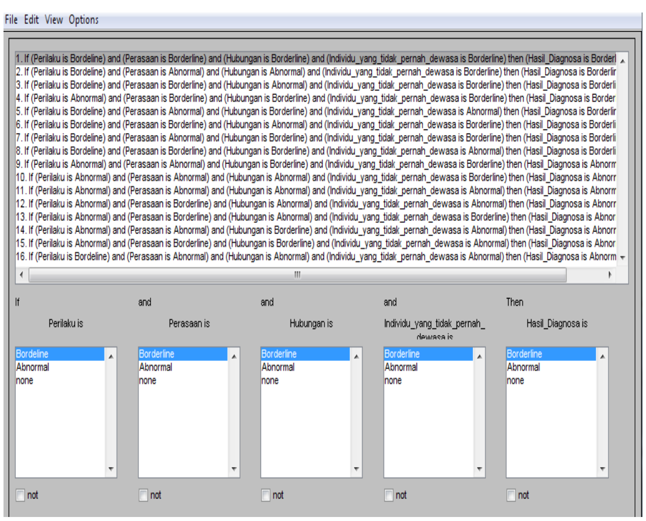


Figure 3. Membership Function Behavior



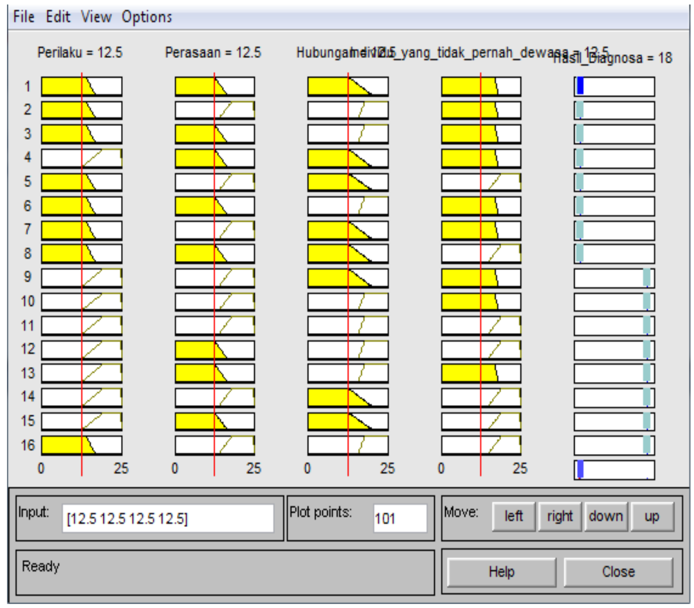
4. Rule Editor

The above picture is the Membership Function (MF) of each variable inputted in the Sugeno type. The use of four input variables with each variable has three MF, then made the rule as much as 16 pieces with the algorithm as follows:

In full, the entire algorithm is created in the Rule Editor by using operand and in determining the result value. As seen in the picture.4 Rule Editor.

Description, the picture above is Rule-rule that will determine the final result in Sugeno type.

The algorithm created in Rule Editor can be visualized by using Rule Viewer, which by using the input parameter values will get the output value.



5. Rule Viewer TipeSugeno

Description, picture above is Rule Viewer in sugeno type. The result of algorithm created in rule editor can be shown in graphical form by using surface viewer. That is shown below.

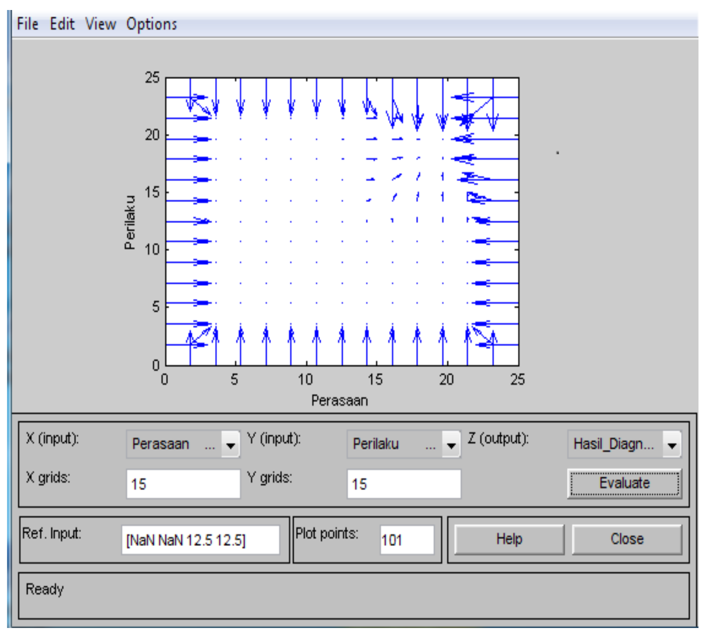


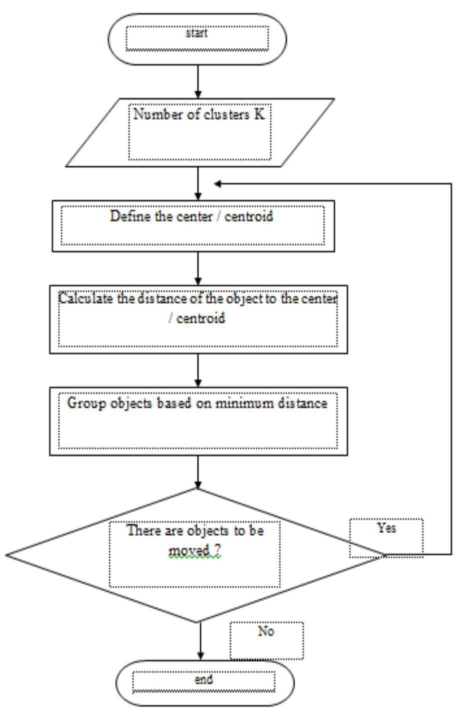
Figure 6. Surface Viewer Tipe Sugeno

The conclusion of the results at the input is:

Table 2. Diagnosis of rule input into *Matlab*

NO	SYMPTOMS				DESCRIPTION OF RESULTS	
	BEHAVIOR	FEELING	RELATIONSHIP	INDIVIDUALS WHO NEVER MATURE		
	IF	AND	AND	AND	THEN	
					Borderline (0,5 = 16)	Abnormal (1 = 25)
1.	5	13	20	14	16	
2.	10	14	16	17	16	
3.	12	12	18	15	16	
4.	13	7	15	18	16	
5.	13	10	20	5	16	
6.	14	5	14	15	16	
7.	14	20	12	24	18,6	
8.	15	24	10	15		21
9.	15	17	18	10		21
10.	16	15	5	18		23.1
11.	17	22	24	16		25
12.	16	15	15	16		25
13.	20	18	17	8		25
14.	18	23	21	20		25
15.	25	14	22	17		25
16.	17	16	16	22		25

Here is a flow chart drawing of K-Means algorithm :



7. Flow chart of K-Means

Information:

1. First select the number of clusters k. The initialization of this cluster center k can be done in various ways. The random way is often used, the cluster centers are scored with random numbers and used as the starting cluster center.
2. Calculate the distance of each data that exists on each cluster center.
3. Determine the cluster with the closest distance to each data.
4. Reassemble each object using the new cluster center. If the cluster center has not changed again, then the clustering process is complete. When changed, then go back to step no.3 until the cluster center does not change anymore.

To form a group of objects, the first thing to do is to measure the euclidean distance between two points or objects or items X and Y defined as follows:

$$\textbf{Deuclidean (X,Y)} = \sqrt{\sum_i (X1 - Y1)^2}$$

Table.3 Distance Determination

Student	Material	Discipline	Attitude
	X	**Y**	**Z**
A	3,15	3,83	3,75
B	3,63	3,88	3,95
C	2,98	3,88	3,75
D	3,63	3,95	3,90
E	3,35	3,88	3,93
F	3,33	3,88	3,95
G	3,83	3,95	3,93
H	3,30	3,95	3,85

From the results of the above data then grouping group can be seen in the table below:

Table.4 Group Determination

Student	Material	Discipline	Attitude	DistanceFrom			Group
	X	Y	Z	C1	C2	C3	
A	3,15	3,83	3,75	0	0,2	0,3	1
B	3,63	3,88	3,95	0,5	0,4	0,3	3
C	2,98	3,88	3,75	0,2	0,3	0,4	1
D	3,63	3,95	3,90	0,5	0,3	0,3	2
E	3,35	3,88	3,93	0,3	0,1	0	3
F	3,33	3,88	3,95	0,3	0,1	0	3
G	3,83	3,95	3,93	0,7	0,5	0,5	2
H	3,30	3,95	3,85	0,7	0,6	0,5	3

Group Search / Determination:

1. If the shortest distance is in C1 then Will be entered into group 1
2. If the shortest distance is in C2 then Will be entered into group 2
3. If the shortest distance is in C3 then Will be entered into group 3

From the results of the above table, the centroid for the second iteration is determined as follows:

Centroid 1

X=(3,15+3,63)/2 = 3,39

Y=(3,83+3,88)/2 = 3,86

Z=(3,75+3,95)/2 = 3,85

Centroid 2

X=(2,98+3,30)/2= 3,14

Y=(3,88+2,95)/2= 3,42

Z=(3,75+3,85)/2= 3,80

Centroid 3

X=(3,63+ 3,35+ 3,33+ 3,83)/4= 3,54

Y=(3,95+3,88+ 3,88 + 3,95)/4= 3,92

Z=(3,90+3,93+ 3,95 + 3,93)/4= 3,93

From the results of the above analysis, the results obtained as shown in Figure 8 below:

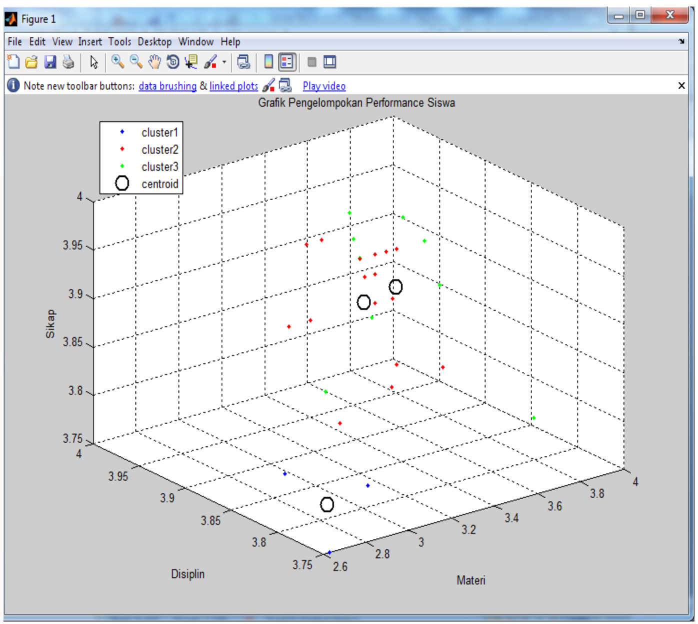


Figure 8 Display Clustering Results

5. Conclusions

From the results of testing the system has been done, it can be taken several conclusions including the following:

1. The system is expected to help users to analyze developmental learning disorders in children.
2. By using the method of data mining will be easy in doing grouping ability or student performance.
3. Data processing is done with the help of matlab program (matrix laboratory) which can calculate the calculation with decludean formula.
4. Utilization of this technology as auxiliary tool to diagnose developmental learning disorders enough to assist the task of a teacher expert (guidance and counseling) and other teachers.

6. References

[1] Yusuf Munawir dkk , *Pendidikan Bagi Anak Dengan Problem Belajar*, Surakarta : Tiga Serangkai Pustaka Mandiri, 2003.

[2] Prof. Dr. Martini Jamaris, M.Sc.Ed, *Kesulitan Belajar*, Jakarta : Ghalia Indonesia, 2013.

[3] Prof. Dr. H. Prayitno, M.Sc.Ed.dkk, *Dasar-dasar Bimbingan dan Konseling*, Jakarta : Rineka Cipta, 1994.

[4] H. Mohammad Asrori, .Pd, *Penelitian Tindakan Kelas*, Bandung : CV Wacana Prima, 2008.

[5] Jogiyanto HM, *Analisis & Desain Sistem Informasi : Pendekatan Terstruktur Teori dan Praktik Aplikasi Bisnis*, Yogyakarta : Andi, 2005.

[6] Kusrini dan Luthfi Emha Taufiq, *Algoritma Data Mining*, Yogakarta: Andi, 2009

[7] Berry dan Linoff, *Data Mining Tecnicque for marketing Sales and Customer Suppor*t, Yogyakarta:Andi Publisher, 2008

[8] Kusuma Dewi Sri, *Artificial Intelligence* (*Teknik dan Aplikasinya*), Yogyakarta: GRAHA ILMU, 2003.

[9] Naba Eng Agus, *Belajar Cepat Fuzzy Logic Menggunakan MATLAB*, Yogyakarta:
Andi, 2009.

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A Theoretical and Experimental Comparison of One Time Pad Cryptography using Key and Plaintext Insertion and Transposition (KPIT) and Key Coloumnar Transposition (KCT) Method

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A B S T R A C T

One Time Pad (OTP) is a cryptographic algorithm that is quite easy to be implemented. This algorithm works by converting plaintext and key into decimal then converting into binary number and calculating Exclusive-OR logic. In this paper, the authors try to make the comparison of OTP cryptography using KPI and KCT so that the ciphertext will be generated more difficult to be known. In the Key and Plaintext Insertion (KPI) Method, we modify the OTP algorithm by adding the key insertion in the plaintext that has been splitted. Meanwhile in the Key Coloumnar Transposition (KCT) Method, we modify the OTP algorithm by dividing the key into some parts in matrix of rows and coloumns. Implementation of the algorithms using PHP programming language.

1. Introduction

In terms about security especially Information Technology (IT), security is very important to be applied. There are many methods that used for this security. But there is no security method that guaranteed reliability. All of them must have weakness. So, for decreasing every weakness we need to make experiment about security method. One of this method is enhancement of cryptography algorithm.

In cryptography are known some algorithms, one of them is One Time Pad (OTP) Algorithm. OTP includes flow ciphers. Discovered by Major J Maugborne and G Vernam in 1971. Each key is only used for a single message.

The encryption process with One Time Pad (OTP) Algorithm or One Time Pad Cryptography is essentially an Exclusive-OR algorithm that is consistent with a less complicated implementation [4].

In transposition cipher, plaintext is similar, but its order was changed. On the other words, the algorithm do transposition of the character set in the text. The other name for this method is permutation because of doing the transposition each character in the text is similar with make permutation the characters.

In this paper, modified the One Time Pad (OTP) algorithm by splitting each binary plaintext into 4 sections and then inserting a key between separate plaintexts.

In transposition process, there is a key in which is mostly used in order to make the cryptanalist be difficult to guess the ciphertext. Transposition would be solution before doing insertion process. Transposition method in this process using matrix [4x2].

After that, Exclusive-OR process will be done so that the ciphertext will be more difficult to be known.

2. Materials and Methods

The underlying mathematical basis of the process of encryption and decryption is the relation between two sets, plaintext and ciphertext. Encryption and decryption are functions of transformation between the sets.

In principle, cryptography has 4 main components:

1.Plaintext : Readable messages
2.Ciphertext : Random messages that can not be read
3. Key : The key that's doing cryptographic
4. Technique Algorithm: Methods for encryption and decryption

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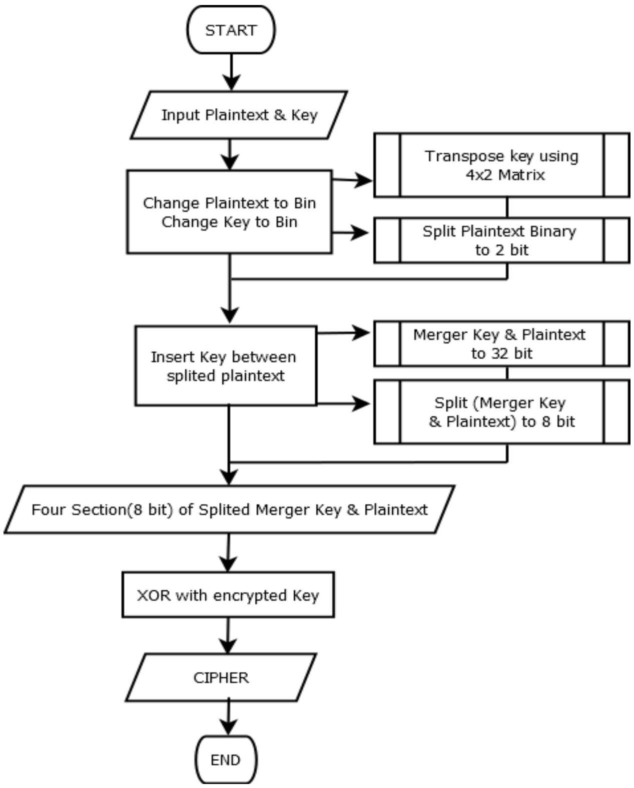

Figure 1. Flowchart of Encrypt Process

In cryptography, the main process used is encryption and decryption. Encryption is formed based on an algorithm that will randomize an information into a form that cannot be read or cannot be seen. Decryption is a process with the same algorithm to return random information to its original form. The algorithm used should consist of carefully planned arrangement of procedures that must effectively produce an encrypted form that cannot be returned by someone, even if they have the same algorithm.

Encryption is the process used to encode plaintext by converting plaintext into ciphertext. While decryption is the reverse process, which is to change the ciphertext to plaintext [2].

Formula of the encrypt and decrypt process as follows:

$$Encrypt:$$

$$plaintext \oplus key = ciphertext$$

$$Decrypt:$$

$$ciphertext \oplus key = plaintext$$

Here the flowcharts of encrypt and decrypt process:

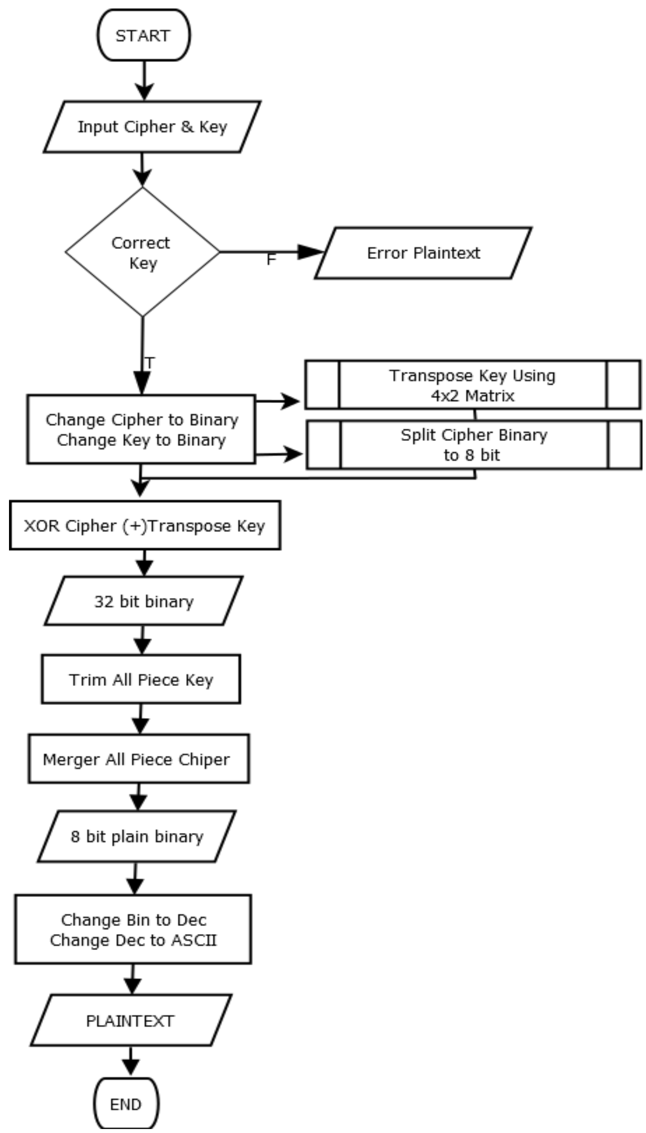

Figure 2. Flowchart of Decrypt Process

Table 1. Encryption Proces

a	*PLAINTEXT*=K	01001011				**FLOW** 8 bit binary
b	=(a1/4)	01	00	10	11	Split 8 bit binary@2bit
c	*KEY*=T=	01010100				8 bit of key
d	*TRANSPOSE Key*	00001110				Transpose Key using 4x2 Matrix
e	*MIX*(**d**) *to* (**b**)	0100001110	0000001110	1000001110	11	Insert key to splitedplaintextupto 32 bit
f	*MERGER*(**e**)	01000011100000001110100000111011				unallocated merger result
g	=(f/4)	01000011	10000000	11101000	00111011	Splitedunallocated merger result @ 8 bitbinary
h	XOR KEY(**d**) with(**g**)	01001101	10001110	11100110	00110101	XOR result from d1 with f1
i	*DEC*(**h**)	77	142	230	53	Change to dec
j	*ASCII*(**i**)	M	Ä	µ	5	Cipher

Tabel 2. Decryption Process

a	*CIPHER* K=	MÄµ5				**FLOW** Enkripsi of "K"
b	*DEC(***a***)*	77	142	230	53	Change to dec
c	*BINARY CIPHER(***a***)*	01001101	10001110	11100110	00110101	XOR key and plain before it
d	*KEY T*	00001110	00001110	00001110	00001110	Key Transpose
e	XOR KEY(**d**) to CIPHER(**c**)	01000011	10000000	11101000	00111011	Xor result
f	*MERGER(***e***)*	01000011100000001110100000111011				Unallocated 32 bit binary
g	=TRIM(**e**)	01	00	10	11	Trim key result
h	*=CONC(***g***)*	01001011				Mix Trim key result
i	*=DEC(***h***)*	75				Change to dec
e	*ASCII (***i***)*	K				Plaintext

3. Results and Discussions

One Time Pad (OTP) in the encryption and decryption process will perform XOR logic on plaintext-key and chipertex-key. The ciphertextthat is generated by OTP has the exact character length of plaintext and key so it opens up opportunities for cryptanalysis to guess key and plaintext.

But in this case, the authors perform the XOR process after the insertion of key into individual plaintext characters. So the ciphertext will be generated more difficult to be guessed. For more details here is a description of how the KPI algorithm works:

3.1. Encryption Process

We must determine the plain text to be encrypted.

According both of the tables, ciphertext was obtained from plaintext **"K"** with *key* **"T"** is " MÄµ5" and plaintext from *cipher*" MÄµ5**"** with *key* **"T"** is **"K".**

4. Implementation of The Algorithm

To find out the success of this algorithm, we will try to implementation this algorithm at PHP Language. This application is a program that do encryption and decryption process from the text. On this application User will be requested for entering a plain text, algorithm will processing it and make a result of chiper text up to four times from plain text.

For the fruitfulness test of this algorithm, we will encryption a text as follows:

"PLAINTEXT"

The Following is view of encryption process result using the key "KEY".

5. Conclusions

According the Implementation, we have concluded that the modification of *One Time Pad* (OTP). Algorithm using insertion key On the splitting plain that the authors develop In this paper can working properly as an alternative cryptographic algorithm, so this algorithm able to improve the capability One Time Pad (OTP) algorithm for today and future. Based using this algorithm we can see the Compare result from the encryption process with a text as follows: "K" and the key "T".

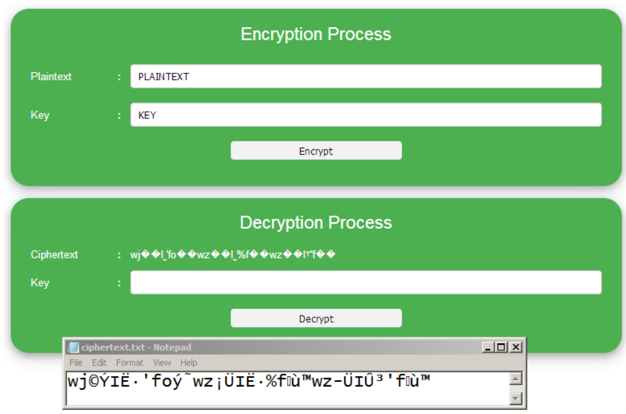

Figure 2. Encryption Interface

The view result of decryption process using the key "KEY" as follows:

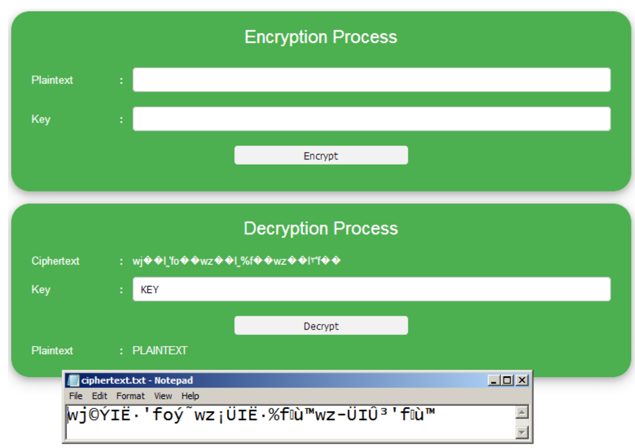

Figure 3. Decryption Interface

Table 3.0 Cipher Comparation, OTP with OTP Enhancementt

CHAR	OTP	OTP with KPI	OTP with KCT
Plaintext	K	K	K
Key	T	T	T
Ciphertext	▼	MÄµ5	**☺Q."**

References

[1] Chen, Z and Xu, J, One-Time-Pads Encryption in the Tile Assembly Model, IEEE Explorer and Conference Proceeding, 2008.
[2] Kurniawan, Y, Criptography: Internet Security and Comunnication System, Informatika Bandung, 2004.
[3] Mezaal, Y. S, OTP Encryption Enhancement Based on Logical Operations, IEEE Explorer and Conference Proceeding, 2016.
[4] Saragih, F. R, Using Chrtography One Time Pad (AlgoritmaVernam) in Information Security,2008.
[5] Pryo Utomo, Enhancement of OTP Cryptography Using Key and Plaintext Insertion, KPI Journal, 2017.

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System Control Device Electronics Smart Home Using Neural Networks

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A B S T R A C T

The use of information technology is very useful for today's life and the next, where the human facilitated in doing a variety of activities in the life day to day. By the development of the existing allows people no longer do a job with difficulty. For that, it takes a system safety home using system technology Web-based and complete video streaming CCTV (video streaming) a person can see the condition of his home whenever and wherever by using handphone, laptops and other tools are connected to the Internet network. This tool can facilitate someone in the monitor at home and control equipment the House as open and close and the lock the gate, turning on and off the lights so homeowners are no longer have to visit their home and fear the state of the House because fully security and control in the House was handled by the system. based on the above problems Writer try to design work system a tool that can control the simulation tools home using two Microcontroller is Attiny 2313 and Atmega16.

1. Introduction

Using of information technology is very useful for today's life and the next, where the human facilitated in doing a variety of activities in the life day to day. With the development of the existing allows people no longer do a job with difficult.

For it takes a system safety home using system technology Web-based and complete video streaming CCTV (video streaming) a person can see the condition of his home whenever and wherever with handphone, laptops and other tools are connected to the Internet network.

This tool can facilitate someone in the monitor the home and control equipment the House as open and close and the lock the gate, turning on and off the lights so homeowners are no longer have to visit his home and fear the state of the House because fully security and control in the House was handled by the system.

Based on the above problems Writer try to design work system a tool that can control the simulation tools home using two microcontroller is ATtiny 2313 and Atmega16.

2. Research Methodology

The previous was conducted testing the use of the ability of NN to receive the input of the sensor Web so that NN, can be decided to do something like that desired. The first test done is using NN as a control the position of a sensor security on an object. Here NN must understand distance sensor door on certain distance with object recipient of the sensor hitch. At the beginning, sensor used is infrared. Testing this fails, because NN can't be receiving put in the form of binary (on/off or detection/not detected). This is because NN only can learn from comparing the mistake of other errors, but with the value should not be too extreme. Testing followed by using the proximity sensor, so that NN can measure the distance sensor with the Wall. in this test, NN can do a good job.

Testing both performed by trying to use the number of entries for information error that is not the same as the number of the output of NN . Generally NN use the information error with the same amount of the output and reference to NN. From this trial obtained, as long as it can be made a formula that can be used to change the amount of information error that there is a certain number same with the output NN, then it can be used. Of the two experiments this simple tried to design control system motion control home using NN can learn own based on information from some of the sensor to drive the two Motor in order to control the door to move forward and can avoid the theft of people who have not been known by the database.

In general the configuration of the desired is such as in figure 2.1. and figure 2.2. Method used in this study in general use the system is common, that NN with learning Backpropagation. Two or sub-system added the counter error and braking, therefore, in

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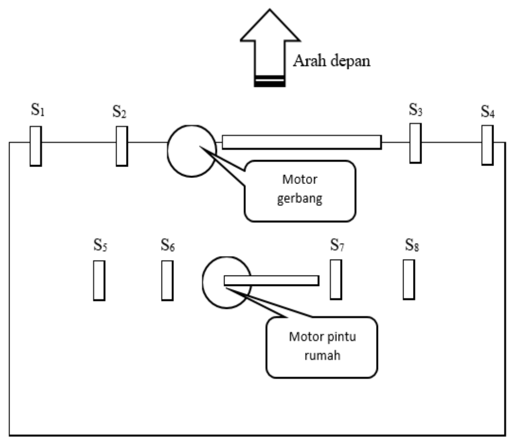
the discussion next will not shown formulaformula, except for the counter error and braking, as well as an explanation system configuration made. rest (for example of NN) has been much discussed in the book text as in the Freeman (1992) or other reference. For sensor, pre-process, drivers, Motor, mechanical will not discussed here, because it depends on the type used or depending needs. That will be discussed here just a mechanism to a control home appliance can have the ability to move to close and open the door by itself through the process of learning themselves without guide from outside or learning process previous off-line.

Configuration mechanical equipment Figure 1. explain the form of mechanical equipment and Figure 2. shows configuration control system using NN and the process of learning on-line.



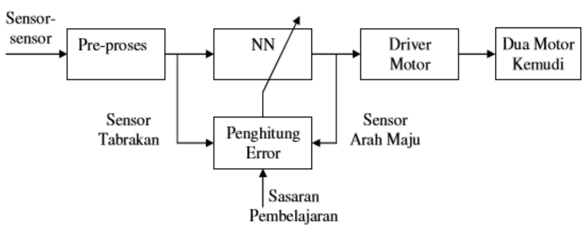


Figure 2. System Configuration

2.1. Architecture NN to controller

In this experiment used NN configuration 10 neurons in the input layer to read 8 sensor collision and two speed sensor, some (10 to 20) neurons in the hidden layer and 2 neurons in the output layer to run the Motor the left and Motor right.

Sensor collision is proximity sensor used to measure the distance objects that exist in front of the sensor. This sensor planned have a range of 0 to 25 centimeters. While the speed sensor is used to determine the amount of Motor speed at that time including forward or backward. Pre-process is used only for the condition the output of the sensor collision and speed sensor to fit the input of NN the value of 0 to 1.

The key of this system is a model or formulas that exist in the counter error. In the model NN, the number of error entered or will be used in the process of learning is the same as the number of neurons output. Where error mean error output of NN to reference value. but in this system, not at all used or no data or reference value given. of course this will make it difficult for learning process NN. The only information that can be used for the learning process is input a number of 10 that point, the actual input is not suitable for error from both sides of the relationship or number of input. Counter error is considered as a major determinant because he should be able to change or calculate than 10 input into 2 data errors can then be used for the learning process NN. Given no relationship direct mathematical between input and error, then made approach functional (operational), that if a sensor has a particular value or detect certain conditions, then are to blame is the Motor next to where. Approach simple this can be made formula a very simple of counter error and even the form of formulas can also be created, as long as it meets the approaches that have been used.

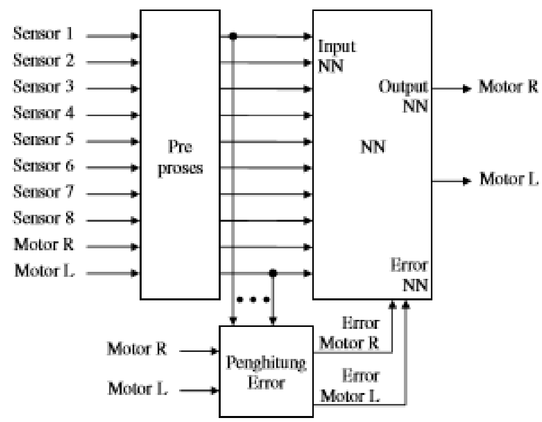


Figure 3. Architecture NN to control system

eL = (-S1 - S2 + S7 + S6 + S5 + sML)

eR = (-S1 - S8 + S3 + S4 + S5 + sMR) (1)

or

eL = (-S1-S2+S7+S6+S5+sML) + (Rnd * 0.5 - 0.25)

eR = (-S1-S8+S3+S4+S5+sMR) + (Rnd * 0.5 - 0.25) (2)

where:

- eL and eR is error output counter error for learning process NN
- S1 to S8 is output proximity sensor after through the pre-process
- sML and sMR is output speed sensor after through the pre-process

Formula (1) is the simplest form to determine the error motion of the Motor, while the formula (2) added functionality random to get the possibility of movement vary. Or can be used other forms.

3. Results and Discussions

After all the design of hardware and software is completed, the next step is testing Circuit has been designed. This test function to show the error writers do when connecting the program is made with a series of which has been designed for each design designed controlled by the system program previously have been made to run the system digital.

3.1. Test Series USB4REL

Serves as a communication between Computers with Devices, and continue execution to atmega16.

Testing in a series of USB4REL this can be done by connecting a series of minimum attinny2313 with adapter power supply as a source voltage. Feet 20th connected by voltage source DC 5 V, while the foot of 10th connected to the ground.

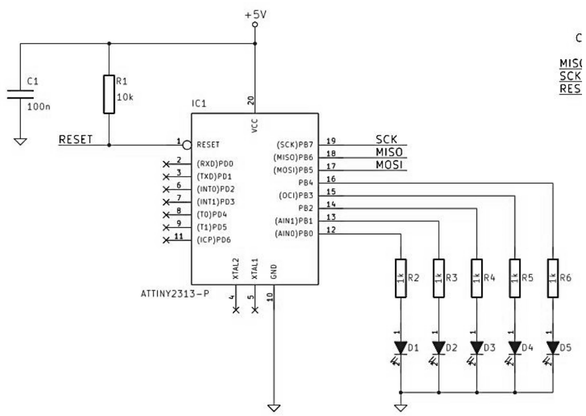


Figure 3 Series USB4REL

Testing program

```
#include <delay.h>

void main(void)

{

while (1)

    {

    // Place your code here

        if (PIND==0xf0)

        { PORTB=0xff; }

    }

}
```

4. Conclusions and Suggestions

4.1. Conclusions

The observations made during the study author can draw conclusions:

1. Home security systems can monitor through Video Systems Live Streaming and controlled in a Web interface using simulation.

2. Using the Web systems, users do not need to install the application in the device controller, the user can also access system using laptop or smartphone connected with the Internet network.

4.2. Suggestions

From observations the authors did, there are some things that can be developed back namely:

1. From the results of the design, manufacture and trial security systems this, it would be a maximum if this system developed back to be able to activate the home appliance other such as water pump, shutters, and air conditioning.
2. Video facilities that exist in the system only limited to live streaming, so the camera Web Cam not record and can not be replay, it would be a maximum if the Web Cam available can also record.

5. References

[1] Adhi Prasetio, Web Design Beginners, PT Mediakita, Jakarta, 2010,

[2] Andi, Build-Based Applications PHP and MySQL, 2nd Edition, Andi offset, Yogyakarta, 2012,

[3] Anhar, PHP & SQL in Self-Taught, PT Mediakita, Jakarta, 2010,

[4] Heri andrianto, Programming AVR Micro controller Atmega16, PT Informatika, Bandung, 2013,

[5] Harmein, Bolton, W, Irzam Harmein, Programable Logic Controller (PLC), Edition 3, Irzam Harmein, PT Gelora Aksara Pratama, Jakarta, 2011.

[6] Muchlas, Digital Circuits, PT Gaya Media, Yogyakarta2013.

[7] Prawiroredjo Kiki, Robot Delivery Auto-Based AVR Microcontroller Atmega16., Journal of electronics. Lecturer Department of Electrical Engineering-FTI, Trisakti University, 2010.

[8] Taufik DWI Septian Suyadhian, The Microcontroller As a Computer System or Microcomputers Single Chip, Edition-1, 406, Gramedia, Jakarta, 2010.

[9] Toni Kun, Website Advanced Design With JQuery, PT Mediakita, Jakarta, 2010.

[10] Rides computers, My SQL Database Server, PT Mediakita, Jakarta, 2010.

[11] Widodo Budiharto, S.Si., Digital Electronics + Microprocessors, Publisher Andi, Yogyakarta, 2010.

[12] Zul Bahrum C, Applications Microcontroller System For Parking, Journal Gradient 3 (1): 200-203, January 2013.

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Comparison of K-Means and Fuzzy C-Means Algorithms on Simplification of 3D Point Cloud Based on Entropy Estimation

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ABSTRACT

In this article we will present a method simplifying 3D point clouds. This method is based on the Shannon entropy. This technique of simplification is a hybrid technique where we use the notion of clustering and iterative computation. In this paper, our main objective is to apply our method on different clouds of 3D points. In the clustering phase we will use two different algorithms; K-means and Fuzzy C-means. Then we will make a comparison between the results obtained.

1 Introduction

The modern 3D scanners are 3D acquisition tools which are developed in terms of resolution and acquisition speed. The clouds of points obtained from the digitization of the real objects can be very dense. This leads to an important data redundancy. This problem must be solved and optimal point cloud must be found. This optimization of the number of points results in reducing the reconstruction calculation.

The problem of simplifying point cloud can be formalized as follows: given a set of points X sampling a surface S, find a sample points X' with $|X| \leq |X|$, Such that X' sampling a surface S' is close to S. $|X|$ is the cardinality of set X. This objective requires defining a measure of geometric error between the original and simplified surface for which the method will resort to the estimation of the global or local properties of the original surface. There are two main categories of algorithms to sampling points: sub-sampling algorithms and resampling algorithms. The subsampling algorithms produce simplified sample points which are a subset of the original point cloud, while the resampling algorithms rely on estimating the properties of the sampled surface to compute new relevant points.

In the literature, the categories of simplification algorithms have been applied according to three main simplification schemes. The first method is simplification by selection or calculation of points representing subsets of the initial sample. This method consists of decomposing the initial set into small areas, each of which is represented by a single point in the simplified sample [1-4]. The methods of this category are distinguished by the criteria defining the areas and their construction.

The second method is iterative simplification. The principle of iterative simplification is to remove points of the initial sample incrementally per geometric or topologic criteria locally measuring the redundancy of the data [5-10].

The third method is simplification by incremental sampling. Unlike iterative simplification, the simplified sample points can be constructed by progressively enriching an initial subset of points or sampling an implicit surface [11-18].

This paper will present a hybrid simplification technique based on the entropy estimation [19] and clustering algorithm [20].

It is organized as follows: In section 2, we will recall some density function estimators. In section 3, we will present clustering algorithm. Then in section

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4, we will present our 3D point cloud simplification algorithm based on Shannon entropy [21]. Section 5 will show the results and validation. Finally, we will present the conclusion.

2 Defining the Estimation of Density Function and Entropy

There are several methods for density estimation: parametric and nonparametric methods. We will fcus on nonparametric methods which include the kernel density estimator, also known as the Parzen-Rosenblatt method [22,23] and the K nearest neighbors (K-NN) method [24]. We will only use in this article K-NN estimator.

2.1 The K Nearest Neighbors Estimator

The algorithm of the k nearest neighbors (K-NN) [24] is a method of estimating the nonparametric probability of the density function. The degree of estimation is defined by an integer k which is the number of the nearest neighbors, generally proportional to the size of the sample N. For each x we define the estimation of the density. The distances between points of the sample and x are as follows:

$$r_1(x) < \dots < r_{k-1}(x) < r_k(x) < \dots < r_N(x)$$

r_i with $(i = 1 \dots k \dots N)$ are distances sorted by ascending order.
The estimator k-NN in dimension d can be defined as follows:

$$p_{knn}(x) = \frac{\frac{k}{N}}{V_k(x)} = \frac{\frac{k}{N}}{C_d r_k(x)} \quad (1)$$

where $r_k(x)$ is the distance from x to the kth nearest point and $V_k(x)$ is the volume of a sphere of radius. $r_k(x)$ and C_d is the volume of the unit sphere in d dimension.
The number k must be adjusted as a function of the size N of the available sample in order to respect the constraints that ensure the convergence of the estimator. For N observations, the k can be calculated as follows:

$$k = k_0 \sqrt{N}$$

By respecting these rules of adjustment, it is certain that the estimator converges when the number N increases indefinitely whatever is the value of k_0.

2.2 Defining Entropy

Claude Shannon introduced the concept of the entropy which is associated with a discrete random variable X as a basic concept in information theory [21]. The distribution of probabilities $p = p_1, p_2, \dots, p_N$ associated with the realizations of X. The Shannon entropy is calculated by using the following formula:

$$H(p) = -\sum_{i=1}^{N} p(x_i)\log(p(x_i)) \quad (2)$$

Entropy measures the uncertainty associated with a random variable. Therefore, the realization of the rare event provides more information about the phenomenon than the realization of the frequent event.

3 Clustering Algorithms Definition

$X = \{x_i \in \Re^d\}$, $i = 1,,N$ is a set of observations described by d attributes; the objective of clustering is the structuring of data into homogeneous classes. Clustering is unsupervised classification. The objective is to try to group clustered points or classes so that the data in a cluster is as similar as possible. Two types of approaches are possible; hierarchical and non-hierarchical approaches[25].

In this article, we will concentrate on the non-hierarchical approach which is encapsulated in both the Fuzzy C-Means Clustering (FCM) algorithm [26,20] and K-means algorithm (KM)[27].

3.1 Fuzzy C-Means Clustering Algorithm

Fuzzy c-means is a data clustering technique wherein each data point belongs to a cluster to some degree that is specified by a membership grade. This technique was originally introduced by J.C. Dunn[20], and improved by J.C. Bezdek[26] as an improvement on earlier clustering methods. It provides a method that shows how to group data points that populate some multidimensional space into a specific number of different clusters.

$X = \{x_1, x_2, \dots, x_n\}$ is a given data set to be analysed, and $V = \{v_1, v_2, \dots, v_c\}$ is the set of centers of clusters in X data set in p dimensional space $\Re^p$. Where N is the number of objects, p is the number of features and c is the number of partitions or clusters. FCM is a clustering method allowing each data point to belong to multiple clusters with varying degrees of membership.

FCM is based on the minimization of the following objective function

$$J_m = \sum_{i=1}^{c}\sum_{j=1}^{N} u_{ij}^m D_{ijA}^2 \quad (3)$$

Where, D_{ijA}^2 is the distances between i^{th} features vector and the centroid of j^{th} cluster. They are computed as a squared inner-product distance norm in Equation (4):

$$D_{ijA} = \| x_j - v_i \| = (x_j - v_i)^T A (x_j - v_i) \quad (4)$$

In the objective function in Equation (3), U is a fuzzy partition matrix that is computed from data set X:

$$U = u_{ij} \quad (5)$$

m is fuzzy partition matrix exponent for controlling the degree of fuzzy overlap, with $m > 1$. Fuzzy overlap refers to how fuzzy the boundaries between clusters are. That is the number of data points that have significant membership in more than one cluster.

The objective function is minimized with the constraints as follows: $u_{ij} \in [0,1]$; $1 \leq i \leq c$; $1 \leq j \leq N$; $\sum_{i=1}^{c} u_{ij} = 1$; $0 < \sum_{i=1}^{N} u_{ij} < N$;

FCM performs the following steps during clustering:

1. Randomly initialize the cluster membership values, u_{ij}.

2. Calculate the cluster centres:

$$\frac{\sum_{i=1}^{N} u_{ij}^m x_j}{\sum_{i=1}^{N} u_{ij}^m}$$

.

3. Update u_{ij} according to the following:

$$\frac{1}{\sum_{k=1}^{N} (D_{ijA}/D_{kjA})^{2/(m-1)}}$$

4. Calculate the objective function, J_m

5. Compare $U^{(t+1)}$ with $U^{(t)}$, where t is the iteration number.

6. If $\| U^{(t+1)} - U^{(t)} \| < \varepsilon$ then it stop, or else returns to the step 2. (ε is a specified minimum threshold, in this case it uses $\varepsilon = 10^{-5}$).

3.2 K-Means Clustering Algorithm

The K-Means algorithm (KM) iteratively computes the cluster centroids for each distance measurement in order to minimize the sum with respect to the specified measure. The objective of the K-Means algorithm is to minimize an objective function named by squared error function given in equation (6) as follows:

$$J_{km}(X;V) = \sum_{i=1}^{c} \sum_{j=1}^{N} D_{ij}^2 \tag{6}$$

D_{ij}^2 is the chosen distance measure which is in Euclidean norm: $\| x_{ij} - v_i \|^2$, $1 \leq i \leq c$,$1 \leq j \leq N_i$. Where N_i represents the number of data points in i^{th} cluster. For c clusters, the K-Means algorithm is based on an iterative algorithm that minimizes the sum of the distances of each object at its cluster center. The goal is to have a minimum value of the sum of the distance by moving the objects between the clusters. The steps of K-means are as follows:

1. Centroids of c clusters are chosen from X randomly.

2. Distances between data points and cluster centroids are calculated.

3. Each data point is assigned to the cluster whose centroid is close to it.

4. Cluster centroids are updated by using the formula in Equation (7):

$$v_i = \sum_{i=1}^{N_i} \frac{x_{ij}}{N_i} \tag{7}$$

5. Distances from the updated cluster centroids are recalculated.

6. If no data point is assigned to a new cluster, the execution of algorithm is stopped, otherwise the steps from 3 to 5 are repeated taking into consideration probable movements of data points between the clusters.

4 Evaulation of the Simplified Meshes

In order to give a theoretical evaluation for the simplification method, we have based on a metric of mean errors, max error and RMS(root mean square error) used by Cignoni et al.[28]. Where he measured the Hausdorff distance between the approximation and the original model. Hausdorff distance is defined as follow:
Let X and Y be two non-empty subsets of a metric space (M,d). We define their Hausdorff distance

$$d_H = max\{\sup_{x\in X} \inf_{y\in Y}(d(x,y)), \sup_{y\in Y} \inf_{x\in X}(d(x,y))\} \tag{8}$$

In our experiments we will use the symmetric Hausdorff distance calculated with the Metro software tool[28].
In order to calculate the approximate error, we will reconstruct the models from the point clouds. We find in the literature several reconstruction techniques [29] to create a 3D model from a set of points.

In the next section, we will present our simplification approach based on the estimation of Shannon entropy and algorithm clustering. Where we will first use the FCM algorithm and then replace it with the KM algorithm and compare the results obtained from the use of the two algorithms in our simplification method. We have previously two types of nonparametric estimators, the K-NN estimator and the Parzen estimator. Each type has advantages and disadvantages. For Parzen estimator, the bandwidth choice has strong impact on the quality of estimated density [30]. For this reason, we will use K-NN estimator to estimate the density function.

5 Proposed Approach

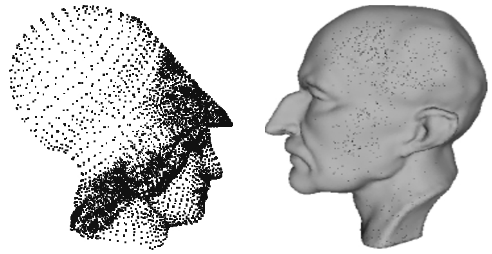
Now, we will propose an algorithm to simplify dense 3D point cloud. First, this algorithm is based on the entropy estimation algorithm. It allows the estimation of the entropy for each 3D point of X as well as making the decision to eliminate or to keep the point. In this pproach we will use the K-NN estimator to estimate the entropy. Moreover, it is based on clustering algorithm (KM or FCM) to subdivide point cloud X into clusters in order to minimize the computation time. The procedures are:

SIMPLIFICATION ALGORITHM

- Input
  - $X = \{x_1, x_2, , x_N\}$: The data simple (point cloud)
  - S: threshold
  - c: the number of clusters
- Begin
- Decomposing the initial set of points X into c small areas denoting R_j $(j = 1, 2, ..., c)$, and using clustering algorithm (FCM or KMA).
- For $j = 1$ to c
  - Calculate global entropy of a cluster j by using all data samples in $R_j = \{y_1, y_2, , y_m\}$ according to the equation (2), Note this entropy $H(R_j)$.
  - Calculate the entropy $H(R_j - y_i)$ of point cloud R_j less point $y_i (i = 1, 2, , m)$
  - Calculate $\triangle H_i = \mid H(R_j) - H(R_j - y_i) \mid$ with $i = 1, 2, , m$
  - If $\triangle H_i \leq S$ Then $R_j = R_j - y_i$.
  - End-if
- End for End.

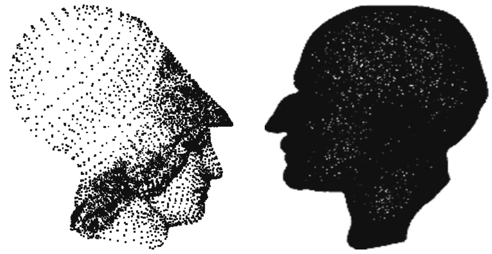
6 Results and Discussion

To validate the efficiency of the use of the two clustering algorithms FCM and KM in our simplification method, we use three 3D models that represent real objects such as Max Planck (fig. 1,b) and Atene (fig. 1,a). Fig. (2,a), (2,b) show simplification results on various point cloud using FCM algorithm. Fig. (3,a), (3,b) show simplification results on various point cloud using KM algorithm.



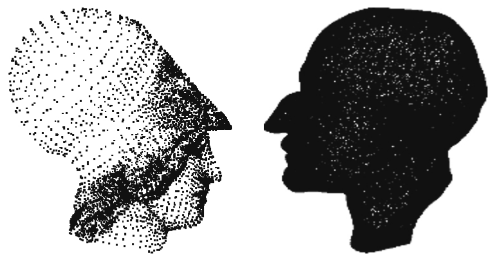
(a) (b)

Figure 1: original point cloud, a) Atene, b) Max Planck



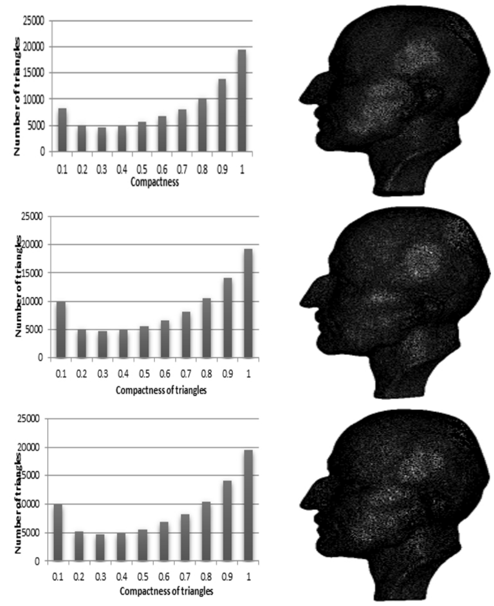
(a) (b)

Figure 2: Simplified point cloud using FCM algorithm, a) Atene, b) Max Planck



(a) (b)

Figure 3: Simplified point cloud using KM algorithm, a) Atene, b) Max Planck

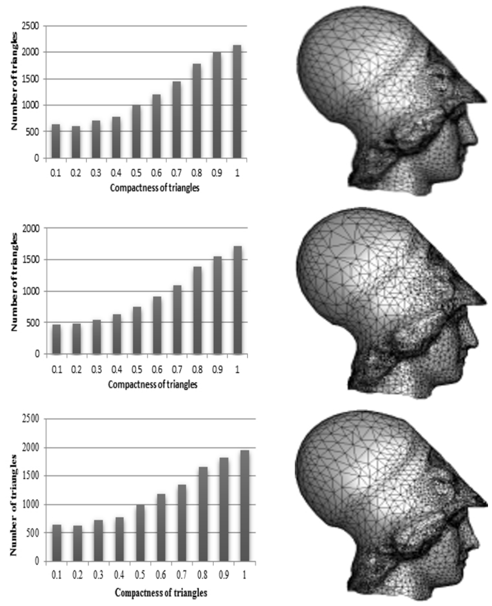


(a)

(b)

(c)

Figure 4: Comparison of Max Planck mesh quality, a) Original point cloud, b) simplified point cloud using FCM, c) simplified point cloud using KM



(a)

(b)

(c)

Figure 5: Comparison of Atene mesh quality, a) Original point cloud, b) simplified point cloud using FCM, c) simplified point cloud using KM

In this section, we will validate the effectiveness of our proposed method. Actually, we have conducted a comparison between the original and simplified point cloud. Accordingly, we will use a comparison between the original and simplified mesh.

Thereafter, we make a comparison between the original mesh and the one created from the simplified point cloud. To reconstruct the mesh, we use ball Pivoting method [31,29] or A.M Hsaini et al. method [32]. Then, to measure the quality of the obtained meshes, we compute the quality of the triangles using the compactness formula proposed by Guziec [33]:

$$c = \frac{4\sqrt{3}a}{l_1^2 + l_2^2 + l_3^2} \tag{9}$$

Where l_i are the lengths of the edges of the triangle. And a is the area of the triangle. We note that this measure is equal to 1 for an equilateral triangle and 0 for a triangle whose vertices are collinear. According to [34], a triangle is an acceptable quality if $c \geq 0.6$.

In figures 4, 5, we have presented the triangles compactness histogram of the two meshes. In each figure, the first line presents the reconstructed mesh from the original point cloud. The second line presents the simplified point cloud using FCM algorithm. The third line presents the simplified point cloud using KM algorithm. Note that, the evaluation of the mesh quality is achieved by the compactness of the triangles.

Depending on [34] meshes are compact if the percentage of the number of triangles, which composes mesh with compactness $c \geq 0.6$ is greater than or equal to 50%. Also, according to the histograms in figures 4 and 5, it is observed that the surfaces obtained from the simplified point cloud are compact surfaces.

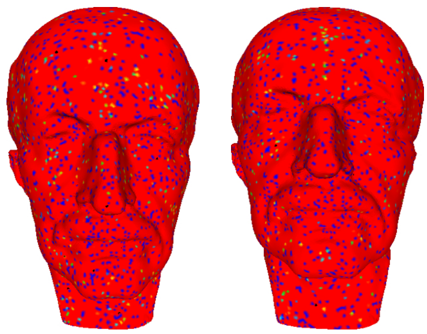
The table 1 also shows that the use of the KM and FCM algorithms retains the compactness of the surfaces. However, the compactness obtained by the FCM algorithm is greater than that obtained by KM for the two surfaces.

Concerning the number of vertices obtained after simplification, we note that this number is higher in the case of FCM for the two models.

It is interesting to note that in the case where FCM is used better results are produced in terms of speed. In contrast, in the other case where KM is used the speed is slow.

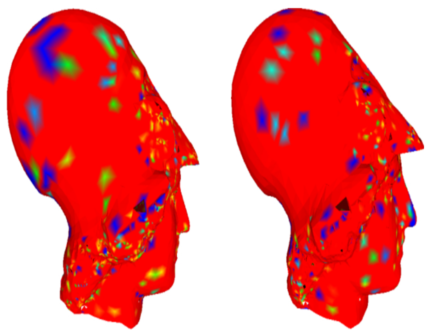
Table 3 and table 4 shows the numerical results obtained by the implementation of the two algorithms FCM and KM in the simplification method. The main results are average error, maximal error and root mean square error (RMS).

Figure 6 and 7 present differences between original and simplified meshes using Hausdorff distance. Note that it is a red-green-blue map, so red is minimal and blue is maximal, so in our case red means zero error and blue high error.



a) b)

Figure 6: difference between original and simplified MaxPlanck mesh



a) b)

Figure 7: difference between original and simplified Atene mesh

Table 3 presents the results relative to the evaluation of the approximation error concerning MaxPlanck model. Table 4 Also presents the same error related to Atene model.

using FCM particularly in simplification does not reach a high level of simplification. Moreover, it records in general the worst result in terms of error

Table 1: simplification results with s=0.001 using KM algorithm

Object	Number of points		Number of cluster	KM computation time (s)	Percentage of triangles with a compactness ≥ 0.6 (%)	
	Original	Simplified using KM			Simplified using KM	Original
Atene	6942	6289	18	1118.98	57.85	59.84
max_planck	49089	44765	123	7752.49	58.54	59.45

Table 2: simplification results with s=0.001 using FCM algorithm

Object	Number of points		Number of cluster	FCM computation time (s)	Percentage of triangles with a compactness ≥ 0.6 (%)	
	Original	Simplified using FCM			Simplified using FCM	Original
Atene	6942	6289	18	775.75	60.19	59.84
max_planck	49089	44812	123	7356.52	58.55	59.45

Table 3: Comparison of FCM and KM algorithm used in simplification method: MaxPlanck mesh (errors are measured as percentages of the datasets bounding box diagonal (699.092499))

Methods	Number of vertex	Number of faces (triangles)	Average Error	Max Error	RMS Error
Entropy-KM	44765	89355	0.000010	0.001799	0.000054
Entropy-FCM	44567	88960	0.000010	0.002241	0.000055

Table 4: Comparison of FCM and KM algorithm used in simplification method: Atene mesh (errors are measured as percentages of the datasets bounding box diagonal (6437.052937))

Methods	Number of vertex	Number of faces (triangles)	Average Error	Max Error	RMS Error
Entropy-KM	6289	11746	0.000153	0.016014	0.000697
Entropy-FCM	6446	9543	0.000331	0.016022	0.000930

shown figures 6.a et figure 7.a. By contrast, it is interesting to note that this method produces the best results when speed is needed (look at table 2).

As expected, KM algorithm in table 1 yields good results in terms of average error, max error and RMS error. Moreover, she recorded in general the worst result in terms of calculation speed.

We have implemented our simplification method under MATLAB. The calculations are performed on a machine with an i3 CPU, 3.4 Ghz, with 2GB of RAM.

7 Conclusion

This work presents a brief overview of two clustering algorithms, K-means and C-means. The results of an empirical comparison are presented to make a comparison between the use of the clustering algorithms. These clustering algorithms are integrated in our method of simplifying 3D point clouds. We have compared the computation time and the precision of the simplified meshes.

From the point of view of accuracy, the results show that K-means gives the best results in terms of error. As for claculation time, the use of Fuzzy C-means algorithm makes simplification faster.

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References

1. M. Pauly, M. Gross, and L. P. Kobbelt, Efficient simplification of point-sampled surfaces, in IEEE Visualization, 2002. VIS 2002., pp. 163170.
2. J. Wu and L. Kobbelt, Optimized Sub-Sampling of Point Sets for Surface Splatting, Comput. Graph. Forum, vol. 23, no. 3, pp. 643652, Sep. 2004.
3. Y. Ohtake, A. Belyaev, and H.-P. Seidel, An integrating approach to meshing scattered point data, in Proceedings of the 2005 ACM symposium on Solid and physical modeling - SPM 05, 2005, pp. 6169.
4. B.-Q. Shi, J. Liang, and Q. Liu, Adaptive simplification of point cloud using K-means clustering, Comput. Des., vol. 43, no. 8, pp. 910922, Aug. 2011.
5. L. Linsen, Point cloud representation, Univ. Karlsruhe, Ger. Tech. Report, Fac. Informatics, pp. 118, 2001.
6. T. K. Dey, T. K. Dey, J. Giesen, and J. Hudson, Decimating Samples for Mesh Simplification, in PROC. 13TH CANADIAN CONF. COMPUT. GEOM, 2001, pp. 85–88.
7. N. Amenta, S. Choi, T. K. Dey, and N. Leekha, A simple algorithm for homeomorphic surface reconstruction, in Proceedings of the sixteenth annual symposium on Computational geometry - SCG 00, 2000, pp. 213222.
8. M. Alexa, J. Behr, D. Cohen-Or, S. Fleishman, D. Levin, and C. T. Silva, Point set surfaces, in Proceedings Visualization, 2001. VIS 01., 2001, pp. 2128.
9. M. Garland and P. S. Heckbert, Surface simplification using quadric error metrics, in Proceedings of the 24th annual conference on Computer graphics and interactive techniques - SIGGRAPH 97, 1997, pp. 209216.
10. R. Allegre, R. Chaine, and S. Akkouche, Convection-driven dynamic surface reconstruction, in International Conference on Shape Modeling and Applications 2005 (SMI 05), pp. 3342.
11. J. Boissonnat and S. Oudot, An Effective Condition for Sampling Surfaces with Guarantees, Symp. A Q. J. Mod. Foreign Lit., pp. 101112, 2004.

12. Boissonnat J. D. and S. Oudot, Provably good surface sampling and approximation, in Proceedings of the 2003 Eurographics/ACM SIGGRAPH symposium on Geometry processing, 2003, pp. 918.

13. J.-D. Boissonnat and S. Oudot, An effective condition for sampling surfaces with guarantees, in Proceedings of the ninth ACM symposium on Solid modeling and applications, 2004, pp. 101112.

14. J.-D. Boissonnat and S. Oudot, Provably good sampling and meshing of surfaces, Graph. Models, vol. 67, no. 5, pp. 405451, Sep. 2005.

15. L. P. Chew and L. Paul, Guaranteed-quality mesh generation for curved surfaces, in Proceedings of the ninth annual symposium on Computational geometry - SCG 93, 1993, pp. 274280.

16. A. Adamson and M. Alexa, Approximating and Intersecting Surfaces from Points, in In Proc. Symposium on Geometry Processing, 2003, pp. 230239.

17. M. Pauly and M. Gross, Spectral processing of point-sampled geometry, in Proceedings of the 28th annual conference on Computer graphics and interactive techniques - SIGGRAPH 01, 2001, pp. 379386.

18. A. P. Witkin and P. S. Heckbert, Using particles to sample and control implicit surfaces, in Proceedings of the 21st annual conference on Computer graphics and interactive techniques - SIGGRAPH 94, 1994, pp. 269277.

19. Jing Wang, Xiaoling Li, and Jianhong Ni, Probability density function estimation based on representative data samples, in IET International Conference on Communication Technology and Application (ICCTA 2011), 2011, pp. 694698.

20. J. C. Dunn, A Fuzzy Relative of the ISODATA Process and Its Use in Detecting Compact Well-Separated Clusters, J. Cybern., vol. 3, no. 3, pp. 3257, Jan. 1973.

21. C. E. Shannon, A mathematical theory of communication, ACM SIGMOBILE Mob. Comput. Commun. Rev., vol. 5, no. 1, p. 3, Jan. 2001.

22. E. Parzen, On Estimation of a Probability Density Function and Mode, Ann. Math. Stat., vol. 33, pp. 10651076.

23. M. Rosenblatt, Remarks On Some Nonparametric Estimates of a Density Function, Ann. Math. Stat., vol. 27, no. 3, pp. 832837, Sep. 1956.

24. B. W. Silverman, Density estimation for statistics and data analysis. Chapman and Hall, 1986.

25. A. K. Jain, Data clustering: 50 years beyond K-means, Pattern Recognit. Lett., vol. 31, no. 8, pp. 651666, Jun. 2010.

26. J. C. Bezdek, Pattern Recognition with Fuzzy Objective Function Algorithms. Boston, MA: Springer US, 1981.

27. G. Gan and M. K.-P. Ng, k -means clustering with outlier removal, Pattern Recognit. Lett., vol. 90, pp. 814, Apr. 2017.

28. P. Cignoni, C. Rocchini, and R. Scopigno, Metro: Measuring Error on Simplified Surfaces, Comput. Graph. Forum, vol. 17, no. 2, pp. 167174, Jun. 1998.

29. A. Mahdaoui, A. Marhraoui Hsaini, A. Bouazi, and E. H. Sbai, Comparative Study of Combinatorial 3D Reconstruction Algorithms, Int. J. Eng. Trends Technol., vol. 48, no. 5, pp. 247251, 2017.

30. H.-G. Mller and A. Petersen, Density Estimation Including Examples, in Wiley StatsRef: Statistics Reference Online, Chichester, UK: John Wiley and Sons, Ltd, 2016, pp. 112.

31. F. Bernardini, J. Mittleman, H. Rushmeier, C. Silva, and G. Taubin, The ball-pivoting algorithm for surface reconstruction, IEEE Trans. Vis. Comput. Graph., vol. 5, no. 4, pp. 349359, Oct. 1999.

32. A. M. Hsaini, A. Bouazi, A. Mahdaoui, E. H. Sbai, and A. R. Bernstein-bezier, Reconstruction and adjustment of surfaces from a 3-D point cloud, nternational J. Comput. Trends Technol., vol. 37, no. 2, pp. 105109, 2016.

33. A. Gueziec and Andr, Locally toleranced surface simplification, IEEE Trans. Vis. Comput. Graph., vol. 5, no. 2, pp. 168189, 1999.

34. R. E. Bank, PLTMG: A Software Package for Solving Elliptic Partial Differential Equations. Society for Industrial and Applied Mathematics, 1998.

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Degenerate $p(x)$-elliptic equation with second membre in L^1

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ABSTRACT

In this paper, we prove the existence of a solution of the strongly nonlinear degenerate $p(x)$-elliptic equation of type:

$$(\mathcal{P})\begin{cases} -div\ a(x,u,\nabla u)+g(x,u,\nabla u) &= f \quad in\ \Omega,\\ \qquad\qquad\qquad\qquad u &= 0 \quad on \quad \partial\Omega,\end{cases}$$

where Ω is a bounded open subset of $I\!R^N$, $N \geq 2$, a is a Carathéodory function from $\Omega \times I\!R \times I\!R^N$ into $I\!R^N$, who satisfies assumptions of growth, ellipticity and strict monotonicity. The nonlinear term g: $\Omega \times I\!R \times I\!R^N \longrightarrow I\!R$ checks assumptions of growth, sign condition and coercivity condition, while the right hand side f belongs to $L^1(\Omega)$.

1 Introduction

Let Ω be a bounded open subset of $I\!R^N$, $N \geq 2$, let $\partial\Omega$ its boundary and $p(x) \in C(\overline{\Omega})$ with $p(x) > 1$.
Let ν be a weight function in Ω, ie: ν measurable and strictly positive a.e. in Ω. We suppose furthermore, that the weight function satisfies also the integrability conditions defined in section 2.
Let us consider the following degenerate $p(x)$-elliptic problem with boundary condition

$$(\mathcal{P})\begin{cases} Au+g(x,u,\nabla u) &= f \quad \text{in}\ \Omega,\\ \qquad\qquad\quad u &= 0 \quad \text{on} \quad \partial\Omega,\end{cases}$$

where A is a Leray-Lions operator defined from $W_0^{1,p(x)}(\Omega,\nu)$ to its dual $W^{-1,p'(x)}(\Omega,\nu^*)$, with $\nu^* = \nu^{1-p'(x)}$, by :

$$Au = -\text{div}\, a(x,u,\nabla u)\,.$$

where a is a Carathéodory function from $\Omega \times I\!R \times I\!R^N \longrightarrow I\!R^N$ who satisfies assumptions of growth, ellipticity and strict monotonicity, while the nonlinear term g: $\Omega \times I\!R \times I\!R^N \longrightarrow I\!R$ checks assumptions of sign and growth. We suppose moreover that g checks the following condition of coercivity:

$$\begin{cases} \qquad \exists\, \rho_1 > 0, \exists\, \rho_2 > 0 \quad \text{such that :}\\ \text{for } |s| \geq \rho_1,\ |g(x,s,\xi)| \geq \rho_2\, \nu(x)|\xi|^{p(x)}\end{cases}$$

We suppose also that the second member f belongs to $L^1(\Omega)$.

Let consider the following degenerate $p(x)$-elliptic problem of Dirichlet:

$$\begin{cases} Au+g(x,u,\nabla u) \ = f \quad \text{in } \mathcal{D}'(\Omega),\\ u \in W_0^{1,p(x)}(\Omega,\nu), \quad g(x,u,\nabla u) \in L^1(\Omega).\end{cases} \tag{1}$$

In the case where p is constant and without weight, there is a wide literature in which one can find existence results for problem (1). When the second member f belongs to $W^{-1,p'}(\Omega)$, A. Bensoussan, L. Boccardo and F. Murat [6] studied the problem and give an existence result. While if $f \in L^1(\Omega)$ the initiated basic works were given by H. Brezis and Strauss [9], L. Boccardo and T. Gallouët [7] also proved an existence result for (1), which was extended to the a unilateral case studied by A. Benkirane and A. Elmahi [5]. When g is not necessarily the null function, T. Del Vecchio [10] proved first existence result for problem (1) in the case where g does not depend on the gradient and then in V. M Monetti and L. Randazzo [16] using, in both works, the rearrangement techniques.
Whoever in [1], Y. Akdim, E. Azroul and A. Benkirane treated the problem (1) within the framework of Sobolev spaces with weight $W_0^{1,p}(\Omega,\omega)$, but while keeping p constant.
E. Azroul, A. Barbara and H. Hjiej [2] studied (1), in the nonclassical case by considering nonstandard Sobolev spaces without weight $W_0^{1,p(x)}(\Omega)$. See as well [3] where existence and regularity of entropy solutions was obtained for equation (1) with degenerated

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second member.
Our objectif, in this paper, is to study equation (1) by adopting Sobolev spaces with weight $\nu(x)$, and to variable exponents $p(x)$, $W_0^{1,p(x)}(\Omega,\nu)$. We prove that the problem (1) admits at least a solution $u \in W_0^{1,p(x)}(\Omega,\nu)$.

2 Functional frame

Throughout this section, we suppose that the variable exponent $p(\cdot) : \overline{\Omega} \to [1,+\infty[$ is log-Hölder continuous on Ω, that is there is a real constant $c > 0$ such that $\forall x, y \in \overline{\Omega}$, $x \neq y$ with $|x-y| < 1/2$ one has:

$$|p(x) - p(y)| \leq \frac{c}{-log|x-y|}$$

and satisfying

$$p^- \leq p(x) \leq p^+ < +\infty$$

where

$$p^- := \operatorname{ess}\inf_{x\in\overline{\Omega}} p(x); \quad p^+ := \operatorname{ess}\sup_{x\in\overline{\Omega}} p(x).$$

We define

$$C_+(\overline{\Omega}) = \{h \text{ log-Hölder continous on } \overline{\Omega}, h(x) > 1\}.$$

Definition 1 *Let ν be a function defined in Ω; we call ν a weight function in Ω if it is measurable and strictly positive a.e. in Ω.*

2.1 Lebesgue spaces with weight and to variable exponents

Let $p \in C_+(\overline{\Omega})$ and ν be a weighted function in Ω.
We define the Lebesgue space with weight and to variable exponents $L^{p(x)}(\Omega,\nu)$, by
$L^{p(x)}(\Omega,\nu) = \{u : \Omega \to I\!R$, measurable : $\int_\Omega \nu(x)|u|^{p(x)}dx < \infty\}$, equipped with the Luxemburg norm:

$$\|u\|_{p(x),\nu} = \inf\left\{\mu > 0 : \int_\Omega \nu(x)|\frac{u}{\mu}|^{p(x)}dx \leq 1\right\}$$

Proposition 1 *The space $\left(L^{p(x)}(\Omega,\nu), \|.\|_{p(x),\nu}\right)$ is of Banach.*

Proof:
Considering the operator

$$M_{\nu^{\frac{1}{p(x)}}} : L^{p(x)}(\Omega,\nu) \longrightarrow L^{p(x)}(\Omega), f \to M_{\nu^{\frac{1}{p(x)}}}(f) = f\,\nu^{\frac{1}{p(x)}}$$

It's clear that $M_{\nu^{\frac{1}{p(x)}}}$ is isomorphism from $L^{p(x)}(\Omega,\nu)$ into $L^{p(x)}(\Omega)$, then $M_{\nu^{\frac{1}{p(x)}}}$ is a continuous biuniformly application. Seeing that $L^{p(x)}(\Omega)$ is a Banach space, then $\left(L^{p(x)}(\Omega,\nu), \|.\|_{p(x),\nu}\right)$ is of Banach.

Let's note $\rho_\nu(u) = \int_\Omega \nu(x)|u|^{p(x)}dx$.

Remark 1 *In simple case $\nu(x) = 1$, we find again the Lebesgue space with variable exponents $L^{p(x)}(\Omega)$; and $\rho_\nu(u) = \rho_1(u) := \rho(u) = \int_\Omega |u|^{p(x)}dx$, (see [12],[13] and [17])*

Lemma 1 *For all function $u \in L^{p(x)}(\Omega,\nu)$. There are the following assertions:*
(i) $\rho_\nu(u) > 1 \quad (= 1; < 1) \Leftrightarrow \|u\|_{p(x),\nu} > 1 \quad (= 1; < 1)$, *respectively.*
(ii) *If* $\|u\|_{p(x),\nu} > 1$ *then* $\|u\|^{p_-}_{p(x),\nu} \leq \rho_\nu(u) \leq \|u\|^{p^+}_{p(x),\nu}$.
(iii) *If* $\|u\|_{p(x),\nu} < 1$ *then* $\|u\|^{p^+}_{p(x),\nu} \leq \rho_\nu(u) \leq \|u\|^{p_-}_{p(x),\nu}$.

Proof:
Seeing that $\rho_\nu(u) = \rho(\nu^{\frac{1}{p(x)}}u)$ and $\|\nu^{\frac{1}{p(x)}}u\|_{p(x)} = \|u\|_{p(x),\nu}$, and using [17], we prove the lemma 2.1 above.
Let ν be a weight function such that the following condition:

$$\textbf{(w1)}\ \nu \in L^1_{loc}(\Omega);\ \nu^{\frac{-1}{p(x)-1}} \in L^1_{loc}(\Omega).$$

Proposition 2 *Let Ω be a bounded open subset of $I\!R^N$, and ν be a weight function on Ω,*
*If **(w1)** is verified then $L^{p(x)}(\Omega,\nu) \hookrightarrow L^1_{loc}(\Omega)$.*

Proof:
Let K be a included compact on Ω. Using Hölder inequality we have

$$\begin{aligned}
\int_K |u|dx &= \int_K |u|\nu^{\frac{1}{p(x)}}\nu^{\frac{-1}{p(x)}}dx \\
&\leq 2\||u|\nu^{\frac{1}{p(x)}}\|_{L^{p(x)}(K)}\|\nu^{\frac{-1}{p(x)}}\|_{L^{p'(x)}(K)}, \\
&\leq 2\|u\|_{p(x),\nu}\left(\int_K \nu^{\frac{-p'(x)}{p(x)}}dx + 1\right)^{\frac{1}{p'_-}}, \\
&\leq 2\|u\|_{p(x),\nu}\left(\int_K \nu^{\frac{-1}{p(x)-1}}dx + 1\right)^{\frac{1}{p'_-}}.
\end{aligned}$$

Thanks to the assumption **(w1)** we deduce that $\int_K |u|dx \leq C\|u\|_{p(x),\nu}$.

2.2 Spaces of Sobolev with weight and to variable exponents

Let $p \in C_+(\overline{\Omega})$ and ν be a weight function in Ω.
We define the space of Sobolev with weight and to variable exponents denoted $W^{1,p(x)}(\Omega,\overrightarrow{\nu})$, by

$$W^{1,p(x)}(\Omega,\nu) = \left\{u \in L^{p(x)}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p(x)}(\Omega,\nu), i = 1,...,N\right\},$$

equipped with the norm

$$\|u\|_{1,p(x),\nu} = \|u\|_{p(x)} + \sum_{i=1}^N \|\frac{\partial u}{\partial x_i}\|_{p(x),\nu}$$

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which is equivalent to the Luxemburg norm

$$|||u||| = \inf\left\{\mu > 0 : \int_\Omega \left(|\frac{u}{\mu}|^{p(x)} + \nu(x)\sum_{i=1}^{N} |\frac{\frac{\partial u}{\partial x_i}}{\mu}|^{p(x)}\right) dx \leq 1\right\}.$$

Proposition 3 *Let ν be a weight function in Ω who checks the condition **(w1)**.*
Then the space $\left(W^{1,p(x)}(\Omega,\nu), \|.\|_{1,p(x),\nu}\right)$ is of Banach.

Proof:
Let us consider $(u_n)_n$ a Cauchy sequence of $\left(W^{1,p(x)}(\Omega,\nu), \|.\|_{1,p(x),\nu}\right)$.
Then $(u_n)_n$ is a Cauchy sequence of $L^{p(x)}(\Omega)$ and the sequence $(\frac{\partial u_n}{\partial x_i})_n$ is also a Cauchy belong $L^{p(x)}(\Omega,\nu)$, $i = 1,...,N$.
According the proposition 2.1, there exists $u \in L^{p(x)}(\Omega)$ such that $u_n \to u$ in $L^{p(x)}(\Omega)$, and there exists $v_i \in L^{p(x)}(\Omega,\nu)$ such that $\frac{\partial u_n}{\partial x_i} \to v_i$ in $L^{p(x)}(\Omega,\nu)$, $i = 1,...,N$.
Seeing that proposition 2.2, we have $L^{p(x)}(\Omega,\nu) \subset L^1_{loc}(\Omega)$ and $L^1_{loc}(\Omega) \subset D'(\Omega)$.
Thus, we obtain $\forall\varphi \in D(\Omega)$,

$$\begin{aligned}\langle T_{v_i},\varphi\rangle &= \lim_{n\to\infty}\langle T_{\frac{\partial u_n}{\partial x_i}},\varphi\rangle,\\ &= -\lim_{n\to\infty}\langle T_{u_n},\frac{\partial\varphi}{\partial x_i}\rangle,\\ &= -\langle T_u,\frac{\partial\varphi}{\partial x_i}\rangle,\\ &= \langle T_{\frac{\partial u}{\partial x_i}},\varphi\rangle.\end{aligned}$$

Hence $T_{v_i} = T_{\frac{\partial u}{\partial x_i}}$, i.e. $v_i = \frac{\partial u}{\partial x_i}$.
Consequently

$$u \in W^{1,p(x)}(\Omega,\nu),$$

and

$$u_n \to u \text{ in } W^{1,p(x)}(\Omega,\nu),$$

We deduce then that $W^{1,p(x)}(\Omega,\nu)$ is a complete space.

• On another side, seeing that ν satisfies the condition **(w1)**, we prove that $C_0^\infty(\Omega)$ is included in $W^{1,p(x)}(\Omega,\nu)$; that enables us to define the following space

$$W_0^{1,p(x)}(\Omega,\nu) = \overline{C_0^\infty(\Omega)}^{\|.\|_{1,p(x),\nu}},$$

who is closed in complete space, then it's complete.

Proposition 4 *(Characterization of the dual space $(W_0^{1,p(x)}(\Omega,\nu))^*$)*
*Let $p(.) \in C_+(\overline{\Omega})$ and ν a vector of weight who satisfies the condition **(w1)**. Then for all $G \in (W_0^{1,p(x)}(\Omega,\nu))^*$, there exist a unique system of functions $(g_0,g_1,...,g_N) \in L^{p'(x)}(\Omega)\times(L^{p'(x)}(\Omega,\nu^{1-p'(x)}))^N$ such that $\forall f \in W_0^{1,p(x)}(\Omega,\nu)$:*

$$G(f) = \int_\Omega f(x)g_0(x)dx + \sum_{i=1}^{N}\int_\Omega \frac{\partial f}{\partial x_i}g_i(x)dx.$$

Proof
The proof of this proposition is similar to that of [14] (theorem3.16).
Besides the **(w1)** assumption, we suppose that the function weight satisfied

$$\textbf{(w2)}\ \ \nu^{-s(x)} \in L^1_{loc}(\Omega)$$

where s is a positive function to specify afterwards.
Let us introduce the function p_s defined by

$$p_s(x) = \frac{p(x)s(x)}{s(x)+1},$$

we have

$$p_s(x) < p(x) \text{ a.e. in } \Omega.$$

and

$$\begin{cases} p_s^*(x) &= \frac{Np_s(x)}{N-p_s(x)} = \frac{Np(x)s(x)}{N(s(x)+1)-p(x)s(x)} \text{ if } p(x)s(x) < N(s(x)+1),\\ p_s^*(x) & \text{arbitrary, else if,}\end{cases}$$

Proposition 5 *Let $p,s \in C_+(\overline{\Omega})$ and ν a function weight which satisfies **(w1)** and **(w2)**. Then $W^{1,p(x)}(\Omega,\nu) \hookrightarrow W^{1,p_s(x)}(\Omega)$.*

Proof
According to the Hőlder inequality, we have

$$\begin{aligned}\int_\Omega |v(x)|^{p_s(x)}dx &= \int_\Omega |v(x)|^{p_s(x)}\nu^{\frac{p_s(x)}{p(x)}}\nu^{\frac{-p_s(x)}{p(x)}}dx\\ \leq& \left(\frac{1}{(\frac{p}{p_s})_-}+\frac{1}{(s+1)^-}\right)\||v(x)|^{p_s(x)}\nu^{\frac{p_s(x)}{p(x)}}\|_{\frac{p(x)}{p_s(x)}}\|\nu^{\frac{-p_s(x)}{p(x)}}\|_{s(x)+1}\\ \leq& C_0\left(\int_\Omega |v(x)|^{p(x)}\nu(x)dx\right)^{\frac{1}{\gamma_1}}\left(\int_\Omega \nu(x)^{-s(x)}dx\right)^{\frac{1}{\gamma_2}}\\ \leq& C_0\left(\int_\Omega |v(x)|^{p(x)}\nu(x)dx\right)^{\frac{1}{\gamma_1}}\left(\int_\Omega \nu(x)^{-s(x)}dx\right)^{\frac{1}{\gamma_2}}\\ \leq& C_0C_1\left(\int_\Omega |v(x)|^{p(x)}\nu(x)dx\right)^{\frac{1}{\gamma_1}}, \text{ according to } \textbf{(w2)}.\end{aligned}$$

If we take $v = \frac{\partial u}{\partial x_i}$, we will obtain then

$$\int_\Omega |\frac{\partial u}{\partial x_i}|^{p_s(x)}dx \leq C_0C_1\left(\int_\Omega |\frac{\partial u}{\partial x_i}|^{p(x)}\nu(x)dx\right)^{\frac{1}{\gamma_1}}$$

where

$$\gamma_1 = \begin{cases} (\frac{p}{p_s})_- & \text{if} \quad \||v(x)|^{p_s(x)}\nu^{\frac{p_s(x)}{p(x)}}\|_{\frac{p(x)}{p_s(x)}} \geq 1,\\ (\frac{p}{p_s})^+ & \text{if} \quad \||v(x)|^{p_s(x)}\nu^{\frac{p_s(x)}{p(x)}}\|_{\frac{p(x)}{p_s(x)}} < 1,\end{cases}$$

consequently

$$\begin{aligned}\|\frac{\partial u}{\partial x_i}(x)\|^{\gamma_2}_{p_s(x)} \leq& C_0C_1\left(\int_\Omega |\frac{\partial u}{\partial x_i}|^{p(x)}\nu(x)dx\right)^{\frac{1}{\gamma_1}}\\ \leq& C_0C_1\|\frac{\partial u}{\partial x_i}(x)\|^{\frac{\gamma_3}{\gamma_1}}_{p(x),\nu}\end{aligned}$$

where

$$\gamma_2 = \begin{cases}(p_s)_- & \text{if} \quad \|\frac{\partial u}{\partial x_i}(x)\|_{p_s(x)} \geq 1,\\ (p_s)^+ & \text{if} \quad \|\frac{\partial u}{\partial x_i}(x)\|_{p_s(x)} < 1,\end{cases}$$

and

$$\gamma_3 = \begin{cases} p^+ & \text{if} \quad \|\frac{\partial u}{\partial x_i}(x)\|_{p(x),\nu} \geq 1, \\ p_- & \text{if} \quad \|\frac{\partial u}{\partial x_i}(x)\|_{p(x),\nu} < 1. \end{cases}$$

thus

$$\|\frac{\partial u}{\partial x_i}\|_{p_s(x)} \leq C_0 C_1 \|\frac{\partial u}{\partial x_i}\|_{p(x),\nu}^{\frac{\gamma_3}{\gamma_1\gamma_2}}, \; i = 1, 2, ..., N. \quad (2)$$

Seeing that

$$p_s(x) < p(x) \text{ a.e. in } \Omega.$$

then, there is a constant

$$C > 0 \text{ such that } \|u\|_{L^{p_s(x)}(\Omega)} \leq C\|u\|_{L^{p(x)}(\Omega)},$$

we conclude then that

$$W^{1,p(x)}(\Omega,\nu) \hookrightarrow W^{1,p_s(x)}(\Omega).$$

Corollaire 1 *Let $p, s \in C_+(\overline{\Omega})$ and ν a fuction weight which satisfies **(w1)** and **(w2)**. Then $W^{1,p(x)}(\Omega,\nu) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$, for $1 \leq r(x) < p_s^*(x)$*

3 Basic assumption

Let Ω be a bounded open subset of $I\!R^N$, $N \geq 2$, let $p(.) \in C_+(\overline{\Omega})$, and ν a function weight in Ω such that:

$$\nu \in L^1_{loc}(\Omega) \quad (3)$$

$$\nu^{\frac{-1}{p(x)-1}} \in L^1_{loc}(\Omega) \quad (4)$$

and

$$\nu^{-s(x)} \in L^1_{loc}(\Omega) \text{ where } s(x) \in \left]\frac{N}{p(x)}, \infty\right[\cap \left[\frac{1}{p(x)-1}, \infty\right[\quad (5)$$

Let A Leray-Lions operator defined from $W_0^{1,p(x)}(\Omega,\nu)$ to its dual $W^{-1,p'(x)}(\Omega,\nu^*)$ by

$$Au = -\text{div}\, a(x,u,\nabla u),$$

where a is a Carathéodory function satisfying the following assumptions:

$$|a_i(x,r,\zeta)| \leq \beta\nu^{\frac{1}{p(x)}}\left[b(x) + |r|^{\frac{p(x)}{p'(x)}} + \nu^{\frac{1}{p'(x)}}|\zeta|^{p(x)-1}\right], i = 1,..,N. \quad (6)$$

$$[a(x,r,\zeta) - a(x,r,\overline{\zeta})](\zeta - \overline{\zeta}) > 0, \quad \forall\, \zeta \neq \overline{\zeta} \in I\!R^N. \quad (7)$$

$$a(x,s,\zeta)\zeta \geq \alpha\nu|\zeta|^{p(x)}. \quad (8)$$

with $b(x)$ be a positive function in $L^{p'(x)}(\Omega)$, and α, β are two strictly positive constants. On another side, let g: $\Omega \times I\!R \times I\!R^N \longrightarrow I\!R$ a Carathéodory function who satisfied a.e. for $x \in \Omega$ and for all $r \in I\!R$, $\zeta \in I\!R^N$ the following conditions:

$$g(x,r,\zeta)r \geq 0. \quad (9)$$

$$|g(x,r,\zeta)| \leq d(|r|)(\nu(x)|\zeta|^{p(x)} + c(x)). \quad (10)$$

with $d : I\!R^+ \longrightarrow I\!R^+$ a continuous, nondecreasing and positive function; whereas $c(x)$ be a positive function in $L^1(\Omega)$.
Moreover, we suppose that g checks:

$$\begin{cases} \exists\, \rho_1 > 0, \exists\, \rho_2 > 0 \text{ such that :} \\ \text{for } |s| \geq \rho_1,\ |g(x,s,\xi)| \geq \rho_2\nu(x)|\xi|^{p(x)} \end{cases} \quad (11)$$

and

$$f \in L^1(\Omega). \quad (12)$$

We have the following theorem

Theorem 1 *Assume that $(3)-(12)$ holds, then the problem (1) admits at least one solution $u \in W_0^{1,p(x)}(\Omega,\nu)$*

Remark 2 *If $f \in W^{-1,p'(x)}(\Omega,\nu^*)$ then we have $ug(x,u,\nabla u) \in L^1(\Omega)$.
But if $f \in L^1(\Omega)$, we necessarily do not have $ug(x,u,\nabla u) \in L^1(\Omega)$.*

Counter example:
Let us consider $\Omega = \{x \in I\!R^2 : |x| < 1\}$, the ball open unit of $I\!R^2$, let $\lambda \in [\frac{1}{4}, \frac{1}{3}[$, we then have $u(x) = ln^{\lambda}(\frac{1}{|x|}) \in H_0^1(\Omega)$ and $-\Delta u \in L^1(\Omega)$, such that $-\Delta u \geq 0, u|\nabla u|^2 \in L^1(\Omega)$ and $|u|^2|\nabla u|^2$ does not belong to $L^1(\Omega)$ see [7].

Remark 3 *The result of theorem 1 is not true when $g(x,r,\zeta) = 0$, seeing that in the non-degenerate case with $p(x) = p$ constant, $p \leq N$, and when $f \in L^1(\Omega)$, a solution of $Au = f$ does not belong to $W_0^{1,p}(\Omega)$ but belongs to $\bigcap_{1<q<\frac{N(p-1)}{N-1}} W_0^{1,q}(\Omega)$, see[8].*

Definition 2 *Let X a Banach space. An operator A from X to its dual X^* is called as type (M) if for all sequence $(u_n)_n \subset X$ satisfying:*

$$\begin{cases} (i) & u_n \rightharpoonup u \text{ weakly in } X \\ (ii) & Au_n \rightharpoonup \chi \text{ weakly in } X^* \\ (iii) & \limsup_{n\to\infty}\langle Au_n, u_n\rangle \leq \langle \chi, u\rangle, \end{cases}$$

then $\chi = Au$.

Remark 4 *The theorem 2.1 (p.171 [15]) remains valid if we replace A monotonous by: A of type (M).*

4 Approximate problem

Let $(f_n)_n$ a sequence of regularly functions such that $f_n \to f$ in $L^1(\Omega)$ and $\|f_n\| \leq \|f\| = C_1$.
Let us consider the following approximate problem:

$$(\mathcal{P}_n)\begin{cases} Au_n + g_n(x,u_n,\nabla u_n) = f_n & \text{in} \quad \Omega, \\ u_n \in W_0^{1,p(x)}(\Omega,\nu), \end{cases}$$

where $g_n(x,r,\zeta) = \frac{g(x,r,\zeta)}{1+\frac{1}{n}g(x,r,\zeta)}\chi_{\Omega_n}$, with χ_{Ω_n} is the characteristic function of Ω_n where Ω_n is a sequence of compact subsets which is increasing towards Ω.

We have $g_n(x,r,\zeta)r \geq 0; |g_n(x,r,\zeta)| \leq |g(x,r,\zeta)|$ and $|g_n(x,r,\zeta)| \leq n$.
Let us consider $G_n : W_0^{1,p(x)}(\Omega,\nu) \longrightarrow W^{-1,p'}(\Omega,\nu^*)$ defined by

$$\langle G_n u, v\rangle = \int_\Omega g_n(x,u,\nabla u)\nabla v dx, u,v \in W_0^{1,p(x)}(\Omega,\nu).$$

Proposition 6 *The operator G_n is bounded.*

Proof:
Let u and v in $W_0^{1,p(x)}(\Omega,\nu)$, according to the Hölder inequality, we have:

$$\begin{aligned}\langle G_n u, v\rangle &= \int_\Omega g_n(x,u,\nabla u)\nabla v dx,\\
&\leq \int_\Omega |g_n(x,u,\nabla u)|\nu(x)^{\frac{-1}{p(x)}}\nabla v \nu(x)^{\frac{1}{p(x)}}dx\\
&\leq (\frac{1}{p_-}+\frac{1}{p'_-})\||g_n(x,u,\nabla u)|\nu(x)^{\frac{-1}{p(x)}}\|_{p'(x)}\|\nabla v\nu(x)^{\frac{1}{p(x)}}\|_{p(x)},\\
&\leq C\bigg(\int_\Omega |g_n(x,u,\nabla u)|^{p'(x)}\nu(x)^* dx\bigg)^{\frac{1}{\gamma_1}}\|v\|_{W_0^{1,p(x)}(\Omega,\nu)}\\
&\leq C\bigg(\int_\Omega n^{p'(x)}\nu(x)^* dx\bigg)^{\frac{1}{\gamma_1}}\|v\|_{W_0^{1,p(x)}(\Omega,\nu)}\\
&\leq Cn^{p'_+}\bigg(\int_\Omega \nu(x)^* dx\bigg)^{\frac{1}{\gamma_1}}\|v\|_{W_0^{1,p(x)}(\Omega,\nu)}\\
&\leq C'\|v\|_{W_0^{1,p(x)}(\Omega,\nu)}\end{aligned}$$

■

Proposition 7 *The operator $A+G_n : W_0^{1,p(x)}(\Omega,\nu) \longrightarrow W^{-1,p'(x)}(\Omega,\nu^*)$ defined by $\forall u,v \in W_0^{1,p(x)}(\Omega,\nu)$,*

$$\langle (A+G_n)u,v\rangle = \int_\Omega a(x,u,\nabla u)\nabla v dx + \int_\Omega g_n(x,u,\nabla u)\nabla v dx,$$

is bounded, coercif, hemicontinous and of type (M).

Thanks to [15], the approximate problem admits at least a solution.

Proof of the proposition 7
Using (6) and Hölder inequality, we conclude that A is bounded, by taking account of the proposition 6, we will obtain that $A+G_n$ is bounded. The coercivity rise from (8) and from (9). It remains to be shown that $A+G_n$ is hemicontinous, i.e. that:

$$\forall u,v,w \in W_0^{1,p(x)}(\Omega,\nu),$$

$$\langle (A+G_n)(u+tv),w\rangle \longrightarrow \langle (A+G_n)(u+t_0v),w\rangle, \text{when} t \to t_0.$$

seeing that for a.e. $x \in \Omega$, we have:

$$a_i(x,u+tv,\nabla(u+tv)) \longrightarrow a_i(x,u+t_0v,\nabla(u+t_0v)), t \to t_0,$$

then (6) combined to the lemma 2, implies that $a_i(x,u+tv,\nabla(u+tv)) \rightharpoonup a_i(x,u+t_0v,\nabla(u+t_0v))$, weakly in $(L^{p'(x)}(\Omega,\nu^*))^N$, when $t \to t_0$. Finally, $\forall w \in W_0^{1,p(x)}(\Omega,\nu)$, $\forall u,v,w \in W_0^{1,p(x)}(\Omega,\nu)$

$$\langle A(u+tv),w\rangle \longrightarrow \langle A(u+t_0v),w\rangle, \text{ when } t \to t_0.$$

On another side $g_n(x,u+tv,\nabla(u+tv)) \to g_n(x,u+t_0v,\nabla(u+t_0v))$, when $t \to t_0$ a.e. $x \in \Omega$, moreover

$$\int_\Omega |g_n(x,u+tv,\nabla(u+tv))|^{p'(x)}dx \leq (\frac{1}{n})^{p'_+}|\Omega| < \infty,$$

while using, still the lemma 2, we obtain:

$$g_n(x,u+tv,\nabla(u+tv)) \rightharpoonup g_n(x,u+t_0v,\nabla(u+t_0v))$$

weakly in $L^{p'(x)}(\Omega)$, when $t \to t_0$.
Seeing that $w \in L^{p'(x)}(\Omega)$, we will have:

$$\langle G_n(u+tv),w\rangle \longrightarrow \langle G_n(u+t_0v),w\rangle, \text{ when } t \to t_0.$$

Now, we will show that $A+G_n$ satisfies the property of type (M), a.e. that, for all sequence $(u_j)_j \subset W_0^{1,p(x)}(\Omega,\nu)$ checking:

$$\begin{cases} (i) & u_j \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega,\nu)\\ (ii) & (A+G_n)u_j \rightharpoonup \chi \text{ weakly in } W^{-1,p'(x)}(\Omega,\nu^*)\\ (iii) & \limsup_{j\to\infty}\langle (A+G_n)u_j, u_j-u\rangle \leq 0,\end{cases}$$

then $\chi = (A+G_n)u$.
Indeed:

$$\begin{aligned}\int_\Omega g_n(x,u_j,\nabla u_j)(u_j-u)dx &\leq \|g_n(x,u_j,\nabla u_j)\|_{p'(x)}\|u_j-u\|_{p(x)}\\
&\leq C\bigg(\int_\Omega |g_n(x,u_j,\nabla u_j)|^{p'(x)}dx\bigg)^{\frac{1}{v}}\|u_j-u\|_{p(x),\nu}\\
&\leq C(\tfrac{1}{n})^{p'_+}|\Omega|\|u_j-u\|_{p(x),\nu} \longrightarrow 0 \text{ when } j\to\infty.\end{aligned}$$

Thus

$$\lim_{j\to\infty}\langle G_n u_j, u_j-u\rangle = 0.$$

consequently, according to (iii), we obtain

$$\limsup_{j\to\infty}\langle Au_j, u_j-u\rangle \leq 0,$$

and seeing that A is pseudo-monotonous [11], we deduce that

$$Au_j \rightharpoonup Au \text{ weakly in } W^{-1,p'(x)}(\Omega,\nu^*)$$

and $\lim_{j\to\infty}\langle Au_j, u_j-u\rangle = 0$.
On another side

$$\begin{aligned}0 &= \lim_{j\to\infty}\int_\Omega a(x,u_j,\nabla u_j)\nabla(u_j-u)dx\\
&= \lim_{j\to\infty}\Big(\int_\Omega\Big(a(x,u_j,\nabla u_j)-a(x,u_j,\nabla u)\Big)\nabla(u_j-u)dx\\
&+\int_\Omega a(x,u_j,\nabla u)\nabla(u_j-u)dx\Big).\end{aligned}$$

Moreover we have $a(x,u_j,\nabla u) \longrightarrow a(x,u,\nabla u)$ strongly in $(L^{p'(x)}(\Omega,\nu^*))^N$, then

$$\lim_{j\to\infty}\int_\Omega a(x,u_j,\nabla u)\nabla(u_j-u)dx = 0.$$

Thus

$$\lim_{j\to\infty}\int_\Omega\Big(a(x,u_j,\nabla u_j)-a(x,u_j,\nabla u)\Big)\nabla(u_j-u)dx = 0.$$

Using the lemma 3, see blow, we conclude that

$$\nabla u_j \longrightarrow \nabla u \text{ a.e. in } \Omega, \text{ when } j \to \infty.$$

Consequently

$$g_n(x, u_j, \nabla u_j) \longrightarrow g_n(x, u, \nabla u) \text{ a.e. in } \Omega, \text{ when } j \to \infty.$$

Seeing that $|g_n(x, u_j, \nabla u_j)| \leq n \in L^{p'(x)}(\Omega)$, then according to the dominated convergence theorem of Lebesgue, we obtain:

$$g_n(x, u_j, \nabla u_j) \longrightarrow g_n(x, u, \nabla u) \text{ in } L^{p'(x)}(\Omega).$$

and seeing that $v \in L^{p(x)}(\Omega)$, then we have: $\int_\Omega g_n(x, u_j, \nabla u_j) v\,dx \longrightarrow \int_\Omega g_n(x, u, \nabla u) v\,dx$ when $j \to \infty$, that means

$$G_n u_j \rightharpoonup G_n u \quad \text{in } W^{-1,p'(x)}(\Omega, \nu^*) \text{ when } j \to \infty.$$

Finally

$Au_j + G_n u_j \rightharpoonup Au + G_n u = \chi$ in $W^{-1,p'(x)}(\Omega, \nu^*)$ when $j \to \infty$.

5 Technical lemmas

Lemma 2 *Let γ a function weight in Ω, $r(.) \in C_+(\overline{\Omega})$, $g \in L^{r(x)}(\Omega, \gamma)$ and $(g_n)_n \subset L^{r(x)}(\Omega, \gamma)$ such that $\|g_n\|_{r(x),\gamma} \leq C$.*
If $g_n \to g$ a.e. in Ω then $g_n \rightharpoonup g$ weakly in $L^{r(x)}(\Omega, \gamma)$.

Proof:
Let $n_0 \geq 1$, let us pose

$$E(n_0) = \left\{ x \in \Omega : |g_n(x) - g(x)| \leq 1, \forall n \geq n_0 \right\}.$$

We have

$$mes(E(n_0)) \to mes(\Omega), \text{ when } n_0 \to \infty.$$

Let

$$F = \left\{ \varphi_{n_0} \in L^{r'(x)}(\Omega, \gamma^*) : \varphi_{n_0} = 0 \text{ a.e. in } \Omega \setminus E(n_0) \right\}.$$

Let us show that F is dense in $L^{r'(x)}(\Omega, \gamma^*)$:
let $f \in L^{r'(x)}(\Omega, \gamma^*)$, let us pose:

$$f_{n_0}(x) = \begin{cases} f(x) & if\ x \in E(n_0), \\ 0 & if\ x \in \Omega \setminus E(n_0). \end{cases}$$

We have

$$\rho_{r'(x),\gamma^*}(f_{n_0}(x) - f(x)) = \int_\Omega |f_{n_0}(x) - f(x)|^{r'(x)} \gamma^* dx$$
$$= \int_{E(n_0)} |f_{n_0}(x) - f(x)|^{r'(x)} \gamma^* dx$$
$$+ \int_{\Omega \setminus E(n_0)} |f_{n_0}(x) - f(x)|^{r'(x)} \gamma^* dx,$$
$$= \int_{\Omega \setminus E(n_0)} |f_{n_0}(x) - f(x)|^{r'(x)} \gamma^* dx,$$
$$= \int_\Omega |f(x)|^{r'(x)} \gamma^* \chi_{\Omega \setminus E(n_0)}(x) dx.$$

Let us pose $\psi_{n_0} = |f(x)|^{r'(x)} \gamma^* \chi_{\Omega \setminus E(n_0)}(x)$. we have

$$\begin{cases} \psi_{n_0} & \to 0 \quad \text{a.e.in } \Omega, \\ & and \\ |\psi_{n_0}| & \leq |f(x)|^{r'(x)} \gamma^*, \end{cases}$$

according to the dominate convergence theorem, we will have

$$\rho_{r'(x),\gamma^*}(f_{n_0}(x) - f(x)) \to 0, \text{ when } n_0 \to \infty,$$

what implies that F is dense in $L^{r'(x)}(\Omega, \gamma^*)$.
let us show, now, that

$$\lim_{n \to \infty} \int_\Omega \varphi(x)(g_n(x) - g(x)) dx = 0, \ \forall \varphi \in F.$$

Seeing that $\varphi = 0$ on $\Omega \setminus E_{n_0}$, it is thus enough to prove that

$$\lim_{n \to \infty} \int_{E_{n_0}} \varphi(x)(g_n(x) - g(x)) dx = 0.$$

Let us pose $\phi_n = \varphi(g_n - g)$, we have

$$\begin{cases} |\varphi(x)||(g_n(x) - g(x))| & \leq |\varphi(x)| \quad \text{in } E_{n_0}, \\ & and \\ \phi_n \longrightarrow 0, & \text{a.e. in } \Omega \end{cases}$$

According to the dominate convergence theorem, we have

$$\phi_n \longrightarrow 0 \quad \text{a.e. in} \quad L^1(\Omega),$$

what implies that

$$\lim_{n \to \infty} \int_\Omega \varphi(x)(g_n(x) - g(x)) dx = 0, \ \forall \varphi \in F,$$

and by density of F in $L^{r'(x)}(\Omega, \gamma^*)$, we conclude that:

$$\lim_{n \to \infty} \int_\Omega \varphi(x) g_n(x) dx = \int_\Omega \varphi(x) g(x) dx, \ \forall \varphi \in L^{r'(x)}(\Omega, \gamma^*),$$

that means

$$g_n \rightharpoonup g \ \text{ weakly in } \ L^{r(x)}(\Omega, \gamma).$$

■

Lemma 3 *Assume that (3.1), (3.3), (3.5) hold, let $(u_n)_n$ a sequence in $W_0^{1,p(x)}(\Omega, \nu)$ and $u \in W_0^{1,p(x)}(\Omega, \nu)$.*

$$If \begin{cases} u_n \rightharpoonup u \text{ weakly in } W_0^{1,p(x)}(\Omega, \nu), \\ and \int_\Omega \Big(a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u) \Big)(\nabla u_n - \nabla u) dx \to 0 \end{cases}$$

then $u_n \longrightarrow u$ strongly in $W_0^{1,p(x)}(\Omega, \nu)$.

Proof:
Let us pose

$$D_n = \Big(a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)\Big)(\nabla u_n - \nabla u),$$

according the (3.5), we have D_n is a positive function.
We have also $D_n \to 0$ in $L^1(\Omega)$.
Seeing that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p(x)}(\Omega,\nu),$$

we have then

$$u_n \longrightarrow u \text{ strongly in } L^{q(x)}(\Omega,\sigma),$$

and consequently

$$u_n \longrightarrow u \text{ a.e. in } \Omega.$$

Thanks to $D_n \to 0$ a.e. in Ω,
there is then $B \subset \Omega$ such that $mes(B) = 0$ and for $x \in \Omega \setminus B$, we have

$$|u(x)| < \infty,\ |\nabla u(x)| < \infty,\ u_n(x) \longrightarrow u(x) \text{ and } D_n(x) \to 0.$$

Let us pose

$$\xi_n = \nabla u_n(x),\ \ \xi = \nabla u(x),$$

we have

$$\begin{aligned} D_n(x) \ &\geq \alpha \sum_{i=1}^{N} \omega_i |\xi_n^i|^{p(x)} + \alpha \sum_{i=1}^{N} \omega_i |\xi^i|^{p(x)} \\ &- \sum_{i=1}^{N} \omega_i^{\frac{1}{p(x)}} [k(x) + \sigma^{\frac{1}{p'(x)}} |u_n|^{\frac{q(x)}{p'(x)}} \\ &+ \sum_{j=1}^{N} \omega_j^{\frac{1}{p'(x)}} |\xi_n^j|^{p(x)-1}]|\xi^i| \\ &- \sum_{i=1}^{N} \omega_i^{\frac{1}{p(x)}} [k(x) + \sigma^{\frac{1}{p'(x)}} |u_n|^{\frac{q(x)}{p'(x)}} \\ &+ \sum_{j=1}^{N} \omega_j^{\frac{1}{p'(x)}} |\xi^j|^{p(x)-1}]|\xi_n^i|, \\ &\geq \alpha \sum_{i=1}^{N} \omega_i |\xi_n^i|^{p(x)} - C(x)[1 + \sum_{j=1}^{N} \omega_j^{\frac{1}{p'(x)}} |\xi_n^j|^{p(x)-1} \\ &+ \sum_{i=1}^{N} \omega_i^{\frac{1}{p'(x)}} |\xi_n^i|] \end{aligned} \tag{13}$$

where $C(x)$ is a function depending on x and not on n.
seeing that $u_n(x) \longrightarrow u(x)$, we have $|u_n(x)| \leq M_x$, where M_x is positive. Then by a standard argument, we will have $|\xi_n|$ is uniformly bounded compared to n; indeed: (4.4) becomes

$$D_n(x) \ \geq \ \sum_{i=1}^{N} \omega_i |\xi_n^i|^{p(x)} (\alpha\omega_i - \frac{C(x)}{N|\xi^i|^{p(x)}} - \frac{C(x)\omega_i^{\frac{1}{p'(x)}}}{|\xi_n^i|} - \frac{C(x)\omega_i^{\frac{1}{p(x)}}}{|\xi_n^i|^{p(x)-1}}).$$

If $|\xi_n| \to \infty$, there is at least i_0 such that$|\xi_n^{i_0}| \to \infty$, what will give us $D_n(x) \to \infty$; what is absurd.
Let ξ^* an adherent point of ξ_n, we have $|\xi^*| < \infty$ and by continuity of the operator a compared to two last variables, we will have:

$$\Big(a(x,u,\xi^*) - a(x,u,\xi)\Big)(\xi^* - \xi) = 0,$$

and according to (3.2), we obtain $\xi^* = \xi$.
the unicity of the adherent point implies $\nabla u_n(x) \to \nabla u(x)$ a.e. in Ω.
Seeing that the sequence $(a(x,u_n,\nabla u_n))_n$ is bounded in $(L^{p(x)}(\Omega,\nu))^N$, and $a(x,u_n,\nabla u_n) \longrightarrow a(x,u,\nabla u)$ a.e. in Ω, then according the lemma 2 we obtain

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \text{ weakly in } (L^{p(x)}(\Omega,\nu))^N.$$

let us pose $\widetilde{y_n} = a(x,u_n,\nabla u_n)\nabla u_n$ and $\widetilde{y} = a(x,u,\nabla u)\nabla u$, in the same way that in [4], we can write
$\widetilde{y_n} = a(x,u_n,\nabla u_n)\nabla u_n \longrightarrow \widetilde{y} = a(x,u,\nabla u)\nabla u$ strongly in $L^1(\Omega)$.
According to (3.3), we have

$$\alpha \sum_{i=1}^{N} \nu |\frac{\partial u_n}{\partial x_i}|^{p(x)} \leq \widetilde{y_n}.$$

let

$$z_n = \sum_{i=1}^{N} \nu |\frac{\partial u_n}{\partial x_i}|^{p(x)},\ z = \sum_{i=1}^{N} \nu |\frac{\partial u}{\partial x_i}|^{p(x)},\ y_n = \frac{\widetilde{y_n}}{\alpha},\ y = \frac{\widetilde{y}}{\alpha}.$$

Thanks to Fatou lemma, we obtain

$$\int_\Omega 2y\,dx \leq \liminf_{n\to\infty} \int_\Omega y + y_n - |z_n - z|\,dx,$$

ie

$$0 \leq -\limsup_{n\to\infty} \int_\Omega |z_n - z|\,dx,$$

then

$$0 \leq \liminf_{n\to\infty} \int_\Omega |z_n - z|\,dx \leq \limsup \int_\Omega |z_n - z|\,dx \leq 0,$$

what implies

$$\nabla u_n \longrightarrow \nabla u \text{ in } (L^{p(x)}(\Omega,\nu))^N,$$

consequently

$$u_n \longrightarrow u \text{ in } W_0^{1,p(x)}(\Omega,\nu).$$

■

Definition 3 *for any $k > 0$ and $s \in I\!R$, the truncature function $T_k(.)$ is defined as:*

$$T_k(s) = \begin{cases} s & if \quad |s| \leq k, \\ k\dfrac{s}{|s|} & if \quad |s| > k. \end{cases}$$

Lemma 4 *Let $(u_n)_n$ a sequence from $W_0^{1,p(x)}(\Omega,\nu)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega,\nu)$. Then $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p(x)}(\Omega,\nu)$.*

Proof
We have $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega,\nu)$, and $W_0^{1,p(x)}(\Omega,\nu) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$, we will have $u_n \to u$ strongly in $L^{p(x)}(\Omega)$ and a.e. in Ω, consequently $T_k(u_n) \to T_k(u_n)$ a.e. in Ω.
On another side

$$\begin{aligned} \|T_k(u_n)\|_{p(x),\nu}^{\theta_1} &\leq \int_\Omega |\nabla T_k(u_n)|^{p(x)}\nu(x)dx, \\ &\leq \int_\Omega |\nabla T_k'(u_n)||\nabla u_n|^{p(x)}\nu(x)dx, \\ &\leq \int_\Omega |\nabla u_n|^{p(x)}\nu(x)dx, \\ &\leq \|u_n\|_{p(x),\nu}^{\theta_2}, \end{aligned}$$

where

$$\theta_1 = \begin{cases} p^+ & \text{if } \|T_k(u_n)\|_{p(x),\nu} \leq 1, \\ p_- & \text{if } \|T_k(u_n)\|_{p(x),\nu} > 1, \end{cases}$$

and

$$\theta_1 = \begin{cases} p^+ & \text{si } \|u_n\|_{p(x),\nu} \geq 1, \\ p_- & \text{si } \|u_n\|_{p(x),\nu} < 1. \end{cases}$$

Thus $(T_k(u_n))_n$ is bounded in $W_0^{1,p(x)}(\Omega,\nu)$, consequently $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p(x)}(\Omega,\nu)$. ■

6 Proof of theorem 1

Step1: a priori estimate
The problem $(\mathcal{P}_n)$ admits at least a solution u_n belonging to $W_0^{1,p(x)}(\Omega,\nu)$. Choosing $T_k(u_n)$ as test function in $(\mathcal{P}_n)$, and seeing that $\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)dx \geq 0$, we obtain:

$$\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u_n)dx \leq \int_\Omega f_n T_k(u_n)dx.$$

Using (5), we deduce that

$$\alpha \int_\Omega \nu(x)|\nabla T_k(u_n)|^{p(x)}dx \leq kC_1,$$

that means

$$\int_\Omega \nu(x)|\nabla T_k(u_n)|^{p(x)}dx \leq \frac{k}{\alpha}C_1.$$

Consequently

$$\|\nabla T_k(u_n)\|_{p(x),\nu}^{\gamma} \leq C_2,$$

where

$$\gamma = \begin{cases} p^+ & if \quad \|\nabla T_k(u_n)\|_{p(x),\nu} \geq 1, \\ p_- & if \quad \|\nabla T_k(u_n)\|_{p(x),\nu} < 1. \end{cases}$$

Onthe other hand, we have

$$\int_\Omega a(x,u_n,\nabla T_k(u_n))\nabla T_k(u_n)dx + \int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)dx = \int_\Omega f_n T_k(u_n)dx.$$

What implies

$$\int_\Omega g_n(x,u_n,\nabla u_n)T_k(u_n)dx \leq \int_\Omega |f_n|kdx.$$

Thus

$$\int_{\{|u_n|>k\}} k\frac{u_n}{|u_n|}g_n(x,u_n,\nabla u_n)dx + \int_{\{|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)u_n dx \leq \int_\Omega |f_n|kdx.$$

Consequently

$$k\int_{\{|u_n|>k\}} g_n(x,u_n,\nabla u_n)dx \leq kC_1$$

However

$$\int_\Omega \nu(x)|\nabla u_n|^{p(x)}dx = \int_{\{|u_n|>k\}} \nu(x)|\nabla u_n|^{p(x)}dx + \int_\Omega \nu(x)|\nabla u_n|^{p(x)}dx.$$

Then for $k > \rho_1$, we will have:

$$\int_\Omega \nu(x)|\nabla u_n|^{p(x)}dx \leq \frac{1}{\rho_2}\int_{\{|u_n|>k\}} g_n(x,u_n,\nabla u_n)dx + C_2 \leq C_4$$

What implies that a sequence $(u_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega,\nu)$.
Step2: Strong convergence of truncations
Seeing that $(u_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega,\nu)$, and $W_0^{1,p(x)}(\Omega,\nu) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega)$, we can extract a subsequence of $(u_n)_n$, still noted $(u_n)_n$, and there is $u \in W_0^{1,p(x)}(\Omega,\nu)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in} \quad W_0^{1,p(x)}(\Omega,\nu), \\ u_n \to u & \text{a.e. in } \Omega, \end{cases}$$

We want to show that $T_k(u_n) \to T_k(u)$ strongly in $W_0^{1,p(x)}(\Omega,\nu)$.
Let $z_n = T_k(u_n) - T_k(u)$, let us pose $v_n = \varphi_\lambda(z_n)$, where $\varphi_\lambda(s) = se^{\lambda s^2}$.
Choosing v_n as test function in $(\mathcal{P}_n)$.
We are $(v_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega,\nu)$, and $v_n \to 0$ a.e. in Ω, then according to lemma 4, we obtain $v_n \rightharpoonup 0$ weakly in $W_0^{1,p(x)}(\Omega,\nu)$,.
thus $\langle f_n, v_n\rangle \to 0$, because $v_n \rightharpoonup 0$ weakly in $L^\infty(\Omega)$, and $f_n \to f$ strongly in $L^1(\Omega)$.
Consequently

$$\eta_{1n} = \langle Au_n, v_n\rangle + \langle G_n u_n, v_n\rangle \longrightarrow 0.$$

Seeing that $g_n(x,u_n,\nabla u_n) \geq 0$ on $\{x \in \Omega : |u_n| \geq k\}$, then we have:

$$\langle Au_n, v_n\rangle + \int_{\{x\in\Omega:|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)v_n dx \leq \eta_{1n}.$$

Thus

$$\langle Au_n, v_n\rangle - |\int_{\{x\in\Omega:|u_n|\leq k\}} g_n(x,u_n,\nabla u_n)v_n dx| \leq \eta_{1n}. \quad (14)$$

however

$$\langle Au_n, v_n\rangle = \int_\Omega a(x,u_n,\nabla u_n)\nabla(T_k(u_n)-T_k(u))\varphi'_\lambda(z_n)dx$$
$$= \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(u))\varphi'_\lambda(z_n)dx$$
$$-\int_{|u_n|>k} a(x,u_n,\nabla u_n)\nabla T_k(u)\varphi'_\lambda(z_n)dx \tag{15}$$

$= \int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\Big]\nabla(T_k(u_n)-T_k(u))\varphi'_\lambda(z_n)dx + \eta_{2n}$,

where

$\eta_{2n} = \int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla(T_k(u_n)-T_k(u))\varphi'_\lambda(z_n)dx - \int_{|u_n|>k} a(x,u_n,\nabla u_n)\nabla T_k(u)\varphi'_\lambda(z_n)dx.$

We have

$$\begin{cases} \nabla T_k(u)\chi_{\{|u_n>k|\}} \to 0 \text{ strongly in } (L^{p(x)}(\Omega,\nu))^N, \\ (a(x,u_n,\nabla u_n))_n \text{ is bounded in } (L^{p'(x)}(\Omega,\nu^*))^N, \end{cases}$$

$\Rightarrow \eta_{2n} \longrightarrow 0$ when $n\to\infty$.

On another side

$|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)v_n dx|$

$\le \int_{\{|u_n|\le k\}} |g_n(x,u_n,\nabla u_n)||v_n|dx$

$$\le \int_{\{|u_n|\le k\}} d(k)(c(x)+\nu(x)|\nabla u_n|^{p(x)}|v_n|dx$$
$$\le d(k)\int_{\{|u_n|\le k\}} c(x)|\varphi_\lambda(z_n)|dx \tag{16}$$
$$+\frac{d(k)}{\alpha}\int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n|\varphi_\lambda(z_n)|dx$$

$\le \eta_{3n} + \frac{d(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi_\lambda(z_n)|dx$

$\le \frac{d(k)}{\alpha}\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u_n))\Big)$

$\times\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]|\varphi_\lambda(z_n)|dx + \eta_{4n}$,

where

$$\eta_{3n} = d(k)\int_{\{|u_n|\le k\}} c(x)|\varphi_\lambda(z_n)|dx \to 0 \text{ when } n\to\infty,$$

and $\eta_{4n} = \frac{d(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)|\varphi_\lambda(z_n)|dx$

$+\frac{d(k)}{\alpha}\int_\Omega a(x,T_k(u),\nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u))|\varphi_\lambda(z_n)|dx$

$+\eta_{3n} \to 0$ when $n\to\infty$,

If $\lambda \ge (\frac{d(k)}{\alpha})^2$ then, by a simple calculation, we have

$$\varphi'_\lambda(s) - \frac{d(k)}{\alpha}|\varphi_\lambda(s)| \ge \frac{1}{2}.$$

By combining this last inequality with (14), (15) and (16), we obtain

$\int_\Omega \Big(a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u_n))\Big)$

$\times\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]$

$\le 2(\eta_{1n} - \eta_{2n} + \eta_{4n}), \longrightarrow 0$; when $n\to\infty$.

Thanks to lemma 3, we deduce that

$$T_k(u_n) \longrightarrow T_k(u) \text{ strongly in } W_0^{1,p(x)}(\Omega,\nu). \tag{17}$$

step3: Equi-integrability of the nonlinearitie $(g_n(x,u_n,\nabla u_n))_n$

We will show that $g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u)$ strongly in $L^1(\Omega)$. Seeing (17), we deduce that

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega, \text{ when } n\to\infty. \tag{18}$$

Let E a measurable part of Ω, and let $m>0$, let us pose

$$X_m^n = \{x\in\Omega : |u_n|\le m\} \text{ et } (X_m^n)^c = \{x\in\Omega : |u_n|>m\}.$$

We have

$$\begin{aligned} \int_E |g_n(x,u_n,\nabla u_n)|dx &= \int_{E\cap X_m^n} |g_n(x,u_n,\nabla u_n)|dx \\ &+ \int_{E\cap (X_m^n)^c} |g_n(x,u_n,\nabla u_n)|dx, \\ &\le d(m)\int_E (\nu(x)|\nabla T_m(u_n)|^{p(x)} \\ &+c(x))dx + \int_{(X_m^n)^c} |g_n(x,u_n,\nabla u_n)|dx. \end{aligned} \tag{19}$$

Thanks to (17) and seeing $c(x)\in L^1(\Omega)$, there is $\delta>0$ such that for all E: $|E|<\delta$, then

$$d(m)\int_E (\nu(x)|\nabla T_m(u_n)|^{p(x)} + c(x))dx \le \frac{\eta}{2}. \tag{20}$$

Let ψ_m a function defined by:

$$\psi_m(x) = \begin{cases} 0 & \text{if } |s|\le m-1, \\ 1 & \text{if } s\ge m, \\ 1 & \text{if } s\le -m, \\ \psi'_m(x) = 1 & \text{if } m-1\le |s|\le m, \end{cases}$$

we have $\psi_m(u_n)\in W_0^{1,p(x)}(\Omega,\nu)$, choosing $\psi_m(u_n)$ as test function in $(\mathcal{P}_n)$, we obtain:

$\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\psi'_m(u_n)dx + \int_\Omega g_n(x,u_n,\nabla u_n)\psi_m(u_n)dx$

$= \int_\Omega f_n\psi_m(u_n)dx.$

According (9) and (10), we deduce that

$$\int_{\{|u_n|>m-1\}} |g_n(x,u_n,\nabla u_n)|dx \le \int_{\{|u_n|>m-1\}} |f_n|dx.$$

what implies

$$\int_{\{|u_n|>m\}} |g_n(x,u_n,\nabla u_n)|dx \le \int_{\{|u_n|>m-1\}} |f_n|dx.$$

however $f_n\to f$ in $L^1(\Omega)$ and $|\{|u_n|>m-1\}|\to 0$ when $m\to\infty$, uniformly in n, then for m big enough we have;

$$\int_{\{|u_n|>m-1\}} |f_n|dx \le \frac{\eta}{2}, \forall n\in I\!N.$$

Thus

$$\int_{\{|u_n|>m\}} |g_n(x,u_n,\nabla u_n)|dx \le \frac{\eta}{2}, \ \forall n \in I\!N. \qquad (21)$$

Combining (19), (20) and (21), we deduce that

$$\int_E |g_n(x,u_n,\nabla u_n)|dx \le \eta, \text{ for all } n \in I\!N.$$

That means that the sequence $(g_n(x,u_n,\nabla u_n))_n$ is equi-integrable, but $g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u)$ a.e. in Ω, then thanks to Vitali theorem, we deduce that

$$g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \text{ strongly in } L^1(\Omega).$$

step4: passing to the limit

Seeing that $(u_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega,\nu)$, and thanks to (6), we conclude that $(a(x,u_n,\nabla u_n))_n$ is bounded in $(L^{p'(x)}(\Omega,\nu^*))^N$, and seeing that $a(x,u_n,\nabla u_n) \to a(x,u,\nabla u)$ a.e. in Ω, then according to the lemma 2, we obtain that

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \text{ weakly in } (L^{p'(x)}(\Omega,\nu^*))^N,$$

while making $n \to \infty$, we obtain $\forall v \in W_0^{1,p(x)}(\Omega,\nu) \cap L^\infty(\Omega)$,

$$\langle Au, v\rangle + \int_\Omega g(x,u,\nabla u) v dx = \int_\Omega f v dx.$$

7 Example of application

Let Ω be a bounded open subset of $I\!R^N$, $N \ge 2$, and let $p(x), q(x) \in C_+(\overline{\Omega})$.
Let us pose:

$$\begin{cases} a_i(x,r,\zeta) = \nu(x)|\zeta_i|^{p(x)-1} sgn(\zeta_i), \ i = 1,...,N, \\ \text{and} \\ g(x,r,\zeta) = \rho r|r|^{q(x)}\nu(x)|\zeta|^{p(x)}, \rho > 0, \end{cases}$$

were $\nu(x)$ a function weight in Ω. The function a_i, $i = 1,...,N$, which satisfies the assumptions of theorem (6), (7) and (8); and well as the function g satisfies (9), (10) and (11), with $|s| \ge \rho_1 = 1$ and $\rho_2 = \rho > 0$, consequently the assumptions of theorem 1 are satisfied; thus for $f \in L^1(\Omega)$, the following theorem, with $\nu(x) = d^\lambda(x)$:

$$(\mathcal{E}) \begin{cases} \displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(d^\lambda(x)|\frac{\partial u}{\partial x_i}|^{p(x)-1} sgn(\frac{\partial u}{\partial x_i})\right) \\ +\rho u|u|^{q(x)} \sum_{i=1}^N \frac{\partial}{\partial x_i} d^\lambda(x)|\frac{\partial u}{\partial x_i}|^{p(x)} = f \text{ in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p(x)}(\Omega, d^\lambda(x)) \text{ and} \\ \rho u|u|^{q(x)} \sum_{i=1}^N \frac{\partial}{\partial x_i} d^\lambda(x)|\frac{\partial u}{\partial x_i}|^{p(x)} \in L^1(\Omega), \end{cases}$$

admits at least one solution $u \in W_0^{1,p(x)}(\Omega, d^\lambda(x))$.

References

1. Y. Akdim, E. Azroul and A. Benkirane, "Existence of solution for quasilinear degenerated elliptic equation.", Electronic J. Equ. **71**, 1-19, 2001.
2. E. Azroul, A. Barbara and H. Hjiej, "Strongly nonlinear p(x)-elliptic problems with second member L^1-dual.", African Diaspora Journal of Mathematics. **16**, Number 2, 11722, 2014.
3. E. Azroul, H. Hjiej and A. Touzani, "Existence and regularity of entropy solutions for strongly nonlinear p(x)-elliptic equations.", Electronic journal of differential equations. **68**, 1-27, 2013.
4. M. B. Benboubker, E. Azroul, A. Barbara, "Quasilinear Elliptic Problems with non standard growth.", EJDE, **62**, 1-16, 2011.
5. A. Benkirane and A. Elmahi, "Strongly nonlinear elliptic unilateral problems having natural growth terms and L^1 data.", Rendiconti di matematica, Serie VII, **18**, 28917303, 1998.
6. A. Bensoussan, L. Boccardo and F. Murat, "On a nonlinear partial differential equation having natural growth terms and unbounded solution.", Ann. Inst. Henri Poincaré, **4**, 347-364, 1988.
7. L. Boccardo and T. Gallouët, "Strongly nonlinear elliptic equations having natural growth terms and L^1.", Nonlinear Analysis Theory methods and applications, **19**(6), 573-579, 1992.
8. L. Boccardo and T. Gallouët, "Nonlinear elliptic equations with right hand side measures.", Comm. P.D.E., **17**, 641-655, 1992 .
9. H. Brezis and W. Strauss, "Semilinear second-order elliptic equations in L^1.", J. Math. Soc. Japan, **25**(4), 565-590, 1973.
10. T. Del Vecchio, "Nonlinear elliptic equations with measure data.", Potential Analysis, **4**, 185-203, 1995.
11. P. Drabek, A. Kufner and V. Mustonen, "Pseudo-monotonicity and degenerated or singular elliptic operators.", Bull. Austral. Math. Soc., **58**, 213-221, 1998.
12. X. L. Fan and Q. H. Zhang, "Existence for p(x)-Laplacien Direchlet problem.", Nonlinear Analysis, **52**, 1843-1852, 2003.
13. X. L. Fan and D. Zhao, "On the generalized Orlicz-Sobolev Space $W^{k,p(x)}(\Omega)$.", J. Gansu Educ. College, **12**(1), 1-6, 1998.
14. O. Kovácik and J. Rákosnik, "On Spaces $L^{p(x)}$ and $W^{k,p(x)}$.", Czechoslovak Math. J. , **41**, 592-618, 1991.
15. J. L. Lions, "Quelques méthodes de résolution des problèmes aux limites non linéaires.", Dunod, 1969.
16. V. M. Monetti and L. Randazzo, "Existence results for non-linear elliptic equations with p-growth in the gradient.", Riceeche di Matimaica, **XLIX**(1), 163-181, 2000.
17. D. Zhao, X. J. Qiang and X. L. Fan, "On generalized Orlicz Spaces $L^{p(x)}(\Omega)$.", J. Gansu Sci., **9**(2), 1-7, 1997.

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Steel heat treating: mathematical modelling and numerical simulation of a problem arising in the automotive industry

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ABSTRACT

We describe a mathematical model for the industrial heating and cooling processes of a steel workpiece representing the steering rack of an automobile. The goal of steel heat treating is to provide a hardened surface on critical parts of the workpiece while keeping the rest soft and ductile in order to reduce fatigue. The high hardness is due to the phase transformation of steel accompanying the rapid cooling. This work takes into account both heating-cooling stage and viscoplastic model. Once the general mathematical formulation is derived, we can perform some numerical simulations.

1 Introduction

In the automative industry, many workpieces such gears, bearings, racks and pinions, are made of steel. Steel is an alloy of iron and carbon. Generally, industrial steel has a carbon content up to about 2 wt%. Other alloying elements may be present, such as Cr and V in tools steels, or Si, Mn, Ni and Cr in stainless steels. Most structural components in mechanical engineering are made of steel. Certain of these components, such as toothed wheels, bevel gears, pinions and so on, engaged each others in order to transmit some kind of (rotational or longitudinal) movement. In this situation, the contact surfaces of these components are particularly stressed. The goal of heat treating of steel is to attain a satisfactory hardness. Prior to heat treating, steel is a soft and ductile material. Without a hardening treatment, and due to the surface stresses, the gear teeth will soon get damaged and they will no longer engage correctly.

In this work we are interested in the mathematical description of the hardening procedure of a car steering rack (see Figure 1). This particular situation is one of the major concerns in the automotive industry. In this case, the goal is to increase the hardness of the steel along the tooth line and at the same time keeping the rest of the workpiece soft and ductile in order to reduce fatigue. This problem is governed by a nonlinear system of partial differential equations coupled with a certain system of ordinary differential equations. Once the full system is set we perform some numerical simulations.

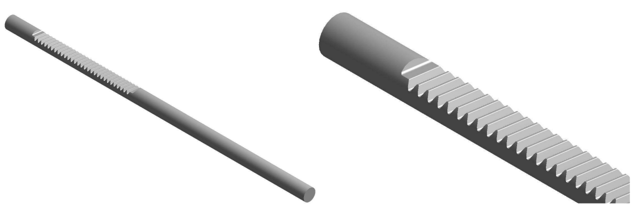

Figure 1: Car steering rack.

Solid steel may be present at different phases, namely austenite, martensite, bainite, pearlite and ferrite. The phase diagram of steel is shown in Figure 2. For a given wt% of carbon content up to 2.11,

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all steel phases are transformed into austenite provided the temperature has been raised up to a certain range. The minimum austenization temperature (727°) is attained for a carbon content of 0.77 wt% (eutectoid steel). Upon cooling, the austenite is transformed back into the other phases (see Figure 3), but its distribution depends strongly on the cooling strategy ([4, 12]).

Martensite is the hardest constituent in steel, but at the same time is the most brittle, whereas pearlite is the softest and more ductile phase. Martensite derives from austenite and can be obtained only if the cooling rate is high enough. Otherwise, the rest of the steel phases will appear.

The hardness of the martensite phase is due to a strong supersaturation of carbon atoms in the iron lattice and to a high density of crystal defects. From the industrial standpoint, heat treating of steel has a collateral problem: hardening is usually accompanied by distortions of the workpiece. The main reasons of these distortions are due to (1) thermal strains, since steel phases undergo different volumetric changes during the heating and cooling processes, and (2) experiments with steel workpieces under applied loading show an irreversible deformation even when the equivalent stress corresponding to the load is in the elastic range. This effect is called transformation induced plasticity.

The heating stage is accomplished by an induction-conduction procedure. This technique has been successfully used in industry since the last century. During a time interval, a high frequency current passes through a coil generating an alternating magnetic field which induces eddy currents in the workpiece, which is placed close to the coil. The eddy currents dissipate energy in the workpiece producing the necessary heating.

2 Mathematical modeling

We consider the setting corresponding to Figure 4. The domain Ω^c represents the inductor (made of copper) whereas Ω^s stands for the steel workpiece to be hardened. Here, the coil is the domain $\Omega = \Omega^s \cup \Omega^c \cup S_0$. In this way, the workpiece itself takes part of the coil.

In order to describe the heating-cooling process, we will distinguish two subintervals forming a partition of $[0,T]$, namely $[0,T] = [0,T_h) \cup [T_h, T_c]$, $T_c > T_h > 0$. The first one $[0, T_h)$ corresponds to the heating process. All along this time interval, a high frequency electric current is supplied through the conductor which in its turn induces a magnetic field. The combined effect of both conduction and induction gives rise to a production term in the energy balance equation (14), namely $b(\theta)|\boldsymbol{\mathcal{A}}_t + \nabla\phi|^2$. This is Joule's heating which is the principal term in heat production. In our model, we will only consider three steel phase fractions, namely austenite (a), martensite (m), and the rest of phases (r). In this way, we have $a + m + r = 1$ and $0 \le a, m, r \le 1$ in $\Omega^s \times [0,T]$. At the initial time we have $r(0) = 1$ in Ω^s. Upon heating only austenite can be obtained. In particular $m = 0$ in $\Omega^s \times [0, T_h]$ and the transformation to austenite is derived at the expense of the other phase fractions (r).

At the instant $t = T_h$, the current is switched off and during the time interval $[T_h, T_c]$ the workpiece is severely cooled down by means of aqua-quenching.

The heating model

The current passing through the set of conductors $\Omega = \Omega^c \cup \Omega^s \cup S_0$ is modeled by the electric potential difference, φ_0, applied on the surface $\Gamma_2 \subset \Omega^c$ (see Figure 4). Notice that the applied potential on Γ_1 is zero. In the sequel, we put $\Gamma = \Gamma_1 \cup \Gamma_2$.

The heating model involves the following unknowns: the electric potential, ϕ; the magnetic vector potential, $\boldsymbol{\mathcal{A}} = (\boldsymbol{\mathcal{A}}_1, \boldsymbol{\mathcal{A}}_2, \boldsymbol{\mathcal{A}}_3)$; the stress tensor, $\boldsymbol{\sigma} = (\sigma_{ij})_{1\le i,j\le 3}$, $\sigma_{ij} = \sigma_{ji}$ for all $1 \le i, j \le 3$; the displacement field $u = (u_1, u_2, u_3)$; the austenite phase fraction, a; and the temperature, θ. Among them, only $\boldsymbol{\mathcal{A}}$ is defined in the domain D containing the set of conductors Ω. On the other hand, since the inductor and the workpiece are in close contact, both ϕ and θ are defined in Ω. Since phase transitions only occur in the workpiece, we may neglect deformations in Ω^c. This implies that $\boldsymbol{\sigma}$, u and a are only defined in the workpiece Ω^s.

Since electromagnetic fields generated by high frequency currents are sinusoidal in time, both the electric potential, ϕ, and the magnetic potential field, $\boldsymbol{\mathcal{A}}$, take the form ([1, 2, 14, 15]) $\mathcal{M}(x,t) = \mathrm{Re}\left[e^{i\omega t} M(x)\right]$, where M is a complex-valued function or vector field, and $\omega = 2\pi f$ is the angular frequency, f being the electric current frequency. In general, M also depends on t, but at a time scale much greater than $1/\omega$. In this way, we may introduce the complex-valued fields φ and $\boldsymbol{A}$ as

$$\phi = \mathrm{Re}[e^{i\omega t}\varphi(x,t)], \quad \boldsymbol{\mathcal{A}} = \mathrm{Re}[e^{i\omega t}\boldsymbol{A}(x,t)]. \tag{1}$$

As a far as the numerical simulation of a system like (2)-(15) is concerned, the introduction of the new variables φ and $\boldsymbol{A}$ is quite convenient since the time scale describing the evolution of both φ and $\boldsymbol{A}$ is much smaller than that of the temperature θ. In the case of steel heat treating, f is about 80 KHz.

The heating model reads as follows ([3, 9, 10, 7]):

$$\nabla \cdot (b(\theta)\nabla\varphi) = 0 \text{ in } \Omega_{T_h} = \Omega \times (0, T_h), \tag{2}$$

$$\frac{\partial \varphi}{\partial n} = 0 \text{ on } (\partial\Omega \setminus \Gamma) \times (0, T_h), \tag{3}$$

$$\varphi = 0 \text{ on } \Gamma_1 \times (0, T_h),\ \varphi = \varphi_0 \text{ on } \Gamma_2 \times (0, T_h), \tag{4}$$

$$\begin{aligned} b_0(\theta) i\omega \boldsymbol{A} + \nabla \times \left(\frac{1}{\mu}\nabla \times \boldsymbol{A}\right) - \delta\nabla(\nabla\cdot\boldsymbol{A}) \\ = -b_0(\theta)\nabla\varphi \text{ in } D \times (0, T_h), \end{aligned} \tag{5}$$

$$\boldsymbol{A} = 0 \text{ on } \partial D \times (0, T_h), \tag{6}$$

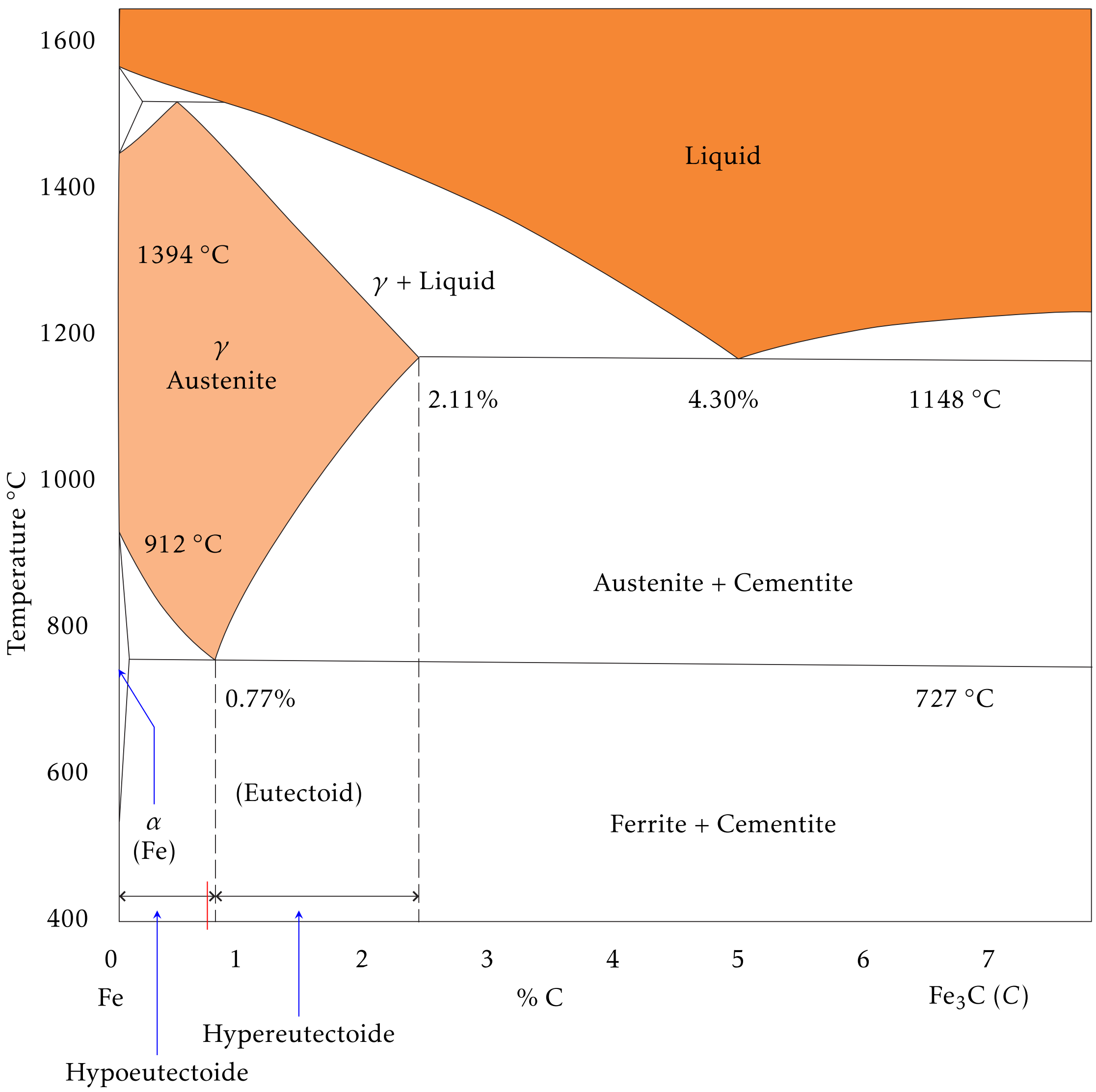


Figure 2: Iron-carbide phase diagram.

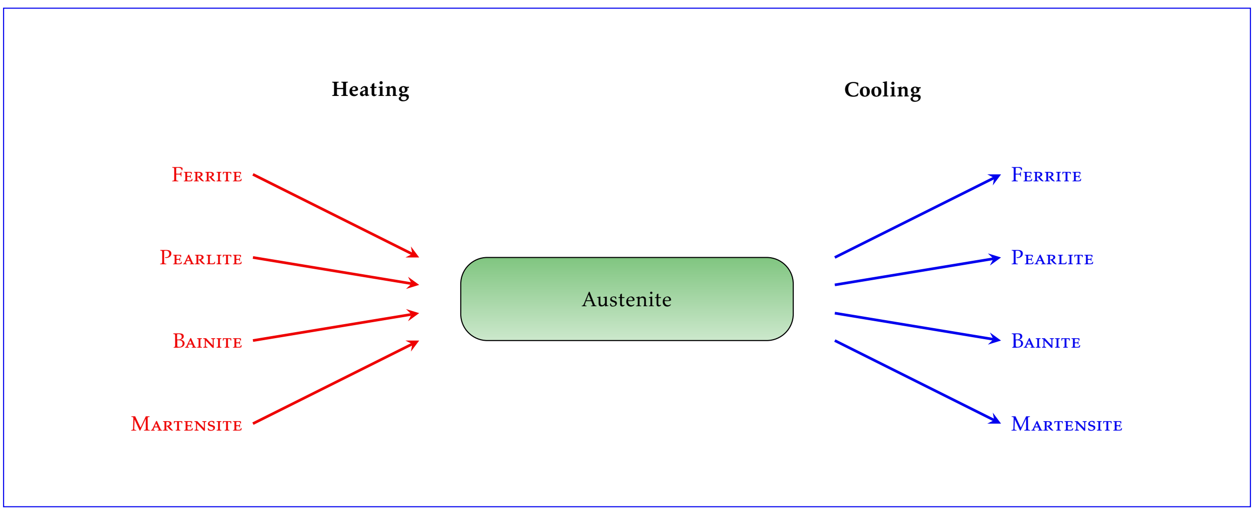


Figure 3: Microconstituents of steel. Upon heating, all phases are transformed into austenite, which is transformed back to the other phases during the cooling process. The distribution of the new phases depends strongly on the cooling strategy. A high cooling rate transforms austenite into martensite. A slow cooling rate transforms austenite into pearlite.

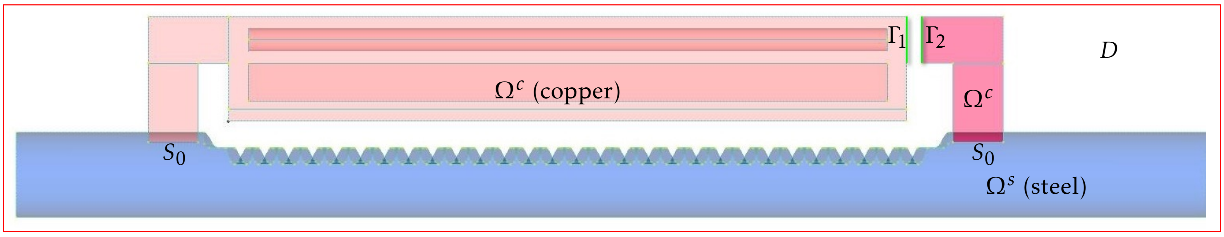


Figure 4: Domains D, $\Omega = \Omega^s \cup \Omega^c \cup S_0$ and the faces $\Gamma_1, \Gamma_2 \subset \Omega^c$. The inductor Ω^c is made of copper. The workpiece contains a toothed part to be hardened by means of the heating-cooling process described below. It is made of a hypoeutectoid steel. The domain is taken as a big enough rectangle containing both the inductor and the rack

$$-\nabla \cdot \boldsymbol{\sigma} = F \text{ in } \Omega^s \times (0, T_h), \tag{7}$$

$$\boldsymbol{\sigma} = \boldsymbol{K}\Big(\varepsilon(u) - A_1(a,m,\theta)\boldsymbol{I} - \int_0^t \gamma(a,m,a_t,m_t,\theta)\boldsymbol{S}\,\mathrm{d}\tau\Big), \tag{8}$$

$$u = 0 \text{ on } \Gamma_0 \times (0, T_h), \tag{9}$$

$$\boldsymbol{\sigma} \cdot n = 0 \text{ on } (\partial\Omega^s \setminus \Gamma_0) \times (0, T_h), \tag{10}$$

$$a_t = \frac{1}{\tau_a(\theta)}(a_{\mathrm{eq}}(\theta) - a)\mathcal{H}(\theta - A_s) \text{ in } \Omega^s \times (0, T_h), \tag{11}$$

$$a(0) = 0 \text{ in } \Omega^s, \tag{12}$$

$$\begin{aligned}\alpha(\theta,a,m,\boldsymbol{\sigma})\theta_t &- \nabla\cdot(\kappa(\theta)\nabla\theta) \\ &+ 3\bar{\kappa}q(a,m)\theta(\nabla\cdot u_t - 3A_2(a_t,m_t,\theta)) \\ &= b(\theta)|\boldsymbol{\mathcal{A}}_t + \nabla\phi|^2 - \rho L_a a_t \\ &\quad + A_2(a_t,m_t,\theta)\operatorname{tr}\boldsymbol{\sigma} \\ &\quad + \gamma(a,m,a_t,m_t,\theta)|\boldsymbol{S}|^2 \text{ in } \Omega_{T_h},\end{aligned} \tag{13}$$

$$\frac{\partial\theta}{\partial n} = 0 \text{ on } \partial\Omega \times (0, T_h), \tag{14}$$

$$\theta(0) = \theta_0 \text{ in } \Omega. \tag{15}$$

Here, $b(\theta)$ is the electrical conductivity (by $b(\theta)$ we mean the function $(x,t) \mapsto b(x,\theta(x,t))$, and also for $\kappa(\theta)$, etc.); φ_0 represents the potential external source. The domain D containing the set of conductors is taken big enough so that the magnetic vector potential $\boldsymbol{A}$ vanishes on its boundary ∂D (in our model, is taken to be a 2D rectangle or a 3D cube). Since both $\boldsymbol{\sigma}$ and a are only defined in Ω^s, when they appear in a term referred in Ω, we mean that this term vanishes outside Ω^s (for instance, $-\rho L_a a_t$ appearing in (13)); $b_0(x,s) = b(x,s)$ if $x \in \Omega$, $b_0(x,s) = 0$ elsewhere; $\mu = \mu(x)$ is the magnetic permeability; $\delta > 0$ is a small constant; F is a given external force (usually $F = 0$); $\boldsymbol{K} = \boldsymbol{K}_{ijkl}$, $1 \le i,j,k,l \le 3$ is the stiffness tensor. Steel can be considered as an isotropic and homogenous material so that

$$\boldsymbol{K}_{ijkl} = \bar{\lambda}\delta_{ij}\delta_{kl} + \bar{\mu}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \text{ for all } i,j,k,l \in \{1,2,3\}$$

where $\bar{\lambda} \ge 0$ and $\bar{\mu} > 0$ are the Lamé coefficients of steel; $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the strain tensor; $A_1(a,m,\theta)\boldsymbol{I}$ models the thermal strain, $\boldsymbol{I}$ being the 3×3 unity matrix, whereas $A_1(a,m,\theta)$ is defined as

$$\begin{aligned}A_1(a,m,\theta) &= q_a a(\theta - \theta_a) + q_m m(\theta - \theta_m) \\ &\quad + q_r(1 - a - m)(\theta - \theta_r),\end{aligned}$$

and in its turn q_a, q_m and q_r are the thermal expansion coefficients of the phase fractions a, m and r, respectively, and θ_a, θ_m and θ_r are reference temperatures (notice that during the heating stage is $m = 0$); $\int_0^t \gamma(a,m,a_t,m_t,\theta)\boldsymbol{S}\,\mathrm{d}\tau$ gives the model, through the function γ, of the transformation induced plasticity strain tensor, where $\boldsymbol{S} = \boldsymbol{\sigma} - \frac{1}{3}\operatorname{tr}\sigma\boldsymbol{I}$ is the deviator of σ, that is, the trace free part of the stress tensor; Γ_0 is a certain smooth enough part of $\partial\Omega^s$; n is the unit outer normal vector to the referred boundary; the functions $\tau_a(\theta)$, $a_{\mathrm{eq}}(\theta)$ are given from experimental data (see Figure 5), and $\mathcal{H}$ is the Heaviside function; $\kappa(\theta)$ is the thermal conductivity; the functions appearing in (13) are given as follows

$$\alpha(\theta,a,m,\boldsymbol{\sigma}) = \rho c_\varepsilon - 9\bar{\kappa}q(a,m)^2\theta - q(a,m)\operatorname{tr}\boldsymbol{\sigma},$$

where ρ and c_ε are the steel density and the specific heat capacity at constant strain, respectively, $\bar{\kappa} = \frac{1}{3}(3\bar{\lambda} + 2\bar{\mu})$ is the bulk modulus, and $q(a,m)$ is defined as

$$q(a,m) = q_a a + q_m m + (1 - a - m)q_r;$$

$$\begin{aligned}A_2(a_t,m_t,\theta) &= q_a a_t(\theta - \theta_a) + q_m m_t(\theta - \theta_m) \\ &\quad - q_r(a_t + m_t)(\theta - \theta_r).\end{aligned}$$

Finally, $L_a > 0$ is the latent heat related to the austenite phase fraction. Notice that, in a more general situation ρ, c_ε and L_a may also depend on a, m and/or θ.

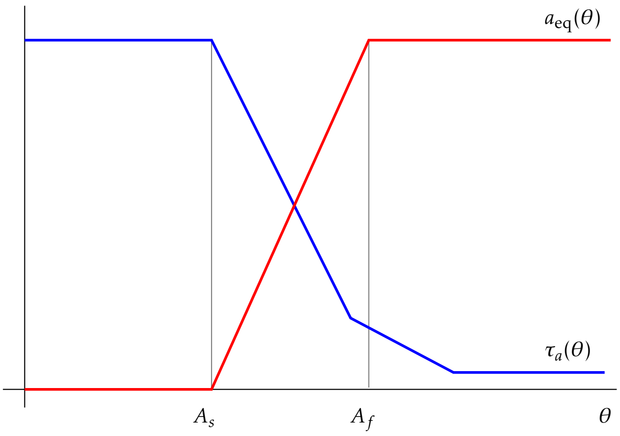


Figure 5: Functions a_{eq} and τ_a.

Equations (2) and (5) derive from Maxwell's equations. In [9], it is assumed the Coulomb gauge condition for the magnetic vector potential, namely, $\nabla \cdot \boldsymbol{A} = 0$. Here, we do not impose this condition since this makes appear an undesired pressure gradient in the equation for $\boldsymbol{A}$. In its turn, we include a penalty term in this equation of the form $-\delta\nabla(\nabla \cdot \boldsymbol{A})$. In doing so, both the theoretical analysis and the numerical simulations are simplified.

Equation (7) is a quasistatic balance law of momentum and (8) is Hooke's law. The transformation to austenite from the initial phase $r(0) = 1$ is described in (11).

Finally, equation (13) derives from the balance law of internal energy. As it has been pointed out above, Joule's heating is the main responsible in heat production. Since $\gamma(a, m, a_t, m_t, \theta)|\mathbf{S}|^2 \geq 0$, the contribution of the transformation induced plasticity to the energy balance is also a production term. On the other hand, during the heating stage we have $a_t \geq 0$ so that $-\rho L_a a_t \leq 0$. This means that the transformation to austenite absorbs energy, which is released during the cooling stage.

The cooling model

The heating process ends, the high frequency current passing through the coil is switched-off and aqua-quenching begins. The quenching is just modeled via the Robin boundary condition given in (25).

We put $a_{T_\mathrm{h}} = a(T_\mathrm{h})$, that is, a_{T_h} is the austenite phase fraction distribution at the final heating instant T_h obtained from (11). In the same way, we define $\theta_{T_\mathrm{h}} = \theta(T_\mathrm{h})$. Obviously, these functions will be taken as the initial phase fraction distribution and temperature, respectively, in the cooling model. Here we use the Koistinen-Marburger model ([11, 13]) for the description of the transformation to martensite from austenite.

The cooling model reads as follows

$$-\nabla \cdot \boldsymbol{\sigma} = F \text{ in } \Omega^\mathrm{s} \times (T_\mathrm{h}, T_\mathrm{c}), \tag{16}$$

$$\boldsymbol{\sigma} = \boldsymbol{K}\Big(\varepsilon(u) - A_1(a, m, \theta)\boldsymbol{I} - \int_0^t \gamma(a, m, a_t, m_t, \theta)\mathbf{S}\,\mathrm{d}\tau\Big), \tag{17}$$

$$u = 0 \text{ on } \Gamma_0 \times (T_\mathrm{h}, T_\mathrm{c}), \tag{18}$$

$$\boldsymbol{\sigma} \cdot n = 0 \text{ on } (\partial\Omega^\mathrm{s} \setminus \Gamma_0) \times (T_\mathrm{h}, T_\mathrm{c}), \tag{19}$$

$$a_t = \frac{1}{\tau_a(\theta)}(a_{\mathrm{eq}}(\theta) - a)\mathcal{H}(\theta - A_s) \text{ in } \Omega^\mathrm{s} \times (T_\mathrm{h}, T_\mathrm{c}), \tag{20}$$

$$a(T_\mathrm{h}) = a_{T_\mathrm{h}} \text{ in } \Omega^\mathrm{s}, \tag{21}$$

$$m_t = c_m(1 - m)\mathcal{H}(-\theta_t)\mathcal{H}(M_s - \theta) \text{ in } \Omega^\mathrm{s} \times (T_\mathrm{h}, T_\mathrm{c}), \tag{22}$$

$$m(T_\mathrm{h}) = 0 \text{ in } \Omega^\mathrm{s}, \tag{23}$$

$$\begin{aligned} &\alpha(\theta, a, m, \boldsymbol{\sigma})\theta_t - \nabla \cdot (\kappa(\theta)\nabla\theta) \\ &\quad + 3\bar{\kappa}q(a, m)\theta\,(\nabla \cdot u_t - 3A_2(a_t, m_t, \theta)) \\ &= -\rho L_a a_t + \rho L_m m_t + A_2(a_t, m_t, \theta)\,\mathrm{tr}\,\boldsymbol{\sigma} \\ &\quad + \gamma(a, m, a_t, m_t, \theta)|\mathbf{S}|^2 \text{ in } \Omega \times (T_\mathrm{h}, T_\mathrm{c}), \end{aligned} \tag{24}$$

$$\frac{\partial\theta}{\theta n} = \beta(x, t)(\theta - \theta_\mathrm{e}) \text{ on } \partial\Omega \times (T_\mathrm{h}, T_\mathrm{c}), \tag{25}$$

$$\theta(T_\mathrm{h}) = \theta_{T_\mathrm{h}} \text{ in } \Omega. \tag{26}$$

In (22) $c_m > 0$ is a constant value. Also, in (24), $L_m > 0$ is the latent heat related to the martensite phase fraction. The function $\beta(x, t)$ in (25) is a heat transfer coefficient and is given by

$$\beta(x, t) = \begin{cases} 0 & \text{on } \partial\Omega \cap \partial\Omega^\mathrm{c}, \\ \beta_0(t) & \text{on } \partial\Omega \cap \partial\Omega^\mathrm{s}. \end{cases}$$

where $\beta_0(t) > 0$ (usually taken to be constant). Finally, θ_e is the temperature of the quenchant.

The mathematical analysis of a system similar to (16)-(26) can be seen in [3]. In this reference, an existence result is shown assuming that the data are smooth enough and $T_\mathrm{c} - T_\mathrm{h}$ is sufficiently small.

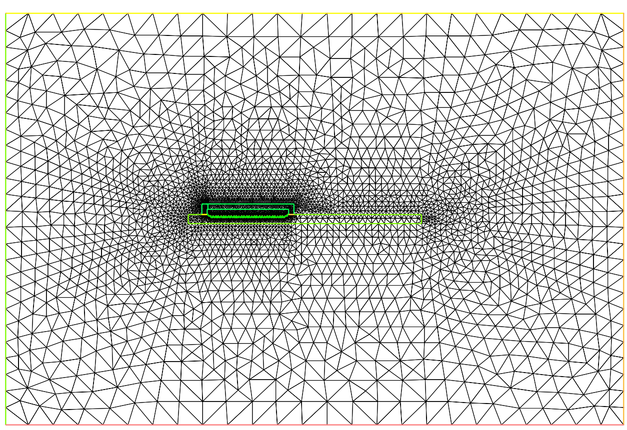


Figure 6: Domain triangulation. The mesh contains 61790 triangles and 30946 vertices.

3 Numerical simulation

Using the Freefem++ package ([8]), we have performed some numerical simulations for the approximation of the solution to the systems (2)-(15) and (16)-(26). We want to describe the hardening treatment of a car steering rack during the heating-cooling process. The goal is to produce martensite along the tooth line together with a thin layer in its neighborhood inside the steel workpiece ([5, 6]).

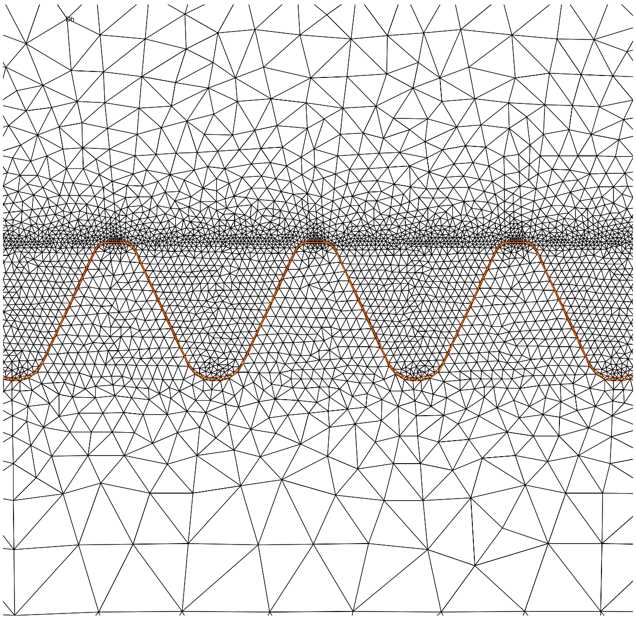


Figure 7: Domain triangulation. Element density near three teeth.

Figure 4 shows the open sets D, $\Omega = \Omega^s \cup \Omega^c \cup S$ and the faces Γ_1 and Γ_2 (they appear stick together in this figure) which intervene in the setting of the problem. The workpiece contains a toothed part to be hardened by means of the heating-cooling process described above. It is made of a hypoeutectoid steel. The open set $D \setminus \bar{\Omega}$ is air. The magnetic permeability μ in (5) is then given by

$$\mu(x) = \begin{cases} \mu_0 & \text{if } x \in D \setminus \bar{\Omega}, \\ 0.99995\mu_0 & \text{if } x \in \Omega^c, \\ 2.24 \times 10^3 \mu_0 & \text{if } x \in \Omega^s, \end{cases}$$

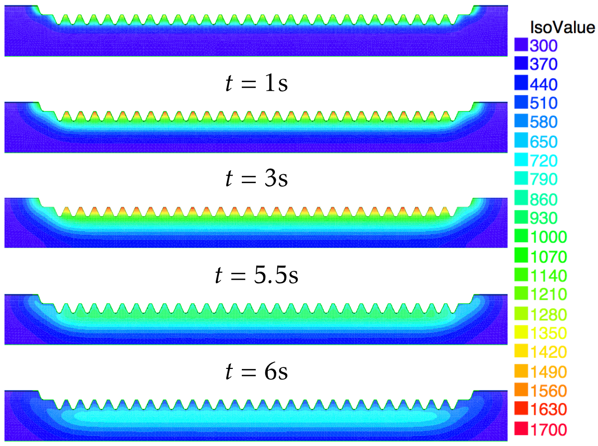
where $\mu_0 = 4\pi \times 10^{-7}$ (N/A^2) is the magnetic constant (vacuum permeability).

The martensite phase can only derive from the austenite phase. Thus we need to transform first the critical part to be hardened (the tooth line) into austenite. For our hypoeutectoid steel, austenite only exists in a temperature range close to the interval [1050, 1670] (in K). During the first stage, the workpiece is heated up by conduction and induction (Joule's heating) which renders the tooth line up to the desired temperature. In order to transform the austenite into martensite, we must cool it down at a very high rate. This second stage is accomplished by aquaquenching.

In this simulation, the final time of the heating process is $T_h = 5.5$ seconds and the cooling process extends also for 5.5 seconds, that is $T_c = 11$.

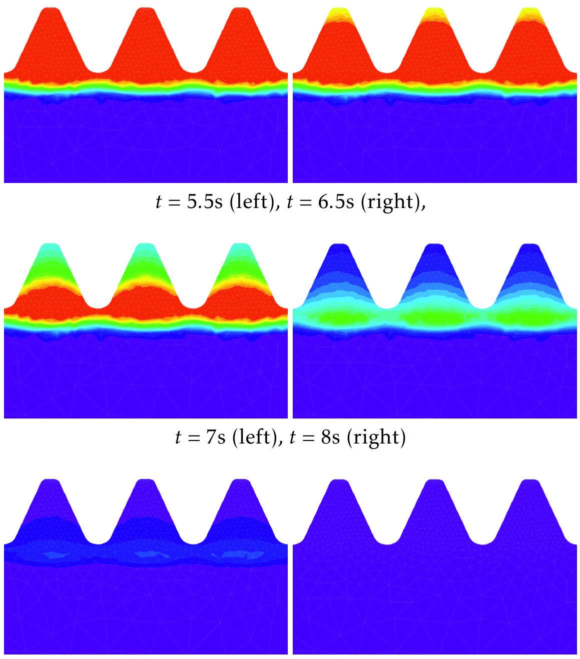
We have used the finite elements method for the space approximation and a Crank-Nicolson scheme for the time discretization. Figures 6 and 7 show the triangulation of D in our numerical simulations. We have used P_2-Lagrange approximation for φ, $\boldsymbol{A}$ and θ and P_1 for a and m.

In Figure 8 we can see the temperature distribution of the rack along the tooth line at different instants of the the heating-cooling process. The initial temperature is $\theta_0 = 300$K. At $t = 5.5$ the heating process ends and the computed temperature shows that the temperature along the rack tooth line lies in the interval [1050, 1670] (K).



$t = 7$s

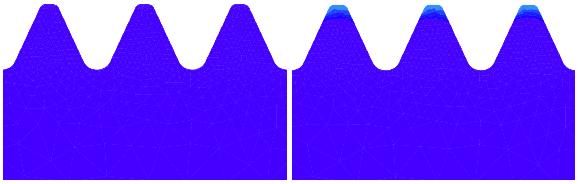
Figure 8: Temperature evolution at instants $t = 1$, $t = 3$, $t = 5.5$ (end of the heating stage, aqua-quenching begins), $t = 6$ and $t = 7$ seconds, respectively. At $t = 5.5$s the temperature along the tooth part has reached the austenization level in this part of the rack. The temperature is measured in Kelvin.



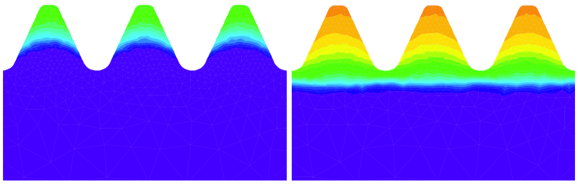
$t = 9$s (left) and $t = 11$s (right).

Figure 9: Transformation of the austenite phase fraction during the aquaquenching at time instants t=5.5, 6.5, 7, 8, 9, and 11 seconds, respectively.

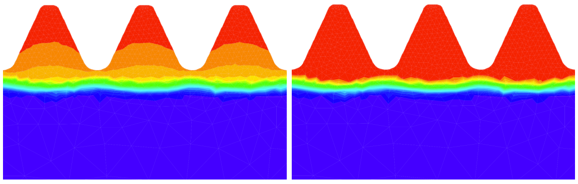
Figure 9 shows the austenite evolution from the beginning of the cooling stage. Blue corresponds to 0% while red is 100%. We observe that martensite starts to appear, approximatively, one second after the beginning of the cooling stage. At the final instant, all the amount of austenite has been transformed into martensite as it is shown in Figure 10 .



$t = 5.5$s (left), $t = 6.5$s (right),



$t = 7$s (left), $t = 8$s (right)



$t = 9$s (left) and $t = 11$s (right).

Figure 10: Transformation of the martensite phase fraction from austenite during the aquaquenching at time instants t=5.5, 6.5, 7, 8, 9, and 11 seconds, respectively.

Figure 11 shows the austenization along the tooth line at the end of the heating process $T = 5.5$ seconds.

Figure 12 shows the final distribution of martensite from austenite along the rack tooth line through the cooling stage $t = 11$ seconds. We have good agreement versus the experimental results obtained in the industrial process.

During the heating-cooling process, the workpiece is deformed so that an industrial rectification is needed (or otherwise the rack would be useless). Figures 13 and 14 shows the different deformations undergone by the workpiece.

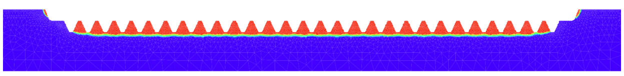


Figure 11: Heating process. Austenite at $t = 5.5$ along the rack tooth line.

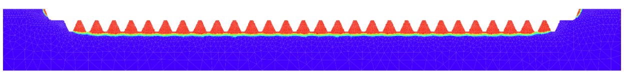


Figure 12: Cooling process. Martensite transformation at the final stage of the cooling process $t = 11$ seconds.

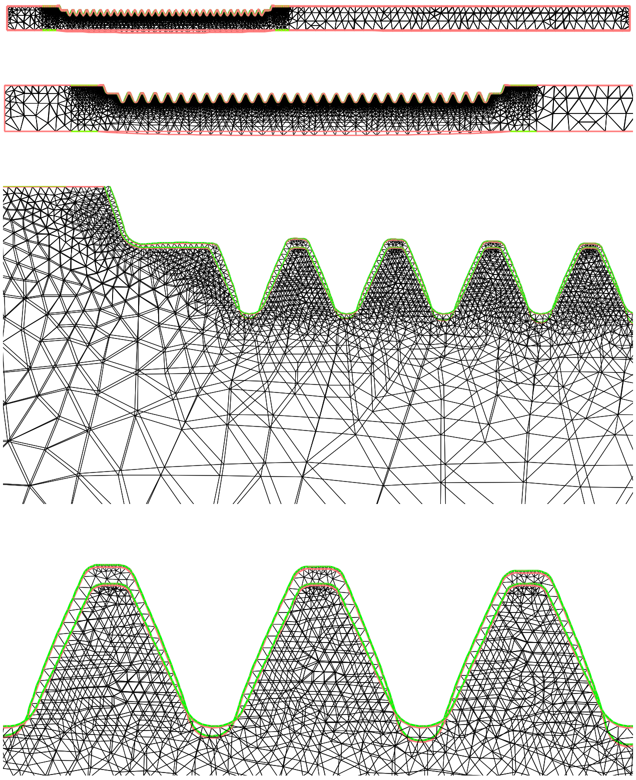


Figure 13: Distorted mesh (with a scale factor of 10) after the heating stage. The austenite transformation along the tooth line changes the original profile.

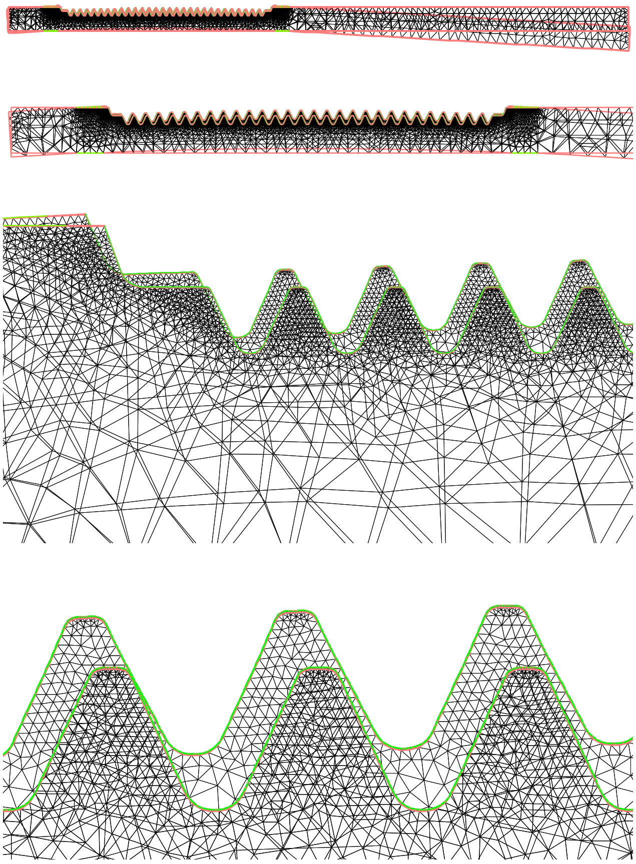


Figure 14: Distorted mesh (with a scale factor of 10) after the cooling stage. The original configuration is partially recovered. Due to the plasticity effect and the lack of the upper supports, the rack bends down.

Conflict of Interest The authors declare no conflict of interest.

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References

[1] A. Bermúdez, J. Bullón, F. Pena and P. Salgado, "A numerical method for transient simulation of metallurgical compound electrodes", Finite Elem. Anal. Des., **39**, 283–299, 2003. https://doi.org/10.1016/S0168-874X(02)00069-0

[2] A. Bermúdez, D. Gómez, M. C. Muñiz and P. Salgado, "Transient numerical simulation of a thermoelectrical problem in cylindrical induction heating furnaces", Adv. Comput. Math., **26**, 39–62, 2007. https://doi.org/10.1007/s10444-005-7470-9

[3] K. Chełminski, D. Hömberg and D. Kern, "On a thermomechanical model of phase transitions in steel", Adv. Math. Sci. Appl., 18, 119–140, 2008.

[4] J. R. Davis *et al.* "ASM Handbook: Heat Treating", vol. **4**, ASM International, USA, 2007.

[5] J. M. Díaz Moreno, C. García Vázquez, M. T. González Montesinos and F. Ortegón Gallego, "Numerical simulation of a Induction-Conduction Model Arising in Steel Hardening model arising in steel hardening", Lecture Notes in Engineering and Computer Science, World Congress on Engineering 2009, Volume II, July 2009, 1251–1255.

[6] J. M. Díaz Moreno, C. García Vázquez, M. T. González Montesinos, F. Ortegón Gallego and G. Viglialoro, "Mathematical modeling of heat treatment for a steering rack including mechanical effects", J. Numer. Math. 20, no. 3-4, 215–231, 2012. https://doi.org/10.1515/jnum-2012-0011

[7] J. Fuhrmann, D. Hömberg and M. Uhle, "Numerical simulation of induction hardening of steel", COMPEL, **18**, No. 3, 482–493, 1999. https://doi.org/10.1108/03321649910275161

[8] F. Hecht, "New development in freeFem++", J. Numer. Math. 20, no. 3-4, 251–265, 2012. https://doi.org/10.1515/jnum-2012-0013

[9] D. Hömberg, "A mathematical model for induction hardening including mechanical effects", Nonlinear Anal.-Real World Appl., **5**, 55–90, 2004. https://doi.org/10.1016/S1468-1218(03)00017-8

[10] D. Hömberg and W. Weiss, "PID control of laser surface hardening of steel", IEEE Trans. Control Syst. Technol., **14**, No. 5, 896–904, 2006. https://doi.org/10.1109/TCST.2006.879978

[11] D. P. Koistinen, R. E. Marburger, "A general equation prescribing the extent of the austenite-martensite transformation in pure iron-carbon alloys and plain carbon steels", Acta Metall., **7**, 59–60, 1959. https://doi.org/10.1016/0001-6160(59)90170-1

[12] G. Krauss, "Steels: Heat Treatment and Processing Principles", ASM International, USA, 2000.

[13] J. B. Leblond and J. Devaux, "A new kinetic model for anisothermal metallurgical transformations in steels including effect of austenite grain size", Acta Metall., **32**, No. 1, 137–146, 1984. https://doi.org/10.1016/0001-6160(84)90211-6

[14] F. J. Pena Brage, "Contribución al modelado matemático de algunos problemas en la metalurgia del silicio", Ph. thesis, Universidade de Santiago de Compostela, 2003.

[15] H. M. Yin, "Regularity of weak solution to Maxwell's equations and applications to microwave heating", J. Differ. Equ., **200**, 137-161, 2004. https://doi.org/10.1016/j.jde.2004.01.010

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Petrov-Galerkin formulation for compressible Euler and Navier-Stokes equations

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The resolution of the Navier-Stokes and Euler equations by the finite element method is the focus of this paper. These equations are solved in conservative form using, as unknown variables, the so-called conservative variables (density, momentum per unit volume and total energy per unit volume). The variational formulation developed is a variant of the Petrov-Galerkin method. The nonlinear system is solved by the iterative GMRES algorithm with diagonal pre-conditioning. Several simulations were carried out, in order to validate the proposed methods and the software developed.

1. Introduction

The mathematical formulation of the conservation laws of mass, momentum and energy is the first step in the modelling of fluid flow problems by the finite element method. Thus, we obtain the Navier-Stokes equations, convection-diffusion transport equations, and the Euler equations, convection transport equations. In this first step of resolution, it is advisable to make a judicious choice of the form and the independent variables. Indeed, there are several possible choices [1], mainly: the non-conservative form, the conservative form in entropic variables and the conservative form. Physically, the conservative formulation is best suited to the fact that equations fluid dynamics are solved by perfectly respecting the laws of physics; in addition the boundary conditions are directly imposed on the physical variables.

The finite element method is one of the most powerful numerical methods conceived to date. The characteristics that have led to its popularity include: ease of modelling complex shape geometries, natural processing of differential-type boundary conditions, and the possibility of being programmed in the form of software adaptable to the processing of a wide range of problems. The classical formulation of the finite element method is based on the Galerkin weighted residual method. Galerkin's formulation has proved to be extremely effective in applying structural mechanics problems and in other situations such as problems governed by diffusion equations. The reason for this success is that in application to problems governed by elliptic or parabolic partial differential equations, Galerkin's finite element method leads to symmetric stiffness matrices. The advantages of the Galerkin's finite element method in solving the problems of structural mechanics and diffusion transport are not directly exploitable for the modelling of fluid flow problems, particularly in the simulation of transport phenomena dominated by convection. A major difficulty arose because of the presence of convective terms in the mathematical formulation when using a kinematic description other than the Lagrangian description as this is practically always the case in fluid mechanics. In practice, Galerkin's numerical solutions for dominant convective problems are frequently polluted by non-physical oscillations which can only be eliminated by considerably refining the mesh, or in the case of transient greatly reducing the time step. This, of course, compromises the very usefulness of the Galerkin method and has motivated the development of alternative methods that exclude the presence of non-physical oscillations independently of any mesh refinement or time step [2-6].

This article proposes elements of answer to these points. The objective analysis of the choice of form and variables independent of convection-diffusion transport equations; justifies our choice of the conservative form using conservative variables to solve the Navier-Stokes and Euler equations. The variational form used is based on Galerkin's weighted residual method. For the flow problems of dominant convection or pure convection fluids, the variational form used is based on a Petrov-Galerkin type formulation. Emphasis is placed on the development and implementation of these two types of variational formulations.

2. Mathematical formulation

We recall in this section the equations of the dynamics of compressible and viscous fluids. These equations express conservation laws of mass, momentum and energy. By identifying conservative variables, namely the density ρ, the momentum per unit volume U, and the total energy per unit volume E, as the

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independent variables, the Navier-Stokes equations, in conservative form and dimensionless, are written:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (U) = 0 = 0 \tag{1a}$$

$$\frac{\partial U}{\partial t} + \nabla \cdot (u \otimes U) + \nabla p = \nabla \cdot \underline{\sigma} + f \tag{1b}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E+p)u] = \nabla \cdot (\underline{\sigma} \cdot u) - \nabla \cdot q + r + f \cdot U \tag{1c}$$

where u, p, q, r, f and $\underline{\sigma}$ represent respectively: velocity, pressure, heat flux, energy source, external force and tensor of viscous stresses. The heat flux and the pressure are expressed, in non-dimensional form, as a function of the temperature and of the density by the Fourier law and the equation of a perfect gas, respectively:

$$q = -\frac{\gamma}{Re\,Pr} \nabla T \tag{2}$$

and

$$p = (\gamma - 1)\rho T \tag{3}$$

where Re, Pe, γ and T represent respectively: Reynolds number, Prandlt number, specific heat ratio and temperature. On the other hand, the tensor of viscous stresses $\underline{\sigma}$ is given by the relation:

$$\underline{\sigma}(\mathbf{u}) = \frac{1}{Re}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^t - \frac{2}{3}\mu(\nabla \cdot \mathbf{u})\underline{I}\right] \tag{4}$$

where $\underline{I}$ is the identity matrix. Temperature and velocity are related to conservative variables by the following relationships:

$$E = \rho\left(T + \tfrac{1}{2}|\mathbf{u}|^2\right) \tag{5}$$

$$\mathbf{u} = \frac{\mathbf{U}}{\rho} \tag{6}$$

The substitution of equation (6) in equation (4) decomposes the viscous stress tensor in the following form:

$$\underline{\sigma}(\mathbf{u}) = \frac{1}{\rho}\underline{\sigma}(\mathbf{U}) + \underline{\sigma}^{\ddagger} \tag{7}$$

where

$$\underline{\sigma}^{\ddagger} = -\frac{\mu}{Re\,\rho^2}\left[\mathbf{U} \cdot (\nabla \rho)^t + (\nabla \rho) \cdot \mathbf{U}^t - \frac{2}{3}(\mathbf{U}^t \cdot \nabla \rho)\underline{I}\right]$$

On the other hand, the substitution of equations (5) and (6) in equations (2) and (3) makes it possible to express the heat flux and the pressure, as a function of conservative variables (ρ, U and E), respectively by the following relationships:

$$\mathbf{q} = -\frac{\gamma}{Re\,Pr} \nabla \left[\frac{E}{\rho} - \frac{|\mathbf{U}|^2}{2\rho^2}\right] \tag{8}$$

and

$$p = (\gamma - 1)\left[E - \frac{|\mathbf{U}|^2}{2\rho}\right] \tag{9}$$

Equations (1) are strongly nonlinear, their mathematical study is a great challenge and their numerical resolution is not easy. The variational formulation as well as the finite element approximation must respect, inter alia, the implicit condition that temperature, pressure and density must be positive.

Let V be the vector of conservative variables:

$$\mathbf{V} = (\rho, \mathbf{U}, E)^t$$

the system of equations (1) can be written in the vector form:

$$\mathbf{V}_{,t} + \mathbf{F}^{\mathrm{conv}}_{\mathrm{i,i}}(\mathbf{V}) = \mathbf{F}^{\mathrm{diff}}_{\mathrm{i,i}}(\mathbf{V}) + \boldsymbol{\mathcal{F}} \tag{10}$$

where $F^{\mathrm{conv}}_{\mathrm{i}}(V)$ and $F^{\mathrm{diff}}_{\mathrm{i}}(V)$ are respectively the convection and diffusion fluxes in the i direction, and $\mathcal{F}$ is the source vector. It is also interesting to write the system (1) in the quasi-linear form:

$$\mathbf{V}_{,t} + \mathbf{A}_{\mathrm{i}}\mathbf{V}_{,\mathrm{i}} = \left(\mathbf{K}_{\mathrm{ij}}\mathbf{V}_{,\mathrm{j}}\right)_{,\mathrm{i}} + \boldsymbol{\mathcal{F}} \tag{11}$$

with A_i the Jacobian matrices of the convection fluxes such that $A_i = F^{\mathrm{conv}}_{i,V}$ and K_{ij} the diffusion matrices such as

$$K_{ij}V_{,j} = F^{\mathrm{diff}}_{i,i}.$$

The flow problems of non-viscous fluids, governed by the Euler equations, are a special case of the flow of viscous fluids. To find the Euler equations, we suppress the viscous terms in the Navier-Stokes equations.

$$F^{\mathrm{diff}}_{i,i}(V) \quad \text{or} \quad \left(K_{ij}V_{,j}\right)_{,i}$$

The Euler equations can be written respectively in the vector form and the quasi-linear form, as follows:

$$V_{,t} + F^{\mathrm{conv}}_{i,i}(V) = \mathcal{F} \qquad \text{and} \quad V_{,t} + A_iV_{,i} = \mathcal{F}$$

2.1. Boundary and Initial Conditions

We consider external and internal flows, the domain of calculation is denoted Ω of boundary Γ.

Inlet boundary condition Γ^-_∞

The flow is considered uniform:

$$\rho = \rho_\infty = 1, \qquad U_\infty = \begin{pmatrix}\cos\beta \\ \sin\beta\end{pmatrix}, \qquad E_\infty = \frac{1}{2} + \frac{1}{\gamma(\gamma-1)}\frac{1}{M^2_\infty}$$

where β is the angle of attack and M_∞ is the Mach number at infinity.

Wall boundary condition Γ_B

The condition of adhesion ($U = 0$) is imposed in the case of viscous fluid flow problems and a sliding condition ($U \cdot n = 0$) in the case of flow problems of non-viscous fluids. In addition, one imposes either a natural condition of Neumann type on the temperature (adiabatic wall): $q \cdot n = 0$, or a uniform distribution of the temperature T_B in order to represent a thermal equilibrium condition to the wall; which amounts to imposing the Dirichlet condition: $E = \rho T_B$. In the case of a uniform temperature distribution, for Mach numbers less than 3, we take T_B equal to the stagnation temperature:

$$T_B = T_\infty\left(1 + \frac{\gamma - 1}{2}\frac{1}{M^2_\infty}\right)$$

Since ρ is unknown, the boundary condition $E = \rho T_B$ is non-linear. During the resolution, the value of the energy at the wall must therefore be continuously updated as a function of the density.

Outlet boundary condition Γ_∞^+

Two cases must be distinguished. If the local Mach number is less than unity, subsonic regime, then at least one condition at the Dirichlet boundary (on density or on pressure or on total energy) is required. If the local Mach number is greater than unity, supersonic regime, then no boundary condition is necessary.

The initial conditions are:

$$\rho(x,0) = \rho_0(x), \quad U(x,0) = U_0(x), \quad E(x,0) = E_0(x)$$

3. Galerkin formulation

Let Ω be a bounded one of the Euclidean space $\mathbb{R}^n$, of boundary Γ. In order to determine a solution numerically, we cancel the projection of the system of equations (1) onto a set of weighting functions according to the Galerkin method. The weighting functions of the conservative variables (ρ, U, E) are denoted by ψ^ρ, ψ^U and ψ^E, respectively. By multiplying the equations (1) in turn by weighting functions, we have:

$$\left(\psi^\rho, \frac{\partial \rho}{\partial t}\right) + \left(\psi^\rho, \nabla \cdot (U)\right) = 0 \tag{12a}$$

$$\left(\psi^U, \frac{\partial U}{\partial t}\right) + \left(\psi^U, \nabla \cdot (u \otimes U)\right) + (\psi^U, \nabla p) + (\psi^U, U, \rho) = (\psi^U, f) + \langle \psi^U, \underline{\sigma} \rangle \tag{12b}$$

$$\left(\psi^E, \frac{\partial E}{\partial t}\right) + (\psi^E, \nabla \cdot [(E+p)u]) + \left(\nabla \psi^E, (\underline{\sigma} \cdot u)\right) - (\nabla \psi^E, q) = (\psi^E, r + f \cdot U) - \langle \psi^E, q \rangle + \langle \psi^E, (\underline{\sigma} \cdot u) \rangle \tag{12c}$$

where $(,)$ denotes the scalar product in $L^2(\Omega)$, and:

$$(\psi^U, U, \rho) = \int_\Omega \nabla \psi^U : \underline{\sigma} \, d\Omega$$

$$\langle \psi^U, \underline{\sigma} \rangle = \int_\Gamma \psi^U (\underline{\sigma} \cdot n) \, d\Gamma$$

$$\langle \psi^E, q \rangle = \int_\Gamma \psi^E (q \cdot n) \, d\Gamma$$

The integration by parts of the terms of diffusion makes it possible to reduce the derivation order from two to one. The appearance of contour integrals can be used to impose natural boundary conditions such as adiabatic wall conditions and sliding conditions. Using vector notation (10), the variational formulation (12) is written as follows:

$$\int_\Omega \left[W \cdot \left(V_{,t} + F_{i,i}^{conv}(V) - \mathcal{F}\right) + W_{,i} F_i^{diff} \right] d\Omega - \int_\Gamma W \cdot \left(F_i^{diff} n_i\right) d\Gamma = 0 \tag{13}$$

with $W = (\psi^\rho, \psi^U, \psi^E)^t$ the vector of the weighting functions. Also, using the quasi-linear form (11), we establish the following variational formulation:

$$\int_\Omega \left[W \cdot \left(V_{,t} + A_i V_{,i} - \mathcal{F}\right) + W_{,i} \left(K_{ij} V_{,j}\right)_{,i} \right] d\Omega - \int_\Gamma W \cdot \left(F_i^{diff} n_i\right) d\Gamma = 0$$

4. Petrov-Galerkin formulation

4.1. Artificial Viscosity

The method of artificial viscosity used consists in replacing the dynamic viscosity μ by:

$$\mu^* = \mu + \mu_{art}$$

where μ_{art} is defined in a similar way as artificial diffusion widely used in finite difference schemes:

$$\mu_{art} = \frac{\|u\| h}{2} \zeta(R_h) C_a$$

where h is the size of the element, C_a is the weighting coefficient of the viscosity and R_h is the local Reynolds number:

$$R_h = \frac{\|u\| h}{2\mu}$$

The function $\zeta(R_h)$ is particularly sensitive to zones with dominant convection; it is defined as follows:

$$\zeta(R_h) = \min\left(\frac{R_h}{3}, 1\right)$$

Thus, the modified system of equations includes an artificial viscosity term whose importance depends on the function $\zeta(R_h)$ and the weighting coefficient C_a. By experience, it is necessary to increase the viscosity in the zones with dominant convection (ie local Reynolds number is large). The effect of this artificial viscosity is to create additional energy dissipation, thus guaranteeing the positivity of temperature, density and pressure. This method has the advantage of being easy to implement and does not require any additional calculations, however it has the disadvantage of containing an arbitrary coefficient C_a whose value influences the quality of the solution. We essentially use this method to quickly obtain a solution, although diffusive, converges towards the final solution. This solution subsequently serves to initialize the stabilized formulation by Petrov-Galerkin finite element method which we describe in the following section.

4.2. SUPG Method

The Petrov-Galerkin formulation that we use is a variant of the Streamline Upwind Petrov-Galerkin (SUPG) method [2-6]. The SUPG method is based on a Petrov-Galerkin formulation, this method is widely used for the resolution of convection-diffusion transport equations. As already mentioned in the introduction, when convection dominates, the use of the Galerkin method generates non-physical oscillations which can only be eliminated by considerably refining the mesh, or by greatly reducing the time step in the transient case. The SUPG method consists in adding an additional perturbation term to the Galerkin standard formulation. On the other hand, the use of the Petrov-Galerkin formulation amounts to modifying the weighting functions of the Galerkin type by weighting functions having more weight upstream; thus introducing artificial diffusion in the direction of the flow. The variational formulation (13) is modified as follows:

$$\int_{\Omega}\left[W\cdot\left(V_{,t}+F_{i,i}^{conv}(V)-\mathcal{F}\right)+W_{,i}F_i^{diff}\right]d\Omega$$
$$-\int_{\Gamma}W\cdot\left(F_i^{diff}n_i\right)d\Gamma=0$$
$$+\sum_e\int_{\Omega^e}\left(A_i^t\cdot W_{,i}\right)\underline{\tau}\left[V_{,t}+F_{i,i}^{conv}-\left(K_{ij}V_{,j}\right)_{,i}-\mathcal{F}\right]d\Omega$$

where Ω^e is an element of the triangulation of the computational domain Ω. The matrix $\underline{\tau}$, for the conservative form using the conservative variables, is written in the following form:

$$\underline{\tau}=\left[\sum_i\left|c_{ij}A_j\right|\right]^{-1}\zeta(Pe) \qquad (14)$$

In equation (14), the coefficients c_{ij} represent the elements of the Jacobian matrix of the geometric transformation of the real element to the reference element. The operator $|\,|$ applied to a matrix $\underline{B}$ defines its absolute value $|\underline{B}|$ by: $|\underline{B}|=S|\Lambda|S$, where S and Λ are respectively the matrices of the eigenvectors and the eigenvalues of $\underline{B}$. In order to compute $\underline{\tau}$ from equation (14) we must obtain the absolute values of matices $\underline{B}_i=\left|c_{ij}A_j\right|$. The definition of the matrix $\underline{\tau}$ therefore requires the solving of a problem with eigenvalues. It should also be recalled that the Jacobian matrices A_i and $\widetilde{A}_i$ of the convection flux in direction i with respect to the conservative and non-conservative variables respectively are connected by the following relation:

$$A_i=M\widetilde{A}_iM^{-1}$$

with

$$M=\frac{\partial V}{\partial\widetilde{V}}$$

In two dimensions, the vectors of conservative and non-conservative variables are written respectively:

$$V=\begin{bmatrix}\rho\\ \rho u_1\\ \rho u_2\\ E\end{bmatrix} \qquad \widetilde{V}=\begin{bmatrix}\rho\\ u_1\\ u_2\\ p\end{bmatrix}$$

Analytically, the matrices M and M^{-1} can be obtained in the two-dimensional case

$$M=\begin{pmatrix}1&0&0&0\\ u_1&\rho&0&0\\ u_2&0&\rho&0\\ u^2/2&\rho u_1&\rho u_2&1/(\gamma-1)\end{pmatrix}$$

and

$$M^{-1}=\begin{pmatrix}1&0&0&0\\ -u_1/\rho&1/\rho&0&0\\ -u_2/\rho&0&1/\rho&0\\ \frac{(\gamma-1)}{2}u^2&(1-\gamma)u_1&(1-\gamma)u_2&(\gamma-1)\end{pmatrix}$$

with $u^2=u_1^2+u_2^2$. The expressions of $\widetilde{A}_1$ and $\widetilde{A}_2$ are given in the two-dimensional case by:

$$\widetilde{A}_1=\begin{pmatrix}u_1&\rho&0&0\\ 0&u_1&0&1/\rho\\ 0&0&u_1&0\\ 0&\rho c^2&0&u_1\end{pmatrix}$$

and

$$\widetilde{A}_2=\begin{pmatrix}u_2&0&\rho&0\\ 0&u_2&0&0\\ 0&0&u_2&1/\rho\\ 0&0&\rho c^2&u_2\end{pmatrix}$$

with $c=\sqrt{\gamma p/\rho}$ the velocity of sound.

5. Resolution algorithm

The spatial-temporal discretization of the variational problem leads to the solution of a system of algebraic equations of the form:

$$[M]\{V_{,t}\}+[K(V)]\{V\}=\{\mathcal{F}\}$$

This system of equations has several types of non-linearities. Some are due to convective terms, others are related to the compressibility of the fluid. In addition, the use of the Petrov-Galerkin method introduces strong non-linearities. Given the set of nonlinear equations, stability of the solution and convergence, the use of a robust algorithm is therefore crucial. Given its convergence properties, the Newton-Raphson algorithm coupled to the chosen implicit time scheme can be effective in solving this problem. However, in the present situation, the implementation of the Newton-Raphson method requires the calculation of the first variations of all the functional ones present in the formulation. These exact analytical expressions are difficult to obtain, especially in the presence of a stabilization method. To solve the problem, we use a variant of the GMRES algorithm [7]. To accelerate the convergence of the GMRES algorithm, pre-conditioning techniques are used.

6. Numerical results

A first series of numerical tests concerning external flows of viscous and non-viscous fluids around a NACA0012 profile was carried out to validate the methods described above. The mesh used comprises 8150 elements. This mesh is illustrated in figure 1, with an enlargement around the profile NACA0012 in figure 2.

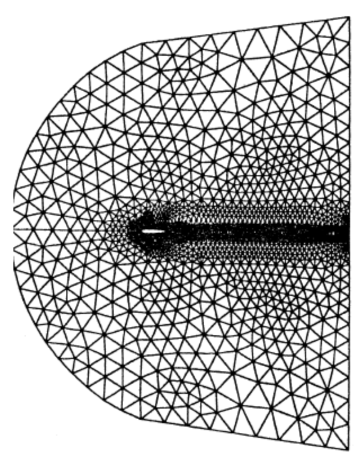

Figure 1: Mesh over the entire domain

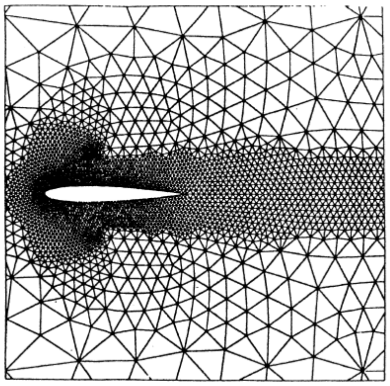

Figure 2: Mesh around profile NAC0012

Viscous flow

The first problem presented is a viscous flow at $M = 0.85$ and $Re = 2000$ with a zero angle of attack. The initial solution used is a uniform field except at the wall. The resolution strategy consists in fixing the Reynolds and Mach numbers to their maximum values and to solve the problem over 200 time steps ($\Delta t = 0.1$) with an artificial viscosity ($Ca = 1$). The resulting solution serves as the initial field for the computation sequence in which the artificial viscosity is canceled by imposing ($Ca = 0$), and the SUPG method is applied until converges of the temporal scheme. The iso-Mach and iso-density curves are shown in figures. 3a and 3b. The pressure coefficients $C_p = 2(p - p_\infty)/(\rho u^2)$ on the profile are compared in figure 3c with those obtained by a finite element method using non-conservative variables [8].

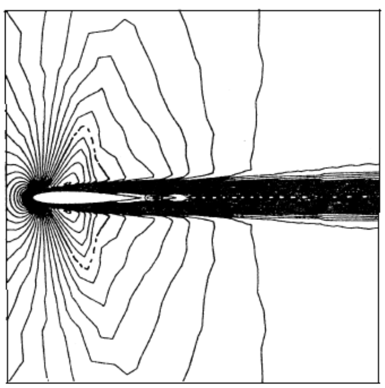

Figure 3a: Iso-Mach lines

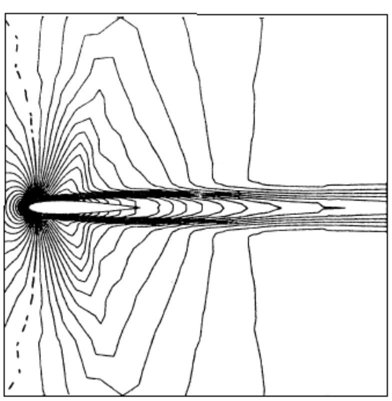

Figure 3b: Iso-density lines

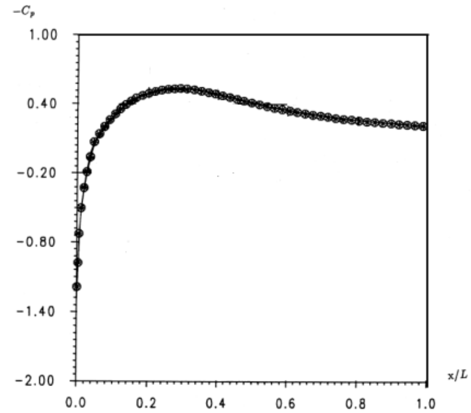

Figure 3c: Pressure coefficient

Non-viscous flow

The Euler equations are now solved for $M = 0.80$ and for an angle of attack of $\beta = 1.25$ degrees; these conditions are similar to those defined in the [6, 9]. The initial solution used is a uniform field except at the wall where we impose the normal component of the velocity at zero. The resolution strategy consists in using the SUPG method from the beginning of the resolution to the convergence in time. The mesh is identical to that shown above. The iso-Mach and iso-pressure curves are shown in FIGS. 4a and 4b. The presence of two zones of low pressure followed by normal shocks is noted, as can be expected conventionally in transonic flow. FIG. 4c shows the evolution of the pressure coefficient along the profile, these results are compared with those obtained by [6, 10] with Roe schemes in finite volumes. We note that for our results the shocks are well defined on the intrados as on the extrados. Obviously, the resolution of shocks can be further improved by using mesh adaptation methods.

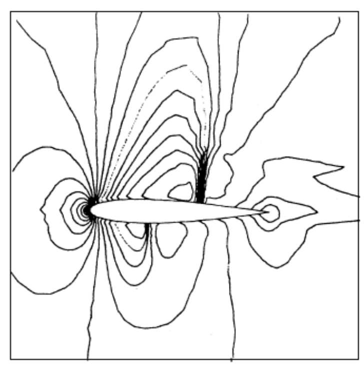

Figure 4a: Iso-Mach lines

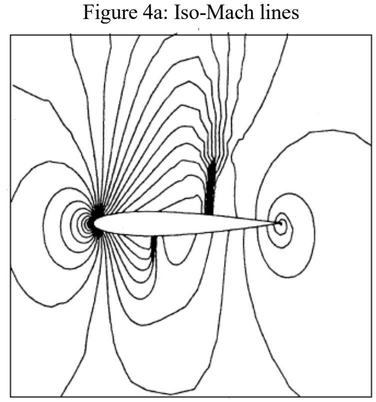

Figure 4b: Iso-pressure lines

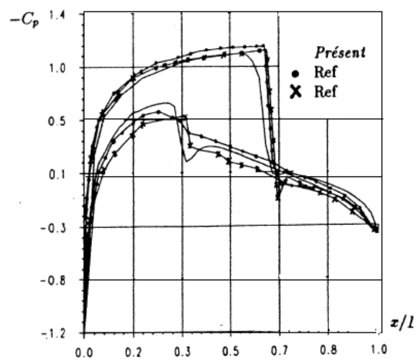

Figure 4c: pressure coefficient

The same problem is then dealt with for an angle of attack of β = 8.34 degrees. The iso-Mach, iso-pressure curves and the distribution of the pressure coefficient for this test case are presented in figures 4d, 4e and 4f. The presence of a strong shock on the extrados is noted.

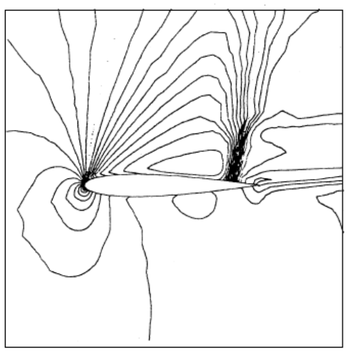

Figure 4d: iso-Mach lines

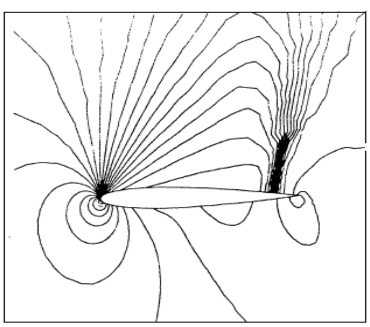

Figure 4e: iso-pressure lines

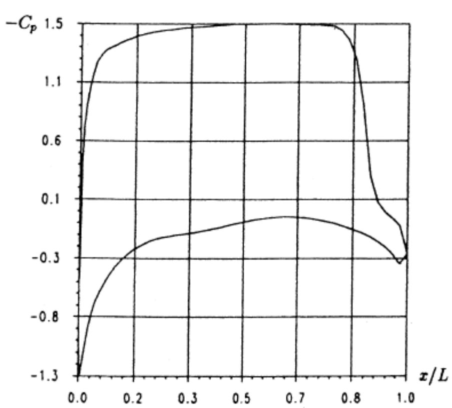

Figure 4f: pressure coefficient

Flow around a half cylinder

The simulation of this flow is aimed at validating the methods described above in the case of supersonic flows. The test consists of simulating the flow around a half cylinder at a Mach number of M = 3.0 without angle of incidence. The calculation domain and the boundary conditions used are illustrated in figure 5a. The resolution strategy is identical to that used in the previous example. In addition, the mesh has been refined during the resolution. The final mesh consists of 15 148 elements. The iso-pressure is shown in figure 5b, this figure is very similar to that presented by [11]. A section of the iso-Machs at the line of symmetry is shown in figure 5c.

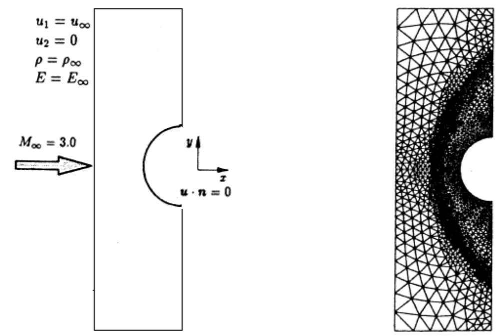

Figure 5a: Meshing around a half cylinder

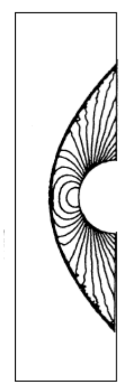

Figure 5b: Iso-pressure lines

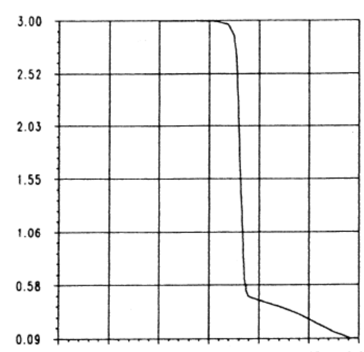

Figure 5c: iso-Mach section at the line of symmetry

7. Conclusion

We have presented a finite element method for solving the Navier-Stokes and Euler equations using conservative variables and a Petrov-Galerkin variational formulation. Particular emphasis was placed on the effective definition and implementation of the Stabilization Matrix. The use of the pre-conditioned GMRES algorithm with an adequate resolution strategy allows a relatively fast convergence. The numerical results obtained show that the present finite element method gives results at least as good as those obtained with other methods. However, it seems imperative for shock problems to use a conservative method; the use of conservative variables is indeed an essential ingredient. The condition of the conservation of the mass at the level of each element is also respected.

References

[1] T.J.R. Hughes, G. Scovazzi and T.E. Tezduyar. "Stabilized methods for compressible flows." Journal of Scientific Computing, 43, 343-368, 2010.

[2] de Frutos J., B. Garcia-Archilla, V. John and J. Novo, "An adaptive SUPG method for evolutionary convection–diffusion equations." Computer Methods in Applied Mechanics and Engineering, 273, 219-237, 2014.

[3] Ganesan S. "An operator-splitting Galerkin/SUPG finite element method for population balance equations: Stability and convergence." ESAIM: Mathematical Modeling and Numerical Analysis (M2AN), 46, 1447-1465, 2012.

[4] John V. and Novo J., ""Error analysis of the SUPG finite element discretization of evolutionary convection-diffusion-reaction equations." SIAM Journal on Numerical Analysis, 49, (3), 1149-1176, 2011.

[5] L. Wang, W. K. Anderson, J. T. Erwin, and S. Kapadia, "Discontinuous Galerkin and Petrov Galerkin methods for compressible viscous flows." Computers & Fluids, (100), 13-29, 2014.

[6] W. K. Anderson, B. R. Ahrabi, and J. C. Newman. "Finite Element Solutions for Turbulent Flow over the NACA 0012 Airfoil." AIAA Journal, 54, (9), 2688-2704, 2016.

[7] Y. Saad and M. H. Schultz, "GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems." SIAM J. Sci. Stat. Comp., 7, 856-869, 1986.

[8] M. Fortin, H. Manouzi and A. Soulaimani. "On finite element approximations and stabilization methods for compressible viscous flows." International Journal of Numerical Methods in fluids, 17, (6), 477-499, 1993.

[9] A. Rizzi and H. Viviand editors, "Numerical methods for the computation of inviscid transonic flows with shock waves, a GAMM-Workshop." Volume 3 of notes on numerical fluid mechanics, Vieweg Verlag, 1981.

[10] T. J. Barth. "Computational fluid dynamics, on unstructured grids and solvers." von Karman Institute for fluid dynamics, NASA Ames Research Center, USA, Lecture Series 1990-03, March 5-9, 1990.

[11] Z. Johan, T.J.R. Hughes and F. Shakib, "A globally convergent matrix-free algorithm for implicit time-marching schemes arising in finite element analysis in fluids." Computer Methods in Applied Mechanics and Engineering, 87, 281-304, 1991.

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Simulation of flows in heterogeneous porous media of variable saturation

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A B S T R A C T

We develop a resolution of the Richards equation for the porous media of variable saturation by a finite element method. A formulation of interstitial pressure head and volumetric water content is used. A good conservation of the global and local mass is obtained. Some applications in the case of heterogeneous media are presented. These are examples which make it possible to demonstrate the capillary barrier effect.

1. Introduction

The study of the flow of fluids in porous media of variable saturations is of practical interest in many fields, in agriculture, civil engineering as well as environment. Control of the movement of fluids in industrial waste, tailings and municipal waste sites requires a thorough knowledge of the dynamics of these fluids; either to assess the extent of the damage, or to design retention or containment structures or to implement a rehabilitation strategy.

It is not always technically easy to represent the physical reality of these problems in the laboratory or to design models at a reasonable cost. The mathematical and numerical modeling of the flow of fluids in porous media has for some decades made considerable progress in the wake of the rapid development of the means of calculation. The flow is modeled by nonlinear partial differential equations. These equations rarely have analytical solutions, or at the cost of simplifications which make the model without practical interest. It is rather by the various numerical approximation methods that these equations are solved. The nonlinear character of the properties of the porous media of variable saturation, the high heterogeneity of these media and the mixed nature of certain boundary conditions such as those prevailing on the soil-atmosphere interface, make the problem singularly difficult and require the implementation of robust and efficient approximation methods.

In this study, we propose a mixed formulation in volumetric water content and in interstitial pressure head. The time discretization is performed with an implicit Euler scheme with variable time step. In space, a finite element method is used. The nonlinear problem is solved by the method of successive iterations. Although the method is of the first order, taking into account the variations of the volumetric water content in the iterative scheme makes it possible to obtain the convergence with a good conservation of the mass.

As applications, we first propose the study of the classic problem of infiltrations of water in a very dry environment. It is a question of testing the convergence and the robustness of the proposed method in comparison with the results of the literature. Subsequently, the case of flow in a heterogeneous medium is examined. We study more precisely the flow of water in a medium formed of several layers of materials of different textures. The flow conditions through these layers are often complex due mainly to the non-linear behavior of the various hydraulic variables of the medium and their discontinuity in the passage from one layer to the other. Due to the contrast of the saturation state between the layers, a capillary barrier effect occurs at their interface. This can greatly reduce the movement of fluids (water and air) between layers. This mechanical phenomenon has been extensively exploited in the design of recovery systems for storage sites of various types of waste (industrial, mining, domestic, etc.). These systems contribute to the isolation of these releases from their immediate environment by restricting the movement of water and oxygen. The general characteristics of multi-layer barriers have been dealt with extensively in the mining and geotechnical literature see [1-4].

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2. Equations of flow

Unsaturated subsurface media are triphase: the medium skeleton is the solid phase, water and air form the liquid and gaseous phases, respectively. When the skeleton is assumed to be rigid and immobile, and air is at atmospheric pressure, the flow of water is governed by Richards' equation [5] . It is a simple (seemingly) equation model describing the evolution of volumetric water content in unsaturated soils. The Richards equation is obtained from the law of conservation of the mass of water and a law of behavior, the law of Darcy [6]. The Richards equation in a mixed formulation, volumetric water content and interstitial pressure head, is:

$$\begin{cases} \dfrac{\partial \theta}{\partial t} + \nabla \cdot u = f \\ & \text{in } \Omega \\ u = -K(\nabla h + \nabla z) \end{cases} \tag{2.1}$$

Where Ω is a domain of the plane $\mathbb{R}^2$ (or the space $\mathbb{R}^3$) representing (geometrically) the porous medium, θ is the volumetric water content ($[L^3\,L^{-3}]$ dimensionless), u is the Darcy velocity $[L\,T^{-1}]$, h is the interstitial pressure head [L], f is a source function $[T^{-1}]$ and z is the elevation [L]. K is the hydraulic conductivity tensor $[L\,T^{-1}]$ of the medium that is written in the form:

$$K = K^A K = K^A k_s k_r$$

Where K^A is the anisotropy tensor, K is the hydraulic conductivity, k_s is the hydraulic conductivity of the medium at water saturation and k_r is the relative conductivity.

To the equation (2.1) are associated constitutive relationships between the volumetric water content and the interstitial pressure head on the one hand, and between the hydraulic conductivity and the interstitial pressure head on the other hand. These are empirical relationships such as those proposed by [7], Van Genuchten at 1980:

$$\theta(h) = \begin{cases} \theta_r + \dfrac{\theta_s - \theta_r}{[1 + |\alpha h|^n]^m} & h < 0 \\ \\ \theta_s & h \geq 0 \end{cases} \tag{2.2}$$

and

$$K(h) = \begin{cases} k_s\, k_r & h < 0 \\ \\ k_s & h \geq 0 \end{cases} \tag{2.3}$$

with:

$$k_r = {S_c}^{1/2} \left[1 - \left(1 - {S_c}^{1/m}\right)^m\right]^2, \quad m = 1 - \frac{1}{n} \quad \text{for } n > 1, \quad \text{and} \quad S_c = \frac{\theta - \theta_r}{\theta_s - \theta_r}$$

Where: θ_r is the residual water content and θ_s is the water content at saturation.

Note 1:

In the literature, the Richards equation is also written and studied in a so-called interstitial pressure formulation. To obtain this formulation, we put: $C(h) = \frac{\partial \theta}{\partial h}$, thus equation (2.1) becomes:

$$\begin{cases} C(h)\dfrac{\partial \theta}{\partial t} + \nabla \cdot u = f \\ & \text{in } \Omega \\ u = -K(\nabla h + \nabla z) \end{cases}$$

where: C(h) is the hydraulic capacity function of the medium. In order to solve mathematically the partial differential equation (2.1), it is associated with initial and boundary conditions.

Initial conditions

We suppose that:

$$h(x, z, 0) = h_0(x, z) \qquad \text{in } \Omega$$

where: h_0 is a given function on Ω.

Boundary conditions

We can associate to the equation (2.1) several types of boundary conditions according to the physical model studied:

Dirichlet boundary conditions

$$h(x, z, t) = h_D(x, z, t) \qquad \text{on } \Gamma_D \times R_+$$

***Neumann* boundary conditions**

$$-[K(\nabla h + \nabla z)] \cdot n = \sigma_n(x, z, t) \qquad \text{on } \Gamma_N \times R_+$$

Gradient boundary conditions

$$-[K\,\nabla h)] \cdot n = \sigma_G(x, z, t) \qquad \text{on } \Gamma_G \times R_+$$

where: Γ_D, Γ_N and Γ_G are respectively parts of the boundary of the domain Ω and h_D, σ_n and σ_G are given functions and n is the normal external to Γ_N and Γ_G. In flow models across soils of varying saturation, boundary conditions are not reduced to the previous types. Non-standard conditions are imposed as seepage boundary conditions, atmospheric boundary conditions, and free drainage boundary conditions.

The seepage boundary conditions consist in imposing the interstitial pressure head at atmospheric pressure on parts Γ_S, a priori unknown, of a surface, subject to a potential seepage.

$$h = 0 \qquad \text{on } \Gamma_S \times R_+$$

The atmospheric boundary conditions are applied to the surface of the soil to take into account the physical phenomena of precipitation and evaporation on the surface Γ_A. These conditions must respect the constraints of the state of saturation or not of the soil in the vicinity of the surface in the form of the two inequalities:

$$|K(\nabla h + \nabla z) \cdot n| \leq E_S \qquad \text{on } \Gamma_A \times R_+$$

and

$$h_A \leq h \leq 0 \qquad \text{on } \Gamma_A \times R_+$$

where: E_S is the maximum potential flux on the surface and h_A is the minimum pressure head under current soil conditions.

The boundary conditions of free drainage consist in imposing the Neumann condition with: $\sigma_G(x, z, t) = 0$.

3. Discretization

There is a vast literature on the numerical study of the Richards equation, both with the finite difference method and with finite element method see [8-15].

The finite element method has the advantage of taking into account general boundary conditions and allows to study the equation of the flow in complex geometries. In addition, the finite element method is flexible and practical.

The main difficulty of the numerical resolution of the Richards equation is its nonlinearity in h by the volumetric water content $\theta(h)$ and the hydraulic conductivity $K(h)$ on the one hand, and the discontinuities due to heterogeneity of the environment on the other hand. Numerical methods present convergence difficulties as soon as these functions undergo sudden transitions in the vicinity of certain values of the interstitial pressure head. This is the case, for example, during the propagation of the wetting front in dry soil. The solutions exhibit instabilities with bad mass conservation see [9, 16, 17].

In the literature, the Richards equation is written and discretized under three types of formulations: in h, in θ and in h and θ. These formulations are formally identical, but sometimes produce different numerical results [18-20]. The formulation in h does not ensure a good conservation of the mass unless we consider a fine mesh and a small time step. The formulation in θ gives a very good conservation of the mass [9], but it does not make it possible to treat correctly the flow in the saturated zones, because the equation degenerates. The so-called mixed form in h and θ, for its part, ensures a good approximation of the solution and a better conservation of the mass. A discussion of this subject in the one-dimensional case is found in [9], [18]. The choice of iterative methods, such as Newton-Raphson or the method of successive approximations (Picard) as well as their various variants, also depends on the considerations of the cost in computation time.

In this section, we study the approximation of the solutions of the Richards equation. We use a finite difference scheme for time discretization with a variable time step. On each time step, we obtain a nonlinear stationary problem, which is solved after linearization by an iterative method. At each iteration, a linear problem is solved by a standard finite element method.

3.1. Time discretization

In the following, we present the time discretization of the Richards equation in the so-called mixed form h and θ:

$$\frac{\partial\theta}{\partial t} - \nabla\cdot[K(h)(\nabla h + \nabla z)] = f \tag{3.1}$$

We denote by Δt the time step and by $t_0, t_1, \dots, t_n, t_{n+1}, \dots$ the points of the discretization, with $t_{n+1} = t_n + \Delta t$. We also denote by h^n, θ^n and f^n respectively the values of h, θ and f at time t_n, ie $h^n = h(\cdot, t_n)$, $\theta^n = \theta(h(\cdot, t_n))$ and $f^n = f(\cdot, t_n)$. The discretization of (3.1) by the implicit Euler scheme leads us to solve the nonlinear problem, finding h^{n+1} satisfying:

$$\begin{cases} \dfrac{\theta^{n+1} - \theta^n}{\Delta t} - \nabla\cdot\left[K\left(h^{n+1}\right)\left(\nabla h^{n+1} + \nabla z\right)\right] = f^{n+1} & \text{in } \Omega \\ h^{n+1}(x,z) = h_D(x,z,t_{n+1}) & \text{on } \Gamma_D \\ -K\left(h^{n+1}\right)\left(\nabla h^{n+1} + \nabla z\right)\cdot n = \sigma_N(x,z,t_{n+1}) & \text{on } \Gamma_N \end{cases} \tag{3.2}$$

To be able to write an iterative scheme of successive approximations of this problem, we must first linearize the function $\theta(h)$. If we denote by $h^{n+1,\vartheta}$, $h^{n+1,\vartheta+1}$ the $\vartheta+1$ th and ϑ th approximation of h^{n+1}, we then write using the Taylor formula :

$$\begin{aligned} \theta^{n+1,\vartheta+1} - \theta^{n+1,\vartheta} &= \theta\left(h^{n+1,\vartheta+1}\right) - \theta\left(h^{n+1,\vartheta}\right) \\ &\approx \frac{d\theta}{dh}\left(h^{n+1,\vartheta}\right)\left(h^{n+1,\vartheta+1} - h^{n+1,\vartheta}\right) \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} \theta^{n+1,\vartheta+1} - \theta^{n+1,\vartheta} &= \theta\left(h^{n+1,\vartheta+1}\right) - \theta\left(h^{n+1,\vartheta}\right) \\ &\approx C\left(h^{n+1,\vartheta}\right)\left(h^{n+1,\vartheta+1} - h^{n+1,\vartheta}\right) \end{aligned} \tag{3.4}$$

The linearization of (3.2) is written in the form

$$\begin{cases} C(h^{n+1,\vartheta})\dfrac{h^{n+1,\vartheta+1} - h^{n+1,\vartheta}}{\Delta t} - \nabla\cdot[K(h^{n+1,\vartheta})(\nabla h^{n+1,\vartheta+1} + \nabla z)] = f^{n+1} - \dfrac{\theta^{n+1,\vartheta} - \theta^n}{\Delta t} & \text{in } \Omega \\ h^{n+1,\vartheta+1}(x,z) = h_D(x,z,t_{n+1}) & \text{on } \Gamma_D \\ -K(h^{n+1,\vartheta})(\nabla h^{n+1,\vartheta+1} + \nabla z)\cdot n = \sigma_N(x,z,t_{n+1}) & \text{on } \Gamma_N \end{cases} \tag{3.5}$$

The iterative process (3.5) is called by some authors a method of modified Picard iterations [9, 18].

Note 2:

In the equation (3.2) we could have linearized the function $K(h^{n+1})$, as we did for $\theta(h^{n+1})$, by writing its Taylor expansion to the order 1 in the neighborhood of a given approximation $h^{n+1,\vartheta}$, which will naturally involve the derived from dK/dh. This would lead to an iterative second order scheme of the Newton type. From this point of view, scheme (3.5) appears in fact as an approximation of Newton's method. The use of the Newton method is advantageous in the case of very dry soils where it converges faster, but with a higher cost [18].

3.2. Finite element discretization

We decompose the domain of the flow Ω into a finite number of elements. The approximate resolution of (3.5) by the finite element method in a standard Galerkin type formulation consists in determining the solution $h^{n+1,\vartheta+1}$ in the form:

$$h^{n+1,\vartheta+1} = \sum_{j=1}^{nn} \varphi_j\, h_j^{n+1,\vartheta+1}$$

Where: nn denotes the number of interpolation nodes, $\{\varphi_1, \dots, \varphi_{nn}\}$ are the basic functions of the Lagrange interpolation space, and $h_j^{n+1,\vartheta+1}$ is the degree of freedom of $h_i^{n+1,\vartheta+1}$ at the jth node. Galerkin's method amounts to writing that $h_i^{n+1,\vartheta+1}$ verifies (3.5) in the weak sense, that is, the integral identity:

$$\begin{aligned} &\int_\Omega \left\{ C\left(h^{n+1,\vartheta}\right)\frac{h^{n+1,\vartheta+1} - h^{n+1,\vartheta}}{\Delta t}\varphi_i + K\left(h^{n+1,\vartheta}\right)\nabla h^{n+1,\vartheta+1}\cdot\nabla\varphi_i \right. \\ &\left. + K\left(h^{n+1,\vartheta}\right)\nabla z\cdot\nabla\varphi_i + \frac{\theta^{n+1,\vartheta} - \theta^n}{\Delta t}\varphi_i - f^{n+1}\varphi_i \right\} d\Omega \\ &= \int_{\Gamma_N} \sigma_N\, \varphi_i\, d\Gamma \end{aligned} \tag{3.6}$$

for any basic function φ_i.

By introducing the explicit form of the terms of identity (3.6), we obtain the linear systems.

$$\left[\frac{[M^{n+1}]}{\Delta t} + [A^{n+1}]\right]\left\{h^{n+1,\vartheta+1}\right\} = \{Q^{n+1}\}$$

where: $[M^{n+1}]$ is the mass matrix given by:

$$M_{ij}^{n+1} = \int_{\Omega} C\left(h^{n+1,\vartheta}\right) \varphi_i \, \varphi_j \, d\Omega$$

$[A^{n+1}]$ is the stiffness matrix

$$A_{ij}^{n+1} = \int_{\Omega} K\left(h^{n+1,\vartheta}\right) \nabla\varphi_i \cdot \nabla\varphi_j \, d\Omega$$

end the vector Q^{n+1} is given by

$$Q_{ij}^{n+1} = \int_{\Omega} \left\{ f^{n+1}\varphi_i - K\left(h^{n+1,\vartheta}\right) \nabla z \cdot \nabla\varphi_i \right\} d\Omega + \frac{1}{\Delta t}\int_{\Omega} \left\{ C\left(h^{n+1,\vartheta}\right)\left(h^{n+1,\vartheta}\right) - \theta^{n+1,\vartheta} + \theta^{n} \right\} \varphi_i \, d\Omega + \int_{\Gamma_N} \sigma_N \, \varphi_i \, d\Gamma$$

for : $i, \; j = 1, \; nn$.

We have thus reduced the approximation of the problem (2.1) to the resolution of a linear system of type (3.7) on each time step.

Resolution algorithm

1. Given t_{max} the maximum time of the simulation, and $\Delta t_{min} \leq \Delta t \leq \Delta t_{max}$ the interval of the time steps
2. Given N the maximum number of non-linear iterations
3. Given ε and δ respectively the tolerances on the interstitial pressure head and the volumetric water content
4. Loop on the time steps: $t = t_{n+1}, t_{n+2}, \dots, t_{max}$
5. Loop on non-linear iterations: $\vartheta = 0, 1, \dots, N$

   For $\vartheta = 0$, put $h^{n+1,0} = h^n$
6. Calculate $h^{n+1,\vartheta+1}$ solution of the linear system $\left[\frac{[M^{n+1}]}{\Delta t} + [A^{n+1}]\right] \left\{h^{n+1,\vartheta+1}\right\} = \left\{Q^{n+1}\right\}$
7. If $\left\|h^{n+1,\vartheta+1} - h^{n+1,\vartheta}\right\| \leq \varepsilon$ and $\left\|\theta^{n+1,\vartheta+1} - \theta^{n+1,\vartheta}\right\| \leq \delta$

   Convergence achieved on non-linear iterations

   To pose : $h^{n+1} = h^{n+1,\vartheta+1}$

   Return to step 3 with $t = t_{n+2}$
8. If the maximum number of iterations on N is reached

   Convergence not achieved in N iterations

   Reduce time step

   Return to step 3 with $t = t_{n+1}$

Note:

The test on θ (step (7)) allows a better control of the volumetric water content and ensures a better convergence and conservation of the mass.

3.3. Linear system resolution

The linear system obtained in (3.7) is of the form: $Ax = b$, where A is a symmetric and definite positive sparse matrix $N \times N$, and b is a vector of $\mathbb{R}^N$ given, x is the unknown vector of $\mathbb{R}^N$. When Matrix A is small, the system can be solved by a direct method such as the Gauss or Choleski elimination method. If, on the other hand, A is large, it is more advantageous to use iterative methods such as the conjugate gradient method or the preconditioned conjugate gradient method. In our case, we use the conjugate gradient method preconditioned by incomplete LU factorization [21].

4. Numerical simulations

In this section, some examples of flow tests in porous media are presented. The main aim is to demonstrate the effectiveness of the formulation used and the robustness of the finite element method. This is particularly illustrated in the case of water infiltration in very dry porous media as well as in the case of drainage in heterogeneous media.

4.1. Infiltration in a homogeneous column

The classical problem of infiltration of water into a homogeneous porous column is considered. This example has been widely used by several authors to validate their numerical methods for solving flow problems in porous media of variable saturation. See [9, 18, 20, 22, 23, 24]. The hydraulic properties of the medium are:

θ_r	θ_s	α	k_s (cm/s)
0,102	0,368	0,0335	0,00922

Figures 1 and 2 show the characteristics of the medium, θ(h) and K(h).

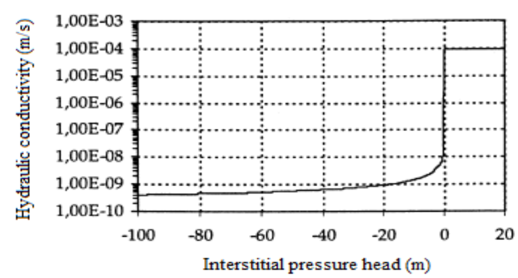

Figure 1: Hydraulic conductivity

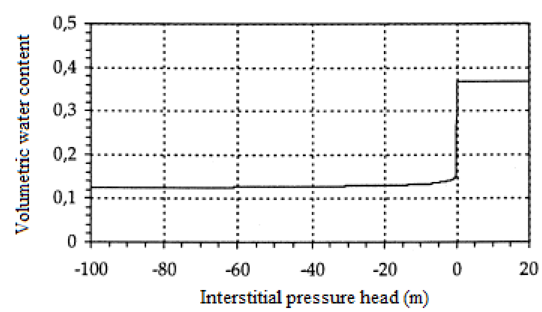

Figure 2: Volumetric water content

In this first test, the height of the column is L = 30 cm. The initial condition is $h(.,0) = -1\,000$ cm, while the boundary conditions are $h(L,t) = -75$ cm at the surface of the column and $h(.,0) = -1\,000$ cm on its base. The simulation time is 6 hours. A regular mesh has been produced with the height of the elements

$\Delta z = 0{,}25$ cm. It has been used with several time steps, from the finest $\Delta t = 0{,}1$ s to the widest $50 \text{ s} \leq \Delta t \leq 500$ s. The solutions obtained are quite comparable to those of Lehmann in [18], and to the semi-analytical solution of Philip [25] as well as those obtained by [9]. We also considered a fine mesh with elements of size $\Delta z = 0{,}1$ cm and a time step $\Delta t = 0{,}1$ s. This mesh was used to calculate the comparison solution. In the case of the fine mesh and the small time step, the same results are obtained with a good conservation of the mass. In figures 3 and 4, the curves of the interstitial pressure head and the curves of the volumetric water content as a function of the elevation for a regular mesh, whereas in figures 5 and 6 the same representations in the case of fine mesh.

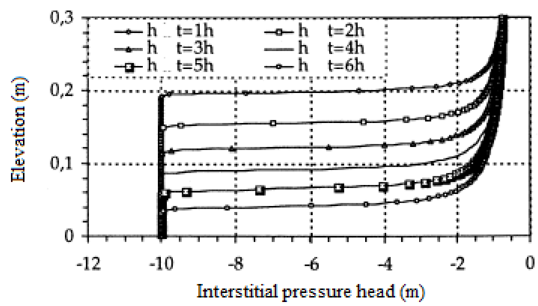

Figure 3: Interstitial pressure head

(regular mesh $\Delta z = 0{,}25$ cm)

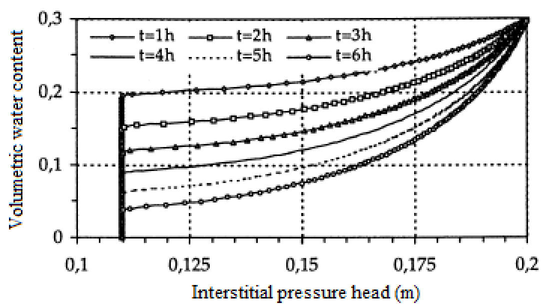

Figure 4: Interstitial pressure head

(regular mesh $\Delta z = 0{,}25$ cm)

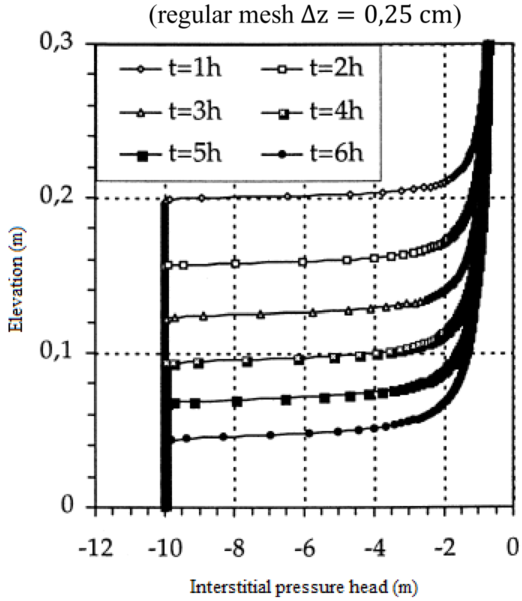

Figure 5: Interstitial pressure head

(fine mesh $\Delta z = 0{,}1$ cm)

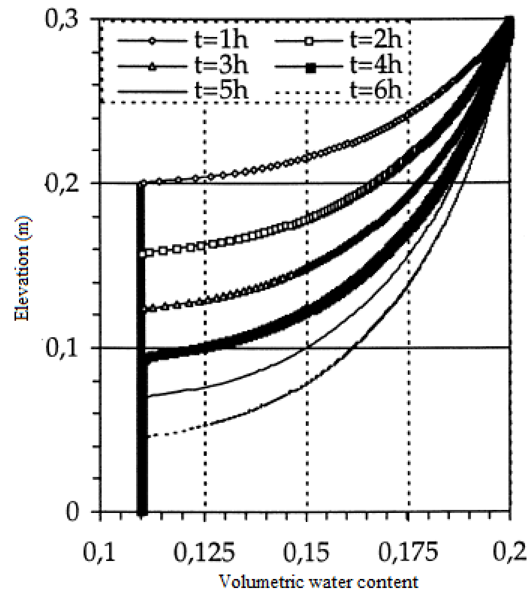

Figure 6: Volumetric water content

(fine mesh $\Delta z = 0{,}1$ cm)

Figure 7 shows the curve of the interstitial pressure head as a function of the calculated elevation and the curve of the semi-analytical solution of Philip [25].

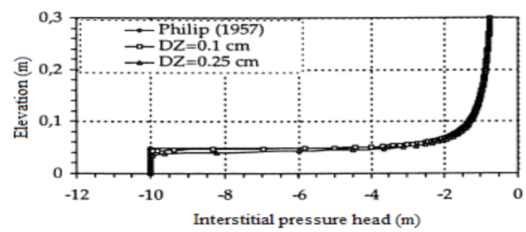

Figure 7: Interstitial pressure head (regular mesh, fine mesh and Philip [25])

4.2. Infiltration in a heterogeneous (multilayer) column

An example given by Lehman in [18]. It is a heterogeneous column of height $L = 180$ cm. In this example, it is sought to test the hydraulic behavior of a heterogeneous medium with boundary conditions relating to a wetting and drainage cycle. The porous medium consists of three layers each 60 cm thick. The hydraulic parameters of the first and the third layer (Berino loamy fine sand) and those of the second layer (Glendale clay loam) are given by:

Table 1: Hydraulic parameters of the first and the third layer

	Berino loamy fine sand	Glendale clay loam
θ_r	0,0286	0,1060
θ_s	0,3658	0,4686
α cm^{-1}	0,0280	0,0104
n	2,239	1,3954
k_s (cm/j)	541	13,1

The characteristics of the medium, $\theta(h)$ and $K(h)$ are shown in figures 8 and 9.

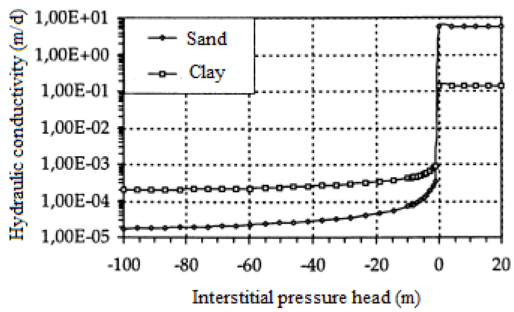

Figure 8: Hydraulic conductivity

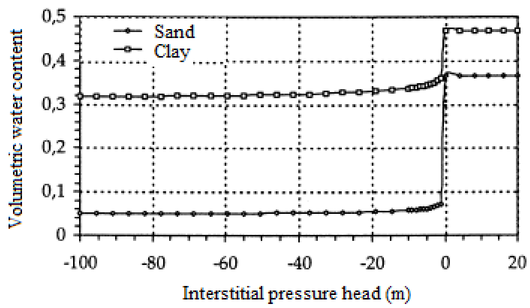

Figure 9: Volumetric water content

At the top of the column, atmospheric boundary conditions are predicted with precipitation and evaporation, figure 10.

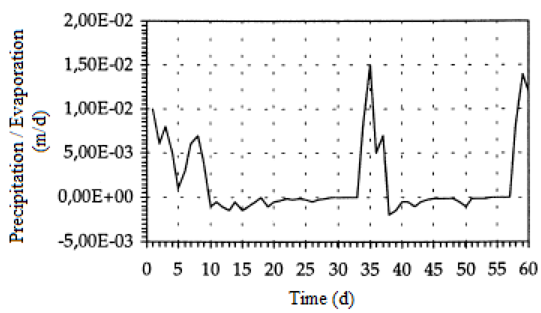

Figure 10: Precipitation and Evaporation

Free drainage conditions are imposed at the bottom of the column. Two types of tests were carried out: a first test with the initially wet medium and a second initially dry test.

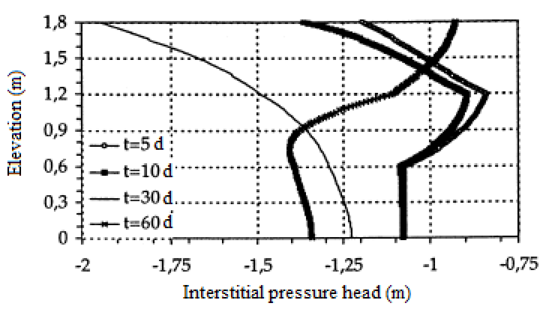

Figure 11: Interstitial pressure head

(fine mesh $h(\cdot,0) = -100$ cm)

I) Initially wet medium

The initial interstitial pressure head is taken as $h(.,0) = -100$ cm. We first considered a regular mesh, with the height of elements $\Delta z = 5,0$ cm. For the time steps, $0,01 \leq \Delta t \leq 1,0$ was used. Thereafter, a mesh with element height $\Delta z = 1,0$ cm was used. In both cases, we find the same results as Lehman and al. [18], figures 11 and 12. In figure 12, it is noted that the middle layer has been saturated rapidly, while the top and bottom layers desaturate. Thus we are in the presence of an effect of capillary barrier between layers.

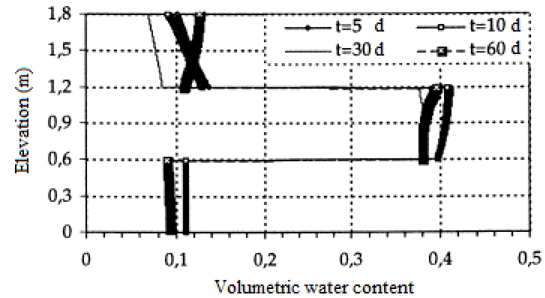

Figure 12: Volumetric water content

(fine mesh $h(\cdot,0) = -100$ cm)

II) Initially dry medium

For this test, it is assumed that the three media are initially very dry. The initial interstitial pressure head is taken as $h(.,0) = -10\,000$ cm and as a time step, successively $\Delta t = 0,1$ days and $\Delta t = 1,0$ days. To obtain good convergence, we have considered the fine mesh ($\Delta z = 1,0$ cm) associated with a rather small time step ($\Delta t = 0,0001$ days). Figures 13 and 14 shows the interstitial pressure head and the volumetric water content. As in the previous case, a capillary barrier effect is highlighted. Although the medium was initially dry $h = -10\,000$ cm, the layer of fine material in the middle of the column saturates quite rapidly, while that of the top remains slightly saturated and that of the bottom always dry, figure 14.

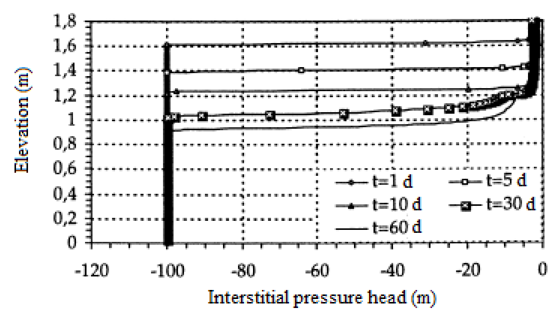

Figure 13: Interstitial pressure head

(fine mesh $h(\cdot,0) = -10\,000$ cm)

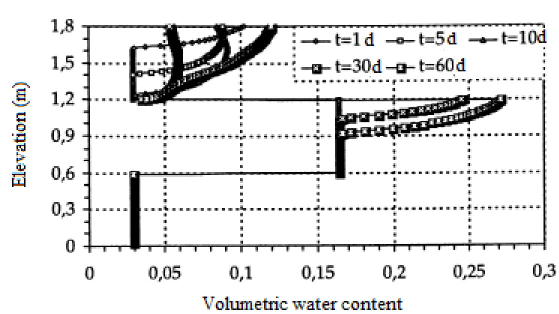

Figure 14: Volumetric water content

(fine mesh $h(\cdot,0) = -10\,000$ cm)

4.3. Drainage in a heterogeneous column (dry barriers)

This is an example of covers with capillary barrier effects from the authors' work in [26]. An initially saturated multilayer medium is drained. The layers are of materials with highly contrasted hydraulic properties. The values of the parameters characterizing the covering materials are:

	Sand	Unreactive mining rejection
θ_r	0,0490	0,0456
θ_s	0,39	0,41
α cm^{-1}	0,0290	0,0017
n	10,2100	2,1366
k_s (cm/j)	2,10x10-2	6,58x10-5

The geometry is a vertical column of height $n = 110$ cm containing a layer of unreactive mining rejection (fine silty material) 60 cm thick confined between two layers of sand (coarse material) 30 cm thick for the bottom layer and of 20 cm thick for the top layer. The medium was initially saturated (the interstitial pressure head was zero everywhere), thereafter a free drainage condition was imposed at the base of the column. The simulation is 60 days. Both layers of sand drain very quickly, figure 15. On the second day, the water content and pressure curves stabilized. The volumetric water content of the fine layer remains substantially at its saturation value, figure 16.

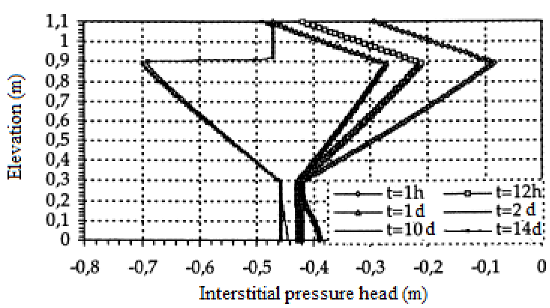

Figure 15. Interstitial pressure head

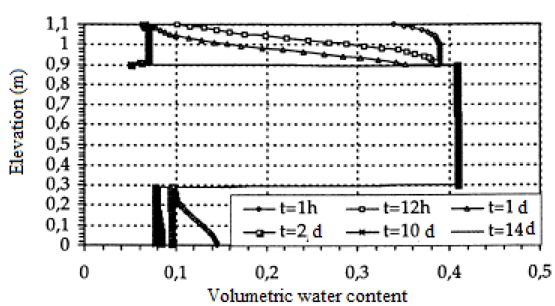

Figure 16. Volumetric water content

5. Conclusion

A finite element method has been developed for calculating the distribution of the interstitial pressure head and the volumetric water content in a water flow through a heterogeneous porous medium such as soil with variable saturation and initially dry. In the resolution, boundary conditions close to the actual conditions are taken into account. In particular, mixed-type conditions corresponding to atmospheric conditions on the soil surface. Conventional flow tests in the columns have proved the robustness and stability of the method with good conservation of the mass.

References

[1] B. Bussière, M. Idrissi, N.E. Elkadri, M. Aubertin, "Simulation des écoulements dans les milieux poreux de saturation variable à l'aide de la formation mixte de l'équation de Richard" Proceedings 1st joint IAH-CNC-CGS groundwater speciality conference, 335-342, 2000.

[2] H. J. Morel-Seytoux, "Capillary Barrier at the Interface of Two Layers" Water Flow and Solute Transport in Soils, **20**, 136-151, 1993.

[3] T. Pabst, M. Aubertin, B. Bussière, J. Molson, "Experimental and numerical evaluation of single-layer covers placed on acid-generating tailings" Geotechnical and Geological Engineering. **35**(4), 1421-1438, 2017.

[4] B. Ross, "The diversion capacity of capillary barriers", Water Resources research, **26**(10), 2625-2629,1990.

[5] L. A. Richards, "Capillary conduction of liquids through porous medium. Physics", **1**, 318-333, 1931.

[6] J. Bear, "Dynamics of porous media", Dover Publications, New York. 1988.

[7] Y. Mualem, "A new model for predicting the hydraulic conductivity of unsaturated porous media", Water Resources research, **12**(3), 573-522, 1976.

[8] B. Belfort, F. Lehmann, "Comparison of equivalent conductivities for numerical simulation of onedimensional unsaturated flow", Vadose Zone Journal, **4**, 1191-1200, 2005.

[9] M. A. Celia, E.T. Bouloutas, R.L. Zarba, "A general mass-conservative numerical solution for the unsaturated flow equation", Water resources research, **26**(7), 1483-1496, 1990.

[10] R. Haverkamp, M. Vauclin, J. Touma, P.J. Wierenga, G. Vachaud, "A comparison of numerical simulation models for one-dimensional infiltration", Soil Science Society of America Journal, **47**, 285-294, 1977.

[11] M. W. Farthing, C. E. Kees, C. T. Miller, "Mixed finite element methods and higher order temporal approximations for variably saturated groundwater flow", Advances in Water Resources, **26**(4), 373-394, 2003.

[12] M. Mousavi Nezhad, A. A. Javadi, F. Abbasi,"Stochastic finite element modelling of water flow in variably saturated heterogeneous soils", International journal for numerical and analytical methods in geomechanics, **35**, 1389-1408, 2011.

[13] I. Rees, I. Masters, A.G. Malan, R.W. Lewis, "An edge-based finite volume scheme for saturated unsaturated groundwater flow", Computer Methods in Applied Mechanics and Engineering, **193**, 4741-4759, 2004.

[14] S. E. Serrano "Modeling infiltration with approximate solutions to Richard's equation", Journal of Hydrologic Engineering, **9**(5), 421-432, 2004.

[15] J. Vanderborght, R. Kasteel, M. Herbst, M. Javaux, D. Thiery, M. Vanclooster, C. Mouvet , H. Vereecken, "A set of analytical benchmarks to test numerical models of flow and transport in soils", Vadose Zone Journal, **4**(1), 206-221, 2005.

[16] X. Hao, R. Zhang, A. Kravchenko, "A mass-conservative switching method for simulating saturated-unsaturated flow", Journal of Hydrology, **311**(1-4), 254-265, 2005.

[17] K. Kosugi, "Comparison of three methods for discretizing the storage term of the Richards equation", Vadose Zone Journal, **7**(3), 957-965, 2008.

[18] F. Lehmann, PH. Ackerer, "Comparison of iterative methods for improved solutions of fluid flow equation in partially saturated porous media", Transport in Porous Media, **31**, 275-292, 1998.

[19] C. Paniconi, M. Putti, "A comparison of Picard and Newton iteration in the numerical solution of multidimensional variably saturated flow", Water Resources research, **30** (12), 3357-3374, 1994.

[20] K. Rathfelder, L. M. Abriola "Mass conservative numerical solutions of the head based Richards equation", Water Resources research, **39**(9), 2579-2586, 1994.

[21] P. G. Ciarlet, "Introduction à l'analyse numérique matricielle et à l'optimisation. Masson", 1985.

[22] R. Haverkamp, M. Vauclin, J. Touma, P. J. Wierenga, G. Vachaud, "A comparison of numerical simulation models for one-dimensional infiltration", Soil Science Society of America Journal, **47**, 285-294, 1977.

[23] J. Touma, M. Vauclin, "Experimental and numerical analysis of two-phase infiltration in a partially saturated soil", Transport in Porous Media, **1**, 22-55, 1986.

[24] M. Vauclin, "Flow of water and air in soils: Theoretical and experimental aspects, in unsaturated flow in hydrologic modeling, Theory and Practice", edited by H. Morel-Seytoux, 53-91, Kluwer Academic, Dordrecht, Netherlands, 1989.

[25] I. R. Philip, "The theory of infiltration: 1. The infiltration equation and its solution", Soil Sci., **83**, 435-448, 1957.

[26] M. Aachib, M. Aubertin, R. P. Chapuis, "Essais en colonne sur des couvertures avec effets de barrière capillaire", 51ème conférence canadienne de géotechnique, **2**, Edmonton, 1998.

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Models accounting for the thermal degradation of combustible materials under controlled temperature ramps

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A B S T R A C T

The purpose of this conference is to present and analyze different models accounting for the thermal degradation of combustible materials (biomass, coals, mixtures...), when submitted to a controlled temperature ramp and under non-oxidative or oxidative atmospheres. Because of the possible rarefaction of fossil fuels, the analysis of different combustible materials which could be used as (renewable) energy sources is important. In industrial conditions, the use of such materials in energy production is performed under very high temperature ramps (up to 10^5 K/min in industrial pulverized boilers). But the analysis of the thermal degradation of these materials usually first starts under much lower temperature ramps (less than 100 K/min), in order to avoid diffusional limitations which modify the thermal degradation.

1 Introduction

Before their use as combustible in industrial furnaces for energy production, materials have to be tested at a laboratory scale. Their physical and chemical properties have to be determined, together with their energy power. Very small amounts of such materials (few milligrams) are tested in a thermobalance where the surrounding gas is injected and for which the evolution of the temperature, from ambient temperature to 900 °C generally under a low heating ramp, is controlled with high accuracy. This thermobalance measures the remaining mass with a high precision along the experiment. During a thermogravimetric analysis, the chemical products which are emitted may also be analyzed. Figure 1 presents an example of mass loss and mass loss rate curves during the combustion process of a Cameroonian woody biomass in a thermobalance.

Two peaks appear on the mass loss rate curve (red), with shoulders on both sides of the main peak, all occurring in successive temperature ranges and corresponding to more intensive emissions of chemical species. The thermal degradation is indeed obtained through a progressive destruction of the material structure under the influence of the gas (here synthetic air) which is injected and of course of the increasing temperature. While different molecules are released, the carbon structure is progressively destroyed. Under non-oxidative atmospheres, the mass associated to this carbon structure is almost preserved, while under air, this carbon structure is totally destroyed and removed. Only ash containing minerals remain at the end of the experiment. It is important to notice that even under (synthetic) air, such experiments do not lead, up to very rare and limited occurrences, to the presence of flames inside the thermobalance.

When modeling the thermal degradation of combustible materials in a thermobalance, as the diameter of the pan is few millimeters long, the space variables may be omitted and one can reasonably consider the evolution of the mass with respect to only the time parameter. All biomass are composed of three constituents: hemicellulose, cellulose and lignin, plus the carbon structure (and ash, moisture...) which are

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known to degrade in different but superimposing temperature ranges. The relative fractions of these constituents depend on the material and the bonds between them influence the thermal degradation of the material.

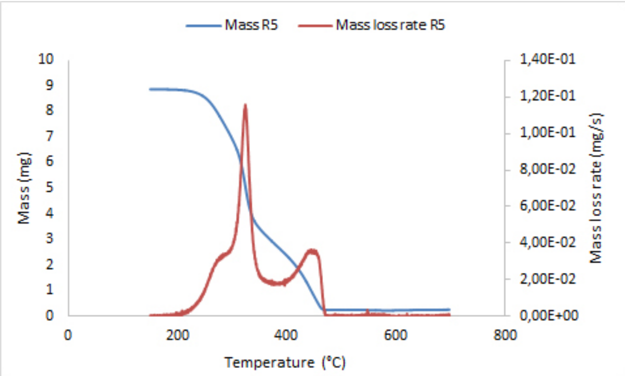

Figure 1: Mass and mass loss rate curves for a Cameroonian wood residue, temperature ramp of 5 °C/min, under synthetic air.

Based on this assumption and observation, different models have already been proposed for a representation of the thermal degradation in a thermobalance of a material considered either as global or as the sum of different constituents which are being degraded in an almost independent way, as their chains are intricate with the carbon structure.

More complex models are now being proposed which are based on balances which are written: masses of the different chemical species which are emitted during the experiments, momentum, energy. Of course, such complex models involve many equations and one usually needs a dedicated software for their numerical resolution.

In the different available models, constants have to be determined from the experimental results. They are called kinetic parameters and they are characteristics of the material and of the experimental conditions. The resolution of a model has thus to be coupled with an optimization procedure which determines the optimal set of kinetic parameters.

The purpose of this talk is to present a review of such models, from the simplest ones to the more complicated ones.

2 Kinetic modeling through a differential isoconversional method

The differential isoconversional method determines the values of the kinetic parameters associated to the thermal degradation of a material performed in a thermobalance, considering the material as global. It starts with the ordinary differential equation

$$\frac{d\alpha}{dt}(t) = k(T(t))f(\alpha(t)), \tag{1}$$

see [1], which expresses the variations versus time of the extent of conversion α taken as

$$\alpha(t) = \frac{m_{ini} - m(t)}{m_{ini} - m_{fin}}, \tag{2}$$

where m_{ini} (resp. $m(t)$, m_{fin}) is the initial (resp. remaining at time t, final) mass of the sample. In (1), $k(T)$ is given an Arrhenius expression $k(T) = Aexp(-Ea/RT)$, T being the temperature expressed in Kelvin (K) and which evolves with respect to the time parameter t from the ambient temperature T_0 with a constant rate a: $T(t) = at + T_0$. Here A is the frequency factor and Ea is the activation energy associated to the thermal degradation process of the material under consideration. Many functions $f(\alpha)$ have been proposed, starting from Mampel's first-order function $f_1(\alpha) = 1-\alpha$. Another one is the 3D diffusion function $f_3(\alpha) = 1.5(1-\alpha)^{2/3}/(1-(1-\alpha)^{1/3})$. The unique ordinary differential equation (1) is supposed to simulate the overall thermal degradation process.

It is impossible to solve the equation (1) in an explicit way. Many researchers already proposed different approximations of the right-hand side of (1) in order to obtain approximate solutions to this simple ordinary differential equation.

Instead of solving this equation, the differential isoconversional method consists to find the values of the kinetic parameters A and Ea, dividing (1) by $f(\alpha(t))$ and taking the logarithm of the two members of (1). This leads to

$$\ln\left(\frac{1}{f(\alpha(t)}\frac{d\alpha}{dt}(t)\right) = \ln(k(T(t))) = \ln(A) - \frac{Ea}{RT(t)}. \tag{3}$$

The values of the left-hand side of the preceding equality (3) are plotted for successive values of the extent of conversion α and for different temperature ramps (at least three), in terms of $1/T$. The ICTAC recommendations suggest to take values of the extent of conversion α with steps not larger than 0.05, see [1]. The parameters of the straight lines which are plotted for different values of the extent of conversion using a linear regression lead to the determination of $\ln(A)$ (hence of A) and of Ea. Figure 2 presents such an example

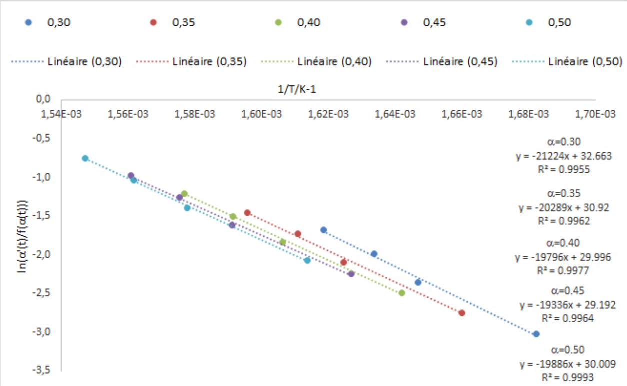

Figure 2: Straight lines for successive values of the extent of conversion and for four temperature ramps.

One obtains values of the kinetic parameters A and Ea which are global for the material but which depend on the extent of conversion α.

Mampel's function		
α	A (1/s)	Ea (kJ/mol)
0.05	2.2	23.0
0.10	3.5×10^{11}	134.7
0.15	1.2×10^{9}	110.0
0.20	7.0×10^{7}	97.8
0.25	1.7×10^{7}	92.6
...	...	...

3D diffusion function		
α	A (1/s)	Ea (kJ/mol)
0.05	3.2	23.0
0.10	5.4×10^{11}	134.7
0.15	1.3×10^{9}	110.0
0.20	1.3×10^{8}	97.7
0.25	3.3×10^{7}	92.5
...	...	...

What can be done with such lists of values of the kinetic parameters? First, take mean values and standard deviations in order to obtain a unique couple of kinetic parameters valid for the material. But large variations of the kinetic parameters may be observed through the different lines. For small or large values of α, the precision of the experimental measurements is not really ensured, as very low amounts of the material are being degraded. But even when removing these lines, large variations may still be observed. A second tool consists to propose a reconstruction process which solves (1) stepwise, according to the time intervals associated to the stepwise changes of the extent of conversion. On two examples proposed in the GRE lab, this reconstruction process led to poor simulations. Probably because the isoconversional method uses logarithms.

This differential isoconversional method gives global values of the kinetic parameters for the material, but which change according to the extent of conversion, hence of the time (or temperature) parameter.

3 Kinetic modeling through the EIPR model

3.1 Description of the model

The EIPR model superimposes the thermal degradations of three or four constituents of the material (in the case of a biomass: the H,C,L constituents of the sample, plus its char under an oxidative atmosphere), whose masses are supposed to evolve in an almost independent way. It leads to a unique set of kinetic parameters for each constituent of the sample. The initial mass m_{ini} of the sample is decomposed as $m_{ini} = m(0) + m_{ash} + m_{hum}$, where $m(0)$ (resp. m_{ash}, m_{hum}) is the mass of volatiles which will be emitted and of char which will be produced (resp. ashes, moisture) in the sample. At time t, the remaining mass $m(t)$ of the sample which can produce volatiles and char is given by

$$\begin{aligned} m(t) &= \textstyle\sum_{i=H,C,L} m_i(t) \\ &= \textstyle\sum_{i=H,C,L} \left(m_i(0) - m^e_{vol,i}(t) - m^p_{char,i}(t) \right), \end{aligned} \tag{4}$$

where $m_i(t)$ is the mass of volatiles and of char contained in the constituent i ($i = H, C, L$) of the sample at time t, $m_i(0)$ is the initial mass of the constituent i, which may be computed as a fraction of the overall mass of the sample: $m_i(0) = c_i m(0)$ (with $c_H + c_C + c_L = 1$), $m^e_{vol,i}(t)$ is the mass of volatiles emitted by the constituent i of the sample ($i = H, C, L$), at time t and $m^p_{char,i}(t)$ is the mass of char produced from the constituent i of the sample ($i = H, C, L$), at time t. The fraction coefficients c_i ($i = H, C, L$) for hemicellulose, cellulose and lignin have to be determined, which is not always an easy task. Chemical protocols have been defined for the determination of these fraction coefficients, but they currently lead to quite large uncertainties.

Under a non-oxidative atmosphere, only the volatiles are emitted during the thermal degradation of the material. Assuming first-order reactions for each constituent, the mass of volatiles emitted from the constituent i ($i = H, C, L$) evolves with respect to the time parameter according to

$$\begin{cases} \dfrac{dm^e_{vol,i}}{dt}(t) = k_i(T(t)) \left(m_i(0) - m^e_{vol,i}(t) \right), \\ m^e_{vol,i}(0) = 0, \end{cases} \tag{5}$$

where $T(t)$ is the temperature at time t in the sample (expressed in K) and which evolves with respect to the time parameter t from the ambient temperature T_0

with a constant rate a: $T(t) = at + T_0$. This leads to a non-coupled system of three equations, which can be compared to (1), in the case of Mampel's function.

In (5), the kinetic constant $k_i(T(t))$ obeys an Arrhenius law: $k_i(T) = A_i \exp(-Ea_i/RT)$, where A_i (resp. Ea_i) is the frequency factor (resp. the activation energy) for the constituent i.

Under an oxidative atmosphere, the material looses its volatiles and its char is also consumed. For each constituent, the mass of volatiles emitted is supposed to be proportional to the mass of char produced, that is

$$\frac{m^e_{vol,i}(t)}{m^p_{char,i}(t)} = \frac{\tau_{vol,i}}{\tau_{char,i}},$$

with $\tau_{vol,i} + \tau_{char,i} = 1$, $\tau_{vol,i}$ (resp. $\tau_{char,i}$) being the fraction of volatiles (resp. char) contained in the constituent i at time t. The fractions $\tau_{vol,i}$ of volatiles in H, C, L are specific to each material. The mass of volatiles emitted from the constituent i ($i = H, C, L$) here evolves with respect to the time parameter according to

$$\begin{cases} \dfrac{dm^e_{vol,i}}{dt}(t) = k_i(T(t)) \left(m_i(0) - \dfrac{m^e_{vol,i}(t)}{\tau_{vol,i}} \right) \\ m^e_{vol,i}(0) = 0, \end{cases} \quad (6)$$

Under an oxidative atmosphere, the mass of char remaining at time t in the constituent i of the sample at time t can be decomposed as $m_{char,i}(t) = m^p_{char,i}(t) - m^c_{char,i}(t)$, where $m^c_{char,i}(t)$ represents the mass of char consumed at time t among that ($m^p_{char,i}(t)$) which is produced from the constituent i of the sample ($i = H, C, L$). The mass of char is supposed to evolve with respect to the time parameter t according to a first-order equation, as for the volatiles, but written as

$$\begin{aligned} &\frac{dm^c_{char,i}}{dt}(t) \\ &\quad = k_{comb}(T(t))\, m_{char,i}(t)\, P_{O_2}, \\ &\quad = k_{comb}(T(t)) \\ &\qquad \times \left(m^p_{char,i}(t) - m^c_{char,i}(t) \right) P_{O_2}, \\ &\quad = k_{comb}(T(t)) \\ &\qquad \times \left(\frac{\tau_{char,i}}{\tau_{vol,i}} m^e_{vol,i}(t) - m^c_{char,i}(t) \right) P_{O_2}, \end{aligned} \quad (7)$$

for $i = H, C, L$, where the kinetic constant $k_{comb}(T)$ also obeys an Arrhenius law:

$$k_{comb}(T) = A_{comb} \exp\left(-\frac{Ea_{comb}}{RT} \right)$$

and where P_{O_2} is the oxygen pressure which is constant during the experiment ($P_{O_2} = 2.026 \times 10^4$ Pa).

Under an oxidative atmosphere, the coupled system of six ordinary differential equations has to be solved, with zero initial conditions

$$\begin{cases} \dfrac{dm^e_{vol,i}}{dt}(t) &= k_i(T(t)) \\ &\quad \times \left(m_i(0) - \dfrac{1}{\tau_{vol,i}} m^e_{vol,i}(t) \right), \\ m^e_{vol,i}(0) &= 0, \\ \dfrac{dm^c_{char,i}}{dt}(t) &= k_{comb}(T(t)) \\ &\quad \times \begin{pmatrix} \dfrac{\tau_{char,i}}{\tau_{vol,i}} m^e_{vol,i}(t) \\ -m^c_{char,i}(t) \end{pmatrix} P_{O_2}, \\ m^c_{char,i}(0) &= 0. \end{cases} \quad (8)$$

The problems (5) or (8) are solved using a numerical software (for example Scilab version 6.0.0) with given initial guesses of the kinetic parameters. An error is built which has to be minimized in order to determine the optimal set of kinetic parameters (A_i,Ea_i), $i = H, C, L$, (plus (A_{comb},Ea_{comb}) under an oxidative atmosphere). This error involves the differences between the experimental and the simulated mass loss rates taken at experimental measure times $(t_j)_{j=1,\ldots,J}$ regularly distributed along the time interval $(0, t_{max})$ of the experiment, for example according to

$$error = \sum_{j=1} \left(\left(\frac{dm}{dt} \right)_{exp} (t_j) - \left(\frac{dm}{dt} \right)_{sim} (t_j) \right)^2, \quad (9)$$

where the simulated mass loss rate taken at time t_j is given under a non-oxidative atmosphere as

$$\left(\frac{dm}{dt} \right)_{sim} (t_j) = \sum_{i=H,C,L} \frac{dm^e_{vol,i}}{dt}(t_j) \quad (10)$$

and under an oxidative atmosphere as

$$\begin{aligned} &\frac{dm}{dt}(t) \\ &= \sum_{i=H,C,L} \begin{pmatrix} k_i(T(t)) \\ \times \left(m_i(0) - \dfrac{1}{\tau_{vol,i}} m^e_{vol,i}(t) \right) \\ +k_{comb}(T(t)) \\ \times \begin{pmatrix} \dfrac{\tau_{char,i}}{\tau_{vol,i}} m^e_{vol,i}(t) \\ -m^c_{char,i}(t) \end{pmatrix} P_{O_2} \end{pmatrix}, \end{aligned} \quad (11)$$

$\left(\frac{dm}{dt}\right)_{exp}(t_j)$ being the experimental mass loss rate at time t_j. Instead of taking all the experimental measure times, only around 150 of them are indeed selected, regularly distributed along the overall experiment duration t_{max}. This reduces in a significant way the computing times.

Figure 3 presents an example of the simulated mass loss rate curves obtained through the EIPR method applied to a product derived from wood (hydrolysis lignin, after a torrefaction process at 275 °C), where four (H,C,L+char) constituents are considered, because of the presence of two successive but "connected" peaks. The mass loss rate curve associated to each constituent has been represented together with the sum (in blue) of these four partial curves. This blue curve is supposed to represent the experimental mass loss rate curve of the material (not represented on the figure).

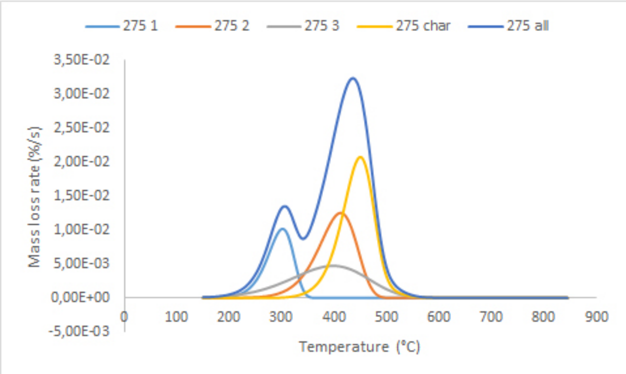

Figure 3: Simulation of the thermal degradation of hydrolysis lignin torrefied at 275 °C under air.

For this material, the optimal values of the kinetic parameters (A in 1/s) and Ea in J/mol) are obtained through the EIPR model as:

A_1	9000000	A_3	1.3
Ea_1	110000	Ea_3	47000
A_2	110000	A_{comb}	10000000
Ea_2	105000	Ea_{comb}	135000

with the following values of the fractions of the three constituents and of the volatile fractions

c_1	0.3	$\tau_{vol,1}$	0.5
c_2	0.3	$\tau_{vol,2}$	0.9
c_3	0.4	$\tau_{vol,3}$	0.5

The maximal difference between the experimental and simulated mass loss rate curves is equal to 0.008 %/s, which has to be compared to the maximal mass loss rate curve, round 0.032 %/s.

3.2 Choice of the reaction function

In the above-indicated presentation of the EIPR model, a first-order reaction function has been used. Different functions may also be introduced, like in the differential isoconversional method, especially when the first-order one does not lead to very good simulations of the thermal degradation of the material. For example, in the case of cotton residue, a unique peak is observed, as cotton is mainly composed of cellulose (more than 92%). But this peak is very narrow, as shown in Figure 4.

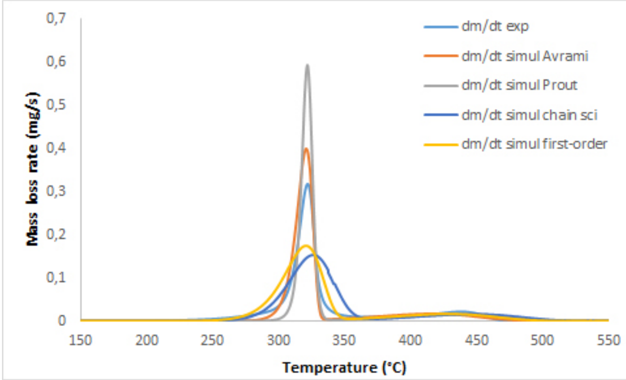

Figure 4: Simulations through the EIPR model with different functions of the thermal degradation under air of cotton residue.

Different functions have been tested for the simulation of the thermal degradation of this cotton sample:

- the classical first-order function: $f(\alpha) = 1 - \alpha$;
- an Avrami-Erofeev reaction function: $f(\alpha) = 4(1-\alpha)(-\ln(1-\alpha))^{3/4}$;
- the Prout-Tompkins reaction function: $f(\alpha) = \alpha(1-\alpha)$;
- a chain scission function $f(\alpha) = 2\left(\alpha^{0.5} - \alpha\right)$.

The different functions which have been tested do not simulate in an appropriate way the very narrow devolatilization peak, see [7] for more details. Other reaction functions are available which should be tested.

In the EIPR model, the kinetic parameters are global but for each constituent of the material and they do not depend on the time parameter. The EIPR model has been tested at GRE lab for different materials, see [2] and [3] for example. Reference articles for the simulation of the thermal degradation of lignocellulosic materials with independent reactions are [4] and [5], among others. The independent decomposition of the three constituents of a lignocellulosic material is questionable. The interactions between the three constituents of a lignocellulosic material have recently been analyzed in [6].

3.3 Influence of the temperature ramp

It has been observed by several authors and for a long time that the temperature ramp influences the thermal degradation of a material. Shifts to higher temperatures indeed appear in the mass loss rate curves, which increase with the temperature ramp, see Figure 5 again in the case of the cotton residue.

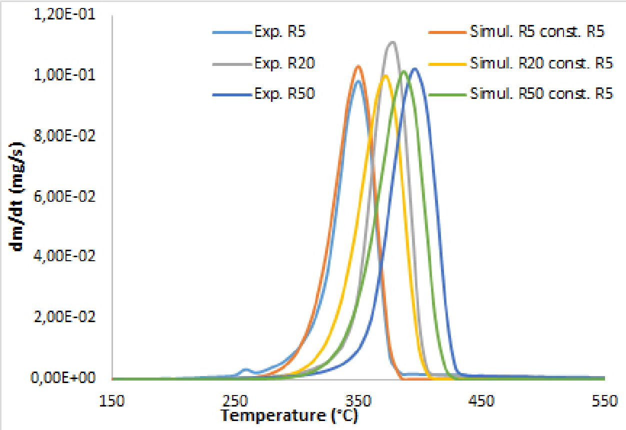

Figure 5: Influence of the temperature ramp on the thermal degradation of cotton residue under nitrogen and on the simulation through the EIPR model.

The simulations through the EIPR model which lead to a quite perfect superimposition of the mass loss rate curves at 5 °C/min fail at higher temperature ramps. This may be the consequence of diffusional limitations which occur in the material and which increase with the temperature ramp. In Figure 5, the mass loss rate curves have been transformed (multiplication by a fixed coefficient) in order to reach almost the same maximal value.

In [7], a heat transfer model has been proposed which brings corrections to the temperature really acting in the material. The temperature is supposed to evolve in the sample according to the classical heat transfer equation

$$\frac{\partial T}{\partial t}(z,t) - a\frac{\partial^2 T}{\partial z^2}(z,t) = 0, \text{ in } [0,e] \times [0,\infty], \quad (12)$$

where e is the thickness of the cotton sample put in the crucible of the thermobalance and the thermal diffusibility coefficient a is given through $a = \frac{\lambda}{\rho c}$ with:

- λ thermal conductivity of cotton, approximately equal to 0.038 W/mK,
- ρ density of cotton, approximately equal to 40 kg/m^3,
- c specific heat coefficient of cotton, approximately equal to 1.725 kJ/kg.

Because of the structure of the TA Q600 thermobalance, the boundary condition

$$\frac{\partial T}{\partial z}(0,t) = -\frac{1}{\lambda}\left(\begin{array}{l} h\left(T_g(t) - T(0,t)\right) \\ +\varepsilon\sigma\left(\left(T_g(t)\right)^4 - \left(T(0,t)\right)^4\right) \end{array}\right)$$

is considered at $z = 0$ (upper surface of the sample in contact with the surrounding heated atmosphere). In this boundary condition, h (resp. ε, σ) is taken equal to 5 W/m^{2}K (resp. 0.9, 5.67×10^{-8} W/m^{2}K^4) and $T_g(t)$ means the temperature of the heated gas surrounding the cotton sample. The temperature $T_g(t)$ is measured in the thermobalance by a thermocouple located just under the crucible.

The boundary condition

$$\frac{\partial T}{\partial z}\left(\frac{e}{2}, t\right) = 0$$

is imposed on the mid-thickness ($z = e/2$, with $e = 0.001$ m) surface of the sample, thus considering a symmetry of the sample layer with respect to its mid-thickness

The temperature T starts at $t = 0$ from the room temperature: $T(z,0) = 20$ °C= 293.15 K.

The above heat transfer problem is solved through an implicit finite difference method with $dt = 8950/896 = 9.99$ s and $dz = (e/2)/100 = 5.0 \times 10^{-6}$ m, using the numerical software Scilab.

When solving this heat transfer model, typical curves representing the difference $diff(t)$ between the measured temperature $T_g(t)$ of the surrounding gas and the mean value $(T(0,t) + T(e/2,t))/2$ of the computed temperatures at the upper surface ($z = 0$) and at the mid-thickness ($z = e/2$) of the bed are given in Figure 6 for temperature ramps of 5, 20 and 50 °C/min.

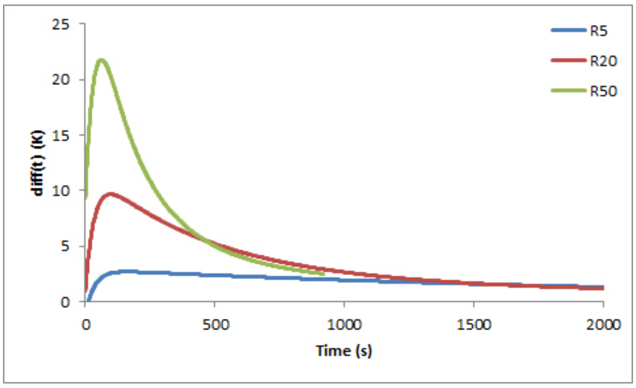

Figure 6: Differences between the temperature T_g of the surrounding gas and the mean value $(T(0,t) + T(e/2,t))/2$ inside the sample, for temperature ramps of, 5, 20 and 50 °C/min.

The difference $diff(t)$ first increases and reaches its maximal value at 22 °C (resp. 10 °C) for a temperature ramp of 50 °C/min (resp. 20 °C/min). The differences between the gas temperature and the solid temperature are really significant and, consequently, have to be taken into account in the determination of the kinetic parameters. In the model (5), $T(t)$, which was initially taken as the measured gas temperature $T_g(t)$, is replaced by the computed mean value

$(T(0,t)+T(e/2,t))/2$. For example, considering a temperature ramp of 50 °C/min, a quite perfect superimposition of the devolatilization peaks may be observed between the experimental and the simulated mass loss rate curves using the above computed values of the kinetic parameters for a temperature ramp of 5 °C/min and considering the above-indicated mean value temperature in the sample instead of $T_g(t)$, see Figure 7.

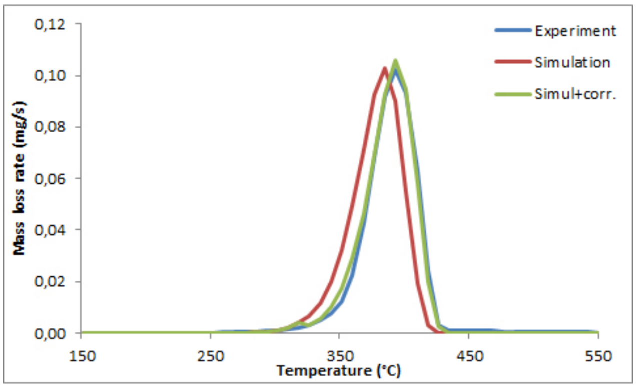

Figure 7: Comparison between the experimental mass loss rate curve (blue) under nitrogen and under temperature ramp equal to 50 °C/min, and the simulation with the values of the kinetic parameters obtained for a temperature ramp of 5 °C/min without correction (red) or with the correction given by the mean value of the temperatures (green).

4 A simplified model accounting for the combustion of coal char particles in a drop tube furnace

4.1 Description of the experiment

Coal char particles are produced from coal particles through a pyrolysis process in a drop tube furnace. Volatiles are emitted from the coal particles and a swelling phenomenon occurs.

In a drop tube furnace, the temperature ramp is much higher than in a thermobalance: around 1500 K/s, against 10 K/min in a thermobalance. The drop tube furnace is 140 cm long and with an internal diameter of 5 cm. It is heated by resistances placed between its outer and inner envelopes. The reactor temperature is held fixed, here 1100, 1200 or 1300 °C. This device simulates the behavior of industrial pulverized boilers.

For each combustion experiment, 2 mg of coal char particles are injected with an oxidative gas flow (88% N_2, 12% O_2, 40 l/hr) through a water cooled injector and entrained by a secondary nitrogen flow (360 l/hr) preheated at 900 °C. During the fall in the drop tube furnace, the temperature of the coal char particle was continuously measured by a pyrometer placed at the bottom of the reactor and pointing vertically to the injection probe of the reactor.

The fall time of the coal char particle is very short: around 1s.

4.2 Description of the model

As a simplifying hypothesis, the structure of the coal char particles may be considered as uniform at each time of the experiment. The temperature of the particle may also be supposed uniform inside the particle. Based on these hypotheses, a simplified model has been elaborated which intends to predict the temperature of the particle during its fall in the drop tube furnace.

The energy balance during the heating up of a spherical char particle of fixed radius R_p is written as

$$\begin{aligned}&\left(\begin{array}{l} m_C^0\left(1-X_C\right)\left(C_p\right)_C\left(T_p\right)\\ +m_p^0\tau_{ash}^0\left(C_p\right)_{ash}\left(T_p\right)\end{array}\right)\frac{\partial T_p}{\partial t}(t)\\ &=4\pi R_p^2\varepsilon_p\sigma\left(T_R^4-T_p^4(t)\right)\\ &+4\pi R_p^2 h\left(T_g-T_p(t)\right)\\ &+\left(\begin{array}{l}\tau\left|\Delta_R H_{CO}^0\left(T_p(t)\right)\right|\\ +\left|\Delta_R H_{CO_2}^0\left(T_p(t)\right)\right|\end{array}\right)\frac{\Phi_{O_2}(t)}{1+\tau/2}.\end{aligned} \tag{13}$$

The equation for oxygen transport within the particle is written as

$$\begin{aligned}&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_{O_2}^e\left(T_p\right)\frac{\partial C_{O_2}}{\partial r}\right)(r,t)\\ &=\frac{k_s(t)A_C(t)\rho_C(t)}{\Lambda(t)}C_{O_2}(r,t),\end{aligned} \tag{14}$$

with the boundary conditions

$$\left\{\begin{array}{l} D_{O_2}^e\left(T_p\right)\frac{\partial C_{O_2}}{\partial r}\left(R_p,t\right)\\ =k_d(t)\left(\left(CO_2\right)_\infty-CO_2\left(R_p,t\right)\right),\\ \frac{\partial C_{O_2}}{\partial r}(0,t)=0.\end{array}\right. \tag{15}$$

The evolution of the conversion ratio X_C defined as $X_C(t)=1-m_C(t)/m_C^0$ is described through

$$\frac{dX_C}{dt}(t)=\frac{\Phi_{O_2}(t)M_C\Lambda(t)}{m_C^0}, \tag{16}$$

the oxygen flux Φ_{O_2} entering the particle being defined through the expression

$$\begin{aligned}&\frac{\left(CO_2\right)_\infty}{\Phi_{O_2}(t)}\\ &=\frac{3\Lambda(t)}{\eta_p k_s\rho_p^0 4\pi R_p^3 A_0\left(1-X_C(t)\right)\sqrt{1-\Psi\ln\left(1-X_C(t)\right)}}\\ &+\frac{1}{k_d(t)4\pi R_p^2}.\end{aligned}$$

The different coefficients which appear in equations (13)-(16) are known or are given known expressions. The coefficient $k_s(t)$ which appears in (14) is given an

Arrhenius expression whose frequency factor and activation energy are deduced from experiments performed on the coal char particles in a thermobalance under an oxidative atmosphere and under a temperature ramp of 5 °C/min, through the EIPR method described in the preceding section.

For the resolution of (13) and (16), an explicit Euler method has been applied (progression with respect to the time parameter). For the resolution of (14), a second-order finite difference method has been built, see [9] for the details of this method, which is slightly better than the classical finite element method but which does not require the introduction of complementary basis functions. Further, observe that the equation (14) is written in spherical coordinates. The equation may thus look as singular. But the boundary condition $(15)_2$ prevents from real singularity. The whole problem has been solved using a Fortran code.

The experimental and simulated temperature curves are gathered in Figure 8.

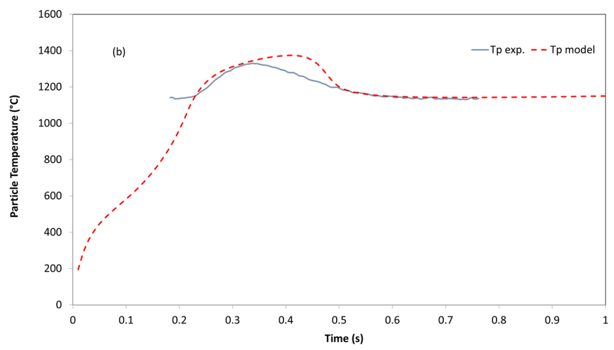

Figure 8: Experimental and simulated temperature curves of a coal char particle injected in a drop tube furnace, with a regulation temperature of 1200 °C.

The maximal difference between the experimental and simulated temperature particle is equal to 43 °C in the interesting time range (0.2-0.8 s). This may seem high. But first the uncertainties of the temperature measurements in a drop tube furnace through a pyrometer are usually taken equal to 50 °C, because of the important radiations from the reactor wall. Second, the accuracy of the pyrometer itself is given at 6 °C. Finally, the relative difference is equal to 3%, which is largely acceptable.

Of course, this simplified model does not simulate the mass loss of the particle during its fall in the drop tube furnace, as it does not involve the evolutions of the chemical species during the combustion process.

A more sophisticated model is currently being developed in Mulhouse, which takes into account local variations of the structure of the particle during its fall in the drop tube furnace (pore opening, progressive destruction of the structure from the exterior of the particle).

5 More complex models accounting for the thermal degradation of materials

For the construction of such models, the chemical species which are emitted during the thermal degradation process have to be determined first. In [10], the authors give a long list of possible chemical reactions which may occur during biomass pyrolysis, together with the corresponding kinetic parameters which are given Arrhenius expressions. Then the balance equations are written, see [11], in the case of 3 (main) chemical reactions and 5 gas species, which lead to a coupled system of 8+7 equations (inside the particle and in a boundary layer around the particle).

5.1 Inside the spherical particle

- Conservation of each of the gaseous species ($k = 1, \ldots, 5$)

$$\frac{\partial\left(\varepsilon\rho_{g,s}m_{k,s}\right)}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2N_{tg,s}M_{av}m_{k,s}\right)$$
$$=\frac{D_{ke,s}\rho_{g,s}}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial m_{k,s}}{\partial r}\right)+\sum_l\varepsilon R_l\gamma_{kl}M_k,$$

with $l = 3$.

- Total molar balance of gas mixture

$$\frac{\partial\left(\varepsilon c_{t,s}\right)}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2N_{tg,s}\right)=\sum_l\sum_k\varepsilon R_l\gamma_{kl}.$$

- Energy balance equation

$$\frac{\partial\left(C_{ps}\rho_sT_s\right)}{\partial t}$$
$$+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2N_{tg,s}M_{av,s}C_{pg,s}T_s\right)$$
$$=\frac{\lambda_e}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_s}{\partial r}\right)+\sum_l\varepsilon R_l\left(-\Delta H_l\right).$$

- Carbon mass balance

$$\frac{\partial W_C}{\partial t}=-\left(2\frac{\eta+1}{\eta+2}R_1+R_2\right)M_C.$$

5.2 In a gas boundary layer

- Conservation of each of the gaseous species ($k = 1, \ldots, 5$)

$$\frac{\partial\left(\rho_gm_k\right)}{\partial t}$$
$$+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2N_{tg}M_{av}m_k\right)$$
$$=\frac{D_{ke}\rho_g}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial m_k}{\partial r}\right)+R_3\gamma_{k3}M_k.$$

- Total molar balance of gas mixture

$$\frac{\partial c_t}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 N_{tg}\right) = \sum_k R_3 \gamma_{k3}.$$

- Energy balance equation

$$\frac{\partial \left(C_{pg}\rho_g T_g\right)}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 N_{tg} M_{av} C_{pg} T_g\right) = \frac{\lambda_g}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T_g}{\partial r}\right) + R_3\left(-\Delta H_3\right).$$

Initial and boundary conditions are added to this problem.

5.3 Other complex models

In [12], the authors introduce a simpler model, starting with the conservation law

$$\varepsilon \frac{\partial \rho_g}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho_g v_g\right) = S_{mass},$$

in the case of a spherical particle, where ε is the porosity of the particle (which may evolve along the combustion process), ρ_g is the gas density, v its velocity and S_{mass} represent the mass sources due to a transfer to the gas phase. The balance of linear momentum is written using Darcy's law as

$$\varepsilon \frac{\partial \rho_g v_g}{\partial t} = -\frac{\partial p}{\partial r} - \frac{\mu}{k} - v_g - C\rho_g v_g \left|v_g\right|,$$

where p is the pressure, μ the viscosity, k the permeability and C is a constant. The convection+diffuse transport of the gaseous species inside the particle is written as

$$\frac{\partial \rho_{i,g}}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho_{i,g} v_g\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{D_{i,eff}}{M_i}\frac{\partial \rho_{i,g}}{\partial r}\right) + \sum_k w_{k,i,gas},$$

where $w_{k,i,gas}$ is a reaction term. Finally the energy conservation is written as

$$\frac{\partial \sum_i \rho_i C_{p,i} T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \lambda_{eff}\frac{\partial T}{\partial r}\right) + \sum_k w_k H_k.$$

Appropriate boundary and initial conditions are also added.

In [13], the authors introduce quite similar balance equations but in a steady regime.

Of course, for the resolution of such coupled and complex systems of evolution equations, numerical codes have to be built or dedicated software have to be used. In their paper [11], the authors indicate that they used Comsol multiphysics software for the resolution of this problem.

Such complex models have not been solved in the GRE lab. Interested readers are referred to the indicated references and to the further references these limited references contain.

To our knowledge, existence and uniqueness results have never been proved for such coupled systems of evolution equations, although first they are based on balance equations and second they are certainly not "singular" systems. A theoretical numerical resolution of such systems should be developed.

6 Conclusion

Throughout this presentation, different methods and models accounting for the thermal degradation of combustible materials under controlled temperature ramps and non-oxidative or oxidative atmospheres have been presented. Each method or model is based on hypotheses and presents limits, although the obtained simulation of the thermal degradation process is roughly acceptable. Mathematical analyses of the models should be done in order to also improve these simulations. The references listed below will help the interested readers for deeper understandings of these concrete experiments.

References

[1] S. Vyazovkin, AK. Burnham, JM. Criado, LA. Perez-Maqueda, C. Popescu and N. Sbirrazuoli, "ICTAC Kinetics Committee recommendations for performing kinetic computations on thermal analysis data," Therm. Acta 520, 1–19, 2011. http://doi.org/10.1016/j.tca.2011.03.034.

[2] M. Valente, A. Brillard, C. Schönnenbeck and JF. Brilhac, "Investigation of grape marc combustion using thermogravimetric analysis. Kinetic modeling using an extended independent parallel reaction (EIPR)," Fuel Proces. Technol. 131, 297–303, 2015. http://dx.doi.org/10.1016/j.fuproc.2014.10.034.

[3] A. Brillard, JF. Brilhac and M. Valente, "Modelization of the grape marc pyrolysis and combustion based on an extended independent parallel reaction and determination of the optimal kinetic constants," Comp. Appl. Math. 36(1), 89-109, 2017. DOI 10.1007/s40314-015-0216-5.

[4] JJM. Orfao, FJA. Antunes and JL. Figueiredo, "Pyrolysis kinetics of lignocellulosic materials - three independent reactions model," Fuel 78, 349-358, 1999. PII: S0016-2361(98)00156-2.

[5] T. Damartzis, D. Vamvuka, S. Sfakiotakis and A. Zabaniotou, "Thermal degradation studies and kinetic modeling of cardoon (Cynara cardunculus) pyrolysis using thermogravimetric analysis (TGA)," Bioresour Technol. 102, 6230–6238, 2011. doi:10.1016/j.biortech.2011.02.060.

[6] J. Yu, N. Paterson, J. Blamey and M. Millan, "Cellulose, xylan and lignin interactions during pyrolysis of lignocellulosic biomass," Fuel 191, 140–149, 2017. doi:10.1016/j.fuel.2016.11.057.

[7] A. Brillard, D. Habermacher and JF. Brilhac, "Thermal degradations of used cotton fabrics and of cellulose: kinetic and heat transfer modeling," Cellulose 24(3), 1579-1595, 2017. DOI 10.1007/s10570-017-1200-6.

[8] P. Gilot, A. Brillard, C. Schönnenbeck and JF. Brilhac, "A simplified model accounting for the combustion of pulverized coal char particles in a drop tube furnace," Energy Fuels, 31(10), 11391–11403, 2017. DOI: 10.1021/acs.energyfuels.7b01756.

[9] A. Brillard, JF. Brilhac and P. Gilot, "A second-order finite difference method for the resolution of a boundary value problem associated to a modified Poisson equation in spherical coordinates," Appl. Math. Model. 49, 182–198, 2017. http://dx.doi.org/10.1016/j.apm.2017.04.034.

[10] A. Anca-Couce and R. Scharler, "Modelling heat of reaction in biomass pyrolysis with detailed reaction schemes," Fuel 206, 572–579, 2017. http://dx.doi.org/10.1016/j.fuel.2017.06.011.

[11] AK. Sadhukhan, P. Gupta and RK. Saha, "Modeling and experimental studies on single particle coal devolatilization and residual char combustion in fluidized bed," Fuel 90, 2132-2141, 2011. doi:10.1016/j.fuel.2011.02.009.

[12] B. Peters, A. Džiugys and R. Navakas, "A shrinking model for combustion/gasification of char based on transport and reaction time scales," Mechanika 18(2), 177-185, 2012.

[13] K. Annamalai and IK. Puri, "Combustion science and engineering," CRC Series in Computational Mechanics and Applied Analysis. CRC Press, Boca Raton, 2007.

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Threshold Multi Split-Row algorithm for decoding irregular LDPC codes

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ABSTRACT
In this work, we propose a new threshold multi split-row algorithm in order to improve the multi split-row algorithm for LDPC irregular codes decoding. We give a complete description of our algorithm as well as its advantages for the LDPC codes. The simulation results over an additive white gaussian channel show that an improvement in code error performance between 0.4 dB and 0.6 dB compared to the multi split-row algorithm.

1 Introduction

Low-density parity check (LDPC) codes are a class of linear block codes, which were first, introduced by gallager in 1963 [1]. As soon as they were re-invented in 1996 by mackay [2], LDPC codes have received a lot of attention because their error performance is very close to the shannon limit when decoded using iterative methods [3]. They have emerged as a viable option for forward error correction (FEC) systems and have been adopted by many advanced standards, such as 10 gigabit ethernet (10GBASET) [4], [5] and digital video broadcasting (DVB-S2) [6], [7]. In addition, the generations of WiFi and WiMAX are considering LDPC codes as part of their error correction systems [8], [9].

In the present paper, we propose the threshold multi split-row method for decoding irregular low-density parity-check (LDPC) codes [10], to have better performance in terms of bit error rate. We will apply the thresholding algorithm in [11] for multi split row of irregular LDPC codes[10].

The proposed split-row decoder [10],[11] splits the row processing into two or multiple nearly independent partitions. Each block is simultaneously processed using minimal information from an adjacent partition. The key idea of split-row is to reduce communication between row and column processors which has a major role in the interconnect complexity of existing LDPC decoding algorithms such as sum product (SPA) [3] and minsum (MS) [12].

This paper introduces threshold multi split-row decoding which significantly reduces wire interconnect complexity and considerably improves the error performance compared to non-threshold multi-split decoding [10]. The paper is organized as follows: Section II reviews the sum product and min-sum, split-row and split-row threshold decoding algorithms. Threshold multi-split-row and its error performance result is presented in section III. Finally, the conclusion is in section IV.

2 Previous algorithms

2.1 Sum Product Decoding(SPD)

The SPD assumes a binary code word $(x_1, x_2, ..., x_N)$ transmitted using a binary phase-shift keying (BPSK) modulation. The sequence is transmitted over an additive white gaussian noise (AWGN) channel and the received symbol is $(y_1, y_2, ..., y_N)$.
We define:
$V_{(i)} = \{j : H_{ij} = 1\}$ as the set of variable nodes which participate in the check equation i.
$C_{(j)} = \{i : H_{ij} = 1\}$ as the set of check nodes which participate in the variable node j update.
Also
$V_{(i)} \setminus j$ denote all variable nodes in $V_{(i)}$ except node j.

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$C_{(j)} \setminus i$ denote all check nodes in $C_{(j)}$ except node i. Moreover, we define the following variables used in this paper.

λ_j : is defined as the information derived from the log-likelihood ratio of received symbol y_j.

$$\lambda_j = log(P(x_i = 0|y_j)/P(x_i = 1|y_j)) = (2y_j)/\sigma^2 \quad (1)$$

where σ^2 is the noise variance. α_{ij} is the message from check node i to variable node j. This is the row processing output. β_{ij} is the message from variable node j to check node i. This is the column processing output.

The sum product algorithm decoding as described by mohsenin and Baas [13], can be summarized in the following four steps:

2.1.1 Initialization

For each i and j, initialize β_{ij} to the value of the log-likelihood ratio of the received symbol y_j, which is λ_j. During each iteration, $\alpha\beta$ messages are computed and exchanged between variables and check nodes through the graph edges according to the following steps numbered 2-4.

2. Row processing or check node update
Compute α_{ij} messages using β messages coming from all other variable nodes connected to check node C_i, excluding the β information from V_j:

$$\alpha_{ij} = \prod_{j' \in V(i) \setminus j} sign(\beta_{ij'}) \times \phi(\Sigma_{j' \in V(i) \setminus j} \phi(|\beta_{ij'}|)) \quad (2)$$

where the non-linear function

$$\phi(x) = -log(tanh(|x|/2)) \quad (3)$$

The first product term in the equation that update the parameter, is called the parity (sign) update and the second product term is the reliability (magnitude) update.

3. Column processing or variable node update
Compute β_{ij} messages using channel information λ_j and incoming, messages α; from all other check nodes connected to variable node V_j, excluding check node C_i.

$$\beta_{ij} = \lambda_j + \Sigma_{i' \in C(j) \setminus i} \alpha_{(i'j)} \quad (4)$$

4. Syndrome check and early termination
When the column processing is finished, every bit in column j is updated by adding the channel information λ_j and α, messages from neighboring check nodes.

$$z_j = \lambda_j + \Sigma_{i' \in C(j)} \alpha_{(i'j)} \quad (5)$$

From the updated vector, an estimated code vector $\widehat{X} = \{\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_N\}$ is calculated by:

$$\widehat{x}_i = \begin{cases} 1 \; if \;\; z_i \le 0 \\ 0 \; if \;\; z_i > 0 \end{cases} \quad (6)$$

If $H \times \widehat{x}^t = 0$ then $\widehat{X}$ is a valid codeword and therefore the iterative process has converged and decoding stops. Otherwise, the decoding repeats from step 2 until a valid codeword is obtained or the number of iterations reaches a maximum number, which terminates the decoding process.

2.2 MinSum Decoding

The check node or row processing stage of SP decoding can be simplified by approximating the magnitude computation in Eq. 2 with a minimum function. The algorithm using this approximation is called min-sum (MS):

$$\alpha_{ij} = \prod_{j' \in V(i) \setminus j} sign(\beta_{ij'}) \times Min_{j' \in V(i) \setminus j}(|\beta_{ij'}|) \quad (7)$$

In MS decoding, the column operation is the same as in SP decoding. The error performance loss of MS decoding can be improved by scaling the check (α) values in Eq. 7 with a scale factor $S \le 1$ which normalizes the approximations [14], [15].

$$\alpha_{ij} = S \times \prod_{j' \in V(i) \setminus j} sign(\beta_{ij'}) \times Min_{j' \in V(i) \setminus j}(|\beta_{ij'}|) \quad (8)$$

2.3 Split Row and Split Row Threshold Decoding

Recall that a standard message passing two-phase algorithm consists of a check node update followed by a variable node update as shown in Fig. 1 (a). The basic idea in split-row [10], [16] and split-row threshold [11], is to divide the check node processing into two or multiple nearly-independent partitions. Each check node processor, simultaneously, computes a new message while using minimal information from its adjacent partitions. Split-row is illustrated in Fig. 1 (b) showing how the check node processing is partitioned into two blocks. A single bit of information (Sign) for each check node processor must be sent between partitions to improve the error performance.

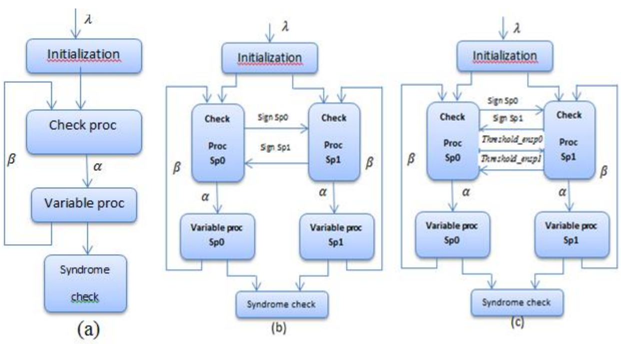

Figure 1. Block diagram of (a) standard two-phase decoding (b) split-row (c) split-row threshold block diagram.

The loss error performance is 0.5-0.7 dB of split-row that is the major drawback, when compared to min-sum normalized and SPA decoders. This performance degradation is dependent on the number of check node partitions. For minsum split-row each partition has no information of the minimum value of the other partition, and a Sign signal is sent to minimize the error due to incorrect sign information of the true check node output α.

The split-row threshold algorithm mitigates the error caused by incorrect magnitude by providing a signal, threshold_en, which indicates whether a partition has a minimum less than a given threshold (T).This causes all check nodes to take the min of their own local minimum or T. Thus, any large deviations from the true minimum because of the partitioning are reduced, which then makes multi-split implementations feasible.

Figure 1 (c) shows the additional single bit signal (threshold_en) added to the original split-row architecture by the split-row threshold algorithm. This signal allows the split-row to remain essentially unchanged while adding some extra logic and minimal wiring to improve error significantly [11].

Therefore, the split-row architectures reduce the communication between check node and variable node processors, which is the major cause of the interconnect complexity found in existing LDPC decoder implementations.In addition, the area of each check node processor will be reduced. However, note that the variable node operation in the SPA, minsum, split-row and split-row threshold algorithms are all identical. Thus, the logic of the variable node processor is left unchanged.

3 Threshold Mutli Split Row Decoding

3.1 Threshold Mutli Split Row Algorithm

In order to increase the parallelism and reduce the complexity of the decoder, the multi-split-row method divides the lines of the systematic submatrix Hs into Spn blocks called split-Spn. This approximation requires of the wire to correctly process the sign bits between the blocks. For the threshold multi split-row method, Each partition sends the status of its local minimum against a threshold to the next partition with a single wire, called $Threshold\_ensp$.

The block diagram of threshold multi split-row decoding with Spn partitions, highlighting the sign and $Threshold\_ensp$ passing signals, is shown in Fig. 2. These are the only wires passing between the partitions [17]. In each partition, local minimums are generated and compared with a threshold T simultaneously. If the local minimum is smaller than T then the $Threshold\_ensp$ signal is asserted high. The magnitudes of the check node outputs are finally computed using local minimums and the $Threshold\_ensp$ signal from neighboring partitions.
If a local partition′s minimums are larger than T , and at least one of the $Threshold\_ensp$ signals is high, then T is used to update its check node outputs. Otherwise, local minimums are used to update check node outputs.

in the threshold multi split-row algorithm, as shown in Eq. (10) and (11), the first and second Mins are compared with a predefined threshold, and a single-bit threshold-enable ($Threshold\_en\_out$) global signal is sent to indicate the presence of a potential global minimum to other partitions.

3.1.1 Initialization

The first and second minimums are determined for each partition according to the following relationships:

$$Min_{j' \in V(i) \setminus j}(|\beta_{ij'}|) = \begin{cases} Min1_i; \ if \ ; \ j \neq argmin(Min1_i) \\ Min2_i; \ if \ ; \ j = argmin(Min1_i) \end{cases} \quad (9)$$

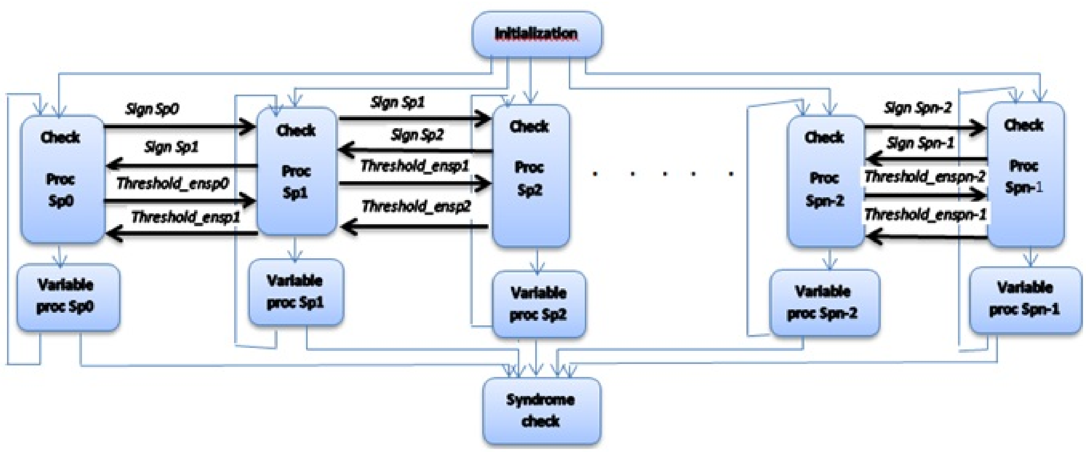

Figure 2. Block diagram of threshold multi split row decoding with Spn partitions.

$$Min1_i = min_{j \in V(i)}(|\beta_{ij'}|) \quad (10)$$

$$Min2_i = min_{j'' \in V(i) \backslash argmin(min1_i)}(|\beta_{ij''}|) \quad (11)$$

to execute the algorithm one needs the number of partitions and the threshold value T. Finds $Threshold_$ensp(i)_out(i∓1) for the ith partition of a spn-decoder.

Algorithm 1 Threshold Multi Split-Row Algorithm

1-if $Min1_i \leq T$ *and* $Min2_i \leq T$ then
$Threshold\_ensp(i)\_out(i \mp 1) = 1$

$$Min_{j' \in V(i) \backslash j}(|\beta_{ij'}|) = \begin{cases} Min1_i & if \quad j \neq argmin(Min1_i) \\ Min2_i & if \quad j = argmin(Min1_i) \end{cases} \quad (12)$$

2- if $Min1_i \leq T$ *and* $Min2_i \geq T$ then
$Threshold\_ensp(i)\_out(i \mp 1) = 1$
$Threshold\_ensp(i+1) == 1$ or
$Threshold\_ensp(i-1) == 1$ then

$$Min_{j' \in V(i) \backslash j}(|\beta_{ij'}|) = \begin{cases} Min1_i & if \quad j \neq argmin(Min1_i) \\ T & if \quad j = argmin(Min1_i) \end{cases} \quad (13)$$

else

$$Min_{j' \in V(i) \backslash j}(|\beta_{ij'}|) = \begin{cases} Min1_i & if \quad j \neq argmin(Min1_i) \\ Min2_i & if \quad j = argmin(Min1_i) \end{cases}$$

end if
3- else if $Min1_i \geq T$ and $(Threshold\_ensp(i+1) == 1$ or $Threshold\_ensp(i-1) == 1)$ then
$Threshold\_ensp(i)\_out(i \mp 1) = 0$

$$Min_{j' \in V(i) \backslash j}(|\beta_{ij'}|) = T \quad (14)$$

4- else
$Threshold\_ensp(i)\_ou(i \mp 1) = 0$

$$Min_{j' \in V(i) \backslash j}(|\beta_{ij'}|) = \begin{cases} Min1_i & if \quad j \neq argmin(Min1_i) \\ Min2_i & if \quad j = argmin(Min1_i) \end{cases} \quad (15)$$

end if.

The kernel of the threshold multi split-row algorithm is given in Algorithm 1. As shown, four conditions will occur:

Condition 1: both *Min1* and *Min2* are less than threshold *T*. In this case these 2 parameters are used to calculate α messages according to Eq.(12). In addition, $Threshold\_ensp(i)\_out$(i∓1), which represents the general threshold-enable signal of a partition with two neighbors, is asserted high indicating that the least minimum *(Min1)* in this partition is smaller than *T*.

Condition 2: only *Min1* is less than *T*. As Condition 1 $Threshold\_ensp$(i)_out(i∓1)= 1. If at least one $Threshold\_ensp(i \mp 1)$ signal from the nearest neighboring partitions is high, indicating that the local minimum in the other partition is less than *T*, then we use *Min1* and *T* to calculate α messages according to Eq.(13). Otherwise, we use Eq.(12).

Condition 3: when the local *Min1* is larger than *T* and at least one $Threshold\_ensp(i \mp 1)$ signal from the nearest neighboring partitions is high; thus, we only use *T* to compute all α messages for the partition using Eq.(14).

Condition 4: when the local *Min1* is larger than *T* and if the $Threshold\_ensp(i \mp 1)$ signals are all low; thus, we again use Eq.(12).

The variable node operation in minsum threshold multi split-row algorithm is identical to the minsum normalized and minsum multi split-row algorithms.

3.2 BER Simulation Results

The error performance simulations presented here assume an additive white gaussian noise (AWGN) channel with binary phase-shift keying (BPSK) modulation. The maximum number of iterations is set to I_{max} = 7 or earlier when the decoder converged [11]. We fix the threshold Imax to satisfy a tradeoff between the performance correction and the speed of the decoder.

The following labelings are used for the figures: "MS Standard" for normalized minsum, "MS Multi Split-Row" for the method minsum multi split-row algorithm, and "S" for the scaling factor. The performances of irregular LDPC codes are illustrated in the figure given below.

The error performance depends strongly on the choice of threshold *T* values and the scaling factor *S*. The optimum values for *T* and *S* are obtained by empirical simulations as illustrated in the following figures.

The figure 3 shows the simulation to determine experimentally the threshold for minsum threshold multi split-row of the LDPC code (6,18) (1536,1152).
with:
Code length N= 1536.
Information length K=1152.
Column weight Wc = 6 which is the number of ones per column.
Row weight Wr =18 which is the number of ones per row.
The threshold optimum value for an SNR ranging from 2.7 dB to 4.2 dB, with the value of the normalization factor S = 0.25 obtained in figure 6, is 1.5.

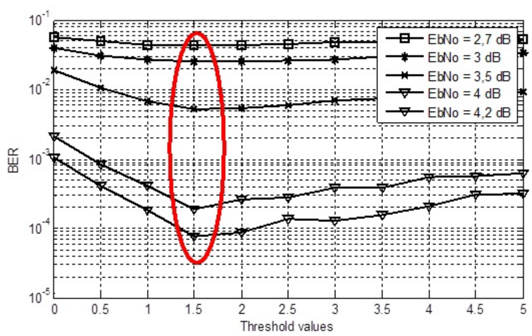

Figure 3. BER (Bit error rate) performance in function of the threshold for different SNR (Signal to noise ratio) values.

Figures 4 and 5 show the simulation to determine experimentally the scaling factor for normalized minsum and minsum multi split-row, respectively, of the LDPC code (6,18) (1536,1152) . Optimal scale factor, for normalized minsum, is 0.6. And optimal scale factor, for minsum multi split-row, is 0.2.

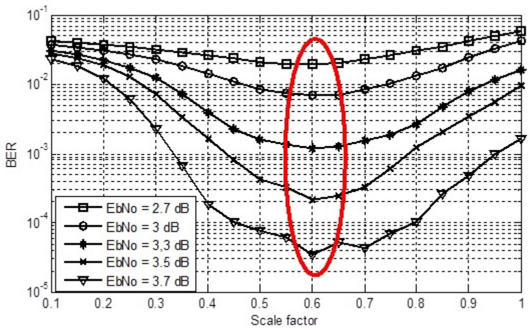

Figure 4. BER (Bit error rate) performance in function of the scale factor using normalized minsum for different SNR (Signal to Noise Ratio) values.

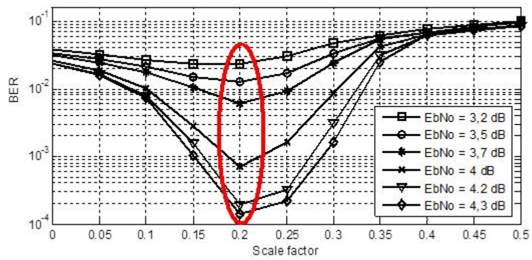

Figure 5. BER (Bit error rate) performance in function of the Scale factor using minsum multi split-row for different SNR (Signal to noise ratio) values.

Figure 6 shows the simulation to determine experimentally the scaling factor for minsum threshold multi split-row of the LDPC code (6,18)(1536,1152). Optimal scale factor is 0.25 with the threshold T=1.5.

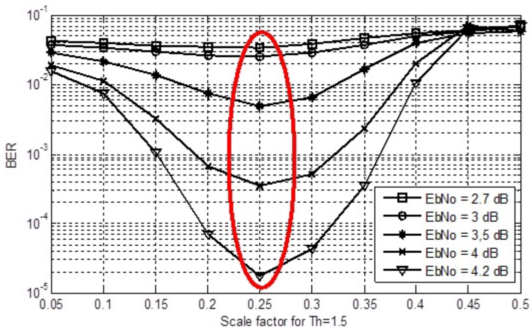

Figure 6. BER (Bit error rate) performance in function of the scale factor using minsum threshold multi split-row for different SNR (Signal to noise ratio) values.

Figure 7 shows the error performance results for a (6,18) (1536,1152) LDPC code for different decoding algorithms (minsum normalized, minsum multi split-row and minsum threshold multi split-row). The code rate is R=0.75 and the maximum number of iteration is set to $I_{max} = 7$, with threshold values obtained in Fig.3 and scale factors obtained in Fig.4, Fig.5 and Fig.6.

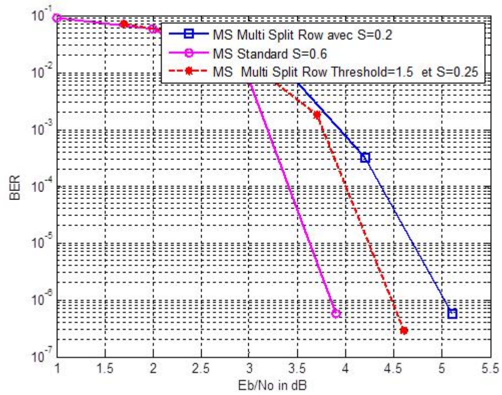

Figure 7. BER performance of (6,18) (1536,1152) irregular code using various decoding algorithms.

As shown in the figure, minsum threshold multi split-row with optimal threshold T = 1.5 performs about 0.6 dB better than minsum multi split-row and is only 0.7 dB away from MinSum normalized at $BER = 6.10^{-7}$.

The minsum threshold multi split-row algorithm utilizes a threshold-enable signal to compensate for the loss of min() interpretation information in multi-split-row algorithm. It provides at least 0.6 dB error performance over the multi-split-row algorithm with Spn = 4. Partitioning the parity matrix in Spn blocs allows us to increase the decoder's parallelism. Therefore, the partitions are performed simultaneously and we obtained a parallel decoder. That means,the complexity decreases and the decoding becomes faster.

As shown in Figure 7, the better performance gap indicated between the minsum threshold multi split-row and minsum multi split-row algorithm revert to the information passed from one partition to another (*Threshold_ensp*) in algorithm threshold multi split-row, which reduces difference between the local minimum in each partition and the first and second global minimums.

4 Conclusion

In this paper we have extended threshold multi split-row decoding method used for regular LDPC code to the irregular LDPC code. The aims was to improve the errors performances of the decoder. Simulation results show that the threshold multi split-row outperforms the multi split-row algorithm for 0.6 dB while maintaining the same level of complexity (only the comparison between the threshold and the first and second minimums is added). Our simulation results show that for a given LDPC code keeping threshold T constant at any SNR does not cause any error performance degradation.

References

1. R. G. Gallager, "Low-Density-Parity-Check,",IRE Trans. Information Theory, vol. II, no. 8, pp. 21-28, Jan 1962.
2. D. MacKay et R. M. Neal, "Near Shanon Limit Performance of Low-Density-Parity-Check Codes," ,Electron. Lett., vol. 32, pp. 1645-1946, Aug. 1996

3. D. J. C. MacKay, “Good error-correcting codes based on very sparse matrices,”,IEEE Trans. Inform. Theory, vol. 45, p. 399-432, Mar. 1999.
4. “IEEE 802.3an, standard for information technology-telecommunications and information exchange between systems”.
5. Cohen, A.E., Parhi and K.K,“A Low-Complexity Hybrid LDPC Code Encoder for IEEE 802.3an (10GBase-T) Ethernet,”,IEEE Transactions on Signal Processing, vol. 57, no. 10, pp. 4085-4094, 2009.
6. “T.T.S.I. digital video broadcasting (DVB) second generation framing structure for broadband satellite applications,”[Online].
7. S.-M. Kim, C.-S. Park and S.-Y. Hwang,“A Novel Partially Parallel Architecture for High-Throughput LDPC Decoder for DVB-S2,”,IEEE Transactions on Con-sumer Electronics, vol. 56, no. 2, pp. 820-825, May 2010.
8. Y.-L. Wang, Y.-L. Ueng, C.-L. Peng and C.-J. Yang, “A Low-Complexity LDPC Decoder Architecture for WiMAX Applications,”,in Proc. of the Int. Symp on VLSI Design, Automation and Test, Hsinchu, April 2011.
9. K.-W. Shin and H.-J. Kim,“A Multi-Mode LDPC Decoder for IEEE 802.16e Mobile WiMAX,”, Journal of Semiconductor Technology and Science, vol. 12, no. 1, pp. 24-33, Mar. 2012.
10. R. ElAlami and M. Mrabti,“Reduced Complexity of Decoding Algorithm for Irregular LDPC Codes Using a Multiple Split Row Method,”, International Journal of Research and Reviews in Mechatronic Design and Simulation, vol. 1, no. 1, pp. 12-17, March 2011.
11. R. ElAlami, H. Qjidaa, Y. Mehdaoui and M. Mrabti,“A Thresholding Algorithm for Improved Split-Row Decoding Method of Irregular LDPC Codes,”,in Proc. of the The IEEE International Conference on Intelligent Systems and Computer Vision 2015, Fez, Morocco, March 2015.
12. M. Fossorier, M. Mihaljevic and H. Imai,“Reduced Complexity Iterative Decoding of Low-Density Parity-Check Codes Based on Belief Propagation,”, IEEE Transactions on Communications, vol. 47, no. 5, pp. 673-680, May 1999.
13. Mohsenin T, Baas B.“Split-row: a reduced complexity, high throughput LDPCdecoder architecture,”,Proceedings of the ICCD, October. 2006.
14. J. Chen and M. Fossorier,“Near optimum universal belief propagation based decoding of low-density parity check codes,”, IEEE Transaction Communications, vol. 50, pp. 406-414, Mar. 2002.
15. J. Chen, A. Dholakia, E. Eleftheriou, and M. Fossorier,“Reduced complexity decoding of LDPC codes,”,IEEE Transaction Communications, vol. 53, pp. 1288-1299, Aug. 2005.
16. R. ElAlami, M. Mrabti and C. B. Gueye,“Reduced Complexity of Decoding Algorithm for Irregular LDPC Codes Using a Split Row Method,”,Journal of Wireless Networking and Communications, vol. 4, no. 2, pp. 29-34, 2012.
17. T. Mohsenin, D. Truong, and B. Baas, “Multi-split-row threshold decoding implementations for LDPC codes,” in Proc. ISCAS, May 2009,pp. 2449-2452.

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A new color image encryption algorithm based on iterative mixing of color channels and chaos

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A B S T R A C T

In this paper, we present a novel secure cryptosystem for direct encryption of color images, based on an iterative mixing spread over three rounds of the R, G and B color channels and three enhanced chaotic maps. Each round includes an affine transformation that uses three invertible matrices of order 2×2, whose parameters are chosen randomly from a chaotic map. The proposed algorithm has a large secret key space and strong secret key sensitivity, which protects our approach from a brutal attack. The simulation results show that our algorithm is better for color images in terms of Peak Signal to Noise Ratio (PSNR), entropy, Unified Average Changing Intensity (UACI) and Number of Pixels Change Rate (NPCR).

1 Introduction

The proliferation of access to information terminals, as well as the implementation of digital personal data transfers, requires the availability of means that ensure a reliable, fast and genuine exchange. In fact, the use of a communications network exposes the transfer of data to certain risks, which require adequate and appropriate security measures. For example, Pirated images can subsequently be the subject of data exchange and illegal digital storage. Data encryption is often the only effective way to meet these requirements. According to Shannon, the basic techniques [1] for an encryption system can be classified into two main categories: permutation of positions (diffusion) and transformation of values (confusion). And the combination between the two classes is also possible. In the literature, several algorithms based on confusion and diffusion have been developed [2-3]. Currently, the use of chaos in cryptosystems, has caught the attention of researchers. This is due to the characteristics of the chaotic signals. To quote, unpredictability, ergodicity, and sensitivity to parameters and initial values. Recently, several chaos-based articles have been proffered [4-5]. Unfortunately, the one-dimensional logistic map has been widely used in several cryptosystems based on chaos [6-7]. Nevertheless, this map has some disadvantages when it is used in cryptography such as not uniform distribution, small space key, chaotic discontinuous ranges, and periodicity in chaotic ranges [8]. Its very important to produce a new chaotic system with better chaotic performance. In this work, we proposed a method based on an improvement of the three chaotic maps, namely the Pseudorandomly Enhanced Logistics Map (PELM) [9], the Pseudorandomly Enhanced Sine Map (PESM) and the Pseudorandomly Enhanced Skew Tent Map (PESTM). Thus, the red channel of the original image is mixed with the ELM, the green channel with the ESM, and the blue channel with the ESTM. Then, a strong avalanche effect will be applied on the three mixed channels, so that a small perturbation on a pixel of the original image will be reflected on the entire image.

The paper is organized as follows. Section 2 makes a new chaotic maps by using the above mentioned three 1D chaotic maps. Section 3 gives a detailed explanation of the proposed image encryption scheme. Section 4 present experimental results demonstrating performance of the proposed method against statistical and sensitivity cryptanalysis. section 5 shows conclusion.

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2 The pseudorandom number generator

The proposed image encryption scheme relies on three improved chaotic maps PELM, PESME and the PESTM, for the generation of pseudo-random number sequences. In this section we show the improvement of the three chaotic maps in terms of chaotic behavior, Lyapunov exponent, bifurcation and distribution.

2.1 Pseudorandomly Enhanced Logistics Map

The logistic map is a simple dynamic nonlinear equation with a complex chaotic behavior, is one of the famous chaotic maps, expressed by the following equation:

$$X_{n+1} = \mu \times X_n \times (1 - X_n) \tag{1}$$

Where $\mu \in [0,4]$ is a control parameter of the logistic map, the variable $X_n \in [0,1]$ with n is the iterations number used to generate the iterative values. The one-dimensional logistics map is characterized by its simple structure and its ease of implementation. Its usually used to encrypt large data in real time [11]. On the other hand, it has several weaknesses [10], including discontinuity, non-uniformity, short periodicity, numerical degradation and weak key space.

We have thus improved the pseudo-randomness of the sequences generated by the logistic map, by a simple multiplication by 10^6 and an application of the modular arithmetic (mod 1). This new pseudo-random generator is indicated in equation (2) [9]:

$$X_{n+1} = mod(((\mu \times X_n \times (1 - X_n))(10^6)) \tag{2}$$

where mod is the operation of module 1. the proposed generator has a Lyapunov exponent higher than that of the logistic map (Fig. 1), and a good distribution (Fig. 2).

2.2 Pseudorandomly Enhanced Sine Map

The sine map is also one of the one-dimensional chaotic maps whose chaotic behavior is similar to that of the logistic map described by the following equation:

$$Y_{n+1} = r \times sin(\pi \times Y_n) \tag{3}$$

Where $r \in]0,1]$ is a control parameter of the sine map, the variable $Y_n \in [0,1]$ with n is the iterations number used to generate the iterative values.
We have also improved the pseudo-randomness of the sequences generated by the Sine map, by a simple multiplication by 10^6 and an application of the modular arithmetic (mod 1). This new pseudo-random generator is indicated in equation (4):

$$Y_{n+1} = mod(((r \times sin(\pi \times Y_n))(10^6)) \tag{4}$$

To test the property of the PESM in terms of the Lyapunov exponent and distribution, several simulations and analysis were performed (see Fig. 1 and Fig. 2).

2.3 Pseudorandomly Enhanced Skew Tent Map

The skew tent map is a one-dimensional simple chaotic system which can be described by [12]:

$$Z_{n+1} = \begin{cases} \frac{Z_n}{b} \;\; if \;\; Z_n < b \\ \frac{1-Z_n}{1-b} \;\; if \;\; Z_n > b \end{cases} \qquad Z_n \in [0,1] \tag{5}$$

Where $Z_n \in [0,1]$ is the state of the chaotic system, and $b \in [0,0.5] \cup [0.5,1]$ is the control parameter. We have also improved the pseudo-randomness of the sequences generated by the skew tent map. This new pseudo-random generator is indicated in equation (6):

$$Z_{n+1} = \begin{cases} mod(((\frac{Z_n}{b})(10^6)),1) \;\; if \;\; Z_n < b \\ mod(((\frac{1-Z_n}{1-b})(10^6)),1) \;\; if \;\; Z_n > b \end{cases} \qquad Z_n \in [0,1] \tag{6}$$

The figure below shows the Lyapunov exponents of the logistic map, the sin map, the skew tent map, and of the pseudo-random generators proposed.

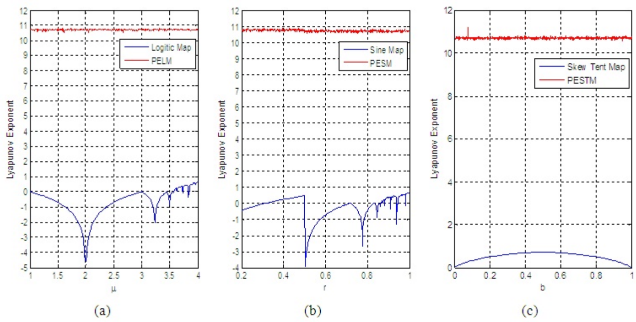

Fig. 1 : Lyapunov exponent of: (a) Logistic Map and PELM; (b) Sine Map and PESM; (c) Skew Tent Map and PESTM.

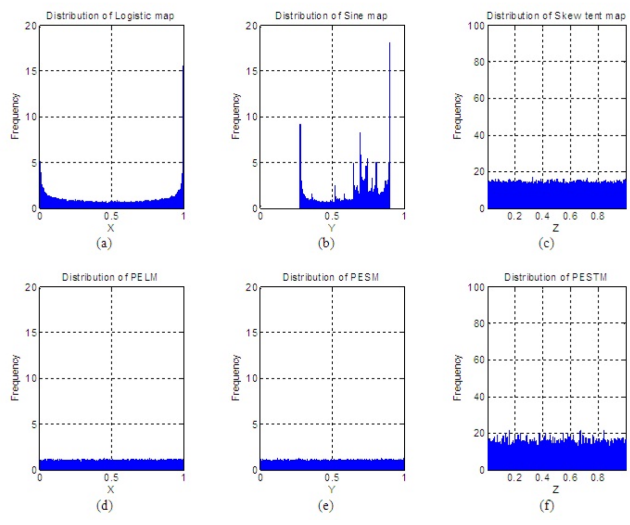

Fig. 2: Distribution of: (a) Logistic Map; (b) Sine Map; (c) Skew Tent Map; (d) PELM; (e) PESM; (f) PESTM.

We can see in the fig. 1 that the Lyapunov exponent is large for the proposed pseudo-random generator. Therefore, this generator presents a higher and faster divergence between two chaotic trajectories having very similar initial conditions. From a cryptographic point of view, a good pseudo-random generator must produce chaotic sequences with uniform distribution. The figure below shows the distribution densities of the logistic map, the sin map, the skew tent map, and of the pseudo-random generators proposed.

It is clear from the fig. 2 that the pseudo-random generators proposed has a uniform distribution density. Therefore, has excellent statistical properties, and is more recommended to use for a robust encryption system.

3 Description of the proposed scheme

Our algorithm is based on three stages of confusion and a diffusion step. Let us consider an $H \times W$ color image P to be encrypted. With H is the height and W the width. The process of our proposed algorithm will be summarized in Fig. 3 :

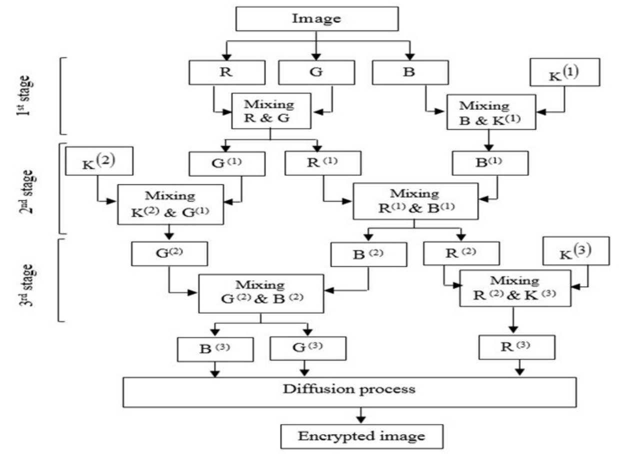

Fig. 3: Flowchart of the proposed algorithm

3.1 Confusion process

The confusion step is described as follows :
Step 1: Load the plain image P of size $W \times H$, and divide it into 3 images with R, G and B channels respectively.
Step 2: Choose six values X_0, Y_0, Z_0, μ, r and b which represent the initial conditions and the control parameters of the pseudo-random generators proposed.
Step 3 : Iterate the PELM, PESM and PESTM respectively for $W \times H$ times to get the state $X = \{X_0, X_1, ..., X_{W\times H-1}\}$, $Y = \{X_0, Y_1, ..., Y_{W\times H-1}\}$ and $Z = \{Z_0, Z_1, ..., Z_{W\times H-1}\}$.
Step 4 : Generate three keys $K^{(1)}$, $K^{(2)}$ and $K^{(3)}$. With $K^{(1)} = mod(floor(X(i) \times 10^{14}), 256)$, $K^{(2)} = mod(floor(Y(i) \times 10^{14}), 256)$, and $K^{(3)} = mod(floor(Z(i) \times 10^{14}), 256)$, With $i = 0, 1 ... W \times H - 1$. Then convert each key into a matrix of dimensions with the size of $W \times H$.
Step 5: Create three invertible matrices $M^{(1)}$, $M^{(2)}$ and $M^{(3)}$ used for iterative mixing. With

$$M^{(1)} = \begin{pmatrix} 1 & p_1 \\ q_1 & 1+p_1q_1 \end{pmatrix}, M^{(2)} = \begin{pmatrix} 1 & p_2 \\ q_2 & 1+p_2q_2 \end{pmatrix},$$

$$and \ M^{(3)} = \begin{pmatrix} 1 & p_3 \\ q_3 & 1+p_3q_3 \end{pmatrix}$$

Where p_1 and q_1 are from $K^{(1)}$, p_2 and q_2 are from $K^{(2)}$, p_3 and q_3 are from $K^{(3)}$.
Step 6 : Mix each color channel with to another or with a key according to the following formulas :

$$\begin{pmatrix} R^{(1)} \\ G^{(1)} \end{pmatrix} = M^{(1)} \times \begin{pmatrix} R \\ G \end{pmatrix} \ (mod\ 256) \quad (7)$$

$$\begin{pmatrix} B^{(1)} \\ Tmp^{(1)} \end{pmatrix} = M^{(1)} \times \begin{pmatrix} B \\ K^{(1)} \end{pmatrix} \ (mod\ 256) \quad (8)$$

$$\begin{pmatrix} R^{(2)} \\ G^{(2)} \end{pmatrix} = M^{(2)} \times \begin{pmatrix} R \\ G \end{pmatrix} \ (mod\ 256) \quad (9)$$

$$\begin{pmatrix} G^{(2)} \\ Tmp^{(2)} \end{pmatrix} = M^{(2)} \times \begin{pmatrix} G^{(1)} \\ K^{(2)} \end{pmatrix} \ (mod\ 256) \quad (10)$$

$$\begin{pmatrix} B^{(3)} \\ G^{(3)} \end{pmatrix} = M^{(3)} \times \begin{pmatrix} B^{(2)} \\ G^{(2)} \end{pmatrix} \ (mod\ 256) \quad (11)$$

$$\begin{pmatrix} R^{(3)} \\ Tmp^{(3)} \end{pmatrix} = M^{(3)} \times \begin{pmatrix} R^{(2)} \\ K^{(3)} \end{pmatrix} \ (mod\ 256) \quad (12)$$

$Tmp^{(1)}$, $Tmp^{(2)}$ and $Tmp^{(3)}$ are variables not taken into consideration in the encryption process.

3.2 Diffusion process

The diffusion step is described as follows :
Step 1: Convert $R^{(3)}$, $G^{(3)}$ and $B^{(3)}$ to a vectors of size $W \times H$, and concatenate them into a single vector $V^{(RGB)} = \{r_0, r_1, ..., r_{W\times H-1}, g_0, g_1, ..., g_{W\times H-1}, b_0, b_1, ..., b_{W\times H-1}\}$. Where $\{r_0, r_1, ..., r_{W\times H-1}\}$, $\{g_0, g_1, ..., g_{W\times H-1}\}$ and $\{b_0, b_1, ..., b_{W\times H-1}\}$ are respectively the components of $R^{(3)}$, $G^{(3)}$ and $B^{(3)}$.
Step 2 : Concatenate the three sequences X, Y and Z into a single vector $V^{(K)} = \{X_0, X_1, ..., X_{W\times H-1}, Y_0, Y_1, ..., Y_{W\times H-1}, Z_0, Z_1, ..., Z_{W\times H-1}\}$. Then by sorting the order of the vector $V^{(K)}$, the chaotic sequence is changed into a sorted order. This sequence is called $V^{(p)}$.
Step 3 : Obtain the permuted image pixel vector $P' = \{p_0, p_1, ..., p_{W\times H-1}\}$, by using the permutation position vector $V^{(p)}$ according to the following formula :

$$P(i) = V^{(RGB)}(V^{(p)}(i)) \quad (13)$$

Step 4 : Obtain the diffusion vector $D =$

$\{d_0, d_1, ..., d_{3\times W\times H-1}\}$, by the following algorithm :

Algorithm 1: Diffusion mechanism

$S \longleftarrow 0$ //Initialisation
For $i \longleftarrow 0$ To $3\times W\times H-1$
  $S \longleftarrow mod(S + P'(i), 256)$
EndFor
$D(0) \longleftarrow S$
For $i \longleftarrow 1$ To $3\times W\times H-1$
  $S \longleftarrow D(i-1) \oplus P'(i)$
  $D(i) \longleftarrow S \oplus V(p)(i)$
EndFor

Step 5 : Convert the diffusion vector D into the R, G and B color image with the size of $W \otimes H$.

3.3 Decryption algorithm

The decryption process is similar to that of encryption procedure in the reversed order.

4 Experimental results and analysis

In this section, we will validate our encryption system in terms of key space, histogram, entropy, correlation coefficient, NPCR, and UACI. All simulations are performed on a personal computer. Table 1 shows the hardware, the software environment and the image source.

Table. 1: Specification table

Processor	Intel Core^{TM} i5-2430M CPU 2.4GHZ
RAM	4GB
Operating system	Windows 8 professional
Programming language	JAVA
Image source	USC-SIPI image data base [13]

4.1 Space of key

The exhaustive attack is to try all possible combinations of keys until obtaining a clear text. Therefore, in a good image cryptosystem, the space of key should be large enough to make brute-force attack infeasible. The secret key of the proposed cryptosystem contains six real numbers (μ, r, b , X_0, Y_0, Z_0). If the precision is 10^{-14}, the key space size can reach to 2^{252}. From a cryptographic point of view, the size of the key space should not be less than 2^{100} to ensure a high level of Security [14]. This means that our algorithm can withstand the brute force attack.

4.2 Statistical Analysis

4.2.1 Histogram analysis

An image histogram is a graphical representation of the number of pixels with the same gray level. Thus, the abscissa axis represents the color level, which starts from zero (color channel off) to 255 (maximum color channel). Fig. 4(a)-(b) show the plain and cipher image. Fig. 4(c)-(d) show the histogram of plain and cipher components' R, G, and B.

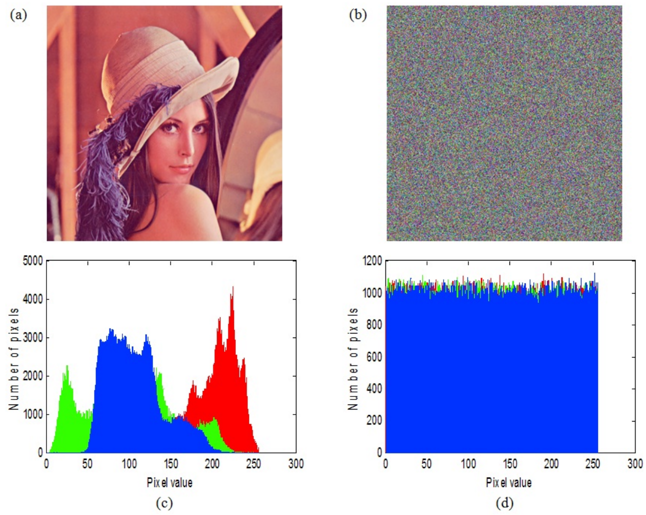

Fig. 4: (a) *Lena* 512×512; (b)encrypted image; (c) Histogram of plain-image; (d) Histogram of encrypted image

Comparing the two histograms of plain and cipher image, we can see that histogram of encrypted image is fairly uniform and is significantly different from that of the original image, and that the encrypted images obtained do not provide any information to the attacker, which can strongly resist statistical attacks.

4.2.2 Information Entropy

Information entropy is designed to evaluate the uncertainty in a random variable as shown in the following equation [1]:

$$H(m) = -\sum_{i=0}^{M} P(m_i) log(P(m_i)) \tag{14}$$

Where, M is the total number of symbols and represents the probability of occurrence of the symbol, for a grayscale image with a data range of, its maximum Information entropy is 8. The results are shown in Table 2.

Table. 2: Entropy of the original image and the encrypted image

mages	Original	Encrypted
House 256×256	7.068625	7.999159
Lena 512×512	7.750197	7.999779
Peppers 512×512	7.669825	7.999776

It is clear from Table 2 that the values of the entropy of the encrypted images are very close to the theoretical value (which is equal to 8).

4.2.3 Correlation analysis

The obvious characteristics of visually meaningful images is redundancy and strong correlation among

adjacent pixels, and it can be used by attackers. In order to test the correlation, we randomly select 1000 pairs of adjacent pixels in three dimension (vertical, horizontal and diagonal). The results are shown in Table 3 and Fig 5.
The correlation coefficient for a sequence of adjacent pixels is given by the following formula:

$$r_{uv} = \frac{cov(u,v)}{\sqrt{D(u)}\sqrt{D(v)}}, \quad (15)$$

Where u and v represent two vectors formed respectively by the values of the pixels of the chosen sequence of the image and the values of their adjacent pixels. The terms $cov(u,v)$, $D(u)$ and $D(v)$ are calculated by the following formulas:

$$E(u) = \frac{1}{N}\sum_{i=1}^{N} u_i, \quad (16)$$

$$D(u) = \frac{1}{N}\sum_{i=1}^{N} [u_i - E(u)]^2, \quad (17)$$

$$Cov(u,v) = \frac{1}{N}\sum_{i=1}^{N} [u_i - E(u)] * [v_i - E(v)] \quad (18)$$

Where N is a large number of adjacent pixel pairs randomly selected in the image (In our case N = 1000), u_i and v_i are respectively the color levels of the collected pixels.

Table. 3: Correlation coefficient of the two adjacent pixels for the original and encrypted image

Image	Image originale			Image cryptée		
	Horizontal	Vertical	Diagonal	Horizontal	Vertical	Diagonal
House256×256	0.978233	0.952932	0.936244	-0.001643	0.008486	-0.001133
Lena512×512	0.961575	0.976149	0.941666	0.002793	-0.003358	0.0048
Pepper512×512	0.965282	0.975921	0.946051	-0.001238	0.004687	-0.003195

We can see in the Table 3, the correlation coefficient values are nearing to zero and negative values which prove that there is no correlation between the plain and cipher image.

4.3 Differential analysis

The Number of Pixels Change Rate (NPCR) [15] and Unified Average Changing Intensity (UACI) [16] are used to analyze the effect of slight change (one bit of single pixel) in the plain image on the cipher image [21]. The formulas used to calculate NPCR and UACI are defined by the following two equations:

$$NPCR = \frac{\sum_{i,j} g(i,j)}{W \times H} \times 100$$

$$UACI = \frac{100}{W \times H}\sum_{i,j} \frac{|C_1(i,j) - C_2(i,j)|}{255} \times 100$$

Where, $C_1(i,j)$ is the encrypted image and $C_2(i,j)$ is the encrypted image after changing a pixel of the clear image. For the pixels at the position (i,j) , if $C_1(i,j) \neq C_2(i,j)$, then $g(i,j) = 1$, otherwise it's equal to zero.
A value of $UACI > 33.4635\%$ and $NPCR > 99.6094\%$ ensures that an image encryption scheme is immune to a differential attack. Table 4 below shows the results of the UACI and NPCR simulations.

Table. 4: NPCR and UACI values after changing the value of a pixel.

Original image	Encrypted image	
	NPCR	UACI
House256×256	100	39.385312
Lena512×512	99.614501	33.473220
Peppers512×512	99.612186	34.536580

It is clear from Table 4 that the value of NPCR is greater than 99.6094% and that of UACI is greater than 33.4635%. That is to say a modification of a single pixel of the original image results in a radical change of all the pixels of the encrypted image.

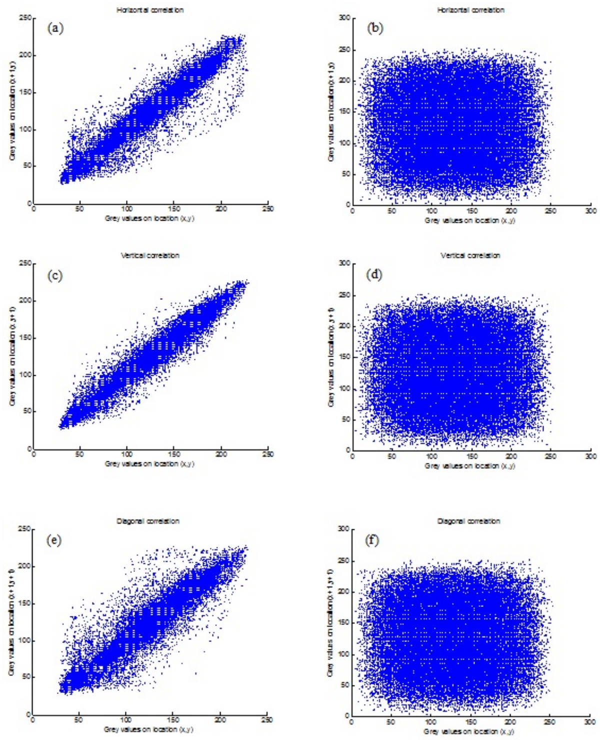

Fig. 5: Correlation distribution of adjacent pixels of the Lena 512×512 in directions, (a) Horizontal, (c) Vertical, (e) diagonal. Correlation distribution of adjacent pixels of the Lena 512×512 encrypted in the directions, (b) Horizontal, (d) Vertical, (f) diagonal.

5 Conclusion

In this work, firstly, we proposed a technique of making an effective pseudorandom number generator by using a difference of the output states of three famous one-dimension chaotic maps namely logistic map, sine map and skew tent map. Simulations evaluations in terms of the Lyapunov exponent and the distribution density showed that the proposed generator is able to generate a one-dimension chaotic system with excellent statistical properties and good chaotic behavior. Secondly, we proposed a novel algorithm based on an iterative mixing of the three components R, G and B with the three chaotic sequences generated from the proposed pseudo-random generator, followed by a strong diffusion. The values of the performance metrics in terms of key space, histogram, entropy, correlation coefficient, NPCR, and UACI, satisfy that the proposed algorithm is extremely difficult to break and withstands on various kinds of security attacks.

References

1. Shannon, C. E. (1949). Communication theory of secrecy systems. Bell Labs Technical Journal, 28(4), 656-715.

2. Chen, G., Mao, Y., and Chui, C. K. (2004). A symmetric image encryption scheme based on 3D chaotic cat maps. Chaos, Solitons and Fractals, 21(3), 749-761.

3. Kanso, A., and Ghebleh, M. (2012). A novel image encryption algorithm based on a 3D chaotic map. Communications in Nonlinear Science and Numerical Simulation, 17(7), 2943-2959.

4. Liu, L., Zhang, Q., and Wei, X. (2012). A RGB image encryption algorithm based on DNA encoding and chaos map. Computers and Electrical Engineering, 38(5), 1240-1248.

5. Hua, Z., and Zhou, Y. (2016). Image encryption using 2D Logistic-Adjusted-Sine map. Information Sciences, 339, 237-253.

6. Wong, K. W., Kwok, B. S. H., and Law, W. S. (2008). A fast image encryption scheme based on chaotic standard map. Physics Letters A, 372(15), 2645-2652.

7. Hanchinamani, G., and Kulakarni, L. (2014). A New Approach for Image Encryption Based on Cyclic Rotations and Multiple Blockwise Diffusions Using Pomeau-Manneville and Sin Maps. JCSE, 8(4), 187-198.

8. Arroyo, D., Alvarez, G., and Fernandez, V. (2008). On the inadequacy of the logistic map for cryptographic applications. arXiv preprint arXiv:0805.4355.

9. Murillo-Escobar, M. A., Cruz-Hernndez, C., Cardoza-Avendao, L., and Mndez-Ramrez, R. (2017). A novel pseudorandom number generator based on pseudorandomly enhanced logistic map. Nonlinear Dynamics, 87(1), 407-425.

10. Arroyo, D., Alvarez, G., and Fernandez, V. (2008). On the inadequacy of the logistic map for cryptographic applications. arXiv preprint arXiv:0805.4355.

11. Mazloom, S., and Eftekhari-Moghadam, A. M. (2009). Color image encryption based on coupled nonlinear chaotic map. Chaos, Solitons and Fractals, 42(3), 1745-1754.

12. Lai, D., Chen, G., and Hasler, M. (1999). Distribution of the Lyapunov exponent of the chaotic skew tent map. International Journal of Bifurcation and Chaos, 9(10), 2059-2067.

13. http://sipi.usc.edu/database/database.php?

14. Alvarez, G., and Li, S. (2006). Some basic cryptographic requirements for chaos-based cryptosystems. International Journal of Bifurcation and Chaos, 16(08), 2129-2151.

15. Akhshani, A., Behnia, S., Akhavan, A., Hassan, H. A., and Hassan, Z. (2010). A novel scheme for image encryption based on 2D piecewise chaotic maps. Optics Communications, 283(17), 3259-3266.

16. Kwok, H. S., and Tang, W. K. (2007). A fast image encryption system based on chaotic maps with finite precision representation. Chaos, solitons and fractals, 32(4), 1518-1529

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Cyclical contractive conditions in probabilistic metric spaces

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The purpose of this paper is to prove a fixed point theorem for a probabilistic k-contraction restricted to two nonempty closed sets of a probabilistic metric spaces, then we prove that these results can be extended to a collection of finite closed sets.

1 Introduction

Fixed points theory plays a basic role in applications of many branches of mathematics. Finding a fixed point of contractive mappings becomes the center of strong research activity. After that, based on this finding, a large number of fixed point results have appeared in recent years. Generally speaking, there usually are two generalizations on them, one is from spaces, the other is from mappings.

Concretely, for one thing, from spaces, for example, the concept of a probabilistic metric spaces was introduced in 1942 by Karl Menger [1], indeed, he proposed replacing the distance $d(p,q)$ by a real function F_{pq} whose value$F_{pq}(x)$ for any real number x is interpreted as the probability that the distance between p and q is less than x.

For another thing, from mappings, for instance, let A and B be nonempty subsets of a metric space (M,d) and let $f : A\cup B \to A\cup B$ be a mapping such that:
(1) $f(A) \subseteq B$ and $f(B) \subseteq A$.
(2) $d(fx,fy) \leq kd(x,y)$, $\forall x \in A$, $\forall y \in B$, where $k \in [0,1)$.
If (1) holds we say that f is a cyclic map and if (1) and (2) hold we say that f is a cyclic contraction [2].

In this work, we show the existence and uniqueness of the fixed point for the cyclic probabilistic k–contraction mapping in a probabilistic metric spaces.

2 Preliminaries

Throughout this work, we adopt the usual terminology, notation and conventions of the theory of probabilistic metric spaces, as in [3].

Definition 2.1. *A distance distribution function (briefly, a d.d.f.) is a nondecreasing function F defined on* $\mathbb{R}^+\cup\{\infty\}$ *that satisfies* $f(0) = 0$ *and* $f(\infty) = 1$*, and is left continuous on* $(0,\infty)$*. The set of all d.d.f's will be noted by* Δ^+*; and the set of all F in* Δ^+ *for which* $\lim_{t\to\infty} f(t) = 1$ *by* D^+*.*

For any a in $\mathbb{R}^+\cup\{\infty\}$, ε_a, the unit step at a, is the function given by:
for $0 \leq a < \infty$

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq a \\ 1 & \text{if} \quad a < x \leq \infty \end{cases}$$

and

$$\varepsilon_\infty(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq \infty \\ 1 & \text{if} \quad x = \infty \end{cases}$$

Note that $\varepsilon_a \leq \varepsilon_b$ if and only if $b \leq a$; that ε_a is in D^+ if $0 \leq a < \infty$; and that ϵ_0 is the maximal element, and ϵ_∞ the minimal element, of Δ^+.

Definition 2.2. *Consider* f *and* g *be in* Δ^+*,* $h \in (0,1]$*, and let* $(f,g;h)$ *denotes the condition*

$$0 \leq g(x) \leq f(x+h) + h,$$

for all x *in* $(0,\frac{1}{h})$*.*
The modified Levy distance is the function d_L *defined on*

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$\Delta^+ \times \Delta^+$ by
$d_L(f,g) = inf\{h :$ both conditions $(f,g;h)$ and $(g,f;h)$ hold$\}$.
Note that for any f and g in Δ^+, both $(f,g;1)$ and $(g,f;1)$ hold, hence d_L is well-defined and $d_L(f,g) \leq 1$.

Lemma 2.1. *The function d_L is a metric on Δ^+.*

Definition 2.3. *A sequence $\{F_n\}$ of d.d.f's is said to converge weakly to a d.d.f. F if and only if the sequence $\{F_n(x)\}$ converges to $F(x)$ at each continuity point x of F.*

Lemma 2.2. *Let $\{F_n\}$ be a sequence of functions in Δ^+, and let F be in Δ^+. Then $\{F_n\}$ converges weakly to F if and only if $d_L(F_n, F) \to 0$.*

Lemma 2.3. *The metric spaces (Δ^+, d_L) is compact, and hence complete.*

Lemma 2.4. *For any F in Δ^+ and $t > 0$,*

$$F(t) > 1 - t \ iff \ d_L(F, \varepsilon_0 < t).$$

Lemma 2.5. *If F and G are in Δ^+ and $F \leq G$ then $d_L(G, \varepsilon_0) \leq d_L(F, \varepsilon_0)$.*

Definition 2.4. *A triangular norm (briefly, a t-norm) is a binary operation T on $[0,1]$ such that:*

$$\begin{aligned} T(x,y) &= T(y,x), \ (commutativity) \\ T(x,y) &\leq T(z,w), \ whenever \ \ x \leq z, \ y \leq w, \\ T(x,1) &= x, \ (1 \ is \ an \ identity \ element) \\ T(T(x,y),z) &= T(x,T(y,z)), \ (associativity). \end{aligned}$$

Example 2.1. *The following t-norms are continuous:*
(i) The t-norm minimum $M(x,y) = Min(x,y)$.
(ii) The t-norm product $\prod(x,y) = xy$.
(iii) The t-norm W, $W(x,y) = Max(x+y-1, 0)$.

Definition 2.5. *A triangle function is a binary operation τ on Δ^+ that is commutative, associative, and nondecreasing in each place, and has ε_0 as identity.*

Example 2.2. *If T is left continuous, then the binary operation τ_T on Δ^+ defined by:*

$$\tau_T(F,G)(x) = \sup\{T(F(u), G(v)) : u + v = x\},$$

is a triangle function.

Lemma 2.6. *If T is continuous, then τ_T is continuous .*

Definition 2.6. *A probabilistic metric space (briefly a bms) is a triple (M, F, τ) where M is a nonempty set, F is a function from $M \times M$ into Δ^+, τ is a triangle function, and the following conditions are satisfied*
for all $p, r; q \in S$
(i) $F_{pp} = \varepsilon_0$,
(ii) $F_{pr} = \varepsilon_0 \ \Rightarrow \ p = r$,
(iii) $F_{pr} = F_{rp}$
(iv) $F_{pr} \geq \tau(F_{pq}, F_{qr})$.
If $\tau = \tau_T$ for some t-norm T, then (M, F, τ_T) is called a Menger space.

It should be noted that if T is a continuous t-norm, then (M,F) satisfies (iv) under τ_T if and only if it satisfies

$$(v) \ \ F_{pr}(x+y) \geq T(F_{pq}(x), F_{qr}(y)),$$

for all $p, r; q \in M$ and for all $x, y > 0$, under T.

Definition 2.7. *Let (M,F) be a probabilistic semimetric space (i.e., (i), (ii) and (iii) of Definition 2.6 are satisfied). For p in M and $t > 0$, the strong t-neighborhood of p is the set*

$$N_p(t) = \{q \in M : \ F_{pq}(t) > 1 - t\}.$$

The strong neighborhood system at p is the collection $\wp_p = \{N_p(t) : \ t > 0\}$, and the strong neighborhood system for M is the union $\wp = \bigcup_{p \in M} \wp_p$.

An immediate consequence of Lemma 2.4 is

$$N_p(t) = \{q \in M : d_L(F_{pq}, \varepsilon_0) < t\}.$$

In probabilistic semimetric space, the convergence of sequence is defined in the way

Definition 2.8. *Let $\{x_n\}$ be a sequence in a probabilistic semimetric space (M,F). Then*
(1) The sequence $\{x_n\}$ is said to be convergent to $x \in M$, if for every $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that $F_{x_n x}(\epsilon) > 1 - \epsilon$ whenever $n \geq N(\epsilon)$.
(2) The sequence $\{x_n\}$ is called a Cauchy sequence, if for every $\epsilon > 0$ there exists a positive integer $N(\epsilon)$ such that $n, \ m \geq N(\epsilon) \Rightarrow F_{x_n x_m}(\epsilon) > 1 - \epsilon$.
(3) (M,F) is said to be complete if every Cauchy sequence has a limit.

The proof of the following result is easy to reproduce.

Proposition 2.1. *Let $\{x_n\}$ be a sequence in a probabilistic semimetric space (M,F) and $x \in M$.*
$1-$ $\{x_n\}$ is convergent to x, if either
- $\lim_{n \to \infty} F_{x_n x}(t) = 1$ for all $t > 0$, or
- for every $\epsilon > 0$ and $\delta \in (0,1)$, there exists a positive integer $N(\epsilon, \delta)$ such that $F_{x_n x}(\epsilon) > 1 - \delta$, whenever $n \geq N(\epsilon, \delta)$.
$2-$ $\{x_n\}$ is Cauchy sequence, if either
- $\lim_{n,m \to \infty} F_{x_n x_m}(t) = 1$ for all $t > 0$, or
- for every $\epsilon > 0$ and $\delta \in (0,1)$, there exists a positive integer $N(\epsilon, \delta)$ such that $F_{x_n x_m}(\epsilon) > 1 - \delta$, whenever $n, \ m \geq N(\epsilon, \delta)$.

Scheizer and Sklar [3] proved that if (M, F, τ) is a probabilistic metric space with τ is continuous, then the family $\mathbb{I}$ consisting of $\emptyset$ and all unions of elements of this strong neighborhood system for M determines a Hausdorff topology for M.
Consequently, in such space we have the following assertions
(a) (M, F, τ) is endowed with the topology $\mathbb{I}$ is a Hausdroff topological space.
(b) There exists a topology Λ on S such that the strong neighborhood system $\wp$ is a basis for Λ.

Let f a self map on M. Power of f at $p \in M$ are defined by $f^0 p = p$ and $f^{n+1} p = f(f^n p)$, $n \geq 0$. We will use the notation $p_n = f^n p$, in particular $p_0 = p$, $p_1 = fp$.
The letter Ψ denotes the set of all function $\varphi : [0, \infty) \to [0, \infty)$ such that

$$0 < \varphi(t) < t \ \ and \ \ lim_{n\to\infty} \varphi^n(t) = 0 \ \ for \ \ each \ \ t > 0$$

Definition 2.9. *[4] We say that a t-norm T is of H-type if the family $\{T^n(t)\}$ is equicontinuous at $t = 1$, that is,*

$\forall \epsilon \in (0,1), \exists \lambda \in (0,1) : t > 1 - \lambda \Rightarrow T^n(t) > 1 - \epsilon, \ \forall n \geq 1$

Where $T^1(x) = T(x,x)$, $T^n(x) = T(x, T^{n-1}(x))$, for every $n \geq 2$.

The t-norm T_M is a trivial example of t-norm of H-type.

Definition 2.10. *[5] Let $\varphi : [0, \infty) \to [0, \infty)$ be a function such that $\varphi(t) < t$ for $t > 0$, and f be a selfmap of a probabilistic metric space (M, F, τ). We say that f is φ-probabilistic contraction if*

$$F_{fpfq}(\varphi(t)) \geq F_{pq}(t).$$

for all $p, q \in M$ and $t > 0$,

Theorem 2.1. *[6] Let (M, F, τ_T) be a complete probabilistic metric space under a continuous t-norm T of H-type such that $RanF \subset D^+$. Let $f : M \to M$ be a φ-probabilistic contraction where $\varphi \in \Psi$. Then f has a unique fixed point $\overline{x}$, and, for any $x \in M$, $\lim_{n\to\infty} f^n(x) = \overline{x}$.*

3 Cyclical contractive conditions in probabilistic metric spaces

Theorem 3.1. *Let $(M, F, \tau_T,)$ be a complete probabilistic metric space under a continuous t-norm T of H-type such that $RanF \subset D^+$. Let $f : M \to M$ be a continuous mapping and satisfies*

$$F_{fpf^2p}(kt) \geq F_{pfp}(t).$$

$for \ \ all \ \ p \in M$ and $t > 0$ where $k \in (0,1)$.
Then f has a fixed point in M.

Proof. Let $p_0 \in M$. Put $p_n = f(p_{n-1}) = f^n(p_0)$ for each $n \in \{0, 1, 2, \ldots\}$. We prove that $\{p_n\}$ is a Cauchy sequence in M. We need to show that for each $\delta > 0$ and $0 < \epsilon < 1$ there exists a positive integer $n_1 = n_1(\delta, \epsilon)$ such that:

$$F_{p_n p_m}(\delta) > 1 - \epsilon \ \ for \ \ all \ \ m > n > n_1(\delta, \epsilon)$$

For each $\delta > 0$, for $m > n$ we have

$$\begin{aligned}
F_{p_n p_m}(\delta) &\geq T(F_{p_n p_{n+1}}(\delta - k\delta), F_{p_{n+1} p_m}(k\delta)) \\
&\geq T(F_{p_{n-1} p_n}((\delta - k\delta)k^{-1}), F_{p_{n+1} p_m}(k\delta)) \\
&\geq T(F_{p_{n-2} p_{n-1}}((\delta - k\delta)k^{-2}), F_{p_{n+1} p_m}(k\delta)) \\
&\geq T(F_{p_{n-3} p_{n-2}}((\delta - k\delta)k^{-3}), F_{p_{n+1} p_m}(k\delta)) \\
&\ \vdots \\
&\geq T(F_{p_0 p_1}((\delta - k\delta)k^{-n}), F_{p_{n+1} p_m}(k\delta))
\end{aligned}$$

It follows that
$F_{p_n p_m}(\delta) \geq T(F_{p_0 p_1}((\delta - k\delta)k^{-n}), T(F_{p_{n+1} p_{n+2}}(k\delta - k^2\delta), Fp_{n+2}p_m(k^2\delta)))$
Then
$F_{p_n p_m}(\delta) \geq T(F_{p_0 p_1}((\delta - k\delta)k^{-n}), T(F_{p_n p_{n+1}}(\delta - k\delta), F_{p_{n+2} p_m}(k^2\delta)))$
Then
$F_{p_n p_m}(\delta) \geq T(F_{p_0 p_1}((\delta - k\delta)k^{-n}), T(F_{p_{n-1} p_n}((\delta - k\delta)k^{-1}), Fp_{n+2}p_m(k^2\delta)))$
Then
$F_{p_n p_m}(\delta) \geq T(F_{p_0 p_1}((\delta - k\delta)k^{-n}), T(F_{p_0 p_1}((\delta - k\delta)k^{-n}), F_{p_{n+2} p_m}(k^2\delta)))$
Using the same argument repeatedly and by definition of the operator $T^{(n)}$, we obtain

$$F_{p_n p_m}(\delta) \geq T^{m-n}(F_{p_0 p_1}((\delta - k\delta)k^{-n})) \qquad (3.1)$$

Since T is a t-norm of H-type, for given $\lambda \in (0,1)$, there exists $\lambda = \lambda(\epsilon) \in (0,1)$ such that if we have $t > 1 - \lambda$ then $T^n(t) > 1 - \epsilon$ for all $n \geq 1$.
Since

$$(\delta - k\delta)k^{-n} \to 0 \ \ as \ \ n \to \infty$$

Then

$$F_{p_0 p_1}((\delta - k\delta)k^{-n}) \to 1 \ \ as \ \ n \to \infty$$

because $F_{p_0 p_1} \in D^+$.
Then there exists $N \in \mathbb{N}$ such that

$$F_{p_0 p_1}((\delta - k\delta)k^{-n}) > 1 - \lambda \ \ for \ \ all \ \ n > N(\lambda(\epsilon))$$

Hence and by (3.1) we conclude that

$$F_{p_n p_m}(\delta) > 1 - \epsilon \ \ for \ \ all \ \ m > n > N$$

Thus we proved that the $\{p_n\}$ is a Cauchy sequence in M.
Since M is complete there is some $q \in M$ such that

$$p_n \to q$$

The continuity of mapping f and the uniqueness of the limit implies that

$$f(q) = q$$

We now state the main fixed point theorem for cyclical contractive conditions.

Theorem 3.2. *Let $(M, F, \tau_T,)$ be a complete probabilistic metric space under a continuous t-norm T of H-type such that $RanF \subset D^+$. Let A and B be nonempty closed subsets of M and let $f : A \cup B \to A \cup B$ be a mapping and satisfies:*
(1) $F(A) \subset B$ and $F(B) \subset A$.
(2) $F_{fpfq}(kt) \geq F_{pq}(t)$, $\forall p \in A$ and $\forall q \in B$, where $k \in (0,1)$.
Then f has a unique fixed point in $A \cap B$.

Proof. For $p \in A \cup B$ we have

$$F_{fpf^2p}(kt) \geq F_{pfp}(t)$$

By theorem 3.1 $\{p_n\}$ is a Cauchy sequence. Consequently $\{p_n\}$ converges to some point $q \in M$. However in view of (2) an infinite number of terms of the sequence $\{p_n\}$ lie in A and an infinite number of terms lie in B, so $A \cap B \neq \emptyset$. (1) implies $f : A \cap B \to A \cap B$ and (2) implies that f restrected to $A \cap B$ is a probabilistic contraction mapping. By theorem 2.1 with $\varphi(t) = kt$, f has a unique fixed point in $A \cap B$.

Corollary 3.1. *Let A and B be two non-empty closed subsets of a complete probabilistic metric space (M, F, τ_T). Let $f : A \to B$ and $g : B \to A$ be two functions such that*

$$F_{fpgq}(kt) \geq F_{pq}(t) \;\; \forall p \in A \;\; and \;\; \forall q \in B \tag{3.2}$$

where $k \in (0,1)$. Then there exists a unique $r \in A \cap B$ such that

$$f(r) = g(r) = r.$$

Proof. Apply theorem 3.1 to the mapping $h : A \cup B \to A \cup B$ defined by

$$h(p) = \begin{cases} f(p) & \text{if} \quad p \in A; \\ g(p) & \text{if} \quad p \in B. \end{cases}$$

Observ that the mapping h is well define because if $p \in A \cap B$, (3.2) implies

$$F_{fpgp}(t) \geq F_{pp}(\frac{t}{k}) \;\; for \;\; all \;\; t > 0$$

Then $F_{fpgp} = \epsilon_0$, so $f(p) = g(p)$

The reasoning of Theorem 3.2 can be extended to a colletion of finite sets.

Theorem 3.3. *Let $\{A_i\}_{i=1}^m$ be nonempty closed subsets of a complete probabilistic metric space, and suppose $f : \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ satisfies the following conditions (where $A_{p+1} = A_1$):*
(1) $F(A_i) \subset A_{i+1}$ for $1 \leq i \leq p$;
(2) $\exists k \in (0,1)$ such that $F_{fpfq}(kt) \geq F_{pq}(t)$ $\forall p \in A_i$, $\forall q \in A_{i+1}$ for $1 \leq i \leq p$.
Then f has a unique fixed point in $\bigcap_{i=1}^m A_i$.

Proof. Let $p_0 \in \bigcup_{i=1}^m A_i$, we observe that, infinitely terms of the Cauchy sequence $\{p_n\}$ lie in each A_i. Thus $\bigcap_{i=1}^m A_i \neq \emptyset$, and the restriction of f to this intersection is a probabilistic contraction mapping. By theorem 2.1 f has a unique fixed point in $\bigcap_{i=1}^m A_i$.

Conflict of Interest The authors declare that they do not have any competing interests.

References

[1] K. Menger, Statistical metrics, Proc. Natl. Acad. Sci. **28** (1942), 535-537.

[2] W. A. Kirk, P. S. Srinivasan, P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions. Fixed Point Theory, volume 4, No. **1**, 2003, 79-89.

[3] Schweizer B. and A.Sklar, Probabilistic Metric Spaces, North-Holland Series in Probability and Applied Mathimatics, **5**, (1983).

[4] O. Hadzić, A fixed point theorem in Menger spaces, Publ. Inst. Math. (Beograd) T **20** (1979) 107–112. http://eudml.org/doc/257517

[5] A. Mbarki, A. Benbrik, A. Ouahab, W. Ahid and T. Ismail, Comments on "Fixed Point Theorems for φ-Contraction in Probabilistic Metric Space", Int. J. Math. Anal. **7**, 2013, no. 13, 625 - 635. doi 10.12988/ijma.2013.13060

[6] L. Ciric, Solving the Banach fixed point principle for nonlinear contractions in probabilistic metric spaces, Nonlinear Analysis **72** (2010) 2009-2018. doi 10.1016/j.na.2009.10.001

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Group law and the Security of elliptic curves on $F_p[e_1,...,e_n]$

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In this paper, we study the elliptic curve $E_{a,b}(A_P)$, with A_P the localization of the ring $A = \mathbb{F}_p[e_1,...,e_n]$ where $e_ie_i = e_i$ and $e_ie_j = 0$ if $i \neq j$, in the maximal ideal $P = (e_1,...,e_n)$. Finally we show that $Card(E_{a,b}(A_P)) \geqslant (Card(E_{a,b}(\mathbb{F}_p))-3)^n + Card(E_{a,b}(\mathbb{F}_p))$ and the execution time to solve the problem of discrete logarithm in $E_{a,b}(A_P)$ is $\Omega(N)$, sch that the execution time to solve the problem of discrete logarithm in $E_{a,b}(\mathbb{F}_p)$ is $O(\sqrt{N})$. The motivation for this work came from search for new groups with intractable (DLP) discrete logarithm problem is therefore of great importance.

1 Introduction

The elliptic curves are a very fashionable subject in mathematics. They are the basis of the demonstration of Fermat's great theorem by Andrew Wiles, it was proposed for cryptographic use independently by Neal Koblitz[1] and Victor Miller in 1985, claim that elliptic curve cryptography requires much smaller keys than those used in conventional public key cryptosystems, while maintaining an equal level of security. In 2008, Virat introduced the elliptic curves over local ring $\mathbb{F}_p[\epsilon] = \mathbb{F}_p[X]/(X^2)$[2], and a proposed a new public key cryptosystem which is a variant of the ElGamal cryptosystem on an elliptic curve, in 2013 Chillali generalized the Virat result for the ring $\mathbb{F}_p[\epsilon] = \mathbb{F}_p[X]/(X^n)$ [3]. Chillali and Abdelalim constructed a ring $\mathbb{F}_p[e_1,e_2,e_3]$, defined an elliptic curve over $E_{a,b}(\mathbb{F}_p[e_1,e_2,e_3])$ and they showed that $Card(E_{a,b}(\mathbb{F}_p[e_1,e_2,e_3])) \geq (Card(E_{a,b}(\mathbb{F}_p)) - 3)^n + Card(E_{a,b}(\mathbb{F}_p))$ [4].

In this work we will generalize the construction of $\mathbb{F}_p[e_1,e_2,e_3]$ to $\mathbb{F}_p[e_1,..,e_n]$, but not a local ring to define a group law in $\mathbb{F}_p[e_1,..,e_n]$,we localized the ring $\mathbb{F}_p[e_1,..,e_n]$ in a maximal ideal, and we give it a group law and show that $Card(E_{a,b}(\mathbb{F}_p[e_1,..,e_n])) \geq (Card(E_{a,b}(\mathbb{F}_p)) - 3)^n + Card(E_{a,b}(\mathbb{F}_p))$, then shows the discrete logarithmic complexity is $\Omega(N)$ Such that $N = E_{a,b}(\mathbb{F}_p)$ by using the attacks baby step/giant step and ρ−Pollard.

Let p be an odd prime number and n be an integer such that $n \geq 1$. we consider the ring $A_n = \mathbb{F}_p[e_1,...,e_n] = \{a_0 + a_1e_1 + ... + a_ne_n / a_0, a_1,...,a_n \in \mathbb{F}_p,\ e_ie_i = e_i$ and $e_ie_j = 0$ if $i \neq j\}$. $\mathbb{F}_p[e_1,...,e_n,]$ is vector space over $\mathbb{F}_p$ with basis $(1, e_1,...,e_n)$. We have
$X + Y = (x_0 + y_0) + (x_1 + x_1)e_1 + ... + (x_n + y_n)e_n$ and $X.Y = t_0 + t_1e_1 + ... + t_ne_n$ with:

$$\begin{cases} t_0 = x_0y_0 & if \quad i = 0 \\ t_i = x_iy_0 + x_0y_i + x_iy_i & if \quad i \neq 0 \end{cases}$$

Proposition 1. *Let $X = x_0 + x_1e_1 + ... + x_ne_n \in \mathbb{F}_p[e_1,...,e_n]$ then X is invertible if and only if $x_0 \neq 0$ and $x_i \neq -x_0$ for all $i \in \{1,...,n\}$.*

Proof. Let $X = x_0 + x_1e_1 + ... + x_ne_n \in \mathbb{F}_p[e_1,...,e_n]$ a invertible element, there $Y = y_0 + y_1e_1 + ... + y_ne_n \in \mathbb{F}_p[e_1,...,e_n]$ such that $X.Y = 1$, this implies that

$$\begin{cases} x_0y_0 = 1 & if \quad i = 0 \\ x_iy_0 + x_0y_i + x_iy_i = 0 & if \quad i \neq 0 \end{cases}$$

therefore, $x_0 \neq 0$ and $x_i \neq -x_0$ for all $i \in \{1,...,n\}$.
In this case:

$$\begin{cases} y_0 = x_0^{-1} & if \quad i = 0 \\ y_i = -(x_0 + x_i)^{-1}x_ix_0^{-1} & if \quad i \neq 0 \end{cases}$$

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The other since is evident.

Proposition 2. *The ideal $P = (e_1, ..., e_n)$ is a maximal (prime) of a ring $\mathbb{F}_p[e_1, ..., e_n]$.*

Proof. We have $\mathbb{F}_p[e_1, ..., e_n]/P \simeq \mathbb{F}_p$ is a field, there P is maximal.

Proposition 3. *Let $S = \mathbb{F}_p[e_1, ..., e_n] - P = \{s_0 + s_1e_1 + ... + s_ne_n/s_0 \in \mathbb{F}_p, s_0 \neq 0\}$. Then the localized of $\mathbb{F}_p[e_1, ..., e_n]$ in P is:*
$A_P = \{\frac{x_0+x_1e_1+...+x_ne_n}{s_0+s_1e_1+...+s_ne_n}/x_0, x_1, ..., x_n, s_0, s_1, ..., s_n \in \mathbb{F}_p, s_0 \neq 0\}$

$$= \{\frac{x_0 + x_1e_1 + ... + x_ne_n}{1 + s_1e_1 + ... + s_ne_n}/x_0, x_1, ..., x_n, s_1, ..., s_n \in \mathbb{F}_p\}$$

A_P is a local ring its maximal ideal is $M = \{\frac{x_1e_1+...+x_ne_n}{1+s_1e_1+...+s_ne_n}/x_1, ..., x_n, s_1, ..., s_n \in \mathbb{F}_p\}$ *and the residual field is $K \simeq \mathbb{F}_p$.*

Proposition 4. *The homomorphism :*

$$\begin{array}{ccc} \pi : A_P & \longrightarrow & \mathbb{F}_p \\ \frac{x_0+x_1e_1+...+x_ne_n}{1+s_1e_1+...+s_ne_n} & \longrightarrow & x_0 \end{array}$$

is a surjective homomorphism of rings.

2 The Elliptic Curve over the ring A_P

Let $n \in \mathbb{N}^*$, $A = \mathbb{F}_p[e_1, ..., e_n]$, p a prime number $p \geq 5$, $P = (e_1, ..., e_n)$ and A_P the localized of A in P.

Definition 1. *An elliptic curve over ring A_P is curve that is given by such Weierstrass equation:*

$$Y^2Z = X^3 + aXZ^2 + bZ^3$$

with $a, b \in A_P$ and $4a^3 + 27b^2$ is invertible on A_P, and the reduction over $\mathbb{F}_p$ is

$$Y^2Z = X^3 + \pi(a)XZ^2 + \pi(b)Z^3$$

$$E_{a,b}(\mathbb{F}_p) = \{[X : Y : Z] \in \mathbb{P}^2(\mathbb{F}_p)/Y^2Z = X^3 + aXZ^2 + bZ^3\}$$

$$E_{a,b}(A) = \{[X : Y : Z] \in \mathbb{P}^2(A)/Y^2Z = X^3 + aXZ^2 + bZ^3\}$$

$$E_{a,b}(A_P) = \{[X : Y : Z] \in \mathbb{P}^2(A_P)/Y^2Z = X^3 + aXZ^2 + bZ^3\}$$

Theorem 1. *The mapping*

$$\begin{array}{ccc} \phi : E_{a,b}(A) & \longrightarrow & E_{a,b}(A_P) \\ [x : y : z] & \longrightarrow & [\frac{x}{1} : \frac{y}{1} : \frac{z}{1}] \end{array}$$

is injective.

Proof. Let $[x : y : z], [x' : y' : z'] \in E_{a,b}(A)$
Suppose that $[x : y : z] = [x' : y' : z']$
$\Rightarrow \exists\lambda \in A$ such that $(x', y', z') = \lambda(x, y, z)$
$\Rightarrow (x', y', z') = (\lambda x, \lambda y, \lambda z)$
$\Rightarrow x' = \lambda x$, $y' = \lambda y$ and $z' = \lambda z$
$\Rightarrow \frac{x'}{1} = \lambda\frac{x}{1}$, $\frac{y'}{1} = \lambda\frac{y}{1}$ and $\frac{z'}{1} = \lambda\frac{z}{1}$
Then $[\frac{x'}{1} : \frac{x'}{1} : \frac{x'}{1}] = [\lambda\frac{x}{1} : \lambda\frac{x}{1} : \lambda\frac{x}{1}] = [\frac{x}{1} : \frac{x}{1} : \frac{x}{1}]$
we deduce that $\phi([x : y : z]) = \phi([x' : y' : z'])$
Then ϕ is well defined.

Note that mapping $\begin{array}{ccc} f : A & \longrightarrow & A_P \\ x & \longrightarrow & \frac{x}{1} \end{array}$ is injective.
We deduce ϕ is injective.

Theorem 2. *Let $a, b \in \mathbb{F}_p$, the mapping*

$$\begin{array}{ccc} \varphi : E_{a,b}(A_P) & \longrightarrow & E_{a,b}(A) \\ [\frac{x}{s_1} : \frac{y}{s_2} : \frac{z}{s_3}] & \longrightarrow & [xs_2s_3 : ys_1s_3 : zs_1s_2] \end{array}$$

is surjective.

Proof. Let $[\frac{x}{s_1} : \frac{y}{s_2} : \frac{z}{s_3}], [\frac{x'}{s'_1} : \frac{y'}{s'_2} : \frac{z'}{s'_3}] \in E_{a,b}(A_P)$
Suppose that $[\frac{x}{s_1} : \frac{y}{s_2} : \frac{z}{s_3}] = [\frac{x'}{s'_1} : \frac{y'}{s'_2} : \frac{z'}{s'_3}]$
Then $\exists\lambda \in A_P^*$ such that $(\frac{x'}{s'_1}, \frac{y'}{s'_2}, \frac{z'}{s'_3}) = \lambda(\frac{x}{s_1}, \frac{y}{s_2}, \frac{z}{s_3})$
$\Rightarrow \frac{x'}{s'_1} = \lambda\frac{x}{s_1}; \frac{y'}{s'_2} = \lambda\frac{y}{s_2}; \frac{z'}{s'_3} = \lambda\frac{z}{s_3}$
$\Rightarrow s_1s_2s_3\frac{x'}{s'_1} = \lambda s_2s_3x; s_1s_2s_3\frac{y'}{s'_2} = \lambda s_1s_3y; s_1s_2s_3\frac{z'}{s'_3} = \lambda s_1s_2z$
Then $\varphi[\frac{x'}{s'_1} : \frac{y'}{s'_2} : \frac{y'}{s'_2}] = \varphi[s_1s_2s_3\frac{x'}{s'_1} : s_1s_2s_3\frac{y'}{s'_2} : s_1s_2s_3\frac{z'}{s'_3}]$
$= \varphi[\lambda s_2s_3x : \lambda s_1s_3y : \lambda s_1s_2z]$
$= \varphi[s_2s_3x : s_1s_3y : s_1s_2z]$
$= \varphi[\frac{x}{s_1} : \frac{y}{s_2} : \frac{z}{s_3}]$
Then φ is well defined.
Let $[x : y : z] \in E_{a,b}(A)$, we have $\varphi[\frac{x}{1} : \frac{y}{1} : \frac{z}{1}] = [x : y : z]$.
Finally the mapping φ is surjective.

Corollary 1. *Let $a, b \in \mathbb{F}_p$, then*

$$Card(E_{a,b}(A_P)) = Card(E_{a,b}(A))$$

Proposition 5. *A Weierstrass equation is defined a elliptic curve over A_P if and only if the reduction ever $\mathbb{F}_p$ is a elliptic curve.*

Proposition 6. *Let $a, b \in A_P$ such that $4a^3 + 27b^2$ is invertible on A_P.The set $E_{a,b}(A_P)$ together with a special point $O = [0 : 1 : 0]$, a commutative binary operation denoted by $+$. It is well known that the binary operation $+$ endows the set $E_{a,b}(A_P)$ with an abelian group with O as identity element.*

Proof. The proof in M.Virat theses[2] page 56, based on [5] page 117 corollary 6.6 and [6] page 63.

Proposition 7. *Let $P = [x : y : z]$ and $P' = [x' : y' : z']$ in $E_{a,b}(A_P)$, we have*

1. *if $\pi_{E_{a,b}(A_P)}(P) \neq \pi_{E_{a,b}(A_P)}(P')$ then $P + P' = [p_1 : q_1 : r_1]$ with*
$p_1 = y^2x'z' - zxy'^2 - a(zx' + xz')(zx' - xz') + (2yy' - 3bzz')(zx' - xz')$
$q_1 = yy'(z'y - zy') - a(xyz'^2 - z^2x'y') + (-2azz' - 3xx')(x'y - xy') - 3bzz'(z'y - zy')$
$r_1 = (zy' + z'y)(z'y - zy') + (3xx' + azz')(zx' - xz')$

2. *if $\pi_{E_{a,b}(R)}(P) = \pi_{E_{a,b}(R)}(P')$ then $P + P' = [p_2 : q_2 : r_2]$ with*
$p_2 = (yy' - bzz')(x'y + xy') + (a^2zz' - 2axx')(zy' + z'y) - 3b(xyz'^2 + z^2x'y') - a(yzx'^2 + x^2y'z')$

$q_2 = y^2y'^2 + 3ax^2x'^2 + (-a^3 - 9b^2)z^2z'^2 - a^2(zx' + xz')^2 - 2a^2zxz'x' + (9bxx' - 3abzz')(zx' + xz')$
$r_2 = (yy' + 3bzz')(zy' + z'y) + (3xx' + 2azz')(x'y + xy') + a(xyz'^2 + z^2x'y')$

Proof. The proof in M.Virat theses[2] page 57 Proposition 2.1.2. based on formulas I, II and III in the article of H.Lange and W.Ruppert[7].

Corollary 2. *The mapping*

$$\begin{array}{cccc} \pi_{E(A_P)} : E(A_P) & \longrightarrow & E(\mathbb{F}_p) \\ [x:y:z] & \longrightarrow & [\pi(x):\pi(y):\pi(z)] \end{array} \text{ is a}$$

homomorphism of groups.

Theorem 3. *Let $a, b \in \mathbb{F}_p$. Then*

$$Card(E_{a,b}(A)) \geqslant (Card(E_{a,b}(\mathbb{F}_p)) - 3)^n + Card(E_{a,b}(\mathbb{F}_p))$$

Proof. The proof for $n = 3$ exist in article of A.Chilali, S.Abdelalim[4].
For $n \in \mathbb{N}^*$ We consider the set:

$$\begin{aligned} T &= \{[x:y:z]/y^2z = x^3 + axz^2 + bz^3\} \\ &= \{[x:0:z]/0 = x^3 + axz^2 + bz^3\} \\ &= \{[x:0:1]/0 = x^3 + ax + b\} \end{aligned}$$

then $Card(T)$ is exactly the number of the solution of the equation $x^3 + ax + b = 0$
therefore $Card(T) \leq 3$.
Let

$$\begin{aligned} G &= E_{a,n}(\mathbb{F}_p) - T \\ &= \{[x:y:z] \in \mathbb{P}^2(\mathbb{F}_p)/y^2z = x^3 + axz^2 + bz^3 \text{ avec } y \neq 0\} \\ &= \{[x:1:z] \in \mathbb{P}^2(\mathbb{F}_p)/z = x^3 + axz^2 + bz^3\} \\ &= \{[x:1:z] \in \mathbb{P}^2(\mathbb{F}_p)/[x:1:z] \in E_{a,n}(\mathbb{F}_p)\} \end{aligned}$$

therefore $Card(G) \geq Card(E_{a,n}(\mathbb{F}_p)) - 3$
We consider the mapping:

$$\begin{array}{ccc} \alpha : G^n & \longrightarrow & E_{a,b}(A) \\ ([x_i:y_i:z_i])_{i=1}^{i=n} & \longrightarrow & [\sum_{i=0}^n x_ie_i : 1 : \sum_{i=0}^n z_ie_i] \end{array}$$

We have:

$$(\sum_{i=0}^n x_ie_i)^3 = \sum_{i=0}^n (x_ie_i)^3$$

$$(\sum_{i=0}^n z_ie_i)^2 = \sum_{i=0}^n (z_ie_i)^2$$

$$a(\sum_{i=0}^n x_ie_i)(\sum_{i=0}^n z_ie_i)^2 = a\sum_{i=1}^n (x_ie_i)(z_ie_i)^2$$

Since $[x_i:1:z_i] \in E_{a,n}(\mathbb{F}_p)$ then $z_i = x_i^3 + ax_iz_i^2 + bz_i^3$. It is clear that :

$$z_ie_i = (x_ie_i)^2 + a(x_ie_i)(z_ie_i)^2 + b(z_ie_i)^3, \ \forall i \in \{1,..,n\}$$

$$\sum_{i=0}^n z_ie_i = \sum_{i=0}^n (x_ie_i)^3 + a\sum_{i=0}^n (x_ie_i)(z_ie_i)^2 + b\sum_{i=0}^n (z_ie_i)^3$$

$$\sum_{i=0}^n z_ie_i = (\sum_{i=0}^n x_ie_i)^3 + a(\sum_{i=0}^n x_ie_i)(\sum_{i=0}^n z_ie_i)^2 + b(\sum_{i=0}^n z_ie_i)^3$$

we deduce that:

$$[\sum_{i=0}^n x_ie_i : 1 : \sum_{i=0}^n z_ie_i] \in E_{a,b}(A)$$

α is injective. Then

$$E_{a,n}(\mathbb{F}_p) \subseteq E_{a,b}(A)$$

$$\alpha(G^n) \subseteq E_{a,b}(A)$$

and $E_{a,n}(\mathbb{F}_p) \cap \alpha(G^n) = \{[0:1:0]\}$
We result that

$$Card(E_{a,b}(A)) \geq Card(\alpha(G^n)) + Card(E_{a,n}(\mathbb{F}_p)) - 1$$

Since α is injective, then

$$Card(G^n) = Card(G)^n \geq Card(E_{a,n}(\mathbb{F}_p) - 3)^n$$

we deduce that:

$$Card(E_{a,b}(A)) \geq Card(E_{a,n}(\mathbb{F}_p) - 1)^n + Card(E_{a,n}(\mathbb{F}_p))$$

Corollary 3. *Let a and b two elements of $\mathbb{F}_p$. Then*

$$Card(E_{a,b}(A_P)) \geqslant (Card(E_{a,b}(\mathbb{F}_p)) - 3)^n + Card(E_{a,b}(\mathbb{F}_p))$$

Proof. In Corollary 1 $Card(E_{a,b}(A_P)) = Card(E_{a,b}(A))$, and in Theorem 3

$$Card(E_{a,b}(A)) \geqslant (Card(E_{a,b}(\mathbb{F}_p)) - 3)^n + Card(E_{a,b}(\mathbb{F}_p))$$

we deduce that:

$$Card(E_{a,b}(A_P)) \geqslant (Card(E_{a,b}(\mathbb{F}_p)) - 3)^n + Card(E_{a,b}(\mathbb{F}_p))$$

3 The discrete logarithm problem: complexity and security

The discrete logarithm problem for G may be stated as: Given $g \in G$ and $h \in < g >$, find an integer x such that $h = g^x$ and $ord(g) = q$ a prime number.

Baby-Step/Giant-Step Method: The idea behind the Baby-Step/Giant-Step method is a standard divide-and-conquer approach found in many areas of computer science. We write

$$x = x_0 + x_1\lceil q \rfloor$$

Now, since $0 \leq x \leq q$, we have that $0 \leq x_0, x_1 \leq \lceil q \rfloor$ We first compute the Baby-Steps

$$g_i \longleftarrow g^i, for 0 \leq i \leq \lceil q \rfloor$$

The pairs (g_i, i) are stored in a table so that one can easily search for items indexed by the first entry in the pair. This can be accomplished by sorting the table on the first entry, or more efficiently by the use of hash tables. To compute and store the Baby-Steps clearly requires $O(\lceil q \rfloor)$ time and a similar amount of storage.
We now compute the Giant-Steps $h_j \longleftarrow h.g^{-j\lceil q \rfloor}$ for

$0 \le j \le \lceil q \rfloor$, and try to find a match in the table of Baby-Steps, i.e. we try to find a value g_i such that $g_i = h_j$. If such a match occurs we have $x_0 = i$ and $x_1 = j$.
since, if $g_i = h_j$, we have $g^i = h.g^{-j\lceil q \rfloor}$
i.e $g^{i+j\lceil q \rfloor} = h$
Notice that the time to compute the Giant-Steps is at most $O(\sqrt{q})$.
Hence, the overall time and space complexity of the Baby-Step/Giant-Step method is $O(\sqrt{q})$[8].
Pollard-Type Methods: We define a partition $G = S_0 \cup S_1 \cup S_2$ and the function

$$f(y,a,b) = \begin{cases} (hy, a, a+b \ mod(q)) & if \quad y \in S_0 \\ (y^2, 2a \ mod(q), 2b \ mod(q)) & if \quad y \in S_1 \\ (gy, a+1 \ mod(q), b) & if \quad y \in S_2 \end{cases}$$

Note that if $y = g^a h^b$, Then, taking $(z,k,l) = (y,a,b)$, $z = g^k h^l$.
Then, it is possible to iterate f until finding two identical results: $(y,a,b) = (z,k,l)$ and

$$g^a h^b = y = z = g^k h^l \Rightarrow g^{l-k} = h^{l-b}$$

As $g^x = h$, we have

$$g^{a-k} = g^{x(l-b)} \Rightarrow x(l-b) = (a-k) \ mod(q)$$

if $l-b$ Is invertible modulo q, We find the solution

$$x = (a-k)(l-b)^{-1} \ mod(q)$$

The time and space complexity of the method is $O(\sqrt{q})$.

Let $Card(E_{a,b}(A_P)) = M$ and $Card(E_{a,b})(\mathbb{F}_p) = N$ $n \ge 3$, and $N \ge 7$[8].

Theorem 4. *The time for solving DLP in $E_{a,b}(A_P)$ is $\Omega(N)$, and $O(\sqrt{N})$ for solving DLP in $E_{a,b}(\mathbb{F}_p)$.*

Proof. Its clear that for solving DLP in $E_{a,b}(\mathbb{F}_p)$ is $O(\sqrt{N})$. We have the time for solving DLP in $E_{a,b}(A_P)$ is $O(\sqrt{M})$. And: $M \ge (N-3)^n + N$
$\Rightarrow M \ge (N-3)^3 + N \ge N^2$ because $n \ge 3$ and $N \ge 7$
$\Rightarrow \sqrt{M} \ge N$
$\Rightarrow \sqrt{M} = \Omega(N)$
Then the time complexity for solving DLP in $E_{a,b}(A_P)$ is $\Omega(N)$ Instead of $O(\sqrt{N})$ for $E_{a,b}(\mathbb{F}_p)$.

Example:
Let $n = 3$, $p = 5$, $a = 1$ and $b = 1$.
$E_{1,1}(\mathbb{F}_5) = \{[0:1:0],[0:1:1],[0:4:1],[2:1:1],[2:4:1],[3:1:1],[3:4:1],[4:2:1],[4:3:1]\}$
We have $Card(E_{1,1}(\mathbb{F}_5)) = 9$. Then
$Card(E_{1,1}(A_P)) \ge (E_{1,1}(\mathbb{F}_5) - 3)^3 + Card(E_{1,1}(\mathbb{F}_5)) = 225$
The following table gives the difficulty of calculating in $A = \mathbb{F}_5[e_1,e_2,e_3]$, and in $E_{1,1}(A)$ as a function of the $\mathbb{F}_5$ addition and the multiplication.

Operations	+ in $\mathbb{F}_5$	$\times$ in $\mathbb{F}_5$
+ in $\mathbb{F}_5[e_1,e_2,e_3]$	3	0
$\times$ in $\mathbb{F}_5[e_1,e_2,e_3]$	6	10
+ in $E_{1,1}(\mathbb{F}_5[e_1,e_2,e_3])$	582	870
+ in $E_{1,1}(\mathbb{F}_5)$	6	4

Note that the computational difficulty in $E_{1,1}(\mathbb{F}_5[e_1,e_2,e_3])$ is much more difficult than in $E_{1,1}(\mathbb{F}_5)$.

4 The fundamental algorithms

4.1 Algorithms in A

Algorithm 1 $sum\_1(p)$

Input:$(X = (x_0,x_1,...,x_n), Y = (y_0,y_1,...,y_n))$;
Output:$(Z = X + Y)$;
for $i = 0$ to n do:
 $z_i = x_i + y_i$;
end for;
return $Z = (z_0,z_1,...,z_n)$;
end;

Algorithm 2 $prod\_1(p)$

Input:$(X = (x_0,x_1,...,x_n), Y = (y_0,y_1,...,y_n))$;
Output:$(Z = X.Y)$;
$z_0 = x_0.y_0$;
for $i = 1$ to n do:
 $z_i = x_0 y_i + x_i y_i + x_i y_0$;
en for;
return $Z = (z_0,z_1,...,z_n)$;
end;

Algorithm 3 $if\_invertible\_1(p)$

Input:$(X = (x_0,x_1,...,x_n))$
Output:(true,false)
for $i = 1$ to n do
 if $(x_0 + x_i == 0 \ mod(p)$ or $x_0 = 0)$
  return false;
 end if
end for;
return true;
end;

Algorithm 4 $invertible\_1(p)$

Input:$(X = (x_0,x_1,...,x_n))$;
Output:$(Z = X^{-1})$;
if $(if\_invertible\_1(X) == false)$;
 print("X is not invertible);
 return null;
end if;
$z_0 = x_0^{-1}$;
for $i = 1$ to n do
 $z_i = -(x_0 + x_i)^{-1} x_i x_0^{-1}$;
end for;
return $Z = (z_0,z_1,...,z_n)$;
end;

Algorithm 5 $projection\_1(p)$

Input:$(X = (x_0,x_1,...,x_n))$;
Output: x_0;
return x_0;
end;

4.2 Algorithms in A_P

Algorithm 6 $prod\_2(p)$

Input: $(X = ((x_0, x_1, ..., x_n), (1, s_1, ..., s_n)),$
$Y = ((y_0, y_1, ..., y_n), (1, t_1, ..., t_n)));$
Output: XY;
$A = pod\_1((x_0, x_1, ..., x_n), (y_0, y_1, ..., y_n));$
$B = pod\_1((1, s_1, ..., s_n), (1, t_1, ..., t_n));$
$Z = (A, B);$
return Z;
end;

Algorithm 7 $sum\_2(p)$

Input: $(X = ((x_0, x_1, ..., x_n), (1, s_1, ..., s_n)),$
$Y = ((y_0, y_1, ..., y_n), (1, t_1, ..., t_n)));$
Output: $X + Y$;
$A = sum\_1(prod\_1((x_0, x_1, ..., x_n), (1, t_1, ..., t_n)),$
$prod\_1((y_0, y_1, ..., y_n), (1, s_1, ..., s_n)));$
$B = pud\_1((1, s_1, ..., s_n), (1, t_1, ..., t_n));$
$Z = (A, B);$
return Z;
end;

Algorithm 8 $projection\_2(p)$

Input:$(X = ((x_0, x_1, ..., x_n), (1, s_1, ..., s_n)));$
Output: x_0;
return x_0;
end;

4.3 Algorithms in $E_{a,b}(A_P)$

Algorithm 9 $projection\_3(p)$

Input:(X, Y, Z);
Output: $(\pi(X), \pi(Y), \pi(Z))$;
return $(projection\_2(X), projection\_2(Y),$
$projection\_2(Z))$;
end;

In this algorithm the + and . in A_P for simplicity

Algorithm 10 $sum\_3(p)$

Input: $(X = (x, y, z), Y = (x', y', z'))$;
Output: $X + Y = (p, q, r)$;
if $projection\_3(X) = prejection\_3(Y)$ then;
$p = y^2x'z' - zxy'^2 - a(zx' + xz')(zx' - xz') + (2yy' - 3bzz')(zx' - xz');$
$q = yy'(z'y - zy') - a(xyz'^2 - z^2x'y') + (-2azz' - 3xx')(x'y - xy') - 3bzz'(z'y - zy');$
$r = (zy' + z'y)(z'y - zy') + (3xx' + azz')(zx' - xz');$
else
$p = (yy' - bzz')(x'y + xy') + (a^2zz' - 2axx')(zy' + z'y) - 3b(xyz'^2 + z^2x'y') - a(yzx'^2 + x^2y'z')$;
$q = y^2y'^2 + 3ax^2x'^2 + (-a^3 - 9b^2)z^2z'^2 - a^2(zx' + xz')^2 - 2a^2zxz'x' + (9bxx' - 3abzz')(zx' + xz');$
$r = (yy' + 3bzz')(zy' + z'y) + (3xx' + 2azz')(x'y + xy') + a(xyz'^2 + z^2x'y');$
end if;
return (p, q, r);
fin ;

5 Conclusion

In this work we study the elliptic curves over a special ring, and we construct a new groups with intractable discrete logarithm problem is therefore of great importance, This allows to define more secure cryptographic cryptosystems.

References

[1] N. Koblitz. *Elliptic Curve Cryptosystems*. Mathematics of Computation, (48): 203-209, 1987.

[2] M. Virat. *Courbe elliptique sur un anneau et applications cryptographiques*. Thése Docteur en Sciences, Nice-Sophia Antipolis. (2009).

[3] Abdelhakim Chillali. *Elliptic Curve Over Special Ideal Ring*. Int. J. Open Problems Compt. Math., Vol. 6, No. 2, June 2013.

[4] A. Chilali, S.Abdelalim. *The Eliptic Curves* $E_{a,b}(\mathbb{F}_p[e_1, e_2, e_3])$ Gulf Journal of Mathematics Vol 3, Issue 2 (2015) 49-53.

[5] D. Mumford, J. Fogarty, and F. Kirwan.*Geometric Invariant Theory* , volume 34 of A Series of Modern Surveys in Mathematics . Springer-Verlag, 3e edition, 1994.

[6] N.M. Katz and B.Mazur. *Arithmetic Moduli of Elliptic Curves*. Number 108 in Annals of Mathematics Studies. Princeton University Press, 1985.

[7] H.Lange and W.Ruppert. *Complete systems of addition laws on abelian varieties*. Invent.Math. 79, 603-610(1985).

[8] Nigel P. Smart. *Cryptography Made Simple*. Springer International Publishing, 2016.

[9] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*. Springer, 1986.

[10] Lawrence C.Washington. *Elliptic Curves Number Theory and Cryptography*. Chapman & HallCRC 2008.

[11] Andreas Enge. *Elleptic Curves And Their Applications To Cryptography, An Introduction*. Kluwers Academic Publishers, 1999.

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Nonlinear parabolic problem with lower order terms in Musielak-Orlicz spaces

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ABSTRACT

We prove an existence result of entropy solutions for the nonlinear parabolic problems: $\frac{\partial b(x,u)}{\partial t}+A(u)-div(\Phi(x,t,u))+H(x,t,u,\nabla u)=f$, *and* $A(u)=-div(a(x,t,u,\nabla u))$ *is a Leary-Lions operator defined on the inhomogeneous Musielak-Orlicz space, the term* $\Phi(x,t,u)$ *is a Crathéodory function assumed to be continuous on u and satisfy only the growth condition* $\Phi(x,t,u)\leq c(x,t)\overline{M}^{-1}M(x,\alpha_0 u)$, *prescribed by Musielak-Orlicz functions M and* $\overline{M}$ *which inhomogeneous and not satisfy* Δ_2*-condition,* $H(x,t,u,\nabla u)$ *is a Crathéodory function not satisfies neither the sign condition or coercivity and* $f\in L^1(Q_T)$.

1 Introduction

Let Ω be a bounded open set of $I\!R^N$ $(N\geq 2)$, T is a positive real number, and $Q_T=\Omega\times(0,T)$. Consider the following nonlinear Dirichlet equation:

$$\begin{cases} \frac{\partial b(x,u)}{\partial t}+A(u)-div(\Phi(x,t,u))+H(x,t,u,\nabla u)=f, \\ u(x,t)=0 \quad on \quad \partial\Omega\times(0,T), \\ b(x,u)(t=0)=b(x,u_0) \quad in \quad \Omega. \end{cases} \tag{1}$$

where $A(u)=-div(a(x,t,u,\nabla u))$ is a Leary-Lions operator defined on the inhomogeneous Musielak-Orlicz-Sobolev space $W_0^{1,x}L_M(Q_T)$, M is a Musielak-Orlicz-function related to the growths of the Carathéodory functions $a(x,t,u,\nabla u)$, $\Phi(x,t,u)$ and $H(x,t,u,\nabla u)$ (see assumptions (12), (15) and (16). $b:\Omega\times I\!R\to I\!R$ is a Carathéodory function such that for every $x\in\Omega$, $b(x,.)$ is a strictly increasing $\mathcal{C}^1(I\!R)$-function, the data f and $b(.,u_0)$ in $L^1(Q_T)$ and $L^1(\Omega)$ respectively.
Starting with the prototype equation:

$$\frac{\partial u}{\partial t}-\triangle_p(u)+div(c(.,t)|u|^{\gamma-1}u)+b|\nabla u|^{\delta}=f, \text{ in } Q_T.$$

In the Classical Sobolev-spaces, the authors in [1] have proved the existence of weak solutions, with $c(.,.)\equiv 0$. For $c(.,.)\in L^2(Q_T)$ and $p=2$, in [2] have proved the existence of entropy solutions, recently in [3] have proved an existence results of renormalized solutions in the case where $p\geq 2$ and $c(.,.)\in L^r(Q_T)$ with $r>\frac{N+p}{p-1}$, and by in [4] for more general parabolic term. For the elliptic version of the problem (1), more results are obtained see e.g. [5-7].

In the degenerate Sobolev-spaces an existence results is shown in [8] without sign condition in $H(x,t,u,\nabla u)$.

In the Orlicz-Sobolev spaces, the existence of entropy solutions of the problem (1) in [9] is proved where $H(x,t,u,\nabla u)\equiv 0$ and the growth of the first lower order Φ prescribed by an isotropic N-function P with $(P<<M)$. To our knowledge, differential equations in general MusielakSobolev spaces have been studied rarely see [10-14], then our aim in this paper is to overcome some difficulties encountered in these spaces and to generalize the result of [4, 9, 15, 16], and we prove an existence result of entropy solution for the obstacle parabolic problem (1), with less restrictive growth, and no coercivity condition in the first lower order term Φ, and without sign condition in the second lower order H, in the framework of inhomogeneous Orlicz-Sobolev spaces $W_0^{1,x}L_M(Q_T)$, and N-function M, defining space does not satisfy the $\triangle_2$-condition.

This paper is organized as follows. In section 2, we recall some definitions, properties and technical lemmas about Musielak Orlicz Sobolev , In section 3 is devoted to specify the assumptions on b,Φ,f, u_0,

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giving the definition of a entropy solution of (1) and we establish the existence of such a solution Theorem 4. In section 4, we give the proof of Theorem 4.

2 Musielak-Orlicz space and a technical lemma

In this part we will define the musielak-Orlicz function which control the growth of our operator.

2.1 Musielak-Orlicz function

Let Ω be an open subset of $I\!R^N$ $(N \geq 2)$, and let M be a real-valued function defined in $\Omega \times I\!R_+$ and satisfying conditions:
(Φ_1):$M(x,.)$ is an N-function for all $x \in \Omega$ (i.e. convex, non-decreasing, continuous, $M(x,0) = 0$, $M(x,0) > 0$ for $t > 0$, $\lim_{t\to 0} \sup_{x\in\Omega} \frac{M(x,t)}{t} = 0$ and $\lim_{t\to\infty} \inf_{x\in\Omega} \frac{M(x,t)}{t} = \infty$).
(Φ_2):$M(.,t)$ is a measurable function for all $t \geq 0$.

A function M which satisfies the conditions Φ_1 and Φ_2 is called a Musielak-Orlicz function. For a Musielak-Orlicz function M we put $M_x(t) = M(x,t)$ and we associate its non-negative reciprocal function M_x^{-1}, with respect to t, that is $M_x^{-1}(M(x,t)) = M(x, M_x^{-1}(t)) = t$. Let M and P be two Musielak-Orlicz functions, we say that P grows essentially less rapidly than M at 0(resp. near infinity, and we write $P << M$, for every positive constant c, we have $\lim_{t\to 0}\Big(\sup_{x\in\Omega} \frac{P(x,ct)}{M(x,t)}\Big) = 0$ (resp.$\lim_{t\to\infty}\Big(\sup_{x\in\Omega} \frac{P(x,ct)}{M(x,t)}\Big) = 0\Big)$.

Remark 1 *[12] If $P << M$ near infinity, then $\forall \epsilon > 0$ there exist $k(\epsilon) > 0$ such that for almost all $x \in \Omega$ we have $P(x,t) \leq k(\epsilon)M(x,\epsilon t) \quad \forall t \geq 0$.*

2.2 Musielak-Orlicz space

For a Musielak-Orlicz function M and a mesurable function $u : \Omega \to I\!R$, we define the functionnal

$$\rho_{M,\Omega}(u) = \int_\Omega M(x, |u(x)|)dx.$$

The set $K_M(\Omega) = \{u : \Omega \to I\!R \text{ mesurable} : \rho_{M,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class. The Musielak-Orlicz space $L_M(\Omega)$ is the vector space generated by $K_M(\Omega)$; that is, $L_M(\Omega)$ is the smallest linear space containing the set $K_M(\Omega)$. Equivalently

$$L_M(\Omega) = \{u : \Omega \to I\!R \quad \text{mesurable} : \quad \rho_{M,\Omega}(\frac{u}{\lambda}) < \infty, \quad \text{for some} \quad \lambda > 0\}.$$

For any Musielak-Orlicz function M, we put $\overline{M}(x,s) = \sup_{t\geq 0}(st - M(x,s))$. $\overline{M}$ is called the Musielak-Orlicz function complementary to M (or conjugate of M) in the sense of Young with respect to s. We say that a sequence of function $u_n \in L_M(\Omega)$ is modular convergent to $u \in L_M(\Omega)$ if there exists a constant $\lambda > 0$ such that $\lim_{n\to\infty} \rho_{M,\Omega}(\frac{u_n - u}{\lambda}) = 0$.

This implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$(see [17]). In the space $L_M(\Omega)$, we define the following two norms

$$\|u\|_M = \inf\Big\{\lambda > 0 : \int_\Omega M(x, \tfrac{|u(x)|}{\lambda})dx \leq 1\Big\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm by

$$|||u|||_{M,\Omega} = \sup_{\|v\|_{\overline{M}}\leq 1} \int_\Omega |u(x)v(x)|dx,$$

where $\overline{M}$ is the Musielak-Orlicz function complementary to M. These two norms are equivalent [17]. $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is by denoted $E_M(\Omega)$. It is a separable space and $(E_M(\Omega))^* = L_M(\Omega)$. We have $E_M(\Omega) = K_M(\Omega)$, if and only if M satisfies the Δ_2−condition for large values of t or for all values of t, according to whether Ω has finite measure or not.
We define

$$W^1L_M(\Omega) = \{u \in L_M(\Omega) : D^\alpha u \in L_M(\Omega), \quad \forall \alpha \leq 1\},$$
$$W^1E_M(\Omega) = \{u \in E_M(\Omega) : D^\alpha u \in E_M(\Omega), \quad \forall \alpha \leq 1\},$$

where $\alpha = (\alpha_1, ..., \alpha_N)$,$|\alpha| = |\alpha_1| + ... + |\alpha_N|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^1L_M(\Omega)$ is called the Musielak-Orlicz-Sobolev space. Let $\overline{\rho}_{M,\Omega}(u) = \sum_{|\alpha|\leq 1} \rho_{M,\Omega}(D^\alpha u)$ and $\|u\|^1_{M,\Omega} = \inf\{\lambda > 0 : \overline{\rho}_{M,\Omega}(\frac{u}{\lambda}) \leq 1\}$ for $u \in W^1L_M(\Omega)$.
These functionals are convex modular and a norm on $W^1L_M(\Omega)$, respectively. Then pair $(W^1L_M(\Omega), \|u\|^1_{M,\Omega})$ is a Banach space if M satisfies the following condition (see [10]),

$$\text{There exists a constant} \quad c > 0 \quad \text{such that} \quad \inf_{x\in\Omega} M(x,1) > c.$$

The space $W^1L_M(\Omega)$ is identified to a subspace of the product $\Pi_{\alpha\leq 1}L_M(\Omega) = \Pi L_M$. We denote by $\mathcal{D}(\Omega)$ the Schwartz space of infinitely smooth functions with compact support in Ω and by $\mathcal{D}(\overline{\Omega})$ the restriction of $\mathcal{D}(I\!R)$ on Ω. The space $W_0^1L_M(\Omega)$ is defined as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$ and the space $W_0^1E_M(\Omega)$ as the(norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.
For two complementary Musielak-Orlicz functions M and $\overline{M}$, we have [17].

- The Young inequality:

$$st \leq M(x,s) + \overline{M}(x,t) \text{ for all } s,t \geq 0 \text{ , } x \in \Omega.$$

- The Holder inequality

$$\Big|\int_\Omega u(x)v(x)dx\Big| \leq \|u\|_{M,\Omega}|||v|||_{\overline{M},\Omega} \text{ for all } u \in L_M(\Omega), v \in L_{\overline{M}}(\Omega).$$

We say that a sequence of functions u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ (respectively in $W_0^1L_M(\Omega)$) if, for some $\lambda > 0$.

$$\lim_{n\to\infty} \overline{\rho}_{M,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0.$$

The following spaces of distributions will also be used

$$W^{-1}L_{\overline{M}}(\Omega) = \Big\{f \in \mathcal{D}'(\Omega) : f = \sum_{\alpha\leq 1}(-1)^\alpha D^\alpha f_\alpha$$

$$\text{where} \quad f_\alpha \in L_{\overline{M}}(\Omega)\Big\},$$

and

$$W^{-1}E_{\overline{M}}(\Omega) = \Big\{f \in \mathcal{D}'(\Omega) : f = \sum_{\alpha \leq 1} (-1)^\alpha D^\alpha f_\alpha$$

$$\text{where} \quad f_\alpha \in E_{\overline{M}}(\Omega)\Big\}.$$

Lemma 1 *[17] Let Ω be a bounded Lipschitz domain in $I\!R^N$ and let M and $\overline{M}$ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

- *There exists a constant $c > 0$ such that $\inf_{x\in\Omega} M(x,1) > c$,*
- *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$, we have*

$$\frac{M(x,t)}{M(y,t)} \leq t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \text{for all} \quad t \geq 1,$$

- *For all $y \in \Omega$, $\int_\Omega M(y,1)dx < \infty$,*
- *There exists a constant $C > 0$ such that*

$$\overline{M}(y,t) \leq C \quad \text{a.e.} \quad \text{in} \quad \Omega.$$

Under this assumptions $\mathcal{D}(\Omega)$ is dense in $L_M(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_M(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W_0^1 L_M(\Omega)$ for the modular convergence.
Consequently, the action of a distribution S in in $W^{-1}L_{\overline{M}}(\Omega)$ on an element u of $W_0^1 L_M(\Omega)$ is well defined. It will be denoted by $< S, u >$.

2.3 Truncation Operator

T_k, $k > 0$, denotes the truncation function at level k defined on $I\!R$ by $T_k(r) = \max(-k, \min(k, r))$. The following abstract lemmas will be applied to the truncation operators.

Lemma 2 *[12] Let $F : I\!R \to I\!R$ be uniformly lipschitzian, with $F(0) = 0$. Let M be an Musielak-Orlicz function and let $u \in W_0^1 L_M(\Omega)$(resp.$u \in W^1 E_M(\Omega)$). Then $F(u) \in W^1 L_M(\Omega)$(resp.$u \in W_0^1 E_M(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(x)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega; u(x) \notin D\} \\ 0 & \text{a.e. in } \{x \in \Omega; u(x) \in D\} \end{cases}$$

Lemma 3 *Suppose that Ω satisfies the segement property and let $u \in W_0^1 L_M(\Omega)$. Then, there exists a sequence $u_n \in \mathcal{D}(\Omega)$ such that $u_n \to u$ for modular convergence in $W_0^1 L_M(\Omega)$. Furthermore, if $u \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.*

Let Ω be an open subset of $I\!R^N$ and let M be a Musielak-Orlicz function satisfying

$$\int_0^1 \frac{M_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt = \infty \quad \text{a.e.} \quad x \in \Omega, \tag{2}$$

and the conditions of Lemma (1). We may assume without loss of generality that

$$\int_0^1 \frac{M_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt < \infty \quad \text{a.e.} \quad x \in \Omega. \tag{3}$$

Define a function $M^* : \Omega \times [0, \infty)$ by $M^*(x,s) = \int_0^s \frac{M_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt$ $x \in \Omega$ and $s \in [0, \infty)$.
M^* its called the Sobolev conjugate function of M (see [18] for the case of Orlicz function).

Theorem 1 *Let Ω be a bounded Lipschitz domain and let M be a Musielak-Orlicz function satisfying (2),(3) and the conditions of lemma (1). Then $W_0^1 L_M(\Omega) \hookrightarrow L_{M^*}(\Omega)$, where M^* is the Sobolev conjugate function of M. Moreover, if Φ is any Musielak-Orlicz function increasing essentially more slowly than M^* near infinity, then the imbedding $W_0^1 L_M(\Omega) \hookrightarrow L_\phi(\Omega)$, is compact.*

Corollaire 1 *Under the same assumptions of theorem (1), we have $W_0^1 L_M(\Omega) \hookrightarrow\hookrightarrow L_M(\Omega)$.*

Lemma 4 *If a sequence $u_n \in L_M(\Omega)$ converges a.e. to u and if u_n remains bounded in $L_M(\Omega)$, then $u \in L_M(\Omega)$ and $u_n \rightharpoonup u$ for $\sigma(L_M(\Omega), E_{\overline{M}}(\Omega))$.*

Lemma 5 *Let $u_n, u \in L_M(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \rightharpoonup u$ for $\sigma(L_M(\Omega), L_{\overline{M}}(\Omega))$.*

Démonstration: Let $\lambda > 0$ such that $\int_\Omega M(x, \frac{u_n - u}{\lambda})dx \to 0$. Thus, for a subsequence, $u_n \to u$ a.e. in Ω. Take $v \in L_{\overline{M}}(\Omega)$. Multiplying v by a suitable constant, we can assume $\lambda v \in \mathcal{L}_{\overline{M}}(\Omega)$. By Young's inequality,

$$|(u_n - u)v| \leq M(x, \frac{u_n - u}{\lambda}) + \overline{M}(x, \lambda v)$$

which implies, by Vitali's theorem, that $\int_\Omega |(u_n - u)v|dx \to 0$.

2.4 Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let Ω an bounded open subset of $I\!R^N$ and let $Q_T = \Omega \times]0, T[$ with some given $T > 0$. Let M be an Musielak-Orlicz function, for each $\alpha \in I\!N^N$, denote by ∇_x^α the distributional derivative on Q_T of order α with respect to the variable $x \in I\!N^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces are defined as follows,

$$W^{1,x}L_M(Q_T) = \{u \in L_M(Q_T) : \nabla_x^\alpha u \in L_M(Q_T),$$
$$\forall \alpha \in I\!N^N, |\alpha| \leq 1\},$$
$$W^{1,x}E_M(Q_T) = \{u \in E_M(Q_T) : \nabla_x^\alpha u \in E_M(Q_T),$$
$$\forall \alpha \in I\!N^N, |\alpha| \leq 1\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm $\|u\| = \sum_{|\alpha|\leq m}\|\nabla_x^\alpha u\|_{M,Q_T}$. We can easily show that they form a complementary system when Ω satisfies the Lipschitz domain [17]. These spaces are considered as subspaces of the product space $\Pi L_M(Q_T)$ which have as many copies as there is α-order derivatives,$|\alpha| \leq 1$. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If $u \in W^{1,x}L_M(Q_T)$ then the function : $t \mapsto u(t) = u(t,.)$ is defined on $(0,T)$ with values $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q_T)$ then the concerned function is a $W^{1,x}E_M(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds $W^{1,x}E_M(\Omega) \subset L^1(0,T,W^{1,x}E_M(\Omega))$. The space $W^{1,x}L_M(Q_T)$ is not in general separable, if $W^{1,x}L_M(Q_T)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However,the scalar function $t \mapsto \|u(t)\|_{M,\Omega}$, is in $L^1(0,T)$ .The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure $W^{1,x}E_M(Q_T)$ of $\mathcal{D}(Q_T)$. We can easily show as in [8],that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q_T)$ with respect of the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is a limit, in $W_0^{1,x}E_M(Q_T)$, of some subsequence $(u_i) \subset \mathcal{D}(Q_T)$ for the modular convergence; i.e. there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$

$$\int_{Q_T} M(x, \frac{\nabla_x^\alpha u_i - \nabla_x^\alpha u}{\lambda})dx\,dt \to 0 \quad \text{as} i \to \infty. \tag{4}$$

This implies that (u_i) converge to u in $W^{1,x}L_M(Q_T)$ for the weak topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.
Consequently,

$$\overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M,\Pi E_{\overline{M}})} = \overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M,\Pi L_{\overline{M}})}. \tag{5}$$

This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore, $W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \cap \Pi E_M$.
We have the following complementary system $\begin{pmatrix} W_0^{1,x}L_M(Q_T) & F \\ W_0^{1,x}E_M(Q_T) & F_0 \end{pmatrix}$ F being the dual space of $W_0^{1,x}E_M(Q_T)$. It is also, except for an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q_T)^\perp$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q_T)$ and it is show that,

$$W^{-1,x}L_{\overline{M}}(Q_T) = \left\{ f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha : \quad f_\alpha \in L_{\overline{M}}(Q_T) \right\}.$$

This space will be equipped with the usual quotient norm $\|f\| = \inf \sum_{|\alpha|\leq 1}\|f_\alpha\|_{\overline{M},Q_T}$ where the infimum is taken on all possible decompositions $f = \sum_{|\alpha|\leq 1}\nabla_x^\alpha f_\alpha, \quad f_\alpha \in L_{\overline{M}}(Q_T)$.

The space F_0 is then given by, $F_0 = \left\{ f = \sum_{|\alpha|\leq 1}\nabla_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q_T) \right\}$ and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q_T)$.

Theorem 2 *[14] Let Ω be a bounded Lipchitz domain and let M be a Musielak-Orlicz function satisfying the same conditions of Theorem (1). Then there exists a constant $\lambda > 0$ such that $\|u\|_M \leq \lambda\|\nabla u\|_M$, $\forall \in W_0^1L_M(Q_T)$.*

Definition 1 *We say that $u_n \to u$ in $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ for the modular convergence if we can write $u_n = \sum_{|\alpha|\leq 1} D_x^\alpha u_n^\alpha + u_n^0$ and $u = \sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$ with $u_n^\alpha \to u^\alpha$ in $L_{\overline{M}}(Q_T)$ for modular convergence for all $|\alpha| \leq 1$ and $u_n^0 \to u^0$ strongly in $L^1(Q_T)$*

Lemma 6 *Let $\{u_n\}$ be a bounded sequence in $W^{1,x}L_M(Q_T)$ such that $\frac{\partial u_n}{\partial t} = \alpha_n + \beta_n$ in $\mathcal{D}'(Q_T)$, $u_n \rightharpoonup u$, weakly in $W^{1,x}L_M(Q_T)$, for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ with $\{\alpha_n\}$ and $\{\beta_n\}$ two bounded sequences respectively in $W^{-1,x}L_{\overline{M}}(Q_T)$ and in $\mathcal{M}(Q_T)$. Then $u_n \to u$ in $L^1_{loc}(Q_T)$. Furthermore, if $u_n \in W_0^{1,x}L_M(Q_T)$, then $u_n \to u$ strongly in $L^1(Q_T)$.*

Theorem 3 *if $u \in W^{1,x}L_M(Q_T) \cap L^1(Q_T)$ (resp. $W_0^{1,x}L_M(Q_T) \cap L^1(Q_T)$) and $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ then there exists a sequence (v_j) in $\mathcal{D}(\overline{Q}_T)$ (resp. $\mathcal{D}(\overline{I}, \mathcal{D}(Q_T))$) such that $v_j \to u$ in $W^{1,x}L_M(Q_T)$ and $\frac{\partial v_j}{\partial t} \to \frac{\partial u}{\partial t}$ in $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ for the modular convergence.*

Démonstration: Let $u \in W^{1,x}L_M(Q_T) \cap L^1(Q_T)$ and $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$, then for any $\epsilon > 0$. Writing $\frac{\partial u}{\partial t} = \sum_{|\alpha|\leq 1} D_x^\alpha u^\alpha + u^0$, where $u^\alpha \in L_{\overline{M}}(Q_T)$ for all $|\alpha| \leq 1$ and $u^0 \in L^1(Q_T)$, we will show that there exits $\lambda > 0$ (depending Only on u and N) and there exists $v \in \mathcal{D}(\overline{Q}_T)$ for which we can write $\frac{\partial u}{\partial t} = \sum_{|\alpha|\leq 1} D_x^\alpha v^\alpha + v^0$ with $v^\alpha, v^0 \in \mathcal{D}(\overline{Q}_T)$ such that

$$\int_{Q_T} M(x, \frac{D_x^\alpha v - D_x^\alpha u}{\lambda})dxdt \leq \epsilon, \forall |\alpha| \leq 1, \tag{6}$$

$$\|v - u\|_{L^1(Q_T)} \leq \epsilon, \tag{7}$$

$$\|v^0 - u^0\|_{L^1(Q_T)} \leq \epsilon, \tag{8}$$

$$\int_{Q_T} \overline{M}(x, \frac{v^\alpha - u^\alpha}{\lambda})dxdt \leq \epsilon, \forall |\alpha| \leq 1 \tag{9}$$

The equation (6) flows from a slight adaptation of the arguments [17],The equations (7),(8) flows also from classical approximation results. For The equation (9) we know that $\mathcal{D}(\overline{Q}_T)$ is dense in $L_{\overline{M}}(Q_T)$ for the modular convergence. The case where $u \in W_0^{1,x}L_M(Q_T) \cap L^1(Q_T))$ can be handled similarly without essential difficulty as it mentioned [17].

Remark 2 *The assumption $u \in L^1(Q_T)$ in theorem (3) is needed only when Q_T has infinite measure, since else, we have $L_M(Q_T)) \subset L^1(Q_T)$ and so $W^{1,x}L_M(Q_T) \cap L^1(Q_T) = W^{1,x}L_M(Q_T)$.*

Remark 3 *If in the statement of theorem (3) above, one takes $I = I\!R$, we have that $\mathcal{D}(\Omega \times I\!R)$ is dense in $\{u \in W_0^{1,x}L_M(\Omega \times I\!R) \cap L^1\Omega \times I\!R) : \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega \times I\!R) + L^1(\Omega \times I\!R)\}$ for the modular convergence. This trivially follows from the fact that $\mathcal{D}(I\!R, \mathcal{D}(\Omega)) = \mathcal{D}(\Omega \times I\!R)$.*

Remark 4 *Let $a < b \in I\!R$ and Ω be a bounded open subset of $I\!R^N$ with the segment property, then* $\{u \in W_0^{1,x}L_M(\Omega\times(a,b)) \cap L^1\Omega\times(a,b)) : \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega\times(a,b)) + L^1(\Omega\times(a,b))\} \subset \mathcal{C}([a,b],L^1(\Omega))$.

Démonstration: Let $u \in W_0^{1,x}L_M(\Omega\times(a,b))$ and $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega\times(a,b)) + L^1(\Omega\times(a,b))$.
After two consecutive reflections first with respect to $t = b$ and then with respect to $t = a$, $\hat{u}(x,t) = u(x,t)\chi_{(a,b)} + u(x,2b-t)\chi_{(b,2b-a)}$ in $\Omega\times(b,2b-a)$ and $\tilde{u}(x,t) = \hat{u}(x,t)\chi_{(a,2b-a)} + \hat{u}(x,2a-t)\chi_{(3a-2b,a)}$ in $\Omega\times(3a-2b,2b-a)$. We get function $\tilde{u} \in W_0^{1,x}L_M(\Omega\times(3a-2b,2b-a))$ with $\frac{\partial \tilde{u}}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega\times(3a-2b,2b-a)) + L^1(\Omega\times(3a-2b,2b-a))$. Now by letting a function $\eta \in \mathcal{D}(I\!R)$ with $\eta = 1$ on $[a,b]$ and $supp\eta \subset (3a-2b,2b-a)$, we set $\overline{u} = \eta\tilde{u}$, therefore,by standard arguments (see [19]), we have $\overline{u} = u$ on $(\Omega\times(a,b))$, $\overline{u} \in W_0^{1,x}L_M(\Omega\times I\!R) \cap L^1(\Omega\times I\!R)$ and $\frac{\partial \overline{u}}{\partial t} \in W_0^{-1,x}L_{\overline{M}}(\Omega\times I\!R) + L^1(\Omega\times I\!R)$.
Let now v_j the sequence given by theorem (3) corresponding to $\overline{u}$, that is,

$$v_j \to \overline{u} \quad \text{in} \quad W_0^{1,x}L_M(\Omega\times I\!R)$$

and

$$\frac{\partial v_j}{\partial t} \to \frac{\partial \overline{u}}{\partial t} \quad \text{in} \quad W_0^{-1,x}L_{\overline{M}}(\Omega\times I\!R) + L^1(\Omega\times I\!R)$$

for the modular convergence.
If we denote $S_k(s) = \int_0^s T_k(t)dt$ the primitive of T_k. We have, $\int_\Omega S_1(v_i - v_j)(\tau)dx = \int_\Omega \int_{-\infty}^{r} T_1(v_i - v_j)(\frac{\partial v_i}{\partial t} - \frac{\partial v_j}{\partial t})dxdt \to 0$ as $i,j \to 0$, from which, one deduces that v_j is a Cauchy sequence in $C(I\!R;L^1(\Omega))$ and hence $\overline{u} \in C(I\!R,L^1(\Omega))$. Consequently, $u \in C([a;b];L1(\Omega))$.

3 Formulation of the problem and main results

Let Ω be an open subset of $I\!R^N$ $(N \geq 2)$ satisfying the segment property,and let M and P be two Musielak-Orlicz functions such that M and its complementary $\overline{M}$ satisfies conditions of Lemma 1, M is decreasing in x and $P \ll M$.
$b : \Omega\times I\!R \to I\!R$ is a Carathéodory function such that for every, $x \in \Omega$, $b(x,.)$ is a strictly increasing $\mathcal{C}^1(I\!R)$-function and

$$b \in L^\infty(\Omega\times I\!R) \quad \text{with} \quad b(x,0) = 0, \tag{10}$$

There exists a constant $\lambda > 0$ and functions $A \in L^\infty(\Omega)$ and $B \in L_M(\Omega)$ such that

$$\lambda \leq \frac{\partial b(x,s)}{\partial s} \leq A(x) \quad \text{and} \quad \left|\nabla_x\left(\frac{\partial b(x,s)}{\partial s}\right)\right| \leq B(x) \tag{11}$$

a.e. $x \in \Omega$ and $\forall\ |s| \in I\!R$.
$A : D(A) \subset W_0^1L_M(Q_T) \to W^{-1}L_{\overline{M}}(Q_T)$ defined by $A(u) = -\text{div}\ a(x,t,u,\nabla u)$, where $a : Q_T\times I\!R\times I\!R^N \to I\!R^N$ is caratheodory function such that for a.e. $x \in \Omega$ and for all $s \in I\!R, \xi, \xi^* \in I\!R^N, \xi \neq \xi^*$

$$|a(x,t,s,\xi)| \leq \nu(a_0(x,t) + \overline{M}_x^{-1}P(x,|s|)), \tag{12}$$

with $a_0(.,.) \in E_{\overline{M}}(Q_T)$,

$$(a(x,t,s,\xi) - a(x,t,s,\xi^*))(\xi - \xi^*) > 0, \tag{13}$$

$$a(x,t,s,\xi).\xi \geq \alpha M(x,|\xi|) + M(x,|s|). \tag{14}$$

$\Phi(x,s,\xi) : Q_T\times I\!R\times I\!R^N \to I\!R^N$ is a Carathéodory function such that

$$|\Phi(x,t,s)| \leq c(x,t)\overline{M}_x^{-1}M(x,\alpha_0|s|), \tag{15}$$

where $c(.,.) \in L^\infty(Q_T)$, $\|c(.,.)\|_{L^\infty(Q_T)} \leq \alpha$, and $0 < \alpha_0 < \min(1,\frac{1}{\alpha})$.
$H(x,t,s,\xi) : Q_T\times I\!R\times I\!R^N \to I\!R$ is a Carathéodory function such that

$$|H(x,t,s,\xi)| \leq h(x,t) + \rho(s)M(x,|\xi|), \tag{16}$$

$\rho : I\!R \to I\!R^+$ is continuous positive function which belong $L^1(I\!R)$ and $h(.,.)$ belong $L^1(Q_T)$.

$$f \in L^1(\Omega), \tag{17}$$

and

$$u_0 \in L^1(\Omega) \text{ such that } b(x,u_0) \in L^1(\Omega). \tag{18}$$

Note that $<,>$ means for either the pairing between $W_0^{1,x}L_M(Q_T) \cap L^\infty(Q_T))$ and $W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ or between $W_0^{1,x}L_M(Q_T)$ and $W^{-1,x}L_{\overline{M}}(Q_T)$.
Weak entropy solution: The definition of a entropy solution of Problem (1) can be stated as follows,

Definition 2 *A measurable function u defined on Q_T is a entropy solution of Problem* (1), *if it satisfies the following conditions:*

$$b(x,u) \in L^\infty(0,T;L^1(\Omega)), b(x,u)(t=0) = b(x,u_0) \quad in \quad \Omega,$$

$$T_k(u) \in W_0^{1,x}L_M(Q_T), \quad \forall k > 0, \quad \forall t \in]0,T],$$

$$\begin{cases} \int_0^T \left\langle \frac{\partial v}{\partial s}; \int_0^u \frac{\partial b(x,z)}{\partial s} T_k'(z-v)dz \right\rangle ds \\ + \int_\Omega \int_0^{u_0} \frac{\partial b(x,s)}{\partial s} T_k(s - v(0))dsdx \\ + \int_{Q_T} a(u,\nabla u)\nabla T_k(u-v)dxds + \int_{Q_T} \Phi(u)\nabla T_k(u-v)dxds \\ + \int_{Q_T} H(u,\nabla u)T_k(u-v)dxds \leq \int_{Q_T} fT_k(u-v)dxds \\ \forall k > 0, \quad and \quad \forall v \in W^{1,x}L_M(Q_T) \cap L^\infty(Q_T) \quad with \\ \quad v(T) = 0, \quad such\ that \quad \frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T) \end{cases} \tag{19}$$

Theorem 4 *Assume that* (11) − (18) *hold true . Then there exists at least one solution u of the following problem* (19).

4 Proof of theorem 4

Truncated problem.
For each $n > 0$, we define the following approximations

$$b_n(x,s) = b(x,T_n(s)) + \frac{1}{n}\,s \quad \forall\ r \in I\!R, \tag{20}$$

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \quad \text{a.e. } (x,t) \in Q_T, \tag{21}$$

$\forall\ s \in I\!R$, $\forall\ \xi \in I\!R^N$,

$$\Phi_n(x,t,s) = \Phi(x,t,T_n(s)) \quad \text{a.e. } (x,t) \in Q_T,\ \forall\ s \in I\!R, \tag{22}$$

$$H_n(x,t,s,\xi) = \frac{H(x,t,s,\xi)}{1+\frac{1}{n}|H(x,t,s,\xi)|}, \tag{23}$$

$$f_n \in L^1(Q_T) \text{ such that } f_n \to f \text{ strongly in } L^1(Q_T), \tag{24}$$

and $\|f_n\|_{L^1(Q_T)} \leq \|f\|_{L^1(Q_T)}$,
and

$$u_{0n} \in \mathcal{C}_0^\infty(\Omega) \text{ such that } b_n(x,u_{0n}) \to b(x,u_0) \tag{25}$$

strongly in $L^1(\Omega)$.
Let us now consider the approximate problem :

$$\begin{cases} \dfrac{\partial b_n(x,u_n)}{\partial t} - div(a_n(x,t,u_n,\nabla u_n)) - div(\Phi_n(x,t,u_n)) \\ +H_n(x,u_n,\nabla u_n) = f_n \ in\ Q_T, \\ u_n(x,t) = 0 \quad on \quad \partial\Omega \times (0,T), \\ b_n(x,u_n)(t=0) = b_n(x,u_{0n}) \quad in \quad \Omega. \end{cases} \tag{26}$$

Since H_n is bounded for any fixed $n > 0$, there exists at last one solution $u_n \in W_0^{1,x}L_M(Q_T)$ of (26)(see [20]).

Remark 5 *the explicit dependence in x and t of the functions a, Φ and H will be omitted so that $a(x,t,u,\nabla u) = a(u,\nabla u)$, $\Phi(x,t,u) = \Phi(u)$ and $H(x,t,u,\nabla u) = H(u,\nabla u)$.*

Proposition 1 *let u_n be a solution of approximate equation* (26)*such that*

$$\begin{cases} T_k(u_n) \to T_k(u) \quad weakly\ in \quad W^{1,x}L_M(Q_T), \\ u_n \to u \quad a.e.\ in \quad Q_T, \\ b_n(x,u_n) \to b(x,u) \quad a.e.\ in \quad Q_T \quad and \quad b(x,u) \in L^\infty(Q_T), \\ a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) \rightharpoonup a(T_k(u),\nabla T_k(u))\nabla T_k(u) \\ \quad weakly\ in \quad L^1(Q_T), \\ \nabla u_n \to \nabla u \quad a.e.\ in \quad Q_T, \\ H_n(u_n,\nabla u_n) \to H(u,\nabla u) \quad strongly\ in \quad L^1(Q_T). \end{cases} \tag{27}$$

then u be a solution of problem (19).

Démonstration: Let $v \in W_0^1L_M(Q_T) \cap L^\infty(Q_T)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$ with $v(T) = 0$, then by theorem 3 we can take $\overline{v} = v$ on Q_T, $\overline{v} \in W^{1,x}L_M(\Omega \times I\!R) \cap L^1(\Omega \times I\!R) \cap L^\infty(\Omega \times I\!R)$, $\frac{\partial \overline{v}}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$, and there exists $v_j \in \mathcal{D}(\Omega \times I\!R)$ such that $v_j(T) = 0$, $v_j \to \overline{v}$ in $W_0^{1,x}L_M(\Omega \times I\!R)$ and

$$\frac{\partial v_j}{\partial t} \to \frac{\partial \overline{v}}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T), \tag{28}$$

for the modular convergence in $W_0^1L_M(Q_T)$, with $\|v_j\|_{L^\infty(Q_T)} \leq (N+2)\|v\|_{L^\infty(Q_T)}$.

Pointwise multiplication of the approximate equation (26) by $T_k(u_n - v_j)$, we get

$$\begin{cases} \displaystyle\int_0^T < \frac{\partial b_n(x,u_n)}{\partial s}; T_k(u_n - v_j) > ds \\ \displaystyle + \int_{Q_T} a_n(u_n,\nabla u_n)\nabla T_k(u_n - v_j)dxds \\ \displaystyle + \int_{Q_T} \Phi_n(u_n)\nabla T_k(u_n - v_j)dxds \\ \displaystyle + \int_{Q_T} H_n(u_n,\nabla u_n)\nabla T_k(u_n - v_j)dxds \\ \displaystyle = \int_{Q_T} f_n T_k(u_n - v_j)dxds \end{cases} \tag{29}$$

We pass to the limit as in (29), n tend to $+\infty$ and j tend to $+\infty$:
Limit of the first term of (29):
The first term can be written

$$\int_0^T < \frac{\partial b_n(x,u_n)}{\partial s}; T_k(u_n - v_j) > ds$$

$$= \int_0^T < \frac{\partial v}{\partial s}; \int_0^{u_n} \frac{\partial b_n(x,z)}{\partial s} T_k'(z - v_j) > ds$$

$$+ \int_\Omega \int_0^{u_n(T)} \frac{\partial b_n(x,s)}{\partial s} T_k(s - v_j(T))dsdx$$

$$- \int_\Omega \int_0^{u_{0n}} \frac{\partial b_n(x,s)}{\partial s} T_k(s - v_j(0))dsdx,$$

the fact that $\frac{\partial b_n(x,s)}{\partial s} \geq 0$ and $v_j(T) = 0$, we get

$$\int_\Omega \int_0^{u(T)} \frac{\partial b(x,s)}{\partial s} T_k(s - v_j(T))dsdx =$$

$$\int_\Omega \int_0^{u(T)} \frac{\partial b(x,s)}{\partial s} T_k(s)dsdx \geq 0$$

On the other hand, we have u_{0n} converge to u_0 strongly in $L^1(\Omega)$, then

$$\lim_{n\to+\infty} \int_\Omega \int_0^{u_{0n}} \frac{\partial b_n(x,s)}{\partial s} T_k(s - v_j(0))dsdx$$

$$= \int_\Omega \int_0^{u_0} \frac{\partial b(x,s)}{\partial s} T_k(s - v_j(0))dsdx,$$

with $M = k + (N+2)\|v\|_\infty$ and $T_M(u_n)$ converges to $T_M(u)$ strongly in $E_M(Q_T)$, we obtain

$$\lim_{n\to+\infty} \int_0^T < \frac{\partial v_j}{\partial t}; \int_0^{u_n} \frac{\partial b_n(x,z)}{\partial s} T_k'(z - v_j)dz > ds$$

$$= \lim_{n\to+\infty} \int_0^T < \frac{\partial v_j}{\partial s}; \int_0^{T_M(u_n)} \frac{\partial b_n(x,z)}{\partial s} T_k'(z - v_j)dz > ds$$

$$= \int_0^T < \frac{\partial v_j}{\partial s}; \int_0^{T_M(u)} \frac{\partial b(x,z)}{\partial s} T_k'(z - v_j)dz > ds,$$

then

$$\int_0^T < \frac{\partial v_j}{\partial s}; \int_0^{T_M(u)} \frac{\partial b(x,z)}{\partial s} T_k'(z - v_j)dz > ds$$

$$\leq \lim_{n\to+\infty} \int_0^T < \frac{\partial b(x,u_n)}{\partial s} T_k(u_n - v_j) > ds$$

$$+ \int_\Omega \int_0^{u_0} < \frac{\partial b(x,s)}{\partial s} T_k(s - v_j(0)) > dsdx,$$

using (28), the definition of T_k and pass to limit as $j \to +\infty$, we deduce

$$\int_0^T < \frac{\partial v}{\partial s}; \int_0^{T_M(u)} \frac{\partial b(x,z)}{\partial s} T_k'(z-v)dz > ds$$
$$\leq \lim_{j\to+\infty}\lim_{n\to+\infty} \int_0^T < \frac{\partial b(x,u_n)}{\partial s} T_k(u_n - v_j) > ds$$
$$+ \int_\Omega \int_0^{u_0} < \frac{\partial b(x,t)}{\partial s} T_k(s - v(0)) > dsdx.$$

- We can follow same way in [21]to prove that
$$\liminf_{j\to\infty}\liminf_{n\to\infty} \int_{Q_T} a(u_n,\nabla u_n)\nabla T_k(u_n - v_j)dxds$$
$$\geq \int_{Q_T} a(u,\nabla u)\nabla T_k(u-v)dxds.$$
- For $n \geq k+(N+2)\|v\|_{L^\infty(Q_T)}$ $\Phi_n(u_n)\nabla T_k(u_n - v_j) = \Phi(T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u_n))\nabla T_k(u_n - v_j)$. The pointwise convergence of u_n to u as n tends to $+\infty$ and (15), then $\Phi(T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u_n))\nabla T_k(u_n - v_j) \rightharpoonup \Phi(T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u))\nabla T_k(u - v_j)$ weakly for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.
In a similar way, we obtain
$$\lim_{j\to\infty}\int_{Q_T} \Phi(T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u))\nabla T_k(u-v_j)dxds$$
$$= \int_{Q_T} \Phi(T_{k+(N+2)\|v\|_{L^\infty(Q_T)}}(u))\nabla T_k(u-v)dxds$$
$$= \int_{Q_T} \Phi(u)\nabla T_k(u-v)dxds.$$
- Limit of $H_n(u_n,\nabla u_n)T_k(u_n - v_j)$:
Since $H_n(u_n,\nabla u_n)$ converge strongly to $H(x,s,u,\nabla u)$ in $L^1(Q_T)$ and the pointwise convergence of u_n to u as $n \to +\infty$, it is possible to prove that $H_n(u_n,\nabla u_n)T_k(u_n - v_j)$ converge to $H(u,\nabla u)T_k(u - v_j)$ in $L^1(Q_T)$ and $\lim_{j\to\infty}\int_{Q_T} H(u,\nabla u)T_k(u - v_j)dxds = \int_{Q_T} H(u,\nabla u)T_k(u-v)dxds.$
- Since f_n converge strongly to f in $L^1(Q_T)$, and $T_k(u_n - v_j) \to T_k(u_n - v_j)$ weakly* in $L^\infty(Q_T)$, we have $\int_{Q_T} f_n T_k(u_n - v_j)dxds \to \int_{Q_T} f T_k(u-v_j)dxds$ as $n \to \infty$ and also we have $\int_{Q_T} f T_k(u - v_j)dxds \to \int_{Q_T} f T_k(u - v)dxds$ as $j \to \infty$
Finally, the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in equation (29) and to conclude that u satisfies (19).

It remains to show that $b(x,u)$ satisfies the initial condition. In fact, remark that, $B_M(x,u_n) = \int_0^{u_n} \frac{\partial b(x,s)}{\partial s} T_M(s-v)ds$ is bounded in $L^\infty(Q_T)$. Secondly, by (69) we show that $\frac{\partial B_M(x,u_n)}{\partial t}$ is bounded in $W^{-1,x}L_{\overline{M}}(Q_T)) + L^1(Q_T)$. As a consequence, a Lemma 4 implies that $B_M(x,u_n)$ lies in a compact set of $C^0([0,T];L^1(\Omega))$. It follows that, $B_M(x,u_n)(t=0)$ converges to $B_M(x,u)(t=0)$ strongly in $L^1(\Omega)$. On the order hand, the smoothness of B_M imply that $B_M(x,u_n)(t=0)$ converges to $B_M(x,u)(t=0)$ strongly in $L^1(\Omega)$, we conclude that $B_M(x,u_n)(t=0) = B_M(x,u_{0n})$ converges to $B_M(x,u)(t=0)$ strongly in $L^1(\Omega)$, we obtain $B_M(x,u)(t=0) = B_M(x,u_0)$ a.e. in Ω and for all $M > 0$, now letting M to $+\infty$, we conclude that $b(x,u)(t=0) = b(x,u_0)$ a.e. in Ω.

Remark 6 *We focus our work to show the conditions of the proposition 27, then for this we go through 4 steps to arrive at our result.*

Step 1: In this step let us begin by showing

Lemma 7
Let $\{u_n\}_n$ be a solution of the approximate problem (26), then for all $k>0$, there exists a constants C_1 and C_2 such that
$$\int_{Q_T} a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \leq kC_1, \quad (30)$$
and
$$\int_{Q_T} M(x,|\nabla T_k(u_n)|)dxdt \leq kC_2, \quad (31)$$
where C_1 and C_2 does not depend on the n and k.

Démonstration: Fixed $k>0$,
Let $\tau \in (0,T)$ and using $\exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)}$ as a test function in problem (26), where $G(s) = \int_0^s \frac{\rho(r)}{\alpha'}dr$ and $\alpha' > 0$ is a parameter to be specified later, we get:
$$\int_{Q_\tau} \frac{\partial b_n(x,u_n)}{\partial s}\exp(G(u_n))T_k(u_n)^+\chi_{(0,t)}dxdt \quad (32)$$
$$+\int_{Q_\tau} a(u_n,\nabla u_n)\frac{\rho(u_n)}{\alpha'}\exp(G(u_n))\nabla u_n T_k(u_n)^+dxdt \quad (33)$$
$$+\int_{\{0\leq u_n\leq k\}} a(u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)dxdt \quad (34)$$
$$+\int_{Q_\tau} \Phi_n(u_n)\nabla\Big(\exp(G(u_n))T_k(u_n)^+\Big)dxdt \quad (35)$$
$$+\int_{Q_\tau} H(u_n,\nabla u_n)\exp(G(u_n))T_k(u_n)^+dxdt \quad (36)$$
$$\leq k\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\|f_n\|_{L^1(Q_T)}. \quad (37)$$

For the (32), we have
$$\int_{Q_\tau} \frac{\partial b_n(x,u_n)}{\partial s}\exp(G(u_n))T_k(u_n)^+\chi_{(0,\tau)}dxdt$$
$$= \int_\Omega B_{n,k}(x,u_n(\tau))dx - \int_\Omega B_{n,k}(x,u_n(0))dx,$$

where

$$B_{n,k}(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s} \exp(G(T_k(s)))T_k(s)^+ ds.$$

By (11), we have $\int_\Omega B_{n,k}(x,u_n(\tau))dx \geq 0$ and

$$\int_{Q_\tau} B_{n,k}(x,u_n(0))dx \leq k \exp(\frac{\|\rho\|_{L^1}}{\alpha'})\|b(x,u_0\|_{L^1(\Omega)}.$$

For the (35), if we use (15) and Young inequality , we get

$$\int_{Q_\tau} \Phi_n(u_n))\nabla(\exp(G(u_n))T_k(u_n)^+)dxdt \leq$$

$$\frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha'}\Big[\alpha_0 \int_{Q_\tau} M(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt$$

$$+\int_{Q_\tau} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt\Big]$$

$$+\|c(.,.)\|_{L^\infty(Q_T)}\alpha_0 \int_{Q_\tau} M(x,u_n)\exp(G(u_n))dxdt$$

$$+\|c(.,.)\|_{L^\infty(Q_T)} \int_{Q_\tau} M(x,|\nabla T_k(u_n)^+|)\exp(G(u_n))dxdt.$$

For the (36), we have,

$$\int_{Q_\tau} H_n(u_n,\nabla u_n)\exp(G(u_n))T_k(u_n)^+dxdt$$

$$\leq k\exp(\frac{\|\rho\|_{L^1}}{\alpha'}) \int_{Q_T} |h(x,t)|dxdt$$

$$+\int_{Q_\tau} \rho(u_n)\exp(G(u_n))M(x,\nabla T_k(u_n))T_k(u_n)^+dxdt.$$

finally using the previous inequalities and (14), we obtain

$$\begin{cases} \frac{1}{\alpha'}\int_{Q_\tau} M(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +\frac{\alpha}{\alpha'}\int_{Q_\tau} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +\int_{Q_\tau} a(u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt \\ \leq \frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha'}[\alpha_0\int_{Q_\tau} M(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +\int_{Q_\tau} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt] \\ +\alpha_0\|c(.,.)\|_{L^\infty(Q_T)}\int_{\{0\leq u_n\leq k\}} M(x,u_n)\exp(G(u_n))dxdt \\ +\|c(.,.)\|_{L^\infty(Q_T)}\int_{Q_\tau} M(x,\nabla T_k(u_n)^+)\exp(G(u_n))dxdt \\ +\int_{Q_\tau} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +k[\exp(\frac{\|\rho\|_{L^1}}{\alpha'}(\|f\|_{L^1(Q_T)}+\|b(x,u_0\|_{L^1(\Omega)} \\ +\int_{Q_T}|h(x,t)|dxdt], \end{cases} \tag{38}$$

which becomes after simplification,

$$\begin{cases} [\frac{1-\alpha_0\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha'}]\int_{Q_\tau} M(x,u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +[\frac{\alpha-\|c(.,.)\|_{L^\infty(Q_T)}-\alpha'}{\alpha'}]\int_{Q_\tau} M(x,\nabla u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dxdt \\ +\int_{Q_\tau} a(u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt \\ \leq \frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha}[\alpha_0\alpha\int_{\{0\leq u_n\leq k\}} M(x,u_n)\exp(G(u_n))dxdt \\ +\alpha M(x,\nabla T_k(u_n)^+)\exp(G(u_n))dxdt]+kC. \end{cases} \tag{39}$$

If we choose α' such that $\alpha' < \alpha - \|c(.,.)\|_{L^\infty(Q_T)}$ and using again (14) in (39) we get

$$[1-\frac{\|c(.,.)\|_{L^\infty(Q_T)}}{\alpha}]\int_{Q_\tau} a(u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+dxdt \leq kC. \tag{40}$$

we deduce,

$$\int_{\{0\leq u_n\leq k\}} a(u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)dxdt \leq kc_1.$$

one has $\exp(G(u_n)) \geq 1$ for in $\{(x,t)\in Q_T : 0\leq u_n\leq k\}$ then

$$\int_{\{0\leq u_n\leq k\}} a(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kc_1. \tag{41}$$

and by (14) another again

$$\int_{Q_t} M(x,|\nabla T_k(u_n)^+|)dxdt \leq kc_2. \tag{42}$$

Similarly, taking $\exp(-G(u_n)T_k(u_n)^-\chi_{(0,\tau)}$ as a test function in problem (26), we get

$$\int_{Q_\tau} \frac{\partial b_n(x,u_n)}{\partial t}\exp(-G(u_n))T_k(u_n)^-dxdt \tag{43}$$

$$+\int_{Q_\tau} a_n(u_n,\nabla u_n)\nabla(\exp(-G(u_n))\nabla T_k(u_n)^-)dxdt \tag{44}$$

$$+\int_{Q_\tau} \Phi_n(u_n)\nabla(\exp(-G(u_n))\nabla T_k(u_n)^-)dxdt \tag{45}$$

$$+\int_{Q_\tau} H(u_n,\nabla u_n)\exp(-G(u_n))T_k(u_n)^-dxdt \tag{46}$$

$$\geq \int_{Q_\tau} f_n\exp(-G(u_n))T_k(u_n)^-dxdt. \tag{47}$$

and using same techniques above, we obtain

$$\int_{\{-k\leq u_n\leq 0\}} a(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kc_1. \tag{48}$$

since $\exp(-G(u_n)) \geq 1$ in $\{(x,t)\in Q_T : -k\leq u_n\leq 0\}$ and

$$\int_{Q_\tau} M(x,|\nabla T_k(u_n)^-|)dxdt \leq kc_2. \tag{49}$$

Combining now (41) and (48) we get,

$$\int_{Q_\tau} a(u_n,\nabla u_n)\nabla T_k(u_n)dxdt \leq kC_1, \tag{50}$$

in the same with (42) and (49) we get,

$$\int_{Q_\tau} M(x,|\nabla T_k(u_n)|)dxdt \leq kC_2. \tag{51}$$

we conclude that $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$ independently of n, and for any $k>0$, so there exists a subsequence still denoted by u_n such that

$$T_k(u_n) \rightharpoonup \xi_k \quad \text{weakly in} \quad W_0^{1,x}L_M(Q_T). \tag{52}$$

On the other hand, using (51), we have

$$M(x,\frac{k}{\delta})meas\{|u_n|>k\} \leq \int_{\{|u_n|>k\}} M(x,\frac{|T_k(u_n)|}{\delta})dxdt$$

$$\leq \int_{Q_T} M(x,|\nabla T_k(u_n)|)dxdt \leq kC_2,$$

then

$$meas\{|u_n|>k\} \leq \frac{kC_2}{M(x,\frac{k}{\delta})},$$

for all n and for all k.
Assuming that there exists a positive function M such that $\lim_{t\to\infty}\frac{M(t)}{t}=+\infty$ and $M(t)\leq ess\inf_{x\in\Omega}M(x,t)$, $\forall t\geq 0$. thus, we get

$$\lim_{k\to\infty} meas\{|u_n|>k\}=0.$$

Now we turn to prove the almost every convergence of u_n, $b_n(x,u_n)$ and $a_n(x,t,T_k(u_n),\nabla T_k(u_n))$.

Proposition 2 *Let u_n be a solution of the approximate problem, then*

$$u_n \to u \quad a.e \text{ in } \quad Q_T, \tag{53}$$

$b_n(x,u_n)\to b(x,u)$ *a.e in* Q_T *and*

$$b(x,u)\in L^\infty(0,T,L^1(\Omega)), \tag{54}$$

$$a_n(T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varpi_k \quad in \quad (L_{\overline{M}}(Q_T))^N, \tag{55}$$

$$for \quad \sigma(\Pi L_{\overline{M}},\Pi E_M),$$

for some $\varpi_k\in(L_{\overline{M}}(Q_T))^N$.

Démonstration: :
Proof of (53) **and** (54)**:**
Proceeding as in [22], we have for any $S\in W^{2,\infty}(I\!R)$, such that S', has a compact support ($suppS'\subset[-K,K]$).

$$B_S^n(x,u_n) \quad \text{is bounded in} \quad W_0^{1,x}L_M(Q_T), \tag{56}$$

and

$$\frac{\partial B_S^n(x,u_n)}{\partial t} \quad \text{is bounded in} \quad L^1(Q_T)+W^{-1,x}L_{\overline{M}}(Q_T), \tag{57}$$

independently of n.
Indeed, we have first

$$|\nabla B_S^n(x,u_n)| \leq \|A_K\|_{L^\infty(\Omega)}|DT_k(u_n)|\|S'\|_{L^\infty(\Omega)}+K\|S'\|_{L^\infty(\Omega)}B_K(x) \tag{58}$$

a.e. in Q_T.
As a consequence of (58) and (51) we then obtain (56). To show that (57) holds true, we multiply the equation (26) by $S'(u_n)$, to obtain

$$\frac{\partial B_S^n(x,u_n)}{\partial t} = div(S'(u_n)a_n(u_n,\nabla u_n)) - S''(u_n)a_n(u_n,\nabla u_n)\nabla u_n \tag{59}$$

$$+div(S'(u_n)\Phi_n(u_n)) - S''(u_n)\Phi_n(u_n)\nabla u_n$$

$$+H_n(u_n,\nabla u_n)S'(u_n)+f_nS'(u_n) \quad \text{in} \quad \mathcal{D}(Q_T).$$

where $B_S^n(x,r)=\int_0^r S'(s)\frac{\partial b_n(x,s)}{\partial s}ds$. Since $suppS'$ and $suppS''$ are both included in $[-K,K]$, u_n may be replaced by $T_k(u_n)$ in each of these terms. As a consequence, each term in the right hand side of (59) is bounded either in $W^{-1,x}L_{\overline{M}}(Q_T)$ or in $L^1(Q_T)$ which shows that (57) holds true.
Arguing again as in [22] estimates (56), (57) and the following remark (1), we can show (53) and (54).
Proof of (55) **:**
The same way in [15], we deduce that $a_n(T_k(u_n),\nabla T_k(u_n))$ is a bounded sequence in $(L_{\overline{M}}(Q_T))^N$, and we obtain (55).
Step 2: This technical lemma will help us in the step 3 of the demonstration,

Lemma 8 *If the subsequence u_n satisfies* (26), *then*

$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(u_n,\nabla u_n)\nabla u_n dxdt = 0. \tag{60}$$

Démonstration: Taking the function $Z_m(u_n)=T_1(u_n-T_m(u_n))^-$ and multiplying the approximating equation (26) by the test function $\exp(-G(u_n))Z_m(u_n)$ we get

$$\begin{cases} \int_{Q_T} B_{n,m}(x,u_n(T))dx \\ +\int_{Q_T} a_n(u_n,\nabla u_n)\nabla(\exp(-G(u_n))Z_m(u_n))dx\,dt \\ +\int_{Q_T} \Phi_n(u_n)\nabla(\exp(-G(u_n))Z_m(u_n))dx\,dt \\ +\int_{Q_T} H_n(u_n,\nabla u_n)\exp(-G(u_n))Z_m(u_n)dx\,dt \\ =\int_{Q_T} f_n\exp(-G(u_n))Z_m(u_n)\,dx\,dt+\int_{Q_T} B_{n,m}(x,u_{0n})dx \end{cases} \tag{61}$$

where $B_{n,m}(x,r)=\displaystyle\int_0^r \frac{\partial b_n(x,s)}{\partial s}\exp(-G(s))Z_m(s)ds.$
Using the same argument in step 2, we obtain

$$\int_{Q_T} M(x,|\nabla Z_m(u_n)|)dxdt \leq C(\int_{Q_T}|h(x,t)|Z_m(u_n)dxdt$$

$$+\int_{Q_T} f_nZ_m(u_n)dxdt+\int_{|u_{0n}|>m}|b_n(x,u_{0n})|dx).$$

where

$$C=\exp(\frac{\|\rho\|_{L^1}}{\alpha'})(\frac{\alpha}{\alpha-\|c(.,.)\|_{L^\infty(Q_T)}}).$$

Passing to limit as $n\to+\infty$, since the pointwise convergence of u_n and strongly convergence in $L^1(Q_T)$ of f_n and $b_n(x,u_{0n})$ we get

$$\lim_{n\to+\infty}\int_{Q_T} M(x,|\nabla Z_m(u_n)|)dxdt \leq C(\int_{Q_T} fZ_m(u)dxdt$$

$$+\int_{Q_T}|h(x,t)|Z_m(u)dxdt+\int_{\{|u_0|>m\}}|b(x,u_0)|dx).$$

By using Lebesgue's theorem and passing to limit as $m \to +\infty$, in the all term of the right-hand side, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} M(x,|\nabla Z_m(u_n)|)dxdt = 0, \tag{62}$$

On the other hand, by (15) and Young inequality,for $n > m+1$ we obtain
$\int_{Q_T} |\Phi_n(x,t,u_n)\exp(-G(u_n))\nabla Z_m(u_n)|dxdt$

$$\leq \exp(\frac{\|\rho\|_{L^1}}{\alpha'})[\int_{\{-(m+1)\leq u_n\leq -m\}} M(x,\alpha_0|T_{m+1}(u_n)|)dxdt$$

$$+\int_{Q_T} M(x,|\nabla Z_m(u_n)|)dxdt].$$

Using the pointwise convergence of u_n and by Lebesgues theorem, it follows,
$\lim_{n\to+\infty}\int_{Q_T} |\Phi_n(u_n)\exp(-G(u_n))\nabla Z_m(u_n)|dxdt$

$$\leq \exp(\frac{\|\rho\|_{L^1}}{\alpha'})[\int_{\{-(m+1)\leq u\leq -m\}} M(x,\alpha_0|T_{m+1}(u)|)dxdt$$

$$+\lim_{n\to+\infty}\int_{Q_T} M(x,|\nabla Z_m(u_n)|)dxdt]$$

passing to the limit in as $m \to +\infty$, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{Q_T} \Phi_n(u_n)\exp(-G(u_n))\nabla Z_m(u_n)dxdt = 0.$$

Finally passing to the limit in (61), we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{-(m+1)\leq u_n\leq -m\}} a_n(u_n,\nabla u_n)\nabla u_n dxdt = 0,$$

In the same way we take $Z_m(u_n) = T_1(u_n - T_m(u_n))^+$ and multiplying the approximating equation (26) by the test function $\exp(G(u_n))Z_m(u_n)$ and we also obtain

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\leq u_n\leq m+1\}} a_n(u_n,\nabla u_n)\nabla u_n dxdt = 0,$$

on the above we get (60).

Step 3: Almost everywhere convergence of the gradients.

This step is devoted to introduce a time regularization of the $T_k(u)$ for $k > 0$ in order to perform the monotonicity method.

Lemma 9 *(See [23]) Under assumptions* (11)-(18), *and let* (z_n) *be a sequence in* $W_0^{1,x}L_M(Q_T)$ *such that:*

$$z_n \rightharpoonup z \quad for \quad \sigma(\Pi L_M(Q_T),\Pi E_{\overline{M}}(Q_T)), \tag{63}$$

$$(a(x,t,z_n,\nabla z_n)) \quad is\ bounded\ in\ (L_{\overline{M}}(Q_T))^N, \tag{64}$$

$$\int_{Q_T} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]dxdt \to 0, \tag{65}$$

as n and s tend to $+\infty$, *and where* χ_s *is the characteristic function of* $Q_s = \{(x,t) \in Q_T; |\nabla z| \leq s\}$ *then,*

$$\nabla z_n \to \nabla z \quad a.e.\ in\ Q_T, \tag{66}$$

$$\lim_{n\to+\infty}\int_{Q_T} a(x,t,z_n,\nabla z_n)\nabla z_n dxdt = \int_{Q_T} a(x,t,z,\nabla z)\nabla z dxdt, \tag{67}$$

$$M(x,|\nabla z_n|) \to M(x,|\nabla z|) \quad in\ L^1(Q_T). \tag{68}$$

Let $v_j \in \mathcal{D}(Q_T)$ be a sequence such that $v_j \to u$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence.
This specific time regularization of $T_k(v_j)$ (for fixed $k \geq 0$) is defined as follows.
Let $(\alpha_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$\alpha_0^\mu \in L^\infty(\Omega)\cap W_0^1L_M(\Omega) \quad \text{for all} \quad \mu > 0, \tag{69}$$

$$\|\alpha_0^\mu\|_{L^\infty(\Omega)} \leq k, \text{ for all } \quad \mu > 0,$$

and α_0^μ converges to $T_k(u_0)$ a.e. in Ω and $\frac{1}{\mu}\|\alpha_0^\mu\|_{M,\Omega}$ converges to 0 as $\mu \to +\infty$.
For $k \geq 0$ and $\mu > 0$, let us consider the unique solution $(T_k(v_j))_\mu \in L^\infty(Q_T)\cap W_0^{1,x}L_M(Q_T)$ of the monotone problem:

$$\frac{\partial (T_k(v_j))_\mu}{\partial t} + \mu((T_k(v_j))_\mu - T_k(v_j)) = 0 \text{ in } D'(\Omega),$$

$$(T_k(v_j))_\mu(t=0) = \alpha_0^\mu \text{ in } \Omega.$$

Remark that due to

$$\frac{\partial (T_k(v_j))_\mu}{\partial t} \in W_0^{1,x}L_M(Q_T).$$

We just recall that, $(T_k(v_j))_\mu \to T_k(u)$ a.e. in Q_T, weakly-$*$ in $L^\infty(Q_T)$,

$(T_k(v_j))_\mu \to (T_k(u))_\mu$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence as $j \to +\infty$ and
$(T_k(u))_\mu \to T_k(u)$ in $W_0^{1,x}L_M(Q_T)$, for the modular convergence as $\mu \to +\infty$.

$$\|(T_k(v_j))_\mu\|_{L^\infty(Q_T)} \leq max(\|(T_k(u))\|_{L^\infty(Q_T)}, \|\alpha_0^\mu\|_{L^\infty(\Omega)}) \leq k,$$

$\forall\ \mu > 0$,$\forall\ k > 0$.
We introduce a sequence of increasing $\mathbf{C}^1(I\!R)$-functions S_m such that $S_m(r) = 1$ for $|r| \leq m$, $S_m(r) = m+1-|r|$, for $m \leq |r| \leq m+1$, $S_m(r) = 0$ for $|r| \geq m+1$ for any $m \geq 1$ and we denote by $\epsilon(n,\mu,\eta,j,m)$ the quantities such that

$$\lim_{m\to+\infty}\lim_{j\to+\infty}\lim_{\eta\to+\infty}\lim_{\mu\to+\infty}\lim_{n\to+\infty}\epsilon(n,\mu,\eta,j,m) = 0,$$

the main estimate is

For fixed $k \geq 0$, let $W_{\mu,\eta}^{n,j} = T_\eta(T_k(u_n) - T_k(v_j)_\mu)^+$ and $W_{\mu,\eta}^{j} = T_\eta(T_k(u) - T_k(v_j)_\mu)^+$.
Multiplying the approximating equation by $\exp(G(u_n)))W_{\mu,\eta}^{n,j}S_m(u_n)$ and using the same technique in step 2 we obtain:

$$\left\{\begin{array}{l}\displaystyle\int_{Q_T} < \frac{\partial b_n(x,u_n)}{\partial t} \exp(G(u_n))W^{n,j}_{\mu,\eta} S_m(u_n)\,dx\,dt\\ \displaystyle+\int_{Q_T} a_n(u_n,\nabla u_n)\exp(G(u_n))\nabla(W^{n,j}_{\mu,\eta})S_m(u_n)\,dx\,dt\\ \displaystyle+\int_{Q_T} a_n(u_n,\nabla u_n)\nabla u_n \exp(G(u_n))W^{n,j}_{\mu,\eta}S'_m(u_n)\,dx\,dt\\ \displaystyle-\int_{Q_T} \Phi_n(u_n)\exp(G(u_n))\nabla(W^{n,j}_{\mu,\eta})S_m(u_n)\,dx\,dt\\ \displaystyle-\int_{Q_T} \Phi_n(u_n)\nabla u_n \exp(G(u_n))W^{n,j}_{\mu,\eta}S'_m(u_n)\,dx\,dt\\ \displaystyle\le \int_{Q_T} f_n \exp(G(u_n))W^{n,j}_{\mu,\eta}S_m(u_n)\,dx\,dt\\ \displaystyle+\int_{Q_T} h(x,t)\exp(G(u_n))W^{n,j}_{\mu,\eta}S_m(u_n)\,dx\,dt.\end{array}\right. \tag{70}$$

Now we pass to the limit in (70) for k real number fixed.

In order to perform this task we prove below the following results for any fixed $k \ge 0$:

$$\int_{Q_T} \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))W^{n,j}_{\mu,\eta}S_m(u_n)\,dx\,dt \ge \epsilon(n,\mu,\eta,j) \tag{71}$$

for any $m \ge 1$,

$$\int_{Q_T} \Phi_n(u_n)S_m(u_n)\exp(G(u_n))\nabla(W^{n,j}_{\mu,\eta})\,dx\,dt = \epsilon(n,j,\mu) \tag{72}$$

for any $m \ge 1$,

$$\int_{Q_T} \Phi_n(u_n)\nabla u_n S'_m(u_n)\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt = \epsilon(n,j,\mu) \tag{73}$$

for any $m \ge 1$,

$$\int_{Q_T} a_n(u_n,\nabla u_n)\nabla u_n S'_m(u_n)\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt \le \epsilon(n,m), \tag{74}$$

$$\int_{Q_T} a_n(u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla(W^{n,j}_{\mu,\eta})\,dx\,dt \le C\eta + \epsilon(n,j,\mu,m), \tag{75}$$

$$\int_{Q_T} f_n S_m(u_n)\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt + \int_{Q_T} h(x,t)\exp(G(u_n))W^{n,j}_{\mu,\eta}S_m(u_n)\,dx\,dt \le C\eta + \epsilon(n,\eta), \tag{76}$$

$$\int_{Q_T} [a(T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))] \times[\nabla T_k(u_n) - \nabla T_k(u)]dx\,dt \to 0. \tag{77}$$

<u>**Proof of** (71)</u>:

Lemma 10

$$\int_{Q_T} \frac{\partial b_n(x,u_n)}{\partial t}\exp(G(u_n))W^{n,j}_{\mu,\eta}S_m(u_n)dxdt \ge \epsilon(n,\mu,\eta,\eta,j) \tag{78}$$

$m \ge 1$.

Démonstration: We adopt the same technics in the proof in [8].

<u>**Proof of** (72)</u>: If we take $n > m+1$, we get $\Phi_n(u_n)\exp(G(u_n))S_m(u_n) =$

$$\Phi(T_{m+1}(u_n))\exp(G(T_{m+1}(u_n)))S_m(T_{m+1}(u_n)),$$

then $\Phi_n(u_n)\exp(G(u_n))S_m(u_n)$ is bounded in $L_{\overline{M}}(Q)$, thus, by using the pointwise convergence of u_n and Lebesgue's theorem we obtain

$$\Phi_n(u_n)\exp(G(u_n))S_m(u_n) \to \Phi(u)\exp(G(u))S_m(u),$$

with the modular convergence as $n \to +\infty$, then $\Phi_n(u_n)\exp(G(u_n))S_m(u_n) \to \Phi(u)\exp(G(u))S_m(u)$ for $\sigma(\Pi L_{\overline{M}}, \Pi L_M)$.

On the other hand $\nabla W^{n,j}_{\mu,\eta} = \nabla T_k(u_n) - \nabla(T_k(v_j))_\mu$ for $|T_k(u_n) - (T_k(v_j))_\mu| \le \eta$ converge to $\nabla T_k(u) - \nabla(T_k(v_j))_\mu$ weakly in $(L_M(Q_T))^N$, then $\int_{Q_T} \Phi_n(u_n)\exp(G(u_n))S_m(u_n)\nabla W^{n,j}_{\mu,\eta}\,dx\,dt \to \int_{Q_T} \Phi(u)S_m(u)\exp(G(u))\nabla W^j_{\mu,\eta}\,dx\,dt$, as $n \to +\infty$.

using the modular convergence of $W^j_{\mu,\eta}$ as $j \to +\infty$ and letting μ tends to infinity, we get (72).

<u>**Proof of** (73)</u>: For $n > m+1 > k$, we have $\nabla u_n S'_m(u_n) = \nabla T_{m+1}(u_n)$ a.e. in Q_T. By the almost every where convergence of u_n we have $\exp(G(u_n))W^{n,j}_{\mu,\eta} \to \exp(G(u))W^j_{\mu,\eta}$ in $L^\infty(Q_T)$ weak-* and since the sequence $(\Phi_n(T_{m+1}(u_n)))_n$ converge strongly in $E_{\overline{M}}(Q_T)$ then $\Phi_n(T_{m+1}(u_n))\exp(G(u_n))\ W^{n,j}_{\mu,\eta} \to \Phi(x,t,T_{m+1}(u))\ \exp(G(u))W^j_{\mu,\eta}$, converge strongly in $E_{\overline{M}}(Q_T)$ as $n \to +\infty$. By virtue of $\nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u)$ weakly in $(L_M(Q_T))^N$ as $n \to +\infty$ we have

$$\int_{\{m\le|u_n|\le m+1\}} \Phi_n(T_{m+1}(u_n))\nabla u_n S'_m(u_n)\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt \to \int_{\{m\le|u|\le m+1\}} \Phi(u)\nabla u\exp(G(u))W^j_{\mu,\eta}\,dx\,dt$$

as $n \to +\infty$.

with the modular convergence of $W^j_{\mu,\eta}$ as $j \to +\infty$ and letting $\mu \to +\infty$ we get (73).

<u>**Proof of** (74)</u>: we have

$$\begin{aligned}&\int_{Q_T} a_n(u_n,\nabla u_n)S'_m(u_n)\nabla u_n \times\exp(G(u_n))\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt\\ &= \int_{\{m\le|u_n|\le m+1\}} a_n(u_n,\nabla u_n)S'_m(u_n)\nabla u_n \times\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,dt\\ &\le \eta C\int_{\{m\le|u_n|\le m+1\}} a_n(u_n,\nabla u_n)\nabla u_n\,dx\,dt.\end{aligned}$$

Using (60), we get

$$\int_{Q_T} a_n(u_n,\nabla u_n)S'_m(u_n)\nabla u_n\exp(G(u_n))W^{n,j}_{\mu,\eta}\,dx\,ds \le \epsilon(n,\mu,m).$$

Proof of (76): Since $S_m(r) \leq 1$ and $W_{\mu,\eta}^{n,j} \leq \eta$ we get

$$\int_{Q_T} f_n S_m(u_n)\exp(G(u_n))W_{\mu,\eta}^{n,j} \quad dx\,dt \leq \epsilon(n,\eta),$$

$$\int_{Q_T} h(x,t)\exp(G(u_n))W_{\mu,\eta}^{n,j}S_m(u_n)\,dx\,dt \leq C\eta.$$

Proof of (75):

$$\int_{Q_T} a_n(u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla W_{\mu,\eta}^{n,j}\,dx\,dt$$
$$= \int_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(v_j)_\mu)\leq\eta\}} a_n(T_k(u_n),\nabla T_k(u_n))$$
$$\times S_m(u_n)\exp(G(u_n))(\nabla T_k(u_n)-\nabla T_k(v_j)_\mu)\,dx\,dt$$
$$- \int_{\{|u_n|>k\}\cap\{0\leq T_k(u_n)-T_k(v_j)_\mu)\leq\eta\}} a_n(u_n,\nabla u_n)$$
$$\times S_m(u_n)\exp(G(u_n))\nabla T_k(v_j)_\mu\,dx\,dt \tag{79}$$

Since $a_n(T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))$ is bounded in $(L_{\overline{M}}(Q_T))^N$ there exist some $\varpi_{k+\eta} \in (L_{\overline{M}}(Q_T))^N$ such that $a_n(T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n)) \to \varpi_{k+\eta}$ weakly in $(L_{\overline{M}}(Q_T))^N$.
Consequently,

$$\int_{\{|u_n|>k\}\cap\{0\leq T_k(u_n)-T_k(v_j)_\mu)\leq\eta\}} a_n(u_n,\nabla u_n)S_m(u_n)$$
$$\exp(G(u_n))\nabla T_k(v_j)_\mu\,dx\,dt$$
$$= \int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(v_j)_\mu)\leq\eta\}} \varpi_{k+\eta}$$
$$\times S_m(u)\exp(G(u))\nabla T_k(v_j)_\mu\,dx\,dt + \epsilon(n), \tag{80}$$

where we have used the fact that
$S_m(u_n)\exp(G(u_n))\nabla T_k(v_j)_\mu)\chi_{\{|u_n|>k\}\cap\{0\leq T_k(u_n)-T_k(v_j)_\mu)\leq\eta\}}$

$\to S_m(u)\exp(G(u))\nabla T_k(v_j)_\mu)\chi_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(v_j)_\mu)\leq\eta\}}$

strongly in $(E_M(Q_T))^N$.
Letting $j \to +\infty$, we obtain

$$\int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(v_j)_\mu)\leq\eta\}} \varpi_{k+\eta}$$
$$S_m(u)\exp(G(u))\nabla T_k(v_j)_\mu\,dx\,dt$$
$$= \int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(u)_\mu)\leq\eta\}} \varpi_{k+\eta}$$
$$S_m(u)\exp(G(u))\nabla T_k(u)_\mu\,dx\,dt + \epsilon(n,j),$$

One easily has,
$\int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(u)_\mu)\leq\eta\}} S_m(u)\exp(G(u))\nabla T_k(u)_\mu \varpi_{k+\eta}\,dx\,dt$

$$= \epsilon(n,j,\mu).$$

By (70)-(76), (79) and (80) we obtain

$$\int_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(v_j)_\mu)\leq\eta\}} a_n(T_k(u_n),\nabla T_k(u_n))S_m(u_n)$$
$$\exp(G(u_n))(\nabla T_k(u_n)-\nabla T_k(v_j)_\mu)\,dx\,dt$$
$$\leq C\eta + \epsilon(n,j,\mu,m),$$

we know that $\exp(G(u_n)) \geq 1$ and $S_m(u_n) = 1$ for $|u_n| \leq k$ then

$$\int_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(v_j)_\mu)\leq\eta\}} a_n(x,t,T_k(u_n),\nabla T_k(u_n))$$
$$\times(\nabla T_k(u_n)-\nabla T_k(v_j)_\mu)\,dx\,dt \leq C\eta + \epsilon(n,j,\mu,m). \tag{81}$$

Proof of (77): Setting for $s > 0$, $Q^s = \{(x,t) \in Q : |\nabla T_k(u)| \leq s\}$ and $Q_j^s = \{(x,t) \in Q : |\nabla T_k(v_j)| \leq s\}$ and denoting by χ^s and χ_j^s the characteristic functions of Q^s and Q_j^s respectively, we deduce that letting $0 < \delta < 1$, define

$$\Theta_{n,k} = (a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u)))$$
$$\times(\nabla T_k(u_n) - \nabla T_k(u))$$

For $s > 0$, we have
$0 \leq \int_{Q^s} \Theta_{n,k}^\delta\,dx\,dt = \int_{Q^s} \Theta_{n,k}^\delta \chi_{\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}}\,dx\,dt$

$$+ \int_{Q^s} \Theta_{n,k}^\delta \chi_{\{T_k(u_n)-T_k(v_j)_\mu>\eta\}}\,dx\,dt$$

The first term of the right-side hand, with the Hölder inequality,
$\int_{Q^s} \Theta_{n,k}^\delta \chi_{\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}}\,dx\,dt \leq$

$$\Big(\int_{Q^s} \Theta_{n,k}\chi_{\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}}\,dx\,dt\Big)^\delta \Big(\int_{Q^s} dx\,dt\Big)^{1-\delta}$$
$$\leq C_1\Big(\int_{Q^s} \Theta_{n,k}\chi_{\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}}\,dx\,dt\Big)^\delta$$

Also using the Hölder inequality, the second term of the right-side hand is

$$\int_{Q^s} \Theta_{n,k}^\delta \chi_{\{T_k(u_n)-T_k(v_j)_\mu\eta\}}\,dx\,dt \leq \Big(\int_{Q^s} \Theta_{n,k}\,dx\,dt\Big)^\delta$$
$$\times\Big(\int_{\{T_k(u_n)-T_k(v_j)_\mu>\eta\}} dx\,dt\Big)^{1-\delta}$$

since $a(x,t,T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(Q_T))^N$, while $\nabla T_k(u_n)$ is bounded in $(L_M(Q_T))^N$ then $\int_{Q^s} \Theta_{n,k}^\delta \chi_{\{T_k(u_n)-T_k(v_j)_\mu\eta\}}\,dx\,dt \leq C_2 meas\{(x,t) \in Q_T : |T_k(u_n) - T_k(v_j)_\mu| > \eta\}^{1-\delta}$
We obtain,

$$\int_{Q^s} \Theta_{n,k}^\delta\,dx\,dt \leq C_1\Big(\int_{Q^s} \Theta_{n,k}\chi_{\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}}\,dx\,dt\Big)^\delta$$
$$+C_2 meas\{(x,t) \in Q_T : T_k(u_n) - T_k(v_j)_\mu > \eta\}^{1-\delta}$$

On the other hand,

$$\int_{Q^s} \Theta_{n,k}\chi_{\{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta\}}\,dx\,dt$$
$$\leq \int_{0\leq T_k(u_n)-T_k(v_j)_\mu\leq\eta} (a(T_k(u_n),\nabla T_k(u_n))$$
$$-a(T_k(u_n),\nabla T_k(u)\chi_s))(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx\,dt$$

For each $s > r, r > 0$, one has

$$0 \le \int_{Q^r \cap \{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))\,dx\,dt$$

$$\le \int_{Q^s \cap \{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))\,dx\,dt$$

$$= \int_{Q^s \cap \{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)\chi_s))(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx\,dt$$

$$\le \int_{Q \cap \{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)\chi^s))(\nabla T_k(u_n) - \nabla T_k(u)\chi^s)\,dx\,dt$$

$$= \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} (a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(v_j)\chi_j^s))(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)\,dx\,dt$$

$$+ \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \times (\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s)\,dx\,dt$$

$$+ \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} (a(T_k(u_n), \nabla T_k(v_j)\chi_j^s) - a(T_k(u_n), \nabla T_k(u)\chi^s))\nabla T_k(u_n)\,dx\,dt$$

$$- \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(v_j)\chi_j^s) \times \nabla T_k(v_j)\chi_j^s)\,dx\,dt$$

$$+ \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(u)\chi^s)\nabla T_k(u)\chi^s)\,dx\,dt$$

$$= I_1(n,j,s) + I_2(n,j) + I_3(n,j) + I_4(n,j,\mu) + I_5(n,\mu)$$

we will go to the limit as n, j, μ, and $s \to +\infty$ $I_1 = \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(u_n))$

$$\times(\nabla T_k(u_n) - \nabla T_k(v_j)_\mu)\,dx\,dt$$

$$- \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(u_n)) \times(\nabla T_k(v_j)\chi_j^s - \nabla T_k(v_j)_\mu)\,dx\,dt$$

$$- \int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(v_j)\chi_j^s)) \times(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s))\,dx\,dt$$

Using (81), the first term of the right-hand side, we get $\int_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} a(T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j)_\mu)\,dx\,dt$

$$\le C\eta + \epsilon(n,m,j,s)$$

$$- \int_{\{|u|>k\} \cap \{|T_k(u) - T_k(v_j)_\mu| \le \eta\}} a(T_k(u), 0)\nabla T_k(v_j)_\mu\,dx\,dt$$

$$\le C\eta + \epsilon(n,m,j,\mu)$$

The second term of the right-hand side tends to

$$\int_{\{|T_k(u) - T_k(v_j)_\mu| \le \eta\}} \varpi_k(\nabla T_k(v_j)\chi_j^s - \nabla T_k(v_j)_\mu)\,dx\,dt$$

since $a(T_k(u_n), \nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(Q_T))^N$, there exist some $\varpi_k \in (L_{\overline{M}}(Q_T))^N$ such that (for a subsequence still denoted by u_n

$$a(T_k(u_n), \nabla T_k(u_n)) \to \varpi_k \quad \text{in} \quad (L_M(Q_T))^N$$

$$\text{for} \quad \sigma(\Pi L_{\overline{M}}, \Pi E_M)$$

In view of the fact that $(\nabla T_k(v_j)\chi_j^s - \nabla T_k(v_j)_\mu)\chi_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} \to (\nabla T_k(v_j)\chi_j^s - \nabla T_k(v_j)_\mu)\chi_{\{0 \le T_k(u) - T_k(v_j)_\mu \le \eta\}}$ Strongly in $(E_M(Q_T))^N$ as $n \to +\infty$.

the third term of the right-hand side tends to $\int_{\{0 \le T_k(u) - T_k(v_j)_\mu \le \eta\}} a(T_k(u), \nabla T_k(v_j)\chi_j^s))$

$$(\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s))\,dx\,dt$$

Since $a(T_k(u_n), \nabla T_k(v_j)\chi_j^s))\chi_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}} \to a(T_k(u), \nabla T_k(v_j)\chi_j^s))\chi_{|T_k(u) - T_k(v_j)_\mu| \le \eta}$ in $(E_{\overline{M}}(Q_T))^N$ while

$$(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s)) \to (\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s))$$

in $(L_M(Q_T))^N$ for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ Passing to limit as $j \to +\infty$ and $\mu \to +\infty$ and using Lebesgue's theorem, we have

$$I_1 \le C\eta + \epsilon(n,j,s,\mu)$$

For what concerns I_2, by letting $n \to +\infty$, we have

$$I_2 \to \int_{\{0 \le T_k(u) - T_k(v_j)_\mu \le \eta\}} \varpi_k(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s)\,dx\,dt$$

Since $a(T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \varpi_k$ in $(L_{\overline{M}}(Q_T))^N$, for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$, while $(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s)\chi_{\{0 \le T_k(u_n) - T_k(v_j)_\mu \le \eta\}}$

$$\to (\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s)\chi_{\{0 \le T_k(u) - T_k(v_j)_\mu| \le \eta\}}$$

strongly in $(E_M(Q_T))^N$.

Passing to limit $j \to +\infty$, and using Lebesgue's theorem, we have

$$I_2 = \epsilon(n,j)$$

Similar ways as above give

$$I_3 = \epsilon(n,j)$$

$$I_4 = \int_{\{0 \le T_k(u) - T_k(u)_\mu \le \eta\}} a(T_k(u), \nabla T_k(u))\nabla T_k(u)\,dx\,dt + \epsilon(n,j,\mu,s,m)$$

$$I_5 = \int_{\{0 \le T_k(u) - T_k(u)_\mu \le \eta\}} a(T_k(u), \nabla T_k(u)) \nabla T_k(u)\, dx\, dt$$
$$+ \epsilon(n, j, \mu, s, m)$$

Finally, we obtain,

$$\int_{Q^s} \Theta_{n,k}\, dx\, dt \le C_1 (C\eta + \epsilon(n, \mu, \eta, m))^\delta + C_2(\epsilon(n, \mu,))^{1-\delta}$$

Which yields,by passing to the limit sup over n,j,μ ,s and η

$$\int_{Q^r \cap \{0 \le T_\eta(T_k(u) - T_k(u)_\mu)\}} [(a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))$$
$$(\nabla T_k(u_n) - \nabla T_k(u))]^\delta\, dx\, dt = \epsilon(n) \qquad (82)$$

Taking on the hand the function $W^{n,j}_{\eta,\mu} = T_\eta(T_k(u_n) - (T_k(v_j))_\mu)^-$ and $W^{j}_{\eta,\mu} = T_\eta(T_k(u) - (T_k(v_j))_\mu)^-$.
Multiplying the approximating equation by $\exp(G(u_n)))W^{n,j}_{\eta,\mu} S_m(u_n)$, we obtain

$$\int_{Q^r \cap \{T_\eta(T_k(u_n) - (T_k(v_j))_\mu) \le 0\}} [(a(x, T_k(u_n), \nabla T_k(u_n))$$
$$- a(x, T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))]^\delta\, dxdt = \epsilon(n) \qquad (83)$$

by (82) and (83) we get

$$\int_{Q^r} [(a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)))$$
$$\times(\nabla T_k(u_n) - \nabla T_k(u))]^\delta\, dxdt = \epsilon(n)$$

Thus, passing to a subsequence if necessary, $\nabla u_n \to \nabla u$ a.e. in Q^r, and since r is arbitrary,

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad Q_T$$

Step 4: Equi-integrability of the nonlinearity sequence
We shall prove that $H_n(u_n, \nabla u_n) \to H(u, \nabla u)$ strongly in $L^1(Q_T)$.
Consider $g_0(u_n) = \int_0^{u_n} \rho(s)\chi_{\{s>h\}} ds$ and multiply (26) by $\exp(G(u_n))g_0(u_n)$, we get

$$\Big[\int_{Q_T} B^n_h(x, u_n) dx\Big]_0^T + \int_{Q_T} a(, u_n, \nabla u_n)\nabla(\exp(G(u_n))g_0(u_n))dxdt$$
$$+ \int_{Q_T} \Phi_n(u_n, \nabla u_n)\nabla(\exp(G(u_n))g_0(u_n))dxdt$$
$$+ \int_{Q_T} H_n(u_n, \nabla u_n)\exp(G(u_n))g_0(u_n))dxdt$$
$$\le \Big(\int_h^{+\infty} \rho(s)dx\Big)\exp\Big(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha'}\Big)[\|f_n\|_{L^1(Q)} + \|h(x, t\|_{L^1(Q_T)}]$$

where $B^n_h(x, r) = \int_0^r \frac{\partial b(x,s)}{\partial s} g_0(s)\exp(G(T_k(s)))ds \ge 0$
then using same technique in step 2 we can have

$$\int_{\{u_n>h\}} \rho(u_n)M(x, \nabla u_n)dxdt \le C\Big(\int_h^{+\infty} \rho(s)dx\Big)$$

Since $\rho \in L^1(I\!R)$, we get

$$\lim_{h\to 0} \sup_{n\in I\!N} \int_{\{u_n>h\}} \rho(u_n)M(x, \nabla u_n)dxdt = 0$$

Similarly, let $g_0(u_n) = \int_{u_n}^0 \rho(s)\chi_{\{s<-h\}} dx$ in (26), we have also

$$\lim_{h\to 0} \sup_{n\in I\!N} \int_{\{u_n<-h\}} \rho(u_n)M(x, \nabla u_n)dxdt = 0$$

We conclude that

$$\lim_{h\to 0} \sup_{n\in I\!N} \int_{\{|u_n|>h\}} \rho(u_n)M(x, \nabla u_n)dxdt = 0 \qquad (84)$$

Let $D \subset Q_T$ then

$$\int_D \rho(u_n)M(x, \nabla u_n)dxdt \le \max_{\{|u_n|\le h\}} \rho(y) \int_{D\cap\{|u_n|\le h\}} M(x, \nabla u_n)dxdt$$
$$+ \int_{D\cap\{|u_n|>h\}} \rho(u_n)M(x, \nabla u_n)dxdt$$

Consequently $\rho(u_n)M(x, \nabla u_n)$ is equi-integrable. Then $\rho(u_n)M(x, \nabla u_n)$ converge to $\rho(u)M(x, \nabla u)$ strongly in $L^1(I\!R)$. By (16), we get our result.

As a conclusion, the proof of Theorem (4) is complete.

References

1. M. M. Porzio, " Existence of solutions for some noncoercive parabolic equations.", Discrete and Continuous Dynamical Systems. 5.3, 553-568, 1999.
2. L. Boccardo, L. Orsina, A. Porretta, " Some noncoercive parabolic equations with a lower order terms in divergence form.", Journal of Evolution Equations 3.3 Dedicated to philippe Bénilan, 407-418,2003.
3. R. Di Nardo, F. Feo, O. Guibé, " Existence result for nonlinear parabolic equations with lower order terms.", Analysis and Applications, Vol. 9, No. 2, 161186, 2011.
4. A. Aberqi, J. Bennouna, M. Mekkour, H. Redwane, " Nonlinear parabolic inequality with lower order terms.", Applicable Analysis 2016.
5. L. Boccardo, " Dirichlet problems with singular convection terms and applications.", Journal of Differential Equations 258 , 2290-2314,2015.
6. L. Boccardo, G. R. Cirmi, " $W_0^{1,1}$ solutions of some unilateral problems.", Nonlinear Analysis Series A: Theorie, Methods and Applications, 121 , 447-457, 2015.
7. G. Zecca, "Existence and uniqueness for nonlinear elliptic equations with lower order terms.", Nonlinear Analysis Serie A: Theorie, Methos and Applications, 75, 899-912, 2012.
8. Y. Akdim, J. Bennouna, M. Mekkour, H. Redwane, " Strongly nonlinear parabolic inequality in Orlicz spaces via a sequence of penalized equations.", African Mathematical Union 2014.
9. M. Mabdaoui, H. Moussa, M. Rhoudaf, " Entropy solutions for a nonlinear parabolic problems with lower order term in Orlicz spaces.",Analysis and Mathematical Physics DOI 10.1007/s13324-016-0129-5, 2016.

10. J. Musielak, " Modular spaces and Orlicz spaces.", Lecture Notes in Math, 1983.

11. A. Ahmed, A. Benkirane, M.S.B.Elemine Vall, A. Touzani, " Existence of entropy solutions for nonlinear elliptic equations in Musielak framework with L^1 data.", Boletim da Sociedade Paranaense de Matematica (3s.) v. 36 1 : 125150, 2018.

12. A. Benkirane, M. Sidi El Vally, " Some approximation properties in Musielak-Orlicz-Sobolev spaces." , Thai Journal of Mathematics, pp. 371-381, 2012.

13. A. Benkirane, M. Sidi El Vally, " Variational inequalities in Musielak-Orlicz-Sobolev spaces." , Bulletin of the Belgian Mathematical Society, pp. 787-811, 2014.

14. M.L. Ahmed Oubeid , A. Benkirane, M. Sidi El Vally, " Strongly nonlinear parabolic problems in Musiealak-Orlicz-Sobolev spaces.", Boletim da Sociedade Paranaense de Matematica, v. (33) 1 , 193-225,2015.

15. M. Ait Kellou, A. Benkirane, S.M. Douiri, " An inequality of type Poincare in Musielak spaces and applications to some nonlinear elliptic problems with L^1-data." ,Complex Variables and Elliptic Equations 60, pp. 1217-1242, 2015.

16. H. Redwane, " Existence of a solution for a class of nonlinear parabolic systems. ", Electronic Journal of Qualitative Theory of Differential Equations 24, 18pp., 2007.

17. A. Benkirane, J. Bennouna, " Existence of entropy solutions for some nonlinear problems in orlicz spaces.", Nonlinear Analysis 54, 85-102, 2001.

18. R. Adams, " Sobolev spaces.",New York (NY):Academic Press, 1975.

19. H. Brezis, " Analyse fonctionnelle.", Masson, 1987.

20. R. Landes, " On the existence of weak solutions for quasilinear parabolic initial-boundary value problems.", Proceedings of the Royal Society of Edinburgh Section A 89, 217-237, 1981.

21. Y. Akdim, N. El gorch, M. Mekkour, " Existence of renormalized solutions for p(x)-parabolic equations with three unbounded non- linearities.", Boletim da Sociedade Paranaense de Matematica, v (3) 34 , no. 1, 225-252, 2016.

22. D. Blanchard, H. Redwane, " Renormalized solutions for class of nonlinear evolution problems.", Journal de Mathmatiques Pures et Appliques. 77, pp. 117-151, 1998.

23. M.S.B. Elemine Vall, A. Ahmed, A. Touzani, A. Benkirane, " Existence of entropy solutions for nonlinear elliptic equations in Musielak framework with L1 data.", Boletim da Sociedade Paranaense de Matematica, v (3s.) v. 36 1, 125150, 2018.

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S-asymptotically ω-periodic solutions in the p-th mean for a Stochastic Evolution Equation driven by $\mathcal{Q}$-Brownian motion

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In this paper, we study the existence (uniqueness) and asymptotic stability of the p-th mean S-asymptotically ω-periodic solutions for some non-autonomous Stochastic Evolution Equations driven by a $\mathcal{Q}$-Brownian motion. This is done using the Banach fixed point Theorem and a Gronwall inequality.

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(\mathbb{H}, \|.\|)$ a real separable Hilbert space. We are concerned in this paper with the existence and asymptotic stability of p-th mean S-asymptotically ω-periodic solution of the following stochastic evolution equation

$$\begin{cases} dX(t) = A(t)X(t)dt + f(t, X(t))dt \\ \qquad +g(t, X(t))dW(t), \qquad t \geq 0 \\ X(0) = c_0, \end{cases} \tag{1}$$

where $(A(t))_{t\geq 0}$ is a family of densely defined closed linears operators which generates an exponentially stable ω-periodic two-parameter evolutionary family. The functions $f : \mathbb{R}_+ \times L^p(\Omega, \mathbb{H}) \to \mathbb{L}^p(\Omega, \mathbb{H})$, $g : \mathbb{R}_+ \times \mathbb{L}^p(\Omega, \mathbb{H}) \to \mathbb{L}^p(\Omega, L_2^0)$ are continuous satisfying some additional conditions and $(W(t))_{t\geq 0}$ is a $\mathcal{Q}$-Brownian motion. The spaces $\mathbb{L}^p(\Omega, \mathbb{H})$, L_2^0 and the $\mathcal{Q}$-Brownian motion are defined in the next section.

The concept of periodicity is important in probability especially for investigations on stochastic processes. The interest in such a notion lies in its significance and applications arising in engineering, statistics, etc. In recent years, there has been an increasing interest in periodic solutions (pseudo-almost periodic, almost periodic, almost automorphic, asymptotically almost periodic, etc) for stochastic evolution equations. For instance among others, let us mentioned the existence, uniqueness and asymptotic stability results of almost periodic solutions, almost automorphic solutions, pseudo almost periodic solutions studied by many authors, see, e.g. ([1]-[11]). The concept of S-asymptotically ω-periodic stochastic processes, which is the central question to be treated in this paper, was first introduced in the literature by Henriquez, Pierri et al in ([12, 13]). This notion has been developed by many authors.

In the literature, there has been a significant attention devoted this concept in the deterministic case; we refer the reader to ([14]-[20]) and the references therein. However, in the random case, there are few works related to the notion of S-asymptotically ω-periodicity with regard to the existence, uniqueness and asymptotic stability for stochastic processes. To our knowledge, the first work dedicated to S-asymptotically ω-periodicity for stochastic processes is due to S. Zhao and M. Song ([21, 22]) where they show existence of square-mean S-asymptotically ω-periodic solutions for a class of stochastic fractional functional differential equations and for a certain class of stochastic fractional evolution equation driven by Levy noise. But until now and to the best our knowledge,there is no investigations for the exis-

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tence (uniqueness), asymptotic stability of p-th mean S-asymptotically ω-periodic solutions when $p > 2$.

This paper is organized as follows. Section 2 deals with some preliminaries intended to clarify the presentation of concepts and norms used latter. We also give a composition result, see Theorem 1. In section 3 we present theoretical results on the existence and uniqueness of S-asymptotically ω-periodic solution of equation (1), see Theorem 2. We also present results on asymptotic stability of the unique S-asymptotically ω-periodic solution of equation(1), see Theorem 3.

2 Preliminaries

This section is concerned with some notations, definitions, lemmas and preliminary facts which are used in what follows.

2.1 p-th mean S asymptotically omega periodic process

Assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with some filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let $p \geq 2$. Denote by $\mathbb{L}^p(\Omega, \mathbb{H})$ the collection of all strongly measurable p-th integrable $\mathbb{H}$-valued random variables such that

$$\mathbb{E}\|X\| = \int_\Omega \|X(\omega)\|^p d\mathbb{P}(\omega) < \infty.$$

Definition 1 *A stochastic process $X : \mathbb{R}_+ \to \mathbb{L}^p(\Omega, \mathbb{H})$ is said to be continuous whenever*

$$\lim_{t\to s} \mathbb{E}\|X(t) - X(s)\|^p = 0.$$

Definition 2 *A stochastic process $X : \mathbb{R}_+ \to \mathbb{L}^p(\Omega, \mathbb{H})$ is said to be bounded if there exists a constant $C > 0$ such that*

$$\mathbb{E}\|X(t)\|^p \leq C \quad \forall t \geq 0$$

Definition 3 *A continous and bounded stochastic process $X : \mathbb{R}_+ \to \mathbb{L}^p(\Omega, \mathbb{H})$ is said to be p-mean S-asymptotically ω periodic if there exists $\omega > 0$ such that*

$$\lim_{t\to+\infty} \mathbb{E}\|X(t+\omega) - X(t)\|^p = 0, \quad \forall t \geq 0.$$

The collection of p-mean S-asymptotically ω-periodic stochastic process with values in $\mathbb{H}$ is then denoted by $SAP_\omega\Big(\mathbb{L}^p(\Omega, \mathbb{H})\Big)$.

A continuous bounded stochastic process X, which is 2-mean S-asymptotically ω-periodic is also called square-mean S-asymptotically ω-periodic.

Remark 1 *Since any p-mean S-asymptotically ω-periodic process X is $\mathbb{L}^p(\Omega, \mathbb{H})$ bounded and continuous, the space $SAP_\omega\Big(\mathbb{L}^p(\Omega, \mathbb{H})\Big)$ is a Banach space equipped with the sup norm :*

$$\|X\|_\infty = \sup_{t\geq 0}\Big(\mathbb{E}\|X(t)\|^p\Big)^{1/p}$$

Definition 4 *A function $F : \mathbb{R}_+ \times \mathbb{L}^p(\Omega, \mathbb{H}) \to \mathbb{L}^p(\Omega, \mathbb{H})$ which is jointly continuous, is said to be p-mean S-asymptotically ω periodic in $t \in \mathbb{R}_+$ uniformly in $X \in K$ where $K \subseteq \mathbb{L}^p(\Omega, \mathbb{K})$ is bounded if for any $\epsilon > 0$ there exists $L_\epsilon > 0$ such that*

$$\mathbb{E}\|F(t+\omega, X) - F(t, X)\|^p\Big] \leq \epsilon$$

for all $t \geq L_\epsilon$ and all process $X : \mathbb{R}_+ \to K$

Definition 5 *A function $F : \mathbb{R}_+ \times \mathbb{L}^p(\Omega, \mathbb{H}) \to \mathbb{L}^p(\Omega, \mathbb{H})$ which is jointly continuous, is said to be p-mean asymptotically uniformly continuous on bounded sets $K' \subseteq \mathbb{L}^p(\Omega, \mathbb{H})$, if for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that*

$$\mathbb{E}\|F(t, X) - F(t, Y)\|^p \leq \epsilon$$

for all $t \geq \delta_\epsilon$ and every $X, Y \in K'$ with $\mathbb{E}\|X - Y\|^p \leq \delta_\epsilon$.

Theorem 1 *Let $F : \mathbb{R}_+ \times \mathbb{L}^p(\Omega, \mathbb{H}) \to \mathbb{L}^p(\Omega, \mathbb{H})$ be a p-mean S-asymptotically ω periodic in $t \in \mathbb{R}_+$ uniformly in $X \in K$ where $K \subseteq \mathbb{L}^p(\Omega, \mathbb{H})$ is bounded and p-mean asymptotically uniformly continuous on bounded sets. Assume that $X : \mathbb{R}_+ \to \mathbb{L}^p(\Omega, \mathbb{H})$ is a p-mean S asymptotically ω-periodic process. Then the stochastic process $(F(t, X(t)))_{t\geq 0}$ is p-mean S-asymptotically ω periodic.*

Proof 1 *Since $X : \mathbb{R}_+ \to \mathbb{L}^p(\Omega, \mathbb{H})$ is a p-mean S-asymptotically ω-periodic process, for all $\epsilon > 0$, there exists $T_\epsilon > 0$ such that for all $t \geq T_\epsilon$:*

$$\mathbb{E}\|X(t+\omega) - X(t)\|^p \leq \epsilon \tag{2}$$

In addition X is bounded that is

$$\sup_{t\geq 0} \mathbb{E}\|X(t)\|^p < \infty$$

Let $K \subset \mathbb{L}^p(\Omega, \mathbb{H})$ be a bounded set such that $X(t) \in K$ for all $t \geq 0$.
We have :

$$\begin{aligned}
&\mathbb{E}\,\|F(t+\omega, X(t+\omega)) - F(t, X(t))\|^p \\
&\quad \leq 2^{p-1}\mathbb{E}\,\|F(t+\omega, X(t+\omega)) - F(t+\omega, X(t))\|^p \\
&\quad + 2^{p-1}\mathbb{E}\,\|F(t+\omega, X(t)) - F(t, X(t))\|^p
\end{aligned}$$

Taking into account (2) and using the fact that F is p-mean asymptotically uniformly continuous on bounded sets, there exists $\delta_\epsilon = \epsilon$ and $L_\epsilon = T_\epsilon$ such that for all $t \geq T_\epsilon$:

$$\mathbb{E}\,\|F(t+\omega, X(t+\omega)) - F(t+\omega, X(t))\|^p \leq \frac{\epsilon}{2^p} \tag{3}$$

Similarly, using the p-mean S-asymptotically ω periodicity in $t \geq 0$ uniformly on bounded sets of F it follows that for all $t \geq T_\epsilon$:

$$\mathbb{E}\,\|F(t+\omega, X(t)) - F(t, X(t))\|^p \leq \frac{\epsilon}{2^p} \tag{4}$$

Bringing together the inequalities (3) and (4), we thus obtain that for all $t \geq T_\epsilon > 0$

$$\mathbb{E}\,\|F(t+\omega, X(t+\omega)) - F(t, X(t))\|^p \leq \epsilon$$

so that the stochastic process $t \to F(t, X(t))$ is p-mean S-asymptotically ω- periodic.

Lemma 1 *Assume that $F : \mathbb{R}_+ \times \mathbb{L}^p(\Omega,\mathbb{H}) \to \mathbb{L}^p(\Omega,\mathbb{H})$ is p-mean uniformly S-asymptotically ω-periodic in $t \in \mathbb{R}_+$ uniformly on bounded sets and satisfies the Lipschitz condition, that is, there exists constant $L(F) > 0$ such that*

$$\mathbb{E}\|F(t,X) - F(t,Y)\|^p \leq L(F)\,\mathbb{E}\|X - Y\|^p \quad \forall t \geq 0,$$

$\forall X, Y \in \mathbb{L}^p(\Omega,\mathbb{K})$. *Let X be an p-mean S asymptotically ω-periodic proces, then the process $(F(t,X(t)))_{t\geq 0}$ is p-mean S-asymptotically ω-periodic.*

For the proof, the reader can refer to [22] whenever $p = 2$. The case $p > 2$ is similar.

Now let us recall the notion of evolutionary family of operators.

Definition 6 *A two-parameter family of bounded linear operators $\{U(t,s) : t \geq s \text{ with } t,s \geq 0\}$ from $\mathbb{L}^p(\Omega,\mathbb{H}))$ into itself associate with $A(t)$ is called an evolutionary family of operators whenever the following conditions hold:*

(a)

$$U(t,s)U(s,r) = U(t,r) \quad \text{for every } r \leq s \leq t;$$

(b)

$$U(t,t) = I, \text{ where } I \text{ is the identity operator};$$

(c) For all $X \in \mathbb{L}^p(\Omega,\mathbb{H}))$, the function $(t,s) \to U(t,s)X$ is continuous for $s < t$;

(d) The function $t \to U(t,s)$ is differentiable and

$$\frac{\partial}{\partial t}(U(t,s)) = A(t)U(t,s) \quad \text{for every } r \leq s \leq t;$$

For additional details on evolution families, we refer the reader to the book by Lunardi [23].

2.2 $\mathcal{Q}$-Brownian motion and Stochastic integrals

Let $(B_n(t))_{n\geq 1}$, $t \geq 0$ be a sequence of real valued standard Brownian motion mutulally independent on the filtered space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_{\sqcup})$. Set

$$W(t) = \sum_{n\geq 1} \sqrt{\lambda_n} B_n(t) e_n, \quad t \geq 0,$$

where $\lambda_n \geq 0$, $n \geq 1$, are non negative real numbers and $(e_n)_{n\geq 1}$ the complete orthonormal basis in the Hilbert space $(\mathbb{H}, \|.\|)$. Let $\mathcal{Q}$ be a symmetric nonnegative operator with finite trace defined by

$$\mathcal{Q}e_n = \lambda_n e_n \qquad \text{such that} \quad Tr(\mathcal{Q}) = \sum_{n\geq 1} \lambda_n < \infty$$

It is well known that $\mathbb{E}[W_t] = 0$ and for all $t \geq s \geq 0$, the distribution of $W(t) - W(s)$ is a Gaussian distribution $(\mathcal{N}(0,(t-s)\mathcal{Q}))$. The above-mentioned $\mathbb{H}$-valued stochastic process $(W(t))_{t\geq 0}$ is called an $\mathcal{Q}$-Brownian motion.

Let $(\mathbb{K}, \|.\|_K)$ be a real separable Hilbert space.
Let also $\mathcal{L}(\mathbb{K},\mathbb{H})$ be the space of all bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. If $\mathbb{K} = \mathbb{H}$, we denote it by $\mathcal{L}(\mathbb{H})$.

Set $\mathbb{H}_0 = \mathcal{Q}^{1/2}\mathbb{H}$. The space $\mathbb{H}_0$ is a Hilbert space equipped with the norm $\|u\|_{\mathbb{H}_0} = \|\mathcal{Q}^{1/2}u\|$.
Define

$$L_2^0 = \{\Phi \in \mathcal{L}(\mathbb{H}_0,\mathbb{H}) : Tr\Big[(\Phi\mathcal{Q}\Phi^*)\Big] < \infty\}$$

the space of all Hilbert-Schmidt operators from $\mathbb{H}_0$ to $\mathbb{H}$ equipped with the norm

$$\|\Phi\|_{L_2^0} = Tr\Big[(\Phi\mathcal{Q}\Phi^*)\Big] = \mathbb{E}\|\Phi\mathcal{Q}^{1/2}\|^2$$

In the sequel, to prove Lemma 4 and Theorem 2 we need the following Lemma that is a particular case of Lemma 2.2 in [24] (see also [25]).

Assume $T > 0$.

Lemma 2 *Let $G : [0,T] \to \mathcal{L}(\mathbb{L}^p(\Omega,\mathbb{H}))$ be an $\mathcal{F}_t$-adapted measurable stochastic process satisfying $\int_0^T \mathbb{E}\|G(t)\|^2 dt < \infty$ almost surely. Then,*

(i) the stochastic integral $\int_0^t G(s)dW(s)$ is a continuous, square integrable martingale with values in $(\mathbb{H}, \|.\|)$ such that

$$\mathbb{E}\left\|\int_0^t G(s)dW(s)\right\|^2 \leq \mathbb{E}\int_0^t \|G(s)\|^2 ds$$

(ii) There exists some constant $C_p > 0$ such that the following particular case of Burkholder-Davis-Gundy inequality holds :

$$\mathbb{E}\sup_{0\leq t\leq T}\left\|\int_0^t G(s)dW(s)\right\|^p \leq C_p\mathbb{E}\left(\int_0^T \|G(s)\|^2 ds\right)^{p/2}$$

In the sequel, we'll frequently make use of the following inequalities :

$$|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p) \quad \text{for all } p \geq 1 \text{ and any real numbers } a, b.$$

$$\int_{t_0}^t e^{-2a(t-s)}ds \leq \int_{t_0}^t e^{-a(t-s)}ds \leq \frac{1}{a} \quad \forall t \geq t_0, \text{ where } a > 0.$$

3 Main results

In this section, we investigate the existence and the asymptotically stability of the p-th mean S-asymptotically ω-periodic solution to the already defined stochastic differential equation :

$$\begin{cases} dX(t) = A(t)X(t)dt + f(t,X(t))dt + g(t,X(t))dW(t), & t \geq 0 \\ X(0) = c_0, \end{cases} \tag{5}$$

where $A(t), t \geq 0$ is a family of densely defined closed linear operators and

$$f : \mathbb{R}_+ \times \mathbb{L}^p(\Omega, \mathbb{H}) \to \mathbb{L}^p(\Omega, \mathbb{H}),$$

$$g : \mathbb{R}_+ \times \mathbb{L}^p(\Omega, \mathbb{H}) \to \mathbb{L}^p(\Omega, L_2^0)$$

are jointly continuous satisfying some additional conditions and $(W(t))_{t\geq 0}$ is a $\mathcal{Q}$-Brownian motion with values in $\mathbb{H}$ and $\mathcal{F}_t$-adapted.

Throughout the rest of this section, we require the following assumption on $U(t,s)$:

(H1): $A(t)$ generates an exponentially ω-periodic stable evolutionnary process $(U(t,s))_{t\geq s}$ in $\mathbb{L}^p(\Omega, \mathbb{H})$, that is, a two-parameter family of bounded linear operators with the following additional conditions :

1. $U(t+\omega, s+\omega) = U(t,s)$ for all $t \geq s$ (ω-periodicity).
2. There exists $M > 0$ and $a > 0$ such that $\|U(t,s)\| \leq Me^{-a(t-s)}$ for $t \geq s$.

Now, note that if $A(t)$ generates an evolutionary family $(U(t,s))_{t\geq s}$ on $\mathbb{L}^p(\Omega, \mathbb{H}))$ then the function g defined by $g(s) = U(t,s)X(s)$ where X is a solution of equation (1), satisfies the following relation

$$\begin{aligned} dg(s) &= -A(s)U(t,s)X(s) + U(t,s)dX(s) \\ &= -A(s)U(t,s)X(s) + A(s)U(t,s)X(s)ds \\ &+ U(t,s)f(s,X(s))ds + U(t,s)g(s,X(s))dW(s). \end{aligned}$$

Thus

$$dg(s) = U(t,s)f(s,X(s))ds + U(t,s)g(s,X(s))dW(s). \quad (6)$$

Integrating (6) on $[0,t]$ we obtain that

$$\begin{aligned} X(t) - U(t,0)c_0 &= \int_0^t U(t,s)f(s,X(s))ds \\ &+ \int_0^t U(t,s)g(s,X(s))dW(s). \end{aligned}$$

Therefore, we define

Definition 7 *An $(\mathcal{F}_t)$-adapted stochastic process $(X(t))_{t\geq 0}$ is called a mild solution of (1) if it satisfies the following stochastic integral equation :*

$$\begin{aligned} X(t) = U(t,0)c_0 &+ \int_0^t U(t,s)f(s,X(s))ds \\ &+ \int_0^t U(t,s)g(s,X(s))dW(s). \end{aligned}$$

3.1 The existence of p-th mean S-asymptotically ω-periodic solution

We require the following additional assumptions:

(H.2) The function $f : \mathbb{R}_+ \times \mathbb{L}^p(\Omega, \mathbb{H}) \to \mathbb{L}^p(\Omega, \mathbb{H})$ is p-mean S-asymptotically ω periodic in $t \in \mathbb{R}_+$ uniformly in $X \in K$ where $K \subseteq \mathbb{L}^p(\Omega, \mathbb{H})$ is a bounded set. Moreover the function f satisfies the Lipschitz condition, that is, there exists constant $L(f) > 0$ such that

$$\mathbb{E}\|f(t,X) - f(t,Y)\|^p \leq L(f)\mathbb{E}\|X - Y\|^p \quad \forall t \geq 0$$

$\forall X, Y \in \mathbb{L}^p(\Omega, \mathbb{H})$.

(H.3) The function $g : \mathbb{R}_+ \times \mathbb{L}^p(\Omega, L_2^0) \to \mathbb{L}^p(\Omega, L_2^0)$ is p-mean S-asymptotically ω periodic in $t \in \mathbb{R}_+$ uniformly in $X \in K$ where $K \subseteq \mathbb{L}^p(\Omega, L_2^0)$ is a bounded set. Moreover the function g satisfies the Lipschitz condition, that is, there exists constant $L(g) > 0$ such that

$$\mathbb{E}\|g(t,X) - g(t,Y)\|^p_{L_2^0} \leq L(g)\mathbb{E}\|X - Y\|^p \quad \forall t \geq 0,$$

$\forall X, Y \in \mathbb{L}^p(\Omega, \mathbb{H})$.

Lemma 3 *We assume that hypothesis* **(H.1)** *and* **(H.2)** *are satisfied. We define the nonlinear operator $\wedge_1$ by: for each $\phi \in SAP_\omega(\mathbb{L}^p(\Omega, \mathbb{H}))$*

$$(\wedge_1 \phi)(t) = \int_0^t U(t,s)f(s,\phi(s))ds$$

Then the operator $\wedge_1$ maps $SAP_\omega(\mathbb{L}^p(\Omega, \mathbb{H}))$ into itself.

Proof 2 *We define $h(s) = f(s,\phi(s))$. Since the hypothesis* **(H.2)** *is satisfied, using Lemma 1, we deduce that the function h is p-mean S-asymptotically ω-periodic.*
Define $F(t) = \int_0^t U(t,s)h(s)ds$. It is easy to check that F is bounded and continuous. Now we have :
$F(t+\omega) - F(t)$

$$\begin{aligned} &= \int_0^\omega U(t+\omega,s)h(s)ds + \int_0^t U(t,s)\Big(h(s+\omega) - h(s)\Big)ds \\ &= U(t+\omega,\omega)\int_0^\omega U(\omega,s)h(s)ds \\ &+ \int_0^t U(t,s)\Big(h(s+\omega) - h(s)\Big)ds \end{aligned}$$

$$\begin{aligned} &\mathbb{E}\|F(t+\omega) - F(t)\|^p \\ &\leq 2^{p-1}M^{2p}e^{-apt}\mathbb{E}\Big(\int_0^\omega e^{-a(\omega-s)}\|h(s)\|ds\Big)^p \\ &+ 2^{p-1}M^p\mathbb{E}\Big(\int_0^t e^{-a(t-s)}\|h(s+\omega) - h(s)\|ds\Big)^p \end{aligned}$$

Let p and q be conjugate exponents. Using Hölder inequality, we obtain that $\mathbb{E}\|F(t+\omega) - F(t)\|^p$

$$\begin{aligned} &\leq 2^{p-1}M^{2p}e^{-apt}\Big(\int_0^\omega e^{-aq(\omega-s)}ds\Big)^{p/q}\int_0^\omega \mathbb{E}\|h(s)\|^p ds \\ &+ 2^{p-1}M^p\mathbb{E}\Big(\int_0^t e^{-a(t-s)}\|h(s+\omega) - h(s)\|ds\Big)^p \\ &= I(t) + J(t) \end{aligned}$$

where

$$I(t) = 2^{p-1}M^{2p}e^{-apt}\Big(\int_0^\omega e^{-aq(\omega-s)}ds\Big)^{p/q}\int_0^\omega \mathbb{E}\|h(s)\|^p ds$$

$$J(t) = 2^{p-1}M^p\mathbb{E}\Big(\int_0^t e^{-a(t-s)}\|h(s+\omega)-h(s)\|ds\Big)^p$$
$$= 2^{p-1}M^p\mathbb{E}\Big(\int_0^t e^{-\frac{a}{q}(t-s)}\times e^{-\frac{a}{q}(t-s)}\|h(s+\omega)-h(s)\|ds\Big)^p$$

It is obvious that

$$\lim_{t\to+\infty} I(t) = 0.$$

Using Hölder inequality, we obtain that $J(t)$

$$\le 2^{p-1}M^p\Big(\int_0^t e^{-a(t-s)}ds\Big)^{p/q} \int_0^t e^{-a(t-s)}\mathbb{E}\|h(s+\omega)-h(s)\|^p ds \quad (7)$$
$$\le 2^{p-1}M^p\Big(\frac{1}{a}\Big)^{p/q}\int_0^t e^{-a(t-s)}\mathbb{E}\|h(s+\omega)-h(s)\|^p ds \quad (8)$$
$$(9)$$

Let $\epsilon > 0$. *Since* $\lim_{u\to+\infty}\mathbb{E}\|h(u+\omega)-h(u)\|^p = 0$:

$$\exists T_\epsilon > 0,\ u > T_\epsilon \Rightarrow \mathbb{E}\|h(u+\omega)-h(u)\|^p \le \frac{\epsilon a^p}{2^{p-1}M^p}. \quad (10)$$

We have
$J(t)$

$$\le 2^{p-1}M^p\Big(\frac{1}{a}\Big)^{p/q}\int_0^{T_\epsilon} e^{-a(t-s)}\mathbb{E}\|h(s+\omega)-h(s)\|^p ds$$
$$+ 2^{p-1}M^p\Big(\frac{1}{a}\Big)^{p/q}\int_{T_\epsilon}^t e^{-a(t-s)}\mathbb{E}\|h(s+\omega)-h(s)\|^p ds$$
$$= J_1(t)+J_2(t),$$

where

$$J_1(t) = 2^{p-1}M^p\Big(\frac{1}{a}\Big)^{p/q}\int_0^{T_\epsilon} e^{-a(t-s)}\mathbb{E}\|h(s+\omega)-h(s)\|^p ds$$

$$J_2(t) = 2^{p-1}M^p\Big(\frac{1}{a}\Big)^{p/q}\int_{T_\epsilon}^t e^{-a(t-s)}\mathbb{E}\|h(s+\omega)-h(s)\|^p ds$$

Estimation of $J_1(t)$.

$$\begin{aligned} J_1(t) & \\ \le\ & 2^{p-1}M^p\Big(\frac{1}{a}\Big)^{p/q}\int_0^{T_\epsilon} e^{-a(t-s)}\mathbb{E}\|h(s+\omega)-h(s)\|^p ds \\ \le\ & 2^{p-1}M^p\Big(\frac{1}{a}\Big)^{p/q} 2^p \sup_{t\ge0}\mathbb{E}\|h(t)\|^p e^{-at}\int_0^{T_\epsilon} e^{aqs}ds. \end{aligned}$$

It is clear that $\lim_{t\to+\infty} J_1(t) = 0$.

Estimation of $J_2(t)$.

Unsing the Inequality in (10) we have
$J_2(t)$

$$\begin{aligned} =\ & 2^{p-1}M^p\Big(\frac{1}{a}\Big)^{p/q}\int_{T_\epsilon}^t e^{-a(t-s)}\mathbb{E}\|h(s+\omega)-h(s)\|^p ds \\ \le\ & 2^{p-1}M^p\Big(\frac{1}{a}\Big)^{p/q}\Big(\frac{1}{a}\Big)\frac{\epsilon a^p}{2^{p-1}M^p} \\ =\ & 2^{p-1}M^p a^{-p}\frac{\epsilon a^p}{2^{p-1}M^p} \\ \le\ & \epsilon. \end{aligned}$$

Lemma 4 *We assume that hypothesis* **(H.1)** *and* **(H.3)** *are satisfied. We define the nonlinear operator* $\wedge_2$ *by: for each* $\phi \in SAP_\omega(\mathbb{L}^p(\Omega, L_2^0))$

$$(\wedge_2\phi)(t) = \int_0^t U(t,s)g(s,\phi(s))dW(s)$$

Then the operator $\wedge_2$ *maps* $SAP_\omega(\mathbb{L}^p(\Omega, L_2^0))$ *into itself.*

Proof 3 *We define* $h(s) = g(s,\phi(s))$. *Since the hypothesis* **(H.3)** *is satisfied, using Lemma 1, we deduce that the function h is p- mean S asymptotically ω periodic.*
Define $F(t) = \int_0^t U(t,s)h(s)dW(s)ds$. *It is easy to check that F is bounded and continuous. We have :*

$$F(t+\omega)-F(t)$$

$$\begin{aligned} &= \int_0^\omega U(t+\omega,s)h(s)dW(s) + \int_0^t U(t,s)\Big(h(s+\omega)-h(s)\Big)dW(s) \\ &= U(t+\omega,\omega)\int_0^\omega U(\omega,s)h(s)dW(s) + \\ &\qquad \int_0^t U(t,s)(h(s+\omega)-h(s))dW(s) \\ &= U(t+\omega,\omega)F(\omega) + \int_0^t U(t,s)\Big(h(s+\omega)-h(s)\Big)ds \end{aligned}$$

$$\mathbb{E}\|F(t+\omega)-F(t)\|^p$$

$$\begin{aligned} &\le 2^{p-1}M^pe^{-apt}\mathbb{E}\Big\|\int_0^\omega U(\omega,s)h(s)dW(s)\Big\|^p \\ &+ 2^{p-1}\mathbb{E}\Big\|\int_0^t U(t,s)\Big(h(s+\omega)-h(s)\Big)dW(s)\Big\|^p \\ &:= I(t)+J(t) \end{aligned}$$

where

$$I(t) = 2^{p-1}M^pe^{-pat}\mathbb{E}\Big\|\int_0^\omega U(\omega,s)h(s)dW(s)\Big\|^p$$

$$J(t) = 2^{p-1}\mathbb{E}\Big\|\int_0^t U(t,s)\Big(h(s+\omega)-h(s)\Big)dW(s)\Big\|^p$$

It is clear that

$$\lim_{t\to+\infty}\mathbb{E}I(t) = 0.$$

Let $\epsilon > 0$. *Since* $\lim_{t\to+\infty}\mathbb{E}\|h(t+\omega)-h(t)\|^p = 0$

$$\exists T_\epsilon > 0,\ t > T_\epsilon \Rightarrow \mathbb{E}\|h(s+\omega)-h(s)\|^p_{L_0^2} \le \frac{\epsilon\,(2a)^{p/2}}{4^{p-1}M^pC_p}, \quad (11)$$

where the constant C_p will be precised in the next lines. We have
$\mathbb{E}J(t)$

$$\begin{aligned}
&= 2^{p-1}\mathbb{E}\left\|\int_0^t U(t,s)\Big(h(s+\omega)-h(s)\Big)dW(s)\right\|^p \\
&\le 4^{p-1}\mathbb{E}\left\|\int_0^{T_\epsilon} U(t,s)\Big(h(s+\omega)-h(s)\Big)dW(s)\right\|^p \\
&+ 4^{p-1}\mathbb{E}\left\|\int_{T_\epsilon}^t U(t,s)\Big(h(s+\omega)-h(s)\Big)dW(s)\right\|^p \\
&:= \mathbb{E}J_1(t)+\mathbb{E}J_2(t),
\end{aligned}$$

where

$$J_1(t) = 4^{p-1}\left\|\int_0^{T_\epsilon} U(t,s)\Big(h(s+\omega)-h(s)\Big)dW(s)\right\|^p$$

$$J_2(t) = 4^{p-1}\left\|\int_{T_\epsilon}^t U(t,s)\Big(h(s+\omega)-h(s)\Big)dW(s)\right\|^p$$

Note that for all $t \ge 0$,

$$h(t+\omega)-h(t) \in \mathbb{L}^p(\Omega, L_2^0) \subseteq \mathbb{L}^2(\Omega, L_2^0)$$

and

$$\int_0^t \mathbb{E}\|U(t,s)(h(s+\omega)-h(s))\|^2 ds$$

$$\begin{aligned}
&\le M^2\int_0^t e^{-2a(t-s)}\mathbb{E}\|h(s+\omega)-h(s)\|^2_{L_2^0}ds \\
&\le 4M^2\sup_{t\ge0}\mathbb{E}\|h(t)\|^2_{L_2^0}\int_0^t e^{-2a(t-s)}ds \\
&\le 4M^2a^{-2}\sup_{t\ge0}\mathbb{E}\|h(t)\|^2_{L_2^0} \\
&< \infty.
\end{aligned}$$

<u>Estimation of</u> $\mathbb{E}J_1(t)$.

Assume that $p > 2$. Using Hölder inequality between conjugate exponents $\frac{p}{p-2}$ and $\frac{p}{2}$ together with Lemma 2, part (ii), there exists constant C_p such that :
$\mathbb{E}J_1(t)$

$$\begin{aligned}
&\le C_p4^{p-1}M^p\mathbb{E}\left[\int_0^{T_\epsilon} e^{-2a(t-s)}\|h(s+\omega)-h(s)\|^2_{L_2^0}ds\right]^{p/2} \\
&\le C_p4^{p-1}M^p\Big(\int_0^{T_\epsilon} e^{-\frac{2ap(t-s)}{p-2}}ds\Big)^{\frac{p-2}{2}} \\
&\qquad\int_0^{T_\epsilon}\mathbb{E}\|h(s+\omega)-h(s)\|^p_{L_2^0}ds \\
&\le C_p4^{p-1}M^p\, e^{-apt}\Big(\int_0^{T_\epsilon} e^{\frac{2aps}{p-2}}ds\Big)^{\frac{p-2}{2}}\, T_\epsilon 2^p\sup_{s\ge0}\mathbb{E}\|h(s)\|^p_{L_2^0}
\end{aligned}$$

Therefore

$$\lim_{t\to+\infty}\mathbb{E}J_1(t) = 0.$$

Assume that $p = 2$. By Lemma 2, part (i) we get :
$\mathbb{E}J_1(t)$

$$\begin{aligned}
&\le 4M^2\mathbb{E}\left[\int_0^{T_\epsilon} e^{-a(t-s)}\|h(s+\omega)-h(s)\|_{L_2^0}dB(s)\right]^2 \\
&\le 4M^2\mathbb{E}\left[\int_0^{T_\epsilon} e^{-2a(t-s)}\|h(s+\omega)-h(s)\|^2_{L_2^0}ds\right] \\
&\le 16M^2e^{-2at}\sup_{s\ge0}\mathbb{E}\|h(s)\|^2_{L_2^0}\int_0^{T_\epsilon} e^{2as}ds \\
&\le 16M^2e^{-2at}\sup_{s\ge0}\mathbb{E}\|h(s)\|^2_{L_2^0}\int_0^{T_\epsilon} e^{2as}ds
\end{aligned}$$

Thus

$$\lim_{t\to+\infty}\mathbb{E}J_1(t) = 0.$$

<u>Estimation of</u> $\mathbb{E}J_2(t)$.

Assume that $p > 2$. Using again Lemma 2, part (ii), Hölder inequality between conjugate exponents $\frac{p}{p-2}$ and $\frac{p}{2}$ and the inequality in (11) we have
$\mathbb{E}J_2(t)$

$$\begin{aligned}
&= 4^{p-1}\mathbb{E}\left\|\int_{T_\epsilon}^t U(t,s)\Big(h(s+\omega)-h(s)\Big)dW(s)\right\|^p \\
&\le 4^{p-1}M^pC_p\mathbb{E}\left[\int_{T_\epsilon}^t e^{-2a(t-s)}\|h(s+\omega)-h(s)\|^2_{L_2^0}ds\right]^{p/2} \\
&= 4^{p-1}M^pC_p\mathbb{E}\left[\int_{T_\epsilon}^t e^{-2a(t-s)\frac{p-2}{p}}\times e^{-2a(t-s)\frac{2}{p}}\|h(s+\omega)-h(s)\|^2_{L_2^0}ds\right]^{p/2} \\
&\le C_p4^{p-1}M^p\Big(\int_{T_\epsilon}^t e^{-2a(t-s)}ds\Big)^{\frac{p-2}{2}}\int_{T_\epsilon}^t e^{-2a(t-s)}\mathbb{E}\|h(s+\omega)-h(s)\|^p_{L_2^0}ds \\
&\le \frac{C_p4^{p-1}M^p\epsilon(2a)^{p/2}}{C_p4^{p-1}M^p}\Big(\int_{T_\epsilon}^t e^{-2a(t-s)}ds\Big)^{\frac{p}{2}} \\
&\le \epsilon.
\end{aligned}$$

We conclude that

$$\lim_{t\to+\infty}\mathbb{E}J_2(t) = 0.$$

Assume that $p = 2$. By Lemma 2, part (i) and Cauchy-Schwarz inequality we have
$\mathbb{E}J_2(t)$

$$\begin{aligned}
&= 4\mathbb{E}\left\|\int_{T_\epsilon}^t U(t,s)\Big(h(s+\omega)-h(s)\Big)dW(s)\right\|^2 \\
&\le 4M^2\mathbb{E}\left[\int_{T_\epsilon}^t e^{-2a(t-s)}\|h(s+\omega)-h(s)\|^2_{L_2^0}ds\right] \\
&= 4M^2\left[\int_{T_\epsilon}^t e^{-a(t-s)}\times e^{-a(t-s)}\mathbb{E}\|h(s+\omega)-h(s)\|^2_{L_2^0}ds\right] \\
&\le 4M^2\Big(\int_{T_\epsilon}^t e^{-2a(t-s)}ds\Big)^{1/2} \\
&\qquad\Big(\int_{T_\epsilon}^t e^{-2a(t-s)}\Big(\mathbb{E}\|h(s+\omega)-h(s)\|^2_{L_2^0}\Big)^2ds\Big)^{1/2}
\end{aligned}$$

Note also that for $t \ge T_\epsilon$:

$$\mathbb{E}\|h(s+\omega)-h(s)\|^2_{\mathbb{L}_2^0} \le \frac{\epsilon a}{2M^2}$$

so that

$$\mathbb{E}J_2(t) \leq \frac{4M^2\epsilon a}{2M^2}\Big(\int_{T_\epsilon}^{t} e^{-2a(t-s)}ds\Big) \leq \epsilon.$$

This implies that

$$\lim_{t\to+\infty} \mathbb{E}J_2(t) = 0$$

Finally, we conclude that

$$\lim_{t\to+\infty} \mathbb{E}\|F(t+\omega) - F(t)\|^p = 0$$

Theorem 2 *We assume that hypothesis* **(H.1)**, **(H.2)** *and* **(H.3)** *are satisfied and*

(i)

$$\Theta = 2^{p-1}M^p\Big(L(f)a^{-p} + C_pL(g)a^{\frac{-p}{2}}\Big) < 1 \qquad if \quad p > 2 \tag{12}$$

(ii)

$$\Xi = 2M^2\Big(L(f)\frac{1}{a^2} + L(g)\frac{1}{a}\Big) < 1 \qquad if \quad p = 2 \tag{13}$$

Then the stochastic evolution equation (1) has a unique p-mean S-asymptoticaly ω-periodic solution.

Proof 4 *We define the nonlinear operator Γ by the expression*

$$(\Gamma\Phi)(t) = U(t,0)c_0 + \int_0^t U(t,s)f(s,\Phi(s))ds + \int_0^t U(t,s)g(s,\Phi(s))dW(s)$$

Note that

$$(\Gamma\Phi)(t) = U(t,0)c_0 + (\wedge_1\Phi)(t) + (\wedge_2\Phi)(t)$$

According to the hypothesis **(H1)** *we have :*

$$\begin{aligned}\mathbb{E}\|U(t+\omega,0) - U(t,0)\|^p &\leq 2^{p-1}\Big(\mathbb{E}\|U(t+\omega,0\|^p + \mathbb{E}\|U(t,0)\|^p\Big)\\ &\leq 2^{p-1}M^pe^{-ap(t+\omega)} + M^pe^{-apt}\\ &= 2^{p-1}M^pe^{-apt}\Big(e^{-ap\omega}+1\Big)\end{aligned}$$

Therefore

$$\lim_{t\to+\infty} \mathbb{E}\|U(t+\omega,0) - U(t,0)\|^p = 0$$

According to Lemma 3 and Lemma 4, the operators $\wedge_1$ and $\wedge_2$ maps the space of p-mean S-asymptotically ω-periodic solutions into itself. Thus Γ maps the space of p-mean S-asymptotically ω periodic solutions into itself. We have

$$\begin{aligned}\mathbb{E}\|\Gamma\Phi(t) - \Gamma\Psi(t)\|^p &\leq 2^{p-1}\mathbb{E}\Big(\int_0^t \|U(t,s)\|\|f(s,\Phi(s)) - f(s,\Psi(s))\|ds\Big)^p\\ &+ 2^{p-1}\mathbb{E}\Big(\int_0^t \|U(t,s)\|\|g(s,\Phi(s)) - g(s,\Psi(s))\|dW(s)\Big)^p\\ &\leq 2^{p-1}M^p\mathbb{E}\Big(\int_0^t e^{-a(t-s)}\|f(s,\Phi(s)) - f(s,\Psi(s))\|ds\Big)^p\\ &+ 2^{p-1}M^p\mathbb{E}\Big(\int_0^t e^{-a(t-s)}\|g(s,\Phi(s)) - g(s,\Psi(s))\|dW(s)\Big)^p\end{aligned}$$

Case p > 2 : By Lemma 2, part (ii) and Hölder inequality we have

$$\begin{aligned}&\mathbb{E}\|\Gamma\Phi(t) - \Gamma\Psi(t)\|^p\\ &\leq 2^{p-1}M^pL(f)\Big(\int_0^t e^{-a(t-s)}ds\Big)^{p-1}\int_0^t e^{-a(t-s)}\mathbb{E}\|\Phi(s) - \Psi(s)\|^pds\\ &+ 2^{p-1}M^pC_p\mathbb{E}\Big(\int_0^t e^{-2a(t-s)}\|g(s,\Phi(s)) - g(s,\Psi(s))\|^2_{L_2^0}ds\Big)^{p/2}\\ &\leq 2^{p-1}M^pL(f)\sup_{s\geq 0}\mathbb{E}\|\Phi(s) - \Psi(s)\|^p\Big(\int_0^t e^{-a(t-s)}ds\Big)^p\\ &+ 2^{p-1}M^pC_p\mathbb{E}\Big(\int_0^t e^{-a(t-s)}\|g(s,\Phi(s)) - g(s,\Psi(s))\|^2_{L_2^0}ds\Big)^{p/2}\\ &\leq 2^{p-1}M^pa^{-p}L(f)\|\Phi - \Psi\|^p_\infty +\\ &2^{p-1}M^pC_p\mathbb{E}\Big(\int_0^t e^{-\frac{ap(t-s)}{p-2}}e^{-\frac{a(t-s)}{p}}\|g(s,\Phi(s)) - g(s,\Psi(s))\|^2_{L_2^0}ds\Big)^{p/2}\\ &\leq 2^{p-1}M^pa^{-p}L(f)\|\Phi - \Psi\|^p_\infty\\ &+ 2^{p-1}M^pC_p\Big(\int_0^t e^{-a(t-s)}ds\Big)^{\frac{p-2}{2}}\times\\ &\int_0^t e^{-a(t-s)}\mathbb{E}\|g(s,\Phi(s)) - g(s,\Psi(s))\|^p_{L_2^0}ds\\ &\leq 2^{p-1}M^p\Big(L(f)a^{-p} + C_pL(g)a^{-\frac{p}{2}}\Big)\|\Phi - \Psi\|^p_\infty\end{aligned}$$

This implies that

$$\|\Gamma\Phi - \Gamma\Psi\|^p_\infty \leq 2^{p-1}M^p\Big(L(f)a^{-p} + C_pL(g)a^{-\frac{p}{2}}\Big)\|\Phi - \Psi\|^p_\infty$$

Consequently, if Θ < 1, then Γ is a contraction mapping. One completes the proof by the Banach fixed-point principle.

Case p = 2 : using Cauchy-Schwarz inequality and Lemma 2, part (i), we obtain

$$\begin{aligned}&\mathbb{E}\|\Gamma\Phi(t) - \Gamma\Psi(t)\|^2\\ &\leq 2M^2\Big(\int_0^t e^{-a(t-s)}ds\Big)\mathbb{E}\int_0^t e^{-a(t-s)}\|f(s,\Phi(s)) - f(s,\Psi(s))\|^2ds\\ &+ 2M^2\mathbb{E}\int_0^t e^{-2a(t-s)}\|g(s,\Phi(s)) - g(s,\Psi(s))\|^2_{L_2^0}ds\\ &\leq 2M^2L(f)\sup_{s\geq 0}\mathbb{E}\|\Phi(s) - \Psi(s)\|^2\Big(\int_0^t e^{-a(t-s)}ds\Big)^2\\ &+ 2M^2L(g)\sup_{s\geq 0}\mathbb{E}\|\Phi(s) - \Psi(s)\|^2\int_0^t e^{-2a(t-s)}ds\\ &\leq 2M^2\Big(L(f)\frac{1}{a^2} + L(g)\frac{1}{a}\Big)\sup_{s\geq 0}\mathbb{E}\|\Phi(s) - \Psi(s)\|^2\end{aligned}$$

This implies that

$$\|\Gamma\Phi - \Gamma\Psi\|_\infty^2 \leq 2M^2\Big(L(f)\frac{1}{a^2} + L(g)\frac{1}{a}\Big)\|\Phi - \Psi\|_\infty^2$$

Consequently, if $\Xi < 1$, then Γ is a contraction mapping. One completes the proof by the Banach fixed-point principle.

3.2 Stability of p-mean S asymptotically ω periodic solution

In the previous section, for the non linear SDE, we obtain that it has a unique p-mean S-asymptotically ω-periodic solution under some conditions. In this section, we will show that the unique p mean S asymptotically ω periodic solution is asymptotically stable in the p mean sense.
Recall that

Definition 8 *The unique p-mean S asymptotically ω periodic solution $X^*(t)$ of (1) is said to be stable in p-mean sense if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\mathbb{E}\|X(t) - X^*(t)\|^p < \epsilon, \qquad t \geq 0,$$

whenever $\mathbb{E}\|X(0) - X^*(0)\|^p < \delta$, where $X(t)$ stands for a solution of (1) with initial value $X(0)$.

Definition 9 *The unique p-mean S asymptotically ω periodic solution $X^*(t)$ is said to be asymptotically stable in p-mean sense if it is stable in p-mean sense and*

$$\lim_{t\to\infty} \mathbb{E}\|X(t) - X^*(t)\|^p = 0$$

The following Gronwall inequality is proved to be useful in our asymptotical stability analysis.

Lemma 5 *Let $u(t)$ be a non negative continuous functions for $t \geq 0$, and α, γ be some positive constants. If*

$$u(t) \leq \alpha e^{-\beta t} + \gamma\int_0^t e^{-\beta(t-s)}u(s)ds, \quad t \geq 0,$$

then

$$u(t) \leq \alpha\exp\{(-\beta+\gamma)t\}$$

Theorem 3 *Suppose that hypothesis* **(H.1)**, **(H.2)** *and* **(H.3)** *are satisfied and assume that*

(i)

$$3^{p-1}M^p\Big(L(f)a^{1-p} + L(g)C_p a^{\frac{2-p}{2}}\Big) < a \qquad (14)$$

whenever $p > 2$.

(ii)

$$3M^2\Big(L(f)a^{-1} + L(g)\Big) < a \qquad (15)$$

whenever $p = 2$.

Then the p-mean S-asymptotically solution X_t^* of (1) is asymptotically stable in the p-mean sense.

Remark 2 *Note that the above conditions (14) respectively (15) implies conditions (12) respectively (13) of Theorem 2.*

Proof 5 $\mathbb{E}\|X(t) - X^*(t)\|^p$

$$\begin{aligned}
&= \mathbb{E}\Big\|U(t,0)(X(0) - X^*(0)) \\
&\quad + \int_0^t U(t,s)\Big(f(s,X(s)) - f(s,X^*(s))\Big)ds \\
&\quad + \int_0^t U(t,s)\Big(g(s,X(s)) - g(s,X^*(s))\Big)dW(s)\Big\|^p
\end{aligned}$$

Assume that $p > 2$. Using Hölder inequality we have $\mathbb{E}\|X(t) - X^*(t)\|^p$

$$\begin{aligned}
&\leq 3^{p-1}M^p e^{-apt}\mathbb{E}\|X(0) - X^*(0)\|^p \\
&+ 3^{p-1}\mathbb{E}\Big(\int_0^t \|U(t,s)\|\,\|f(s,X(s)) - f(s,X^*(s))\|ds\Big)^p \\
&+ 3^{p-1}\mathbb{E}\Big(\int_0^t \|U(t,s)\|\,\|g(s,X(s)) - g(s,X^*(s))\|dW(s)\Big)^p \\
&\leq 3^{p-1}M^p e^{-apt}\mathbb{E}\|X(0) - X^*(0)\|^p \\
&+ 3^{p-1}M^p\mathbb{E}\Big(\int_0^t e^{-a(t-s)}\|f(s,X(s)) - f(s,X^*(s))\|ds\Big)^p \\
&+ 3^{p-1}M^p\mathbb{E}\Big(\int_0^t e^{-a(t-s)}\|g(s,X(s)) - g(s,X^*(s))\|dW(s)\Big)^p \\
&= 3^{p-1}M^p e^{-apt}\mathbb{E}\|X(0) - X^*(0)\|^p + 3^{p-1}M^p \times \Big(\\
&\mathbb{E}\Big(\int_0^t e^{-\frac{a(p-1)(t-s)}{p}}e^{-\frac{a(t-s)}{p}}\|f(s,X(s)) - f(s,X^*(s))\|ds\Big)^p\Big) \\
&+ 3^{p-1}M^p C_p \times \Big(\\
&\Big(\int_0^t e^{-\frac{2a(p-2)(t-s)}{p}}e^{-\frac{4a(t-s)}{p}}\mathbb{E}\|g(s,X(s)) - g(s,X^*(s))\|_{L_2^0}^2 ds\Big)^{p/2}\Big) \\
&\leq 3^{p-1}M^p e^{-apt}\mathbb{E}\|X(0) - X^*(0)\|^p \\
&+ 3^{p-1}M^p L(f)\Big(\int_0^t e^{-a(t-s)}ds\Big)^{p-1} \times \Big(\\
&\int_0^t e^{-a(t-s)}\mathbb{E}\|X(s) - X^*(s)\|^p ds\Big) \\
&+ 3^{p-1}M^p C_p\Big(\int_0^t e^{-2a(t-s)}\Big)^{\frac{p-2}{2}} \times \Big(\\
&\int_0^t e^{-2a(t-s)}\mathbb{E}\,\|g(s,X(s)) - g(s,X^*(s))\|_{L_2^0}^p ds\Big)
\end{aligned}$$

so that
$\mathbb{E}\|X(t) - X^*(t)\|^p$

$$\begin{aligned}
&\leq 3^{p-1}M^p e^{-apt}\mathbb{E}\|X(0) - X^*(0)\|^p \\
&+ 3^{p-1}M^p L(f)(\frac{1}{a})^{p-1}\int_0^t e^{-a(t-s)}\mathbb{E}\|X(s) - X^*(s)\|^p ds \\
&+ 3^{p-1}M^p C_p L(g)(\frac{1}{a})^{\frac{p-2}{2}}\int_0^t e^{-a(t-s)}\mathbb{E}\,\|X(s) - X^*(s)\|^p ds
\end{aligned}$$

Using Lemma 5, we obtain :
$\mathbb{E}\|X(t) - X^*(t)\|^p$

$$\begin{aligned}
&\leq 3^{p-1}M^p \times \mathbb{E}\|X(0) - X^*(0)\|^p \times \\
&\exp\Big\{\Big(-a + 3^{p-1}M^p\Big(L(f)a^{1-p} + L(g)C_p a^{\frac{2-p}{2}}\Big)\Big)t\Big\}
\end{aligned}$$

Straightforwardly, we obtain that $X(t)$ converges to 0 exponentially fast if

$$-a+3^{p-1}M^p\Big(L(f)(\frac{1}{a})^{p-1}+L(g)C_p\,a^{\frac{2-p}{2}}\Big)<0,$$

which is equivalent to our condition (14). Therefore X^* is asymptotically stable in the p-mean sense.

Assume that $p=2$. We have

$$\begin{aligned}
&\mathbb{E}\|X(t)-X^*(t)\|^2\\
&\le 3M^2e^{-at}\mathbb{E}\|X(0)-X^*(0)\|^2\\
&\quad+3\mathbb{E}\Big(\int_0^t\|U(t,s)\|\|f(s,X(s))-f(s,X^*(s))\|ds\Big)^2\\
&\quad+3\mathbb{E}\Big(\int_0^t\|U(t,s)\|\|g(s,X(s))-g(s,X^*(s))\|dW(s)\Big)^2\\
&\le 3M^2e^{-at}\mathbb{E}\|X(0)-X^*(0)\|^2\\
&\quad+3M^2\mathbb{E}\Big(\int_0^t e^{-a(t-s)}\|f(s,X(s))-f(s,X^*(s))\|ds\Big)^2\\
&\quad+3M^2\mathbb{E}\Big(\int_0^t e^{-a(t-s)}\|g(s,X(s))-g(s,X^*(s))\|dW(s)\Big)^2
\end{aligned}$$

Then using Cauchy-Schwartz inequality and Lemma 2.3 part (i), we have

$$\begin{aligned}
&\mathbb{E}\|X(t)-X^*(t)\|^2\\
&\le 3M^2e^{-at}\mathbb{E}\|X(0)-X^*(0)\|^2\\
&+3M^2\int_0^t e^{-a(t-s)}ds\int_0^t e^{-a(t-s)}\mathbb{E}\|f(s,X(s))-f(s,X^*(s))\|^2ds\\
&+3M^2\int_0^t e^{-2a(t-s)}\,\mathbb{E}\|g(s,X(s))-g(s,X^*(s))\|^2_{L_2^0}ds,\\
&\le 3M^2e^{-at}\mathbb{E}\|X(0)-X^*(0)\|^2\\
&+3M^2L(f)a^{-1}\int_0^t e^{-a(t-s)}\mathbb{E}\|X(s)-X^*(s)\|^2ds\\
&+3M^2L(g)\int_0^t e^{-a(t-s)}\,\mathbb{E}\|X(s)-X^*(s)\|^2ds
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E}\|X(t)-X^*(t)\|^2\\
&\le 3M^2e^{-at}\mathbb{E}\|X(0)-X^*(0)\|^2\\
&+\Big(\frac{3M^2L(f)}{a}+3M^2L(g)\Big)\int_0^t e^{-a(t-s)}\mathbb{E}\|X(s)-X^*(s)\|^2ds
\end{aligned}$$

By Lemma 5 we have

$$\mathbb{E}\|X(t)-X^*(t)\|^2$$

$$\le 3M^2\mathbb{E}\|X(0)-X^*(0)\|^2\exp\Big\{\Big(-a+3M^2(\frac{L(f)}{a}+L(g))\Big)t\Big\}$$

Therefore $\mathbb{E}\|X(t)-X^*(t)\|^2$ converges to 0 exponentially fast whenever condition (15) holds. In particular the unique S-asymptotically ω-periodic solution is asymptotically stable in square mean sense.

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References

[1] P. Bezandry and T. Diagana, *Existence of Almost Periodic Solutions to Some Stochastic Differential Equations. Applicable Analysis, 86 (7), 819-827.*

[2] P. Bezandry and T. Diagana, *Square-Mean Almost Periodic Solutions Nonautonomous Stochastic Differential Equations. Electron. J. Differential Equations (2007), no. 117, 1-10.*

[3] P. Bezandry and T. Diagana, *Existence of Quadratic- Mean Almost Periodic Solutions to Some Stochastic Hyperbolic Differential Equations. Electron. J. Differential Equations, 2009, vol. 2009, no. 111, 1-14.*

[4] P. Bezandry and T. Diagana, *Square-Mean Almost Periodic solutions to Some Stochastic Hyperbolic Differential Equations with Infinite Delay. Commun. Math. Anal. 8 (2010), no.2, 1-22*

[5] P. Bezandry and T. Diagana, *p-th Mean Pseudo Almost Automorphic Mild solutions to Some Nonautonomous Stochastic Differential Equations. Afr. Diaspora. J. Math. Vol. 12 (2011), no.1, 60-79*

[6] J. Cao, Q.Yang, Z.Huang and Q. Liu *Asymptotically almost periodic solutions of stochastic functional differential equations. Applied Mathematics and Computation 218 (2011) 1499-1511*

[7] Y-K. Chang, Z-H. Zhao and G.M. N'Gurkata *Square mean almost automorphic mild solutions to non-autonomous stochastic differential equations in Hilbert spaces. Computers and Mathematics with Applications 61 (2011),384-391.*

[8] M.A. Diop, K.Ezzinbi and M.M. Mbaye. *Existence and global attractiveness of a pseudo almost periodic solution in p-th mean sense for stochastic evolution equation driven by a fractional Brownian motion. Stochastics An International Journal of Probability and Stochastic Processes 87(6):1-33, 2015.*

[9] Xie Liang, *Square-Mean Almost Periodic solutions to Some Stochastic Evolutions Equations. Acta Mathematica Sinica. English Series. (2014) Vol. 30, no.5, 881-898*

[10] ZhenXin Liu and Kai Sun. *Almost automorphic solutions to SDE driven by Levy noise. Journal of Functional Analysis 266 (2014), 1115-1149*

[11] M. Zhang and G. Zong. *Almost Periodic Solutions for stochastic Differential Equations driven by G-Brownian motion. Communications in Statistics. Theory and Methods. 44(11),2371-2384 (2015)*

[12] H. R. Henríquez, M. Pierri and P. Táboas. *On S-asymptotically ω-periodic function on Banach spaces and applications. J. Math. Anal Appl. 343, 1119-1130, 2008.*

[13] H. R. Henríquez, M. Pierri and P. Táboas. *Existence of S-asymptotically ω-periodic solutions for abstract neutral equations. Bull. Aust. Math.Soc. 78, 365-382, 2008.*

[14] J. Blot, P. Cieutat and G. M. N'Guérékata *S-asymptotically ω-periodic functions and applications to evolution equations. African Diaspora J. Math. 12, 113-121, 2011.*

[15] C. Cuevas and J.C. de Souza. *S-asymptotically ω-periodic solutions of semilinear fractional integro-differential equations. Appl. Math. Lett, 22, 865-870, 2009.*

[16] C. Cuevas and C. Lizanna. *S-asymptotically ω-periodic solutions of semilinear Volterra equations. Math. Meth. Appl. Sci, 33, 1628-1636, 2010.*

[17] C. Cuevas and C. Lizanna. *Existence of S-asymptotically ω-periodic solutions for two-times fractional order differential equations. Southeast.Asian Bull.Math. 37, 683-690, 2013.*

[18] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions. Encyclopedia of Mathematics and Its Applications, 44, Cambridge University Press, Cambridge, 1992.*

[19] W. Dimbour J-C. Mado. *S-asymptotically ω-periodic solution for a nonlinear differential equation with piecewise constant argument in a Banach space. CUBO A Mathematical Journal 16(13), 55-65, 2014.*

[20] W. Dimbour, G.Mophou and G.M. N'Guérékata. *S-asymptotically ω-periodic solution for partial differential equations with finite delay. Electron.J.Differ.Equa. 2011, 1-12, 2011.*

[21] Shufen Zhao and Minghui Song. *S-asymptotically ω-periodic solutions in distribution for a class of Stochastic fractional functional differential equations. arXiv : 1609.01453v1 [math.DS]. 6 Sep 2016.*

[22] Shufen Zhao and Minghui Song. *Square-mean S-asymptotically ω-periodic solutions for a Stochastic fractional evolution equation driven by Levy noise with piecewise constant argument. arXiv : 1609.01444v1 [math.DS]. 6 Sep 2016.*

[23] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems. PNLDE Vol. 16, Birkhaauser Verlag, Basel,1995.*

[24] J.Seidler. *Da Prato-Zzabczyk's maximal inequality revisited. I. Math. Bohem 118(1), 67-106, 1993*

[25] J.Seidler and T. Sobukawa. *Exponential Integrability of Stochastic Convolutions. J. London Math. Soc. (2) 67 (2003), 245-258*

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On the Spectrum of problems involving both p(x)-Laplacian and P(x)-Biharmonic

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ABSTRACT

We prove the existence of at least one non-decreasing sequence of positive eigenvalues for the problem

$$\begin{cases} \Delta^2_{p(x)}u - \Delta_{p(x)}u = \lambda|u|^{p(x)-2}u, & \text{in } \Omega \\ u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), \end{cases}$$

Our analysis mainly relies on variational arguments involving Ljusternik-Schnirelmann theory.

1 Introduction

Consider the following nonlinear eigenvalue problem

$$\begin{cases} \Delta^2_{p(x)}u - \Delta_{p(x)}u = \lambda|u|^{p(x)-2}u, & \text{in } \Omega \\ u \in W_0^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 4)$.
The real λ is a parameter which plays the role of eigenvalue. For a function $p(.) \in C(\overline{\Omega})$, we assume the following hypothesis

$$1 < p^- = \min_{x\in\overline{\Omega}} p(x) \leq p^+ = \max_{x\in\overline{\Omega}} p(x) < +\infty. \tag{1.2}$$

$\Delta^2_{p(x)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$, is the $p(x)$-biharmonic operator which is a natural generalization of the p-biharmonic (where the exponent p is constant) and $\Delta_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-harmonic operator.

It is well known that elliptic equations involving the non-standard growth are not trivial generalizations of similar problems studied in the constant case since the non-standard growth operator is not homogeneous and, thus, some techniques which can be applied in the case of the constant growth operators will fail in this new situation, such as the Lagrange multiplier theorem, see, e.g [1, 2]. Problerms with $p(x)$-growth conditions are an interesting topic, which arises from nonlinear electrorheological fluids and elastic mechanics.

Recently for the case $p(x) \equiv p$ constant Giri, Choudhuri and Pradhan [3] proved the existence and concentration phenomena of solutions on the set $V^{-1}\{0\}$ for the following p-biharmonic elliptic equation:
$\Delta^2_p u - \Delta_p u + \lambda V(x)|u|^{p-2}u = f(x,u) \quad x \in \mathbb{R}^{\mathbb{N}}$, as $\lambda \to \infty$ unther some assumptions on the nonlinear function f. By variational methods, Lihua Liu and Caisheng Chen [4] establish the existence of infinitely many high-energy solutions to the equation

$$\Delta^2_p u - \Delta_p u + V(x)|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^{\mathbb{N}},$$

with a concave-convex nonlinearity,i.e.,
$f(x,u) = \lambda h_1(x)|u|^{m-2}u + h_2(x)|u|^{q-2}u$, $1 < m < p < q < p^* = \frac{pN}{N-2p}$.
In our case by using the Ljusternik-Schnirelmann theory we obtain the existence of infinitely many solutions for the problem (1.1).

The outline of the rest of the paper is as follows. In Section 2 we present some definitions and basic resuls that are necessary. In Section 3 we give the proof of our main result about existence of solutions for problem (1.1).

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2 Preliminaries and Useful results

We state some basic properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(.)}(\Omega)$ and $W^{m,p(.)}(\Omega)$. We refer the reader to the monograph by [5] and to the references therein. Define the generalized Lebesgue space by

$$L^{p(.)}(\Omega)=\Big\{u:\Omega\to\mathbb{R}\text{ measurable and }\int_\Omega |u(x)|^{p(x)}dx<\infty\Big\},$$

endowed with the Luxemburg norm

$$|u|_{p(.)}=\inf\Big\{\mu>0:\int_\Omega\Big|\frac{u}{\mu}\Big|^{p(x)}dx\le 1\Big\}$$

To manipulate this spaces better, we use the modular mapping

$$\rho:L^{p(.)}(\Omega)\to\mathbb{R}$$

defined by

$$\rho(u)=\int_\Omega |u|^{p(x)}\,dx$$

Proposition 2.1 (6) *Under the hypothesis* (1.2), *the space $(L^{p(x)}(\Omega),|\cdot|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate dual space is $L^{p'(.)}(\Omega)$ where $p'(.)$ is the conjugate function of $p(.)$, related by*

$$p'(x)=\frac{p(x)}{p(x)-1},\quad \forall x\in\Omega.$$

For $u\in L^{p(.)}(\Omega)$ and $v\in L^{p'(.)}(\Omega)$ we have

$$\Big|\int_\Omega u(x)v(x)\,dx\Big|\le\Big(\frac{1}{p^-}+\frac{1}{p'^-}\Big)|u|_{p(.)}|v|_{p'(.)}\le 2|u|_{p(.)}|v|_{p'(.)}.$$

Sobolev space with variable exponent $W^{m,p(.)}(\Omega)$ are defined as

$$W^{m,p(.)}(\Omega)=\Big\{u\in L^{p(.)}(\Omega):D^\alpha u\in L^{p(.)}(\Omega),|\alpha|\le m\Big\},$$

where $D^\alpha u=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_N^{\alpha_N}}u$, (the derivation in distributions sense) with $\alpha=(\alpha_1,\dots,\alpha_N)$ is a multi-index and $|\alpha|=\sum_{i=1}^N\alpha_i$. The space $W^{m,p(.)}(\Omega)$, equipped with the norm

$$\|u\|_{m,p(x)}=\sum_{|\alpha|\le m}|D^\alpha u|_{p(x)},$$

is a Banach, separable and reflexive space. For more details, we refer the reader to [6, 7, 8] and [9]. We denote by $W_0^{m,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p(x)}(\Omega)$.
Note that the weak solutions of the problem (1.1) are considered in the Sobolev space $W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$ is equiped with the norm

$$\|u\|_{p(x)}=|\triangle u|_{p(x)}+|\nabla u|_{p(x)}$$

In the sequel, we Set

$$X=W_0^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)$$

Then, endowed with the norm $\|u\|_{p(x)}$, X is a separable and reflexive Banach space. Moreover, $\|.\|_{p(x)}$ and $|\triangle u|_{p(x)}$ are two equivalent norms of X by [10, Theorem4.4].
Let

$$\|u\|=\inf\{\mu>0;\int_\Omega\Big(\Big|\frac{\Delta u}{\mu}\Big|^{p(x)}+\Big|\frac{\nabla u}{\mu}\Big|^{p(x)}\Big)dx\le 1\},$$

Then, $\|u\|$ is equivalent to the norms $\|.\|_{p(x)}$ and $|\triangle u|_{p(x)}$ in X.

Lemma 2.2 (6) *For all $p,r\in C_+(\overline{\Omega})$ such that $r(x)\le p_m^*(x)$ for all $x\in\overline{\Omega}$, then there is a continuous and compact embedding $W^{m,p(x)}(\Omega)\hookrightarrow L^{r(x)}(\Omega)$, where*

$$p_m^*(x)=\begin{cases}\frac{Np(x)}{N-mp(x)}, & \text{if } mp(x)<N;\\ +\infty, & \text{if } mp(x)\ge N.\end{cases}$$

Proposition 2.3 *Let $I(u)=\int_\Omega(|\frac{\Delta u}{\mu}|^{p(x)}+|\frac{\nabla u}{\mu}|^{p(x)})dx$, for $u\in L^{p(.)}$, we have*

(1) $\|u\|<(=;>1)\Leftrightarrow I(u)<(=;>1)$

(2) $\|u\|\le 1\Rightarrow\|u\|^{p^+}\le I(u)\le\|u\|^{p^-}$

(3) $\|u\|\ge 1\Rightarrow\|u\|^{p^-}\le I(u)\le\|u\|^{p^+}$

(4) $\|u\|\to 0(resp\to+\infty)\Leftrightarrow I(u)\to 0,(resp\to+\infty)$

The proof of this proposition is similar to the proof of [6, Theorem 1.3] .

Recall that our main result of this work is to show that problem (1.1) has at least one non-decreasing sequence of nonnegative eigenvalues $(\lambda_k)_{k\ge1}$. To attain this objective we will use a variational technique based on Ljusternick-Schnirelmann theory on C^1-manifolds [11]. In fact, we give a direct characterization of λ_k involving a mini-max argument over sets of genus greater than k.

We set

$$\lambda_1=\inf\Big\{\int_\Omega\frac{1}{p(x)}(|\Delta u|^{p(x)}+|\nabla u|^{p(x)})\,dx,u\in X,\int_\Omega\frac{1}{p(x)}|u|^{p(x)}\,dx=1\Big\}\tag{2.1}$$

The value defined in (2.1) can be written as the Rayleigh quotient

$$\lambda_1=\inf\frac{\displaystyle\int_\Omega\frac{1}{p(x)}(|\Delta u|^{p(x)}+|\nabla u|^{p(x)})\,dx,}{\displaystyle\int_\Omega\frac{1}{p(x)}|u|^{p(x)dx}},\tag{2.2}$$

where the infimum is taken over $X\setminus\{0\}$.

Definition 2.4 *Let X be a real reflexive Banach space and let X^* stand for its dual with respect to the pairing $\langle .,. \rangle$. We shall deal with mappings T acting from X into X^*. The strong convergence in X (and in X^*) is denoted by $\rightarrow$ and the weak convergence by $\rightharpoonup$. T is said to belong to the class (S^+), if for any sequence u_n in X converging weakly to $u \in X$ and $\limsup_{n\to+\infty}\langle T, u_n - u\rangle \leq 0$, it follows that u_n converges strongly to u in X. We write $T \in (S^+)$.*

Consider the following two functionals defined on X:

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |\nabla u|^{p(x)})\,dx \text{ and } \varphi(u) = \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx,$$

and set $\mathcal{M} = \{u \in X; \varphi(u) = 1\}$.

Lemma 2.5 *We have the following statements*

(i) Φ and φ are even, and of class C^1 on X.

(ii) $\mathcal{M}$ is a closed C^1-manifold.

Proof. It is clear that φ and Φ are even and of class C^1 on X and $\mathcal{M} = \varphi^{-1}\{1\}$. Therefore $\mathcal{M}$ is closed. The derivative operator φ' satisfies $\varphi'(u) \neq 0\ \forall u \in \mathcal{M}$ (i.e., $\varphi'(u)$ is onto for all $u \in \mathcal{M}$). Hence φ is a submersion, which proves that $\mathcal{M}$ is a C^1-manifold. Let as split Φ on two functionals.

$$\Phi(u) = \Phi_1(u) + \Phi_2(u),$$

where

$$\Phi_1 = \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}\,dx; \quad \Phi_2 = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx.$$

Now we consider the operator $T_1 := \Phi_1' : W_0^{2,p(.)}(\Omega) \to W^{-2,p'(.)}(\Omega)$ is defined as

$$\langle T_1(u), v\rangle = \int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v\,dx, \quad \text{for any } u,v \in W_0^{2,p(.)}(\Omega),$$

and the p(x)-laplace operator $-\Delta_{p(x)} := T2 := \Phi_2' : W_0^{1,p(.)}(\Omega) \to W^{-1,p'(.)}(\Omega)$ as

$$\langle -\Delta_{p(x)}(u), v\rangle = \langle T_2(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\,dx, \quad \text{for } u,v \in W_0^{1,p(.)}(\Omega),$$

Lemma 2.6 *The following statements hold*

(i) T_1 is continuous, bounded and strictly monotone.

(ii) T_1 is of (S_+) type.

(iii) T_2 is a homeomorphism.

Proof.

(i) We recall the following well-known inequalities, which hold for any three real a, b and p

$$\Big(a|a|^{p-2} - b|b|^{p-2}\Big)(a-b) \geq c(p)\begin{cases} |a-b|^p, & \text{if } p \geq 2 \\ \dfrac{|a-b|^2}{(|a|+|b|)^{2-p}}, & \text{if } 1 < p < 2, \end{cases} \tag{2.3}$$

where $c(p) = 2^{2-p}$ when $p \geq 2$ and $c(p) = p-1$ when $1 < p < 2$.

Let $(u_n)_n \subset W_0^{2,p(.)}(\Omega)$ and $u_n \rightharpoonup u$ (weakly) in $W_0^{2,p(.)}(\Omega)$. Therefore we have for $p(\cdot) \geq 2$.

$$\begin{aligned} 2^{2-p^+}&\int_{\{x\in\Omega:\,p(x)\geq 2\}} |\Delta u_n - \Delta u|^{p(x)}\,dx \\ &\leq \int_{\{x\in\Omega:\,p(x)\geq 2\}} \Big(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u\Big)\Big(\Delta u_n - \Delta u\Big)dx \\ &\leq \int_\Omega \Big(|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u\Big)\Big(\Delta u_n - \Delta u\Big)dx \\ &:= \varepsilon(\tfrac{1}{n}). \end{aligned} \tag{2.4}$$

On the set where $1 < p(\cdot) < 2$, we employ (2.3) as follows:

$$\begin{aligned} &\int_{\{x\in\Omega:\,1<p(x)<2\}} |\Delta u_n - \Delta u|^{p(x)}\,dx \\ &\leq \int_{\{x\in\Omega:\,1<p(x)<2\}} \frac{|\Delta u_n - \Delta u|^{p(x)}}{(|\Delta u_n| + |\Delta u|)^{\frac{p(x)(2-p(x))}{2}}}(|\Delta u_n| + |\Delta u|)^{\frac{p(x)(2-p(x))}{2}}\,dx \\ &\leq 2\left\|\frac{|\Delta u_n - \Delta u|^{p(x)}}{(|\Delta u_n|+|\Delta u|)^{\frac{p(x)(2-p(x))}{2}}}\right\|_{L^{2/p(x)}(\Omega)} \times \left\|(|\Delta u_n| + |\Delta u|)^{\frac{p(x)(2-p(x))}{2}}\right\|_{L^{\frac{2}{2-p(x)}}(\Omega)} \\ &\leq 2\max\left\{\left(\int_\Omega \frac{|\Delta u_n - \Delta u|^2}{(|\Delta u_n| + |\Delta u|)^{2-p(x)}}\,dx\right)^{\frac{p^-}{2}}, \left(\int_\Omega \frac{|\Delta u_n - \Delta u|^2}{(|\Delta u_n| + |\Delta u|)^{2-p(x)}}\,dx\right)^{\frac{p^+}{2}}\right\} \\ &\quad\times\max\left\{\left(\int_\Omega (|\Delta u_n| + |\Delta u|)^{p(x)}\,dx\right)^{\frac{2-p^-}{2}}; \left(\int_\Omega (|\Delta u_n| + |\Delta u|)^{p(x)}\,dx\right)^{\frac{2-p^+}{2}}\right\} \\ &\leq 2\max\left\{\Big(p^- - 1\Big)^{\frac{-p^-}{2}}\Big(\varepsilon(\frac{1}{n})\Big)^{\frac{p^-}{2}}; \Big(p^- - 1\Big)^{\frac{-p^+}{2}}\Big(\varepsilon(\frac{1}{n})\Big)^{\frac{p^+}{2}}\right\} \\ &\quad\times\max\left\{\left(\int_\Omega (|\Delta u_n| + |\Delta u|)^{p(x)}\,dx\right)^{\frac{2-p^-}{2}}; \left(\int_\Omega (|\Delta u_n| + |\Delta u|)^{p(x)}\,dx\right)^{\frac{2-p^+}{2}}\right\}. \end{aligned} \tag{2.5}$$

Since u_n is bounded in X, implies that $\varepsilon(\frac{1}{n}) \to 0$ as $n \to \infty$. Hence, sending n to ∞ in (2.4) and (2.5), we obtain

$$\lim_{n\to\infty}\int_\Omega |\Delta u_n - \Delta u|^{p(x)}\,dx = 0.$$

Since T_1 is the Fréchet derivative of Φ_1, it follows that T_1 is continuous and bounded so that we deduce that for all $u,v \in W_0^{2,p(.)}(\Omega)$ such that $u \neq v$,

$$\langle T_1(u) - T_1(v), u - v\rangle > 0.$$

This means that T_1 is strictly monotone.

(ii) Let $(u_n)_n$ be a sequence of X such that $u_n \rightharpoonup u$ weakly in $W_0^{2,p(.)}(\Omega)$ and $\limsup_{n\to+\infty}\langle T_1(u_n), u_n - u\rangle \leq 0$. From (2.3), we have

$$\langle T_1(u_n) - T_1(u), u_n - u\rangle \geq 0, \qquad (2.6)$$

and since $u_n \rightharpoonup u$ weakly in $W_0^{2,p(.)}(\Omega)$, it follows that

$$\limsup_{n\to+\infty}\langle T_1(u_n) - T_1(u), u_n - u\rangle = 0. \qquad (2.7)$$

Thus again from (2.3), we have

$$\begin{cases} \int_{\{x\in\Omega: p(x)\geq 2\}} |\Delta u_n - \Delta u|^{p(x)}\,dx \\ \leq 2^{(p^- - 2)}\int_\Omega A(u_n,u)\,dx, \\ \int_{\{x\in\Omega:1<p(x)<2\}} |\Delta u_n - \Delta u|^{p(x)}dx \\ \leq (p^+ - 1)\int_\Omega (A(u_n,u))^{\frac{p(x)}{2}}(B(u_n,u))^{(2-p(x))\frac{p(x)}{2}}\,dx, \end{cases}$$

where

$$\begin{cases} A(u_n,u) = \\ (|\Delta u_n|^{p(x)-2}\Delta u_n - |\Delta u|^{p(x)-2}\Delta u)(\Delta u_n - \Delta u), \\ B(u_n,u) = (|\Delta u_n| + |\Delta u|)^{2-p(x)}. \end{cases}$$

On the other hand, by (2.6) and since

$$\int_\Omega A(u_n,u)\,dx = \langle T_1(u_n) - T_1(u), u_n - u\rangle,$$

we can consider $0 \leq \int_\Omega A(u_n,u)\,dx < 1$.

We distinguish two cases:

First, If $\int_\Omega A(u_n,u)\,dx = 0$, then $A(u_n,u) = 0$, since $A(u_n,u) \geq 0$ a.e. in Ω.

Second, If $0 < \int_\Omega A(u_n,u)\,dx < 1$. Thus

$$t^{p(x)} := \left(\int_{\{x\in\Omega:1<p(x)<2\}} A(u_n,u)\,dx\right)^{-1} \text{ is positive}$$

and by applying Young's inequality we deduce that

$$\int_{\{x\in\Omega:1<p(x)<2\}} \left[t(A(u_n,u))^{\frac{p(x)}{2}}\right](B(u_n,u))^{(2-p(x))\frac{p(x)}{2}}\,dx$$
$$\leq \int_{\{x\in\Omega:1<p(x)<2\}} \left(A(u_n,u)(t)^{\frac{2}{p(x)}} + (B(u_n,u))^{p(x)}\right)dx$$

The fact that $\frac{2}{p(x)} < 2$, we have

$$\int_{\{x\in\Omega:1<p(x)<2\}} \left(A(u_n,u)(t)^{\frac{2}{p(x)}} + (B(u_n,u))^{p(x)}\right)dx$$
$$\leq \int_{\{x\in\Omega:1<p(x)<2\}} \left(A(u_n,u)t^2 + (B(u_n,u))^{p(x)}\right)dx$$
$$\leq 1 + \int_{\{x\in\Omega:1<p(x)<2\}} (B(u_n,u))^{p(x)}\,dx.$$

Hence.

$$\int_{\{x\in\Omega:1<p(x)<2\}} |\Delta u_n - \Delta u|^{p(x)}dx$$
$$\leq \left(\int_{\{x\in\Omega:1<p(x)<2\}} A(u_n,u)\,dx\right)^{\frac{1}{2}}\left(1 + \int_\Omega (B(u_n,u))^{p(x)}\,dx\right).$$

Since $\int_\Omega (B(u_n,u))^{p(x)}\,dx$ is bounded, then

$$\int_{\{x\in\Omega:1<p(x)<2\}} |\Delta u_n - \Delta u|^{p(x)}dx \to 0 \text{ as } n\to\infty$$

(iii) Note that the strict monotonicity of T_1 implies that T_1 is into operator.

Moreover, T_1 is a coercive operator. Indeed, from Proposition 2.3 and since $p^- - 1 > 0$, for each $u \in W_0^{2,p(x)}(\Omega)$ such that $\|u\| \geq 1$, we have

$$\frac{\langle T_1(u), u\rangle}{\|u\|} = \frac{\Phi'(u)}{\|u\|} \geq \|u\|^{p^- - 1} \to \infty, \quad \text{as } \|u\| \to \infty.$$

Finally, thanks to Minty-Browder Theorem [12], the operator T_1 is an surjection and admits an inverse mapping.

To complete the proof of (iii), it suffices then to show the continuity of T_1^{-1}. Indeed, let $(f_n)_n$ be a sequence of $W^{-2,p'(.)}(\Omega)$ such that $f_n \to f$ in $W^{-2,p'(.)}(\Omega)$. Let u_n and u in $W_0^{2,p(.)}(\Omega)$ such that Since $\int_\Omega (B(u_n,u))^{p(x)}\,dx$ is bounded, then $\int_{\{x\in\Omega:1<p(x)<2\}} |\Delta u_n - \Delta u|^{p(x)}dx \to 0$ as $n\to\infty$

(iii) Note that the strict monotonicity of T_1 implies that T_1 is into operator.

Moreover, T_1 is a coercive operator. Indeed, from Proposition 2.3 and since $p^- - 1 > 0$, for each $u \in W_0^{2,p(x)}(\Omega)$ such that $\|u\| \geq 1$, we have $\frac{\langle T_1(u), u\rangle}{\|u\|} = \frac{\Phi'(u)}{\|u\|} \geq \|u\|^{p^- - 1} \to \infty, \quad \text{as } \|u\| \to \infty.$

$$T_1^{-1}(f_n) = u_n \text{ and } T_1^{-1}(f) = u.$$

By the coercivity of T_1, we deduce that the sequence $(u_n)_n$ is bounded in the reflexive space $W_0^{2,p(.)}(\Omega)$. For a subsequence if necessary, we have $u_n \rightharpoonup \widehat{u}$ in $W_0^{2,p(.)}(\Omega)$, for a some $\widehat{u}$. Then

$$\lim_{n\to+\infty}\langle T_1(u_n) - T_1(u), u_n - \widehat{u}\rangle = \lim_{n\to+\infty}\langle f_n - f, u_n - \widehat{u}\rangle = 0.$$

It follows by the second assertion and the continuity of T_1 that

$$u_n \to \widehat{u} \text{ in } W_0^{2,p(x)}(\Omega) \text{ strongly and}$$
$$T_1(u_n) \to T_1(\widehat{u}) = T_1(u) \text{ in } W^{-2,p'(x)}(\Omega)$$

Further , since T_1 is an into operator, we conclude that $u \equiv \widehat{u}$.

This completes the proof.

Lemma 2.7 *[13] The following statements hold*

(i) $-\triangle_{p(x)} := T_2$ *is continuous, bounded and strictly monotone.*

(ii) $-\triangle_{p(x)} := T_2$ *is of* (S_+) *type.*

(iii) $-\triangle_{p(x)} := T_2$ *is a homeomorphism.*

The following lemma plays a central key to prove our main result related to the existence.

Lemma 2.8 *We have the following statements*

(i) φ' *is completely continuous.*

(ii) The functional Φ *satisfies the Palais-Smale condition on* $\mathcal{M}$*, i.e., for* $\{u_n\} \subset \mathcal{M}$*, if* $\{\Phi(u_n)\}_n$ *is bounded and*

$$\Phi'(u_n) \to 0 \quad as\ n \to \infty. \tag{2.8}$$

$\{u_n\}$ *has a convergent subsequence in* X*.*

Proof (i) First let us prove that φ' is well defined. Let $u, v \in X$. We have

$$\langle \varphi'(u), v\rangle = \int_\Omega |u|^{p(x)-1} v\, dx.$$

By applying Hölder's inequality, we obtain

$$\langle \varphi'(u), v\rangle \leq |u|_{p(x)}^{p(x)-1} |v|_{p(x)}$$

Then

$$|\langle \varphi'(u), v\rangle| \leq C\|u\|^{p(x)-1}\|v\|,$$

where C is the constant given by the embedding of $W_0^{2,p(.)}(\Omega)$ in $L^{p(.)}(\Omega)$. Hence

$$\|\varphi'(u)\|_* \leq C\|u\|^{p(x)-1},$$

where $\|.\|_*$ is the dual norm associated with $\|.\|$.
For the complete continuity of φ', we argue as follow. Let $(u_n)_n \subset X$ be a bounded sequence and $u_n \rightharpoonup u$ (weakly) in X. Due the fact that the embedding $W_0^{2,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is compact u_n converges strongly to u in $L^{p(.)}(\Omega)$, and there exists a positive function $g \in L^{p(.)}(\Omega)$ such that

$$|\, u \,| \leq g \text{ a.e. in } \Omega.$$

Since $g \in L^{p(.)-1}(\Omega)$, it follows from the Dominated Convergence Theorem that

$$|\, u_n \,|^{p(x)-2}\, u_n \to |\, u \,|^{p(x)-2}\, u \text{ in } L^{p'(.)}(\Omega)$$

That is,

$$\varphi'(u_n) \to \varphi'(u) \text{ in } L^{p'(.)}(\Omega).$$

Recall that the embedding

$$L^{p'(.)}(\Omega) \hookrightarrow W^{-2,p'(.)}(\Omega)$$

is compact. Thus

$$\varphi'(u_n) \to \varphi'(u) \text{ in } X^*$$

This proves the assertion (i).
(ii) by the definition of Φ we have

$$\Phi(u) \geq \frac{1}{p^+}\|u\|^{p^-},$$

then u_n is bounded in X. So we deduce that there exists a subsequence, again denoted $\{u_n\}$, and $u \in X$ such that $\{u_n\}$ converges weakly to u in X. On the other hend, by (2.8) we get

$$\lim_{n\to\infty} \langle \Phi'(u_n), (u_n - u)\rangle = 0. \tag{2.9}$$

Then, by (ii) of lemma 2.6 and (ii) of lemma 2.7, we conclude that $\{u_n\}$ converges strongly to $u \in X$. This achieves the proof the lemma.

3 Existence results

Set

$$\Gamma_j = \{K \subset \mathcal{M} : K \text{ symmetric, compact and } \ \gamma(K) \geq j\},$$

where $\gamma(K) = j$ being the Krasnoselskii's genus of set K, i.e., the smallest integer j, such that there exists an odd continuous map from K to $\mathbb{R}^j \setminus \{0\}$.

Now, let us establish some useful properties of Krasnoselskii genus proved by Szulkin [12].

Lemma 3.1 *Let* X *be a real Banach space and* A*,* B *be symmetric subsets of* $E \setminus \{0\}$ *which are closed in* X*. Then*

(a) If there exists an odd continuous mapping $f : A \to B$*, then* $\gamma(A) \leq \gamma(B)$

(b) If $A \subset B$ *then* $\gamma(A) \leq \gamma(B)$*.*

(c) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$*.*

(d) If $\gamma(B) < +\infty$ *then* $\gamma(\overline{A - B} \geq \gamma(A) - \gamma(B)$*.*

(e) If A *is compact then* $\gamma(A) < +\infty$ *and there exists a neighborhood* $\mathcal{N}$ *of* A*,* N *is a symmetric subset of* $X \setminus \{0\}$*, closed in* X *such that* $\gamma(N) = \gamma(A)$*.*

(f) If $\mathcal{N}$ *is a symmetric and bounded neighborhood of the origin in* $\mathbb{R}^k$ *and if* A *is homeomorphic to the boundary of* $\mathcal{N}$ *by an odd homeomorphism then* $\gamma(A) = k$*.*

(g) If X_0 *is a subspace of* X *of codimension* k *and if* $\gamma(A) > k$ *then* $A \cap X_0 \neq \phi$*.*

Let us now state the first our main result of this paper using Ljusternick-Schnirelmann theory:

Theorem 3.2 *For any integer* $j \in \mathbb{N}^*$*,*

$$\lambda_j = \inf_{K\in\Gamma_j} \max_{u\in K} \Phi(u),$$

is a critical value of Φ *restricted on* $\mathcal{M}$*. More precisely, there exist* $u_j \in K$*, such that*

$$\lambda_j = \Phi(u_j) = \sup_{u\in K} \Phi(u),$$

and u_j *is a solution of* (1.1) *associated to positive eigenvalue* λ_j*. Moreover,*

$$\lambda_j \to \infty, \text{ as } j \to \infty$$

Proof We only need to prove that for any $j \in \mathbb{N}^*$, $\Gamma_j \neq \emptyset$ and the last assertion. Indeed, since $W_0^{2,p(.)}(\Omega)$ is separable, there exists $(e_i)_{i\geq 1}$ linearly dense in $W_0^{2,p(.)}(\Omega)$ such that
$\operatorname{supp} e_i \cap \operatorname{supp} e_n = \emptyset$ if $i \neq n$. We may assume that $e_i \in \mathcal{M}$ (if not, we take $e_i' \equiv \frac{e_i}{[p(x)\varphi(e_i)]^{\frac{1}{p(x)}}}$).
Let now $j \in \mathbb{N}^*$ and denote

$$F_j = \operatorname{span}\{e_1, e_2, \dots, e_j\}$$

Clearly, F_j is a vector subspace with $\dim F_j = j$. If $v \in F_j$, then there exist $\alpha_1, \dots \alpha_j$ in $\mathbb{R}$, such that
$v = \sum_{i=1}^{j} \alpha_i e_i$.
Thus

$$\varphi(v) = \sum_{i=1}^{j} |\alpha_i|^{p(.)} \varphi(e_i) = \sum_{i=1}^{j} |\alpha_i|^{p(.)}.$$

It follows that the map

$$v \mapsto (\varphi(v))^{\frac{1}{p(.)}} = |||v|||$$

defines a norm on F_j. Consequently, there is a constant $c > 0$ such that

$$c\|v\| \leq |||v||| \leq \frac{1}{c}\|v\|.$$

This implies that the set

$$\mathcal{V}_j = F_j \cap \left\{ v \in W_0^{2,p(.)}(\Omega) : \varphi(v) \leq 1 \right\},$$

is bounded because $\mathcal{V}_k \subset B(0, \frac{1}{c})$, where

$$B\left(0, \frac{1}{c}\right) = \left\{ u \in W_0^{2,p(.)}(\Omega), \text{ such that } \|u\| \leq \frac{1}{c} \right\}.$$

Thus, $\mathcal{V}_j$ is a symmetric bounded neighborhood of $0 \in F_j$. Moreover, $F_j \cap \mathcal{M}$ is a compact set. By (f) of Lemma 2.7, we conclude that $\gamma(F_j \cap \mathcal{M}) = j$ and then we obtain finally that $\Gamma_j \neq \emptyset$. This completes the proof of first part of the theorem.
Now, we claim that

$$\lambda_j \to \infty, \text{ as } j \to \infty$$

Let $(e_k, e_n^*)_{k,n}$ be a bi-orthogonal system such that $e_k \in W_0^{2,p(.)}(\Omega)$ and $e_n^* \in W^{-2,p'(.)}(\Omega)$, the $(e_k)_k$ are linearly dense in $W_0^{2,p(.)}(\Omega)$ and the $(e_n^*)_n$ are total for the dual $W^{-2,p'(.)}(\Omega)$). For $k \in \mathbb{N}^*$, set

$$F_k = \operatorname{span}\left\{e_1, \dots, e_k\right\} \text{ and } F_k^\perp = \operatorname{span}\left\{e_{k+1}, e_{k+2}, \dots\right\}.$$

By (g) of Lemma 2.7, we have for any $K \in \Gamma_k$, $K \cap F_{k-1}^\perp \neq \emptyset$. Thus

$$t_k = \inf_{K \in \Gamma_k} \sup_{u \in K \cap F_{k-1}^\perp} \Phi(u) \to \infty, \text{ as } k \to \infty$$

Indeed, if not, for k is large, there exists $u_k \in F_{k-1}^\perp$ with $|u_k|_{p(.)} = 1$ such that

$$t_k \leq \Phi(u_k) \leq M,$$

for some $M > 0$ independent of k. Thus $\|u_k\|_{p(.)} \leq M$. This implies that $(u_k)_k$ is bounded in X. For a subsequence of $\{u_k\}$ if necessary, we can assume that $\{u_k\}$ converges weakly in X and strongly in $L^{p(.)}(\Omega)$. By our choice of $F_{k-1}^\perp$, we have $u_k \rightharpoonup 0$ weakly in X, because $\langle e_n^*, e_k \rangle = 0$, for any $k > n$. This contradicts the fact that $|u_k|_{p(.)} = 1$ for all k. Since $\lambda_k \geq t_k$ the claim is proved.

Corrolary 3.3 *we have the following statements:*

(i) $\lambda_1 = \inf\left\{\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |\nabla u|^{p(x)})\,dx, u \in X, \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx = 1\right\}$.,

(ii) $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty$,

(iii) $\lambda_1 = Inf\Lambda$ *(i.e.,* λ_1 *is the smallest eigenvalue in the spectrum of* (1.1)*).*

Proof

(i) For $u \in \mathcal{M}$, set $K_1 = \{u, -u\}$. It is clear that $\gamma(K_1) = 1$, Φ is even and that

$$\Phi(u) = \max_{K_1} \Phi \geq \inf_{K \in \Gamma_1} \max_{u \in K} \Phi(u).$$

Thus

$$\inf_{u \in \mathcal{M}} \Phi(u) \geq \inf_{K \in \Gamma_1} \max_{u \in K} \Phi(u) = \lambda_1.$$

On the other hand, $\forall K \in \Gamma_1$, $\forall u \in K$, we have

$$\sup_{u \in K} \Phi \geq \Phi(u) \geq \inf_{u \in \mathcal{M}} \Phi(u).$$

It follows that

$$\inf_{K \in \Gamma_1} \max_{K} \Phi = \lambda_1 \geq \inf_{u \in \mathcal{M}} \Phi(u).$$

Then

$$\lambda_1 = \inf\left\{\int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + |\nabla u|^{p(x)})\,dx, u \in X, \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx = 1\right\}.$$

(ii) For all $i \geq j$, we have $\Gamma_i \subset \Gamma_j$ and in view of definition of $\lambda_i, i \in \mathbb{N}^*$, we get $\lambda_i \geq \lambda_j$. As regards to $\lambda_n \to \infty$, it is proved before in Theorem 3.2.

(iii) Let $\lambda \in \Lambda$. Thus there exists u_λ an eigenfunction of λ such that

$$\int_\Omega \frac{1}{p(x)}|u_\lambda|^{p(x)}\,dx = 1.$$

Therefore

$$\Delta_{p(x)}^2 u_\lambda - \Delta_{p(x)} u_\lambda = \lambda |u_\lambda|^{p(x)-2} u_\lambda \text{ in } \Omega.$$

Then

$$\int_\Omega \frac{1}{p(x)}(|\Delta u_\lambda|^{p(x)} + |\nabla u_\lambda|^{p(x)})\,dx = \lambda \int_\Omega \frac{1}{p(x)}|u_\lambda|^{p(x)}\,dx.$$

In view of the characterization of λ_1 in (2.1), we conclude that

$$\lambda = \frac{\int_\Omega \frac{1}{p(x)}(|\Delta u_\lambda|^{p(x)} + |\nabla u_\lambda|^{p(x)})\,dx}{\int_\Omega \frac{1}{p(x)}|u_\lambda|^{p(x)}\,dx} = \int_\Omega \frac{1}{p(x)}(|\Delta u_\lambda|^{p(x)} + |\nabla u_\lambda|^{p(x)})\,dx \geq \lambda_1.$$

This implies that $\lambda_1 = \inf \Lambda$.

References

1. A. Ferrero and G. Warnaut, On a solution of second and fourth order elliptic with power type nonlinearities, Nonlinear Analysis TMA, (70), 2889–2902, 2009.

2. T.G. Myers, Thin films with high surface tension, SIAM Review, 40 (3), 441–462, 1998.

3. R. K. Giri, D. Choudhuri and Sh. Pradhan, Existence and Concentration of solutions for a class of elliptic PDEs involving p-biharmonic operator, [math.AP] 8 Jun 2016. http://arxiv.org/abs/ 1606.02512.

4. L. Lieu and C.Chen

5. L. Diening, P. Harjulehto, P. Hästö and M. Růžička, Lecture Notes in Mathematics, issue 2017, Springer Science and business Media, 2011.

6. X. L. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl., (263), 424–446, 2001.

7. X. L. Fan and X. Fan, A Knobloch-type result for p(t) Laplacian systems, J. Math. Anal. Appl., (282), 453–464, 2003.

8. M. Mihăilescu, Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplace operator, Nonlinear Anal. T. M. A., (67), 1419–1425, 2007.

9. J. H. Yao, Solution for Neumann boundary problems involving the $p(x)$-Laplace operators, Nonlinear Anal. T.M.A., (68), 1271–1283, 2008.

10. A. Zang, Y. Fu, Interpolation inequalities for derivatives in variable exponent Lebesgue-Sobolev spaces, Nonlinear Anal. (69), 3629-3636, 2008.

11. A. Szulkin, Ljusternick-Schnirelmann theory on C^1-manifolds, Ann. Inst. Henri Poincaré Anal. Non. (5), 119–39, 1988.

12. E. Zeidler, Nonlinear functional analysis and its applications, II/B, Springer-Verlag, New York, 1990.

13. X.L. Fan and Q.H. Zhang, Existence of solutions for $p(x)$-laplacian Dirichlet problem, Nonlinear Anal. (52), 18431852, 2003.

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Existence and Boundedness of Solutions for Elliptic Equations in General Domains

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ABSTRACT

This article is devoted to study the existence of solutions for the strongly nonlinear $p(x)$-elliptic problem:

$$-\Delta_{p(x)}(u)+\alpha_0|u|^{p(x)-2}u = d(x)\frac{|\nabla u|^{p(x)}}{|u|^{p(x)}+1}+f-\operatorname{div} g(x) \quad in\ \Omega,$$
$$u\in W_0^{1,p(x)}(\Omega),$$

where Ω is an open set of $\mathbb{R}^N$, possibly of infinite measure, also we will give some regularity results for these solutions.

1 Introduction

In recent years, there has been an increasing interest in the study of various mathematical problems with variable exponents. These problems are interesting in applications (see [[1], [2]]). For the usual problems when p is constant, there are many results for existence of solutions when the domain is bounded or unbounded. For p variable, when the domain is bounded, on the results of existence of solutions, we refer to [[3], [4], [5]], when the domain is unbounded, results of existence of solutions are rare we can cite for example [[6], [7]].

In the case where Ω is a bounded, and for $1 < p < N$, In [8] authors studied the problem:

$$-\operatorname{div} a(x,u,\nabla u) = H(x,u,\nabla u)+f-\operatorname{div} g \quad \text{in } D'(\Omega),$$
$$u\in W_0^{1,p}(\Omega),$$

where the right hand side is assumed to satisfy:

$$f\in L^{N/p}(\Omega), g\in (L^{N/(p-1)}(\Omega))^N.$$

Under suitable smallness assumptions on f and g they prove the existence of a solution u which satisfies a further regularity.

In [9] in the case of unbounded domains Guowei Dai By variational approach and the theory of the variable exponent Sobolev spaces establish the existence of infinitely many distinct homoclinic radially symmetric solutions whose $W^{1,p(x)}(\mathbb{R}^N)$-norms tend to zero (to infinity, respectively) under weaker hypotheses about nonlinearity at zero (at infinity, respectively).

The principal objective of this paper is to prove the existence and some regularity of solutions of the following p(x)-Laplacian equation in open set Ω of $\mathbb{R}^N$ (possibly of infinite measure):

$$-\Delta_{p(x)}(u)+\alpha_0|u|^{p(x)-2}u = d(x)\frac{|\nabla u|^{p(x)}}{|u|^{p(x)}+1}+f-\operatorname{div} g(x) \quad \text{in } \Omega,$$
$$u\in W_0^{1,p(x)}(\Omega), \tag{1}$$

where p is log-Hölder continuous function such that $1 < p_- \le p_+ < N$, $\Delta_{p(x)}(u) = div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplace operator, α_0 is a positive constant, d is a function in $L^\infty(\Omega)$. We assume the following hypotheses on the source terms f and g :
$f:\Omega\to\mathbb{R}$, $g:\Omega\to\mathbb{R}^N$ are a measurable function

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satisfying:

$$
\begin{gathered}
f \in L^{N/p(x)}(\{x \in \Omega : 1 < |f(x)|\}), \\
f \in L^{p'(x)}(\{x \in \Omega : |f(x)| \leq 1\}) \\
g \in L^{N/(p(x)-1)}(\Omega; \mathbb{R}^N) \cap L^{p'(x)}(\Omega; \mathbb{R}^N)
\end{gathered} \tag{2}
$$

We will proceed by solving the problem on a sequence Ω_n of bounded sets after that we pass to the limit in the approximating problems by using the a priori estimate (this a priori estimates provide the necessary compactness properties for solutions) from which the desired results are easily inferred. To this aim, we can neither use any embedding theorem between $L^{p(.)}(\Omega)$ nor any argument involving the measure of Ω_n, and under suitable assumptions on f and g we prove some regularity of a solutions u of (1). A similar result has been proved in [7] where p is constant such that $1 < p < N$ but in the present setting such an approach cannot be used directly, because of the variability of p.

The plan of the paper is the following: In Section 2 we recall some important defnitions and results of variable exponent Lebesgue and Sobolev spaces. In Section 3 we will give the precise assumptions and state the main results. In Section 4 we will define the approximate problems, state the a priori estimates that we want to obtain. In the Sections which follow we will prove strong convergence of u_n and their gradients ∇u_n. Section 5 is devoted to conclude the proof of the main existence results. Finally, in Section 6, we prove that, if f and g have higher integrability, then every solution u of (1) is bounded. More precisely, we will assume that (2) are replaced by:

$$
\begin{gathered}
f \in L^{q(x)}(\{x \in \Omega : 1 < |f(x)|\}) \text{ for some } q(x) > N/p(x), \\
f \in L^{p'(x)}(\{x \in \Omega : |f(x)| \leq 1\}) \\
g \in L^{r(x)}(\Omega; \mathbb{R}^N) \cap L^{p'(x)}(\Omega; \mathbb{R}^N) \\
\text{for some } r(x) > N/(p(x)-1)
\end{gathered} \tag{3}
$$

2 Preliminaries

In order to discuss the problem (1), we need to recall some definitions and basic properties of Lebesgue and Sobolev spaces with variable exponents.

Let Ω an open bounded set of $\mathbb{R}^N$ with $N \geq 2$. We say that a real-valued continuous function $p(.)$ is log-Hölder continuous in Ω if:

$$
|p(x) - p(y)| \leq \frac{C}{|\log|x-y||} \quad \forall x, y \in \overline{\Omega} \quad \text{such that } |x-y| < \frac{1}{2},
$$

We denote:

$$
C_+(\overline{\Omega}) = \{\text{log-Hölder continuous function } p : \overline{\Omega} \to \mathbb{R} \text{ with } 1 < p_- \leq p_+ < N\},
$$

where:

$$
p_- = ess\min_{x\in\overline{\Omega}} p(x) \quad p_+ = ess\sup_{x\in\overline{\Omega}} p(x).
$$

We define the variable exponent Lebesgue space for $p \in C_+(\overline{\Omega})$ by:

$$
L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^{p(x)} dx < \infty\},
$$

the space $L^{p(x)}(\Omega)$ under the norm:

$$
\|u\|_{L^{p(x)}(\Omega)} = \inf\Big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\Big\}
$$

is a uniformly convex Banach space, and therefore reflexive.

We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see [10, 11]).

Proposition 1 (Generalized Hölder inequality [10, 11]) *(i) For any functions $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$
|\int_\Omega uv\,dx| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}.
$$

(ii) For all $p_1, p_2 \in C_+(\overline{\Omega})$ such that: $p_1(x) \leq p_2(x)$ a.e. in Ω, we have: $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2 ([10, 11]) *If we denote*

$$
\rho(u) = \int_\Omega |u|^{p(x)} dx \quad \forall u \in L^{p(x)}(\Omega),
$$

then, the following assertions hold

(i) $\|u\|_{L^{p(x)}(\Omega)} < 1$ *(resp, = 1, > 1) if and only if* $\rho(u) < 1$ *(resp, = 1, > 1);*

(ii) $\|u\|_{L^{p(x)}(\Omega)} > 1$ *implies* $\|u\|^{p_-}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq \|u\|^{p_+}_{L^{p(x)}(\Omega)}$*, and* $\|u\|_{L^{p(x)}(\Omega)} < 1$ *implies* $\|u\|^{p_+}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq \|u\|^{p_-}_{L^{p(x)}(\Omega)}$*;*

(iii) $\|u\|_{L^{p(x)}(\Omega)} \to 0$ *if and only if* $\rho(u) \to 0$*, and* $\|u\|_{L^{p(x)}(\Omega)} \to \infty$ *if and only if* $\rho(u) \to \infty$.

Now, we define the variable exponent Sobolev space by:

$$
W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \text{ and } |\nabla u| \in L^{p(x)}(\Omega)\},
$$

with the norm:

$$
\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)} \quad \forall u \in W^{1,p(x)}(\Omega).
$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$.

Proposition 3 ([10, 12]) *(i) If $1 < p_- \leq p_+ < \infty$, then the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) *If $q \in C_+(\bar{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \Omega$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.*

(iii) *Poincaré inequality: There exists a constant $C > 0$, such that:*

$$\|u\|_{L^{p(x)}(\Omega)} \le C\|\nabla u\|_{L^{p(x)}(\Omega)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

(vi) *Sobolev-Poincaré inequality : there exists an other constant $C > 0$, such that:*

$$\|u\|_{L^{p*(x)}(\Omega)} \le C\|\nabla u\|_{L^{p(x)}(\Omega)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

The symbol $\rightharpoonup$ will denote the weak convergence, and the constants C_i, $i = 1,2,\dots$ used in each step of proof are independent.

3 Approximate problems and A priori estimates

In this section we will prove the existence result to the approximate problems. Also we will give a uniform estimate for this solutions u_n.

Approximate problems

For $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ is defined by:

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases} \tag{4}$$

Let $\Omega_n = \Omega \cap B_n(0)$ where $B_n(0)$ is the Ball with center 0 and radius n, we consider the approximate problem:

$$\begin{gathered} -\Delta_{p(.)}(u_n) + c(x,u_n) = H_n(x,u_n,\nabla u_n) + f_n - \operatorname{div} g_n \quad \text{in } \Omega_n, \\ u_n \in W_0^{1,p(.)}(\Omega_n) \cap L^\infty(\Omega_n), \end{gathered} \tag{5}$$

with $c(x,u) = \alpha_0|u|^{p(x)-2}u$, $H_n(x,s,\xi) = T_n(H(x,s,\xi))$, $H(x,s,\xi) = d(x)\frac{|\xi|^{p(x)}}{|s|^{p(x)}+1}$, $f_n(x) = T_n(f(x))$ and $g_n(x) = \frac{g(x)}{1+\frac{1}{n}|g(x)|}$. Let us remark that $|H_n| \le |H|$, $|H_n| \le n$,$|f_n| \le |f|$and $|g_n| \le |g|$.

Lemma 1 ([13]) *Let p be a measurable function and $s > 0$ such that $sp_- > 1$ then $\||f|^s\|_{L^{p(x)}(\Omega)} = \|f\|^s_{L^{p(x)}(\Omega)}$ for every f in $L^{p(x)}(\Omega)$.*

Lemma 2 ([3]) *Let $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function (measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$, and continuous with respect to (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$ for almost every x in Ω) and let us Assume that:*

$$|a(x,s,\xi)| \le \beta(K(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}), \tag{6}$$

$$a(x,s,\xi)\xi \ge \alpha|\xi|^{p(x)}, \tag{7}$$

$$[a(x,s,\xi) - a(x,s,\overline{\xi})](\xi - \overline{\xi}) > 0 \quad \text{for all } \xi \ne \overline{\xi} \text{ in } \mathbb{R}^N, \tag{8}$$

hold, and let $(u_n)_n$ be a sequence in $W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)dx \to 0, \tag{9}$$

then $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$ for a subsequence.

We define the operator:
$R_n : W_0^{1,p(x)}(\Omega_n) \to W^{-1,,p'(x)}(\Omega_n)$, by:

$$\langle R_n u, v\rangle = \int_{\Omega_n} c(x,u)v - H_n(x,u,\nabla u)v dx \quad \forall v \in W_0^{1,p(x)}(\Omega_n).$$

by the Hölder inequality we have that:
for all $u,v \in W_0^{1,p(x)}(\Omega_n)$,

$$\begin{aligned} &\Big|\int_{\Omega_n} c(x,u)v - H_n(x,u,\nabla u)v dx\Big| \\ &\le \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\Big[\|c(x,u)\|_{L^{p'(x)}(\Omega_n)}\|v\|_{L^{p(x)}(\Omega_n)} \\ &+ \|H_n(x,u,\nabla u)\|_{L^{p'(x)}(\Omega_n)}\|v\|_{L^{p(x)}(\Omega_n)}\Big] \\ &\le \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\Big[\Big(\int_{\Omega_n}\Big(|c(x,u)|^{p'(x)}dx + 1\Big)^{\frac{1}{p'_-}} \\ &+ \Big(\int_{\Omega_n}\Big(|H_n(x,u,\nabla u)|^{p'(x)}dx + 1\Big)^{\frac{1}{p'_-}}\Big]\|v\|_{W^{1,p(x)}(\Omega_n)} \end{aligned}$$

Then:

$$\begin{aligned} &\Big|\int_{\Omega_n} c(x,u)v + H_n(x,u,\nabla u)v dx\Big| \\ &\le \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\Big[\Big(\int_{\Omega_n}|u|^{p(x)}\Big)dx + 1\Big)^{\frac{1}{p'_-}} \\ &+ \Big(\int_{\Omega_n} n^{p'(x)}dx + 1\Big)^{\frac{1}{p'_-}}\Big]\|v\|_{W^{1,p(x)}(\Omega_n)} \\ &\le \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\Big[\Big(\int_{\Omega_n}|u|^{p(x)}\Big)dx + 1\Big)^{\frac{1}{p'_-}} \\ &+ \Big(n^{p'_+}.\operatorname{meas}(\Omega_n) + 1\Big)^{\frac{1}{p'_-}}]]\|v\|_{W^{1,p(x)}(\Omega_n)} \\ &\le C_1\|v\|_{W^{1,p(x)}(\Omega_n)}, \end{aligned} \tag{10}$$

Lemma 3 *The operator $B_n = A + R_n$ is pseudo-monotone from $W_0^{1,p(x)}(\Omega_n)$ into $W^{-1,p'(x)}(\Omega_n)$. Moreover, B_n is coercive in the following sense*

$$\frac{\langle B_n v, v\rangle}{\|v\|_{W^{1,p(x)}(\Omega_n)}} \to +\infty \quad as \quad \|v\|_{W^{1,p(x)}(\Omega_n)} \to +\infty$$

$$for \quad v \in W_0^{1,p(x)}(\Omega_n).$$

where $Au = -\Delta_{p(x)}(u)$

Proof: Using Hölder's inequality we can show that the operator A is bounded, and by using (10) we conclude that B_n is bounded. For the coercivity, we have for any $u \in W_0^{1,p(x)}(\Omega_n)$,

$$\begin{aligned}
\langle B_n u, u\rangle &= \langle Au, u\rangle + \langle R_n u, u\rangle \\
&= \int_{\Omega_n} |\nabla u|^{p(x)} dx + \int_{\Omega_n} c(x,u)u - H_n(x,u,\nabla u)u dx \\
&\geq \int_{\Omega} |\nabla u|^{p(x)} dx - C_1.\|u\|_{W^{1,p(x)}(\Omega_n)} \quad (\text{using}(10)) \\
&\geq \|\nabla u\|^{\delta'}_{L^{p(x)}(\Omega_n)} - C_1.\|u\|_{W^{1,p(x)}(\Omega_n)} \\
&\geq \alpha' \|u\|^{\delta'}_{W^{1,p(x)}(\Omega_n)} - C_1.\|u\|_{W^{1,p(x)}(\Omega_n)}
\end{aligned}$$

(using Poincaré's inequality)

With

$$\delta' = \begin{cases} p_- & \text{if } \|\nabla u\|_{L^{p(x)}(\Omega_n)} > 1, \\ p_+ & \text{if } \|\nabla u\|_{L^{p(x)}(\Omega_n)} \leq 1, \end{cases}$$

Then, we obtain:

$$\frac{\langle B_n u, u\rangle}{\|u\|_{W^{1,p(x)}(\Omega_n)}} \to +\infty \quad \text{as } \|u\|_{W^{1,p(x)}(\Omega_n)} \to +\infty.$$

It remains now to show that B_n is pseudo-monotone. Let $(u_k)_k$ a sequence in $W_0^{1,p(x)}(\Omega_n)$ such that:

$$\begin{aligned}
u_k &\rightharpoonup u \quad \text{in } W_0^{1,p(x)}(\Omega_n), \\
B_n u_k &\rightharpoonup \chi \quad \text{in } W^{-1,p'(x)}(\Omega_n), \\
\limsup_{k\to\infty} &\langle B_n u_k, u_k\rangle \leq \langle \chi, u\rangle.
\end{aligned} \tag{11}$$

We will prove that:

$$\chi = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \quad \text{as } k \to +\infty.$$

Firstly, since $W_0^{1,p(x)}(\Omega_n) \hookrightarrow\hookrightarrow L^{p(x)}(\Omega_n)$, then $u_k \to u$ in $L^{p(x)}(\Omega_n)$ for a subsequence still denoted by $(u_k)_k$.

We have $(u_k)_k$ is a bounded sequence in $W_0^{1,p(x)}(\Omega_n)$, then $(|\nabla u_k|^{p(x)-2}\nabla u_k)_k$ is bounded in $(L^{p'(x)}(\Omega_n))^N$, therefore, there exists a function $\varphi \in (L^{p'(x)}(\Omega_n))^N$ such that:

$$|\nabla u_k|^{p(x)-2}\nabla u_k \rightharpoonup \varphi \quad \text{in } (L^{p'(x)}(\Omega_n))^N \text{ as } k \to \infty. \tag{12}$$

Similarly, since $(c(x,u_k) - H_n(x,u_k,\nabla u_k))_k$ is bounded in $L^{p'(x)}(\Omega_n)$, then there exists a function $\psi_n \in L^{p'(x)}(\Omega_n)$ such that:

$$c(x,u_k) - H_n(x,u_k,\nabla u_k) \rightharpoonup \psi_n \quad \text{in } L^{p'(x)}(\Omega_n) \text{ as } k \to \infty, \tag{13}$$

For all $v \in W_0^{1,p(x)}(\Omega_n)$, we have:

$$\begin{aligned}
\langle \chi, v\rangle &= \lim_{k\to\infty} \langle B_n u_k, v\rangle \\
&= \lim_{k\to\infty} \int_{\Omega_n} |\nabla u_k|^{p(x)-2}\nabla u_k \nabla v dx \\
&\quad + \lim_{k\to\infty} \int_{\Omega_n} (c(x,u_k) - H_n(x,u_k,\nabla u_k)) v dx \\
&= \int_{\Omega_n} \varphi \nabla v dx + \int_{\Omega_n} \psi_n v dx.
\end{aligned} \tag{14}$$

Using (11) and (14), we obtain:

$$\begin{aligned}
&\limsup_{k\to\infty} \langle B_n(u_k), u_k\rangle \\
&= \limsup_{k\to\infty} \left\{ \int_{\Omega_n} |\nabla u_k|^{p(x)} dx + \int_{\Omega_n} (c(x,u_k) - H_n(x,u_k,\nabla u_k)) u_k dx \right\} \\
&\leq \int_{\Omega} \varphi \nabla u dx + \int_{\Omega} \psi_n u dx,
\end{aligned} \tag{15}$$

Thanks to (13), we have:

$$\int_{\Omega_n} (c(x,u_k) - H_n(x,u_k,\nabla u_k)) u_k dx \to \int_{\Omega_n} \psi_n u dx; \tag{16}$$

Therefore,

$$\limsup_{k\to\infty} \int_{\Omega_n} |\nabla u_k|^{p(x)} dx \leq \int_{\Omega_n} \varphi \nabla u dx. \tag{17}$$

On the other hand, we have:

$$\int_{\Omega_n} (|\nabla u_k|^{p(x)-2}\nabla u_k - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_k - \nabla u) dx \geq 0, \tag{18}$$

Then

$$\int_{\Omega_n} |\nabla u_k|^{p(x)} dx \geq -\int_{\Omega_n} |\nabla u|^{p(x)} dx + \int_{\Omega_n} |\nabla u_k|^{p(x)-2}\nabla u_k \nabla u dx + \int_{\Omega_n} |\nabla u|^{p(x)-2}\nabla u \nabla u_k dx,$$

and by (12), we get:

$$\liminf_{k\to\infty} \int_{\Omega_n} |\nabla u_k|^{p(x)} dx \geq \int_{\Omega_n} \varphi \nabla u dx,$$

this implies, thanks to (17), that:

$$\lim_{k\to\infty} \int_{\Omega_n} |\nabla u_k|^{p(x)} dx = \int_{\Omega_n} \varphi \nabla u dx. \tag{19}$$

By combining (14), (16) and (19), we deduce that:

$$\langle B_n u_k, u_k\rangle \to \langle \chi, u\rangle \quad \text{as } k \to +\infty.$$

Now, by (19) we can obtain:

$$\lim_{k\to+\infty} \int_{\Omega_n} (|\nabla u_k|^{p(x)-2}\nabla u_k - |\nabla u|^{p(x)-2}\nabla u)(\nabla u_k - \nabla u) dx = 0,$$

In view of the Lemma 2, we obtain:

$$u_k \to u, \quad W_0^{1,p(x)}(\Omega_n), \quad \nabla u_k \to \nabla u \quad \text{a.e. in } \Omega_n,$$

then

$$|\nabla u_k|^{p(x)-2}\nabla u_k \rightharpoonup |\nabla u|^{p(x)-2}\nabla u \qquad \text{in } (L^{p'(x)}(\Omega_n))^N,$$

and

$$c(x,u_k) - H_n(x,u_k,\nabla u_k) \rightharpoonup c(x,u) - H_n(x,u,\nabla u) \quad \text{in } L^{p'(x)}(\Omega_n),$$

we deduce that $\chi = B_n u$, which completes the proof.

By Lemma 3, we deduce that there exists at least one weak solution $u_n \in W_0^{1,p(x)}(\Omega_n)$ of the problem (5), (cf. [14]).

A priori estimates

Proposition 4 *Assuming that $p(.) \in C_+(\overline{\Omega})$ holds, and let u_n be any solution of (5). Then for every $\lambda > 0$ there exists a positive constant $C = C(N, p, \alpha_0, d, f, g, \lambda)$ such that:*

$$\|e^{\lambda|u_n|} - 1\|_{W_0^{1,p(x)}(\Omega_n)} \leq C. \tag{20}$$

Remark 1 *The previous estimate yields an estimate for the functions $e^{\lambda|u_n|}$ in $L^{r(x)}_{loc}(\Omega)$ for every $r \in [1, +\infty)$, every $\lambda > 0$ and every set $\Omega_0 \subset\subset \Omega$, one has*

$$\|e^{|u_n|}\|_{L^{r(x)}(\Omega_0)} \leq C(r_{\mp}, \lambda, \Omega_0)$$

Proof:

For simplicity of notation we will always omit the index n of the sequence. We take $\varphi(G_k(u))$ as test function in (5), where

$$G_k(s) = s - T_k(s) = \begin{cases} s-k & \text{if } s > k, \\ 0 & \text{if } |s| \leq k, \\ s+k & \text{if } s < -k. \end{cases} \tag{21}$$

$$\text{and} \quad \varphi(s) = \left(e^{\lambda|s|} - 1\right) sign(s).$$

we have:

$$\begin{aligned} &\int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) + \alpha_0 \int_\Omega |u|^{p(x)-1} |\varphi(G_k(u))|, \\ &\leq d \int_\Omega |\nabla G_k(u)|^{p(x)} |\varphi(G_k(u))| + \int_\Omega |f||\varphi(G_k(u)) \\ &+ \int_\Omega |g||\nabla G_k(u)| \varphi'(G_k(u)) \\ &= I + J + K, \end{aligned} \tag{22}$$

For every s in $\mathbb{R}$ and if λ satisfies:

$$\lambda \geq 8d \tag{23}$$

we have:

$$d|\varphi(s)| \leq \frac{1}{8} \varphi'(s) \tag{24}$$

then

$$I \leq \frac{1}{8} \int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) \tag{25}$$

Before estimating J, we remark that:

$$\int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u)) = \int_\Omega |\nabla \Psi(G_k(u))|^{p(x)} \tag{26}$$

where

$$\Psi(s) = \int_0^{|s|} (\varphi'(t))^{\frac{1}{p(x)}} dt = \frac{p(x)}{\lambda^{\frac{1}{p'(x)}}} \left(e^{\frac{\lambda|s|}{p(x)}} - 1\right) \tag{27}$$

Moreover, we observe that there exists a positive constant $c_2 = c_2(p, \lambda)$ such that

$$|\varphi(s)| \leq c_2 (\Psi(s))^{p(x)} \text{for every s such that} |s| \geq 1 \tag{28}$$

Now let us observe that p is a continuous variable exponent on $\overline{\Omega}$ then there exists a constant $\delta > 0$ such that:

$$\max_{y \in B(x,\delta)\cap\Omega} \frac{N - p(y)}{Np(y)} \leq \min_{y \in B(x,\delta)\cap\Omega} \frac{p(y)(N - p(y))}{Np(y)} \quad \text{for all } x \in \Omega. \tag{29}$$

while $\overline{\Omega}$ is compact then we can cover it with a finite number of balls B_i for $i = 1, ..., m$ from (29) we can deduce the pointwise estimate:

$$1 < p_{-,i} \leq p_{+,i} \leq \frac{p_{-,i}^2 N}{N - p_{-,i} + p_{-,i}^2} < N. \tag{30}$$

is satisfy for all $i = 1, ..., m$.
$p_{-,i}, p_{+,i}$ denote the local minimum and the local maximum of p on $\overline{B_i \cap \Omega}$ respectively

Estimation of the integral J:

Let $H \geq 1$ be a constant that we will chose later. We can estimate J by splitting it as follows:

$$\begin{aligned} J &= \sum_{i=0}^m \Big[\int_{B_i \cap \{|f|>H, |G_k(u)| \geq 1\}} |f||\varphi(G_k(u))| \Big] \\ &+ \int_{\{|f|>H, |G_k(u)|<1\}} |f||\varphi(G_k(u))| + \int_{\{|f| \leq H\}} |f||\varphi(G_k(u))| \\ &= J_1 + J_2 + J_3 \end{aligned}$$

By (28) J_1, can be estimated as follows

$$J_1 \leq c_2 \sum_{i=0}^m \Big[\int_{B_i \cap \{|f|>H, |G_k(u)| \geq 1\}} |f| \Psi(G_k(u))^{p(x)} \Big]$$

Let ϵ a positive constant to be chosen later. Using Young, Sobolev's embedding and Lemma 1 we have:

$$\begin{aligned} J_1 &\leq C \int_{\{|f|>H\}} |f|^{\frac{\epsilon N}{\epsilon N + p(x) - N}} + \frac{1}{8} \sum_{i=0}^m \|\nabla \Psi(G_k(u))\|_{L^{p(x)}(B_i)}^{\epsilon p_{+,i}^*} \\ &\leq C \int_{\{|f|>H\}} |f|^{\frac{\epsilon N}{\epsilon N + p(x) - N}} + \frac{1}{8} \sum_{i=0}^m \Big[\int_{B_i} |\nabla \Psi(G_k(u))|^{p(x)} \Big]^{\frac{\epsilon p_{+,i}^*}{p_{-,i}}} \end{aligned}$$

where $p^*(x) = \frac{Np(x)}{N - p(x)}$ and $p_{+,i}^* = \frac{Np_{+,i}}{N - p_{+,i}}$ since (30) we can choose ϵ such that:

$$\frac{N - p_{-,i}}{Np_{-,i}} \leq \epsilon \leq \frac{p_{-,i}}{p_{+,i}^*} \tag{31}$$

Then using (31) and (26) we obtain that :

$$J_1 \leq C \int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} + \frac{1}{8} \int_\Omega |\nabla G_k(u)|^{p(x)} \varphi'(G_k(u) \tag{32}$$

Remark 2 *The cases where $\|\Psi(G_k(u))\|_{L^{p*(x)}(\Omega)} \leq 1$ or $\|\nabla \Psi(G_k(u))\|_{L^{p(x)}(\Omega)} \leq 1$ are easy to see that $J_1 \leq C$ (C depend on the data of the problem)*

On the other hand

$$J_2 \le \varphi(1)\int_{\{|f|>H\}}|f| \le \frac{\varphi(1)}{H^{\frac{N-p_+}{p_+}}}\int_{\{|f|>H\}}|f|^{\frac{N}{p(x)}} \tag{33}$$

Finally, if we choose k sufficiently large such that:

$$\alpha_0 k^{p_--1} \ge 4H \tag{34}$$

We obtain:

$$\begin{aligned} J_3 &\le \frac{\alpha_0}{4}\int_\Omega k^{p(x)-1}|\varphi(G_k(u))| \\ &\le \frac{\alpha_0}{4}\int_\Omega |u|^{p(x)-1}|\varphi(G_k(u))| \end{aligned} \tag{35}$$

Estimation of the integral K:

Thanks to Young's inequality, we have:

$$\begin{aligned} K \le& \frac{1}{8}\int_\Omega |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)) \\ &+ C_4\int_\Omega |g|^{p'(x)}\varphi'(G_k(u)), \\ &= K_1 + K_2 \end{aligned} \tag{36}$$

The integral K_2 can be estimated as follows:

$$\begin{aligned} K_2 &\le C_4\int_\Omega |g|^{p'(x)}\varphi'(G_k(u)), \\ &\le C_4\lambda e^\lambda \int_{\{|G_k(u)|<1\}}|g|^{p'(x)} \\ &+ C_4\sum_{i=0}^m\Big[\int_{B_i\cap\{|g|>1,|G_k(u)|>1\}}|g|^{p'(x)}\varphi'(G_k(u))\Big] \\ &+ C_4\int_{\{|g|\le 1,|G_k(u)|>1\}}\varphi'(G_k(u)) \\ &= K_{2,1}+K_{2,2}+K_{2,3} \end{aligned} \tag{37}$$

Since $\varphi'(s) \le C_5(\Psi(s))^{p(x)}$ for every s such that $|s| \ge 1$, we have:

$$K_{2,2} \le C_6\sum_{i=0}^m\Big[\int_{B_i\cap\{|g|>1,|G_k(u)|>1\}}|g|^{p'(x)}\Psi(G_k(u))^{p(x)}\Big]$$

Let ϵ be a positive constant such that (31). Using Young, Sobolev's embedding and Lemma 1 we have:

$$\begin{aligned} K_{2,2} &\le C_7\int_{\{|g|>1\}}|g|^{\frac{\epsilon N p'(x)}{\epsilon N + p(x) - N}} + \frac{1}{8}\sum_{i=0}^m \|\nabla\Psi(G_k(u))\|_{L^{p(x)}(B_i)}^{\epsilon p^*_{+,i}} \\ &\le C_7\int_{\{|g|>1\}}|g|^{\frac{\epsilon N p'(x)}{\epsilon N + p(x) - N}} + \frac{1}{8}\sum_{i=0}^m\Big[\int_{B_i}|\nabla\Psi(G_k(u))|^{p(x)}\Big]^{\frac{\epsilon p^*_{+,i}}{p_{-,i}}} \end{aligned}$$

Then using (31) and (26) we obtain that:

$$K_{2,2} \le C_7\int_{\{|g|>1\}}|g|^{\frac{N}{p(x)-1}} + \frac{1}{8}\int_\Omega |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u) \tag{38}$$

The same as before in the cases where $\|\Psi(G_k(u))\|_{L^{p*(x)}(\Omega)} \le 1$ or $\|\nabla\Psi(G_k(u))\|_{L^{p(x)}(\Omega)} \le 1$ it is easy to check that $K_{2,2} \le C$

Finally, using inequality

$$\varphi'(s) \le C_8|\varphi(s)|, \quad \text{for every s such that} |s| \ge 1 \tag{39}$$

and choosing $k = k(p_-, \alpha_0, \lambda)$ sufficiently large such that:

$$\alpha_0 k^{p_--1} \ge 4C_4 \tag{40}$$

we obtain:

$$\begin{aligned} K_{2,3} &\le \frac{\alpha_0}{4}\int_\Omega k^{p(x)-1}|\varphi(G_k(u))| \\ &\le \frac{\alpha_0}{4}\int_\Omega |u|^{p(x)-1}|\varphi(G_k(u))| \end{aligned} \tag{41}$$

Putting all the inequalities (22), (25), (32), (33), (35), (38), (41), (37) and (36) together, we get an estimate in $W_0^{1,p(x)}(\Omega)$ for $G_k(u)$,when k is large enough:

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)) + \frac{\alpha_0}{2}\int_\Omega |u|^{p(x)-1}|\varphi(G_k(u))| \\ &\le C\int_{\{|f|>H\}}|f|^{\frac{N}{p(x)}} + \frac{\varphi(1)}{H^{\frac{N-p_+}{p_+}}}\int_{\{|f|>H\}}|f|^{\frac{N}{p(x)}} \\ &+ C_4\lambda e^\lambda\int_\Omega |g|^{p'(x)} + C_7\int_\Omega |g|^{\frac{N}{p(x)-1}} \\ &= C_9(N, p_-, p_+, \alpha_0, f, g, \lambda) \end{aligned} \tag{42}$$

For every λ, k satisfying (23), (34), (40) and for every $H \ge 1$. We fix now λ and k such that (42) holds. As before, If we take $\varphi(T_k(u))$ as a test function in (5) we obtain:

$$\begin{aligned} &\int_\Omega |\nabla T_k(u)|^{p(x)}\varphi'(T_k(u)) + \alpha_0\int_\Omega |u|^{p(x)-1}|\varphi(T_k(u))|, \\ &\le d\int_\Omega |\nabla T_k(u)|^{p(x)}|\varphi(T_k(u))| + d\varphi(k)\int_\Omega |\nabla G_k(u)|^{p(x)}| \\ &+ \int_\Omega |f||\varphi(T_k(u))| + \int_\Omega |g||\nabla T_k(u)|\varphi'(T_k(u)) \\ &= L_1 + L_2 + L_3 + L_4, \end{aligned} \tag{43}$$

Using (24), we have:

$$L_1 \le \frac{1}{4}\int_\Omega |\nabla T_k(u)|^{p(x)}\varphi'(T_k(u) \tag{44}$$

By (42),

$$L_2 \le C_{10}(N, p_-, p_+, \alpha_0, f, g, \lambda) \tag{45}$$

Remark 3 *if meas*(Ω) *is finite or if* $f \in L^1(\Omega)$ *it is easy to estimate the integral* L_3

In general case, let ϵ be a positive constant to be chosen later, we write

$$\begin{aligned} L_3 &\le \varphi(k)\int_{\{|f|>1\}}|f| + \int_{\{|f|\le 1\}}|f||\varphi(T_k(u))| \\ &\le \varphi(k)\int_{\{|f|>1\}}|f| + \epsilon\int_\Omega |\varphi(T_k(u))|^{p(x)} \\ &+ c(\epsilon, p'_-)\int_{\{|f|\le 1\}}|f|^{p'(x)} \end{aligned}$$

Since

$$|\varphi(T_k(u))|^{p(x)} \leq C_{11}(\lambda, p_+, p_-, k)|\varphi(T_k(u))||u|^{p(x)-1},$$

choosing ϵ such that $\epsilon C_{11} \leq \frac{\alpha_0}{2}$,we have:

$$L_3 \leq \frac{\alpha_0}{2}\int_\Omega |u|^{p(x)-1}|\varphi(T_k(u))| + C_{12}(\alpha_0, f, \lambda, p_+, p_-, k) \tag{46}$$

Finally, one has

$$L_4 \leq \frac{1}{4}\int_\Omega |\nabla T_k(u)|^{p(x)}\varphi'(T_k(u) + C_{13}(\alpha_0, \lambda, p'_-, g, p_-, k) \tag{47}$$

In conclusion, putting all the estimations ((43) - (47)) together, we get:

$$\frac{1}{2}\int_\Omega |\nabla T_k(u)|^{p(x)}\varphi'(T_k(u)) + \frac{\alpha_0}{2}\int_\Omega |u|^{p(x)-1}|\varphi(T_k(u))| \leq C_{14}(N, p_-, p_+, \alpha_0, f, g, \lambda) \tag{48}$$

In view of (42) and (48), we have:

$$\int_{\{|u|\leq k\}} |\nabla u|^{p(x)}e^{\lambda|u|} \leq C_{15}, \quad \int_{\{|u|>k\}} |\nabla u|^{p(x)}e^{\lambda(|u|-k)} \leq C_{15}$$

For every λ, k large enough (see (23), (34) and (40)), where C_{15} depends on λ, k and the data. Since

$$\begin{aligned}\int_\Omega |\nabla u|^{p(x)}e^{\lambda|u|} &= \int_{\{|u|\leq k\}} |\nabla u|^{p(x)}e^{\lambda|u|} \\ &\quad + e^{\lambda k}\int_{\{|u|>k\}} |\nabla u|^{p(x)}e^{\lambda(|u|-k)} \\ &\leq C_{16}\end{aligned}$$

If we fix the value of k (depending on λ), we obtain an estimate of $|\nabla(e^{\lambda|u|}-1)|$ in $L^{p(x)}(\Omega)$ (depending on λ). This implies, by Sobolev's embedding, that:

$$\int_\Omega (e^{\lambda|u|}-1)^{p*(x)} \leq C_{17} \tag{49}$$

For every λ such that (23) , where C_{17} depends on λ and on the data of the problem. Note that (49) does not imply an estimate in $L^{p(x)}(\Omega)$ for $e^{\lambda|u|}-1$, since $meas(\Omega)$ may be infinite. To obtain such an estimate, we have to combine (48) and (49) , since, for every $k>0$, one has the inequalities

$$\int_{\{|u|\leq k\}} (e^{\lambda|u|}-1)^{p(x)} \leq C_{19}\int_\Omega |u|^{p(x)-1}|\varphi(T_k(u)|,$$

$$\int_{\{|u|>k\}} (e^{\lambda|u|}-1)^{p(x)} \leq C_{20}\int_\Omega (e^{\lambda|u|}-1)^{p*(x)},$$

Therefore, if $k = k(\lambda)$ is such that (48) holds, we can write

$$\begin{aligned}&\int_\Omega (e^{\lambda|u|}-1)^{p(x)} \\ &= \int_{\{|u|\leq k\}} (e^{\lambda|u|}-1)^{p(x)} + \int_{\{|u|>k\}} (e^{\lambda|u|}-1)^{p(x)} \leq C_{21}\end{aligned} \tag{50}$$

where C_{21} depends on λ and the data of the problem.

4 Main results

In this section we will prove the main result of this paper. Let $\{u_n\}$ be any sequence of solutions of problem (5), we extend them to zero in $\Omega \setminus \Omega_n$. By (20), there exist a subsequence (still denoted by u_n) and a function $u \in W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$.

Theorem 1 *There exists at least one solution u of (1); which is such that*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\psi dx + \int_\Omega c(x,u)\psi dx + \int_\Omega H(x,u,\nabla u)\psi dx = \int_\Omega f\psi dx - \int_\Omega g\nabla\psi dx. \tag{51}$$

for every function $\psi \in W_0^{1,p(x)}(\Omega)\cap L^\infty(\Omega)$. Moreover u satisfies

$$e^{\lambda|u|}-1 \in W_0^{1,p(x)}(\Omega) \tag{52}$$

for every $\lambda \geq 0$.

The proof will be made in three steps.

Step 1: An estimate for $\int_\Omega |\nabla G_k(u_n)|^{p(x)}$

In view of (42) we have:

$$\begin{aligned}&\int_\Omega |\nabla G_k(u_n)|^{p(x)} \\ &\leq \frac{2C}{\lambda}\int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} + \frac{2\varphi(1)}{\lambda H^{\frac{N-p_+}{p_+}}}\int_{\{|f|>H\}} |f|^{\frac{N}{p(x)}} + \frac{2C_4\lambda e^\lambda}{\lambda}\int_\Omega |g|^{p'(x)} \\ &+ \frac{2C_7}{\lambda}\int_\Omega |g|^{\frac{N}{p(x)-1}}\end{aligned} \tag{53}$$

If η is an arbitrary positive number, let us choose H such that the right-hand side of (53) is smaller than η. It follows that, for every k satisfying (34), (40) , every λ satisfying (23), and every $n \in \mathbb{N}$

$$\int_\Omega |\nabla G_k(u_n)|^{p(x)} \leq \eta$$

which proves:

$$\sup_n \int_\Omega |\nabla G_k(u_n)|^{p(x)} \to 0 \quad \text{as} \quad k \to \infty \tag{54}$$

Step 2: Strong convergence of $\nabla T_k(u_n)$

In this step, we will fix $k>0$ and prove that $\nabla T_k(u_n) \to \nabla T_k(u)$ strongly in $L^{p(x)}(\Omega_0;\mathbb{R}^N)$ as $n \to \infty$; for k fixed. In order to prove this result we define:

$$z_n(x) = T_k(u_n) - T_k(u)$$

and we choose ψ a cut-off function such that

$$\psi \in C_0^\infty(\Omega), \quad 0 \leq \psi \leq 1, \quad \psi = 0 \quad \text{in} \quad \Omega_0$$

Let us take:

$$v = \varphi(z_n)e^{\delta|u_n|}\psi \tag{55}$$

as a test function in (5), where λ and δ are a positive constant to be chosen later, we obtain:

$$\begin{aligned}
A_n + B_n &= \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla z_n \varphi'(z_n)e^{\delta|u_n|}\psi \\
&+ \int_\Omega c(u_n)\varphi(z_n)e^{\delta|u_n|}\psi \\
&\leq d\int_\Omega |\nabla u_n|^{p(x)}|\varphi(z_n)|e^{\delta|u_n|}\psi + \int_\Omega |f||\varphi(z_n)|e^{\delta|u_n|}\psi \\
&- \delta\int_\Omega |\nabla u_n|^{p(x)}\varphi(z_n)e^{\delta|u_n|}sign(u_n)\psi \\
&+ \int_\Omega |\nabla u_n|^{p(x)-1}|\nabla\psi||\varphi(z_n)|e^{\delta|u_n|} \\
&+ \int_\Omega |g||\nabla z_n|\varphi'(z_n)e^{\delta|u_n|}\psi \\
&+ \delta\int_\Omega |g||\nabla u_n||\varphi(z_n)|e^{\delta|u_n|}\psi + \int_\Omega |g||\nabla\psi||\varphi(z_n)|e^{\delta|u_n|} \\
&= C_n + D_n + E_n + F_n + G_n + H_n + L_n
\end{aligned} \tag{56}$$

Splitting Ω into $\Omega = \{|u_n| \leq k\} \cup \{|u_n| > k\}$ we can write:

$$\begin{aligned}
A_n &= \int_{\{|u_n|\leq k\}} |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\nabla z_n\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi \\
&+ \int_{\{|u_n|>k\}} |\nabla u_n|^{p(x)-2}\nabla u_n\nabla z_n\varphi'(z_n)e^{\delta|u_n|}\psi \\
&= \int_{\{|u_n|\leq k\}} [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)] \\
&* \nabla z_n\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi \\
&+ \int_{\{|u_n|\leq k\}} |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\nabla z_n\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi \\
&+ \int_{\{|u_n|>k\}} |\nabla u_n|^{p(x)-2}\nabla u_n\nabla z_n\varphi'(z_n)e^{\delta|u_n|}\psi \\
&= A_{1,n} + A_{2,n} + A_{3,n}
\end{aligned}$$

since

$$\begin{aligned}
&|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi\chi_{\{|u_n|\leq k\}} \\
&\qquad \to |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(0)e^{\delta|T_k(u)|}\psi\chi_{\{|u|\leq k\}}
\end{aligned}$$

almost everywhere in Ω (on the set where $|u(x)| = k$ we have $|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u) = 0$)and

$$\begin{aligned}
&||\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi\chi_{\{|u_n|\leq k\}}| \\
&\qquad \leq |\nabla u|^{p(x)-1}\varphi'(2k)e^{\delta k}\psi
\end{aligned}$$

which is a fixed function in $L^{p'(x)}(\Omega)$. Therefore by Lebesgue's theorem we have

$$\begin{aligned}
&|\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(z_n)e^{\delta|T_k(u_n)|}\psi\chi_{\{|u_n|\leq k\}} \\
&\qquad \to |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\varphi'(0)e^{\delta|T_k(u)|}\psi\chi_{\{|u|\leq k\}}
\end{aligned}$$

strongly in $L^{p'(x)}(\Omega)$. Indeed, $\nabla z_n \rightharpoonup 0$ weakly in $L^{p(x)}(\Omega;\mathbb{R}^N)$ then $A_{2,n} \to 0$. Similarly, since $\nabla z_n\chi_{\{|u_n|>k\}} = -\nabla T_k(u)\chi_{\{|u_n|>k\}} \to 0$ strongly in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, while $|\nabla u_n|^{p(x)-2}\nabla u_n\varphi'(z_n)e^{\delta|u_n|}\psi$ is bounded in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, by (6), (20) and Remark 1 we obtain $A_{3,n} \to 0$. Therefore, we have proved that:

$$A_n = A_{1,n} + o(1) \tag{57}$$

For the integral B_n while $\varphi(z_n)$ has the same sign as $c(u_n)$ on the set $\{|u_n| > k\}$ we have

$$\begin{aligned}
B_n &= \int_{\{|u_n|\leq k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\psi \\
&+ \int_{\{|u_n|>k\}} c(u_n)\varphi(z_n)e^{\delta|u_n|}\psi \\
&\geq \int_{\{|u_n|\leq k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\psi
\end{aligned}$$

the last integrand converges pointwise and it is bounded then $\int_{\{|u_n|\leq k\}} c(T_k(u_n))\varphi(z_n)e^{\delta|T_k(u_n)|}\psi$ goes to zero. Therefore, we obtain that:

$$B_n \geq o(1) \tag{58}$$

Let us examine C_n and D_n together. We first fix δ such that

$$\delta > d$$

Since $\varphi(z_n)sign(u_n) = |\varphi(z_n)|$ on the set $\{|u_n| > k\}$ we have

$$\begin{aligned}
C_n + E_n &\leq d\int_\Omega |\nabla u_n|^{p(x)}|\varphi(z_n)|e^{\delta|u_n|}\psi \\
&- \delta\int_\Omega |\nabla u_n|^{p(x)}\varphi(z_n)e^{\delta|u_n|}sign(u_n)\psi \\
&\leq (d+\delta)\int_{\{|u_n|\leq k\}} |\nabla T_k(u_n)|^{p(x)}|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \\
&+ (d-\delta)\int_{\{|u_n|>k\}} |\nabla u_n|^{p(x)}\varphi(z_n)e^{\delta|u_n|}\psi \\
&\leq (d+\delta)\int_{\{|u_n|\leq k\}} |\nabla T_k(u_n)|^{p(x)}|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \\
&= (d+\delta)\int_{\{|u_n|\leq k\}} [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) \\
&- |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)]\nabla z_n|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \\
&+ (d+\delta)\int_{\{|u_n|\leq k\}} |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)\nabla T_k(u)| \\
&\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi \\
&+ (d+\delta)\int_{\{|u_n|\leq k\}} |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\nabla z_n|\varphi(z_n)|e^{\delta|T_k(u_n)|}\psi
\end{aligned}$$

The last two integrals converge to zero.If we choose λ such that:

$$\lambda \geq 2(d+\delta)$$

we have:

$$(d+\delta)|\varphi(s)| \leq \frac{\varphi'(s)}{2} \quad \text{for every s in } \mathbb{R}$$

then we can obtain:

$$C_n + E_n \leq \frac{1}{2}A_{1,n} + o(1) \tag{59}$$

Using Remark 1 we can observe that:

$$D_n \to 0 \tag{60}$$

For the term F_n we can see that $|\nabla\psi||\varphi z_n|$ converge strongly to zero in $L^{r(x)}(\Omega)$ for every $r(x) > 1$. by (20) the term $|\nabla u_n|^{p(x)-2}\nabla u_n e^{\delta|u_n|}$ is bounded in $L^{p'(x)}_{loc}(\Omega)$ then we have that:

$$F_n \to 0 \tag{61}$$

For the term G_n like before we have:

$$\begin{aligned} G_n &= \int_{\{|u|\le k\}} |g||\nabla z_n|\varphi'(z_n)e^{\delta|u_n|}\psi \\ &+ \int_{\{|u|>k\}} |g||\nabla z_n|\varphi'(z_n)e^{\delta|u_n|}\psi \\ &= G_{1,n} + G_{2,n} \end{aligned}$$

since

$$|g|\varphi'(z_n)e^{\delta|u_n|}\psi\chi_{\{|u_n|\le k\}} \to |g|\varphi'(0)e^{\delta|T_k u|}\psi\chi_{\{|u|\le k\}}$$

almost everywhere in Ω and

$$|g|\varphi'(z_n)e^{\delta|u_n|}\psi\chi_{\{|u_n|\le k\}} \le |g|\varphi'(2k)e^{\delta k}\psi$$

Therefore by Lebesgue's theorem we have:

$$|g|\varphi'(z_n)e^{\delta|u_n|}\psi\chi_{\{|u_n|\le k\}} \to |g|\varphi'(0)e^{\delta|T_k u|}\psi\chi_{\{|u|\le k\}}$$

strongly in $L^{p'(x)}(\Omega)$. Indeed, $\nabla z_n \rightharpoonup 0$ weakly in $L^{p(x)}(\Omega;\mathbb{R}^N)$ then $G_{1,n} \to 0$. Similarly, since $|\nabla z_n|\chi_{\{|u_n|>k\}} = |\nabla T_k(u)|\chi_{\{|u_n|>k\}} \to 0$ strongly in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, while $|g|\varphi'(z_n)e^{\delta|u_n|}\psi$ is bounded in $L^{p'(x)}(\Omega;\mathbb{R}^N)$, by (20) and remark 1 we obtain $G_{2,n} \to 0$. Therefore, we have proved that:

$$G_n \to 0 \tag{62}$$

Moreover

$$|g||\varphi(z_n)|\psi \to 0$$

almost everywhere in Ω and

$$|g||\varphi(z_n)|\psi \le |g||\varphi(2k)|\psi$$

Therefore by Lebesgue's theorem we have:

$$|g||\varphi(z_n)|\psi \to 0$$

strongly in $L^{p'(x)}(\Omega)$. Indeed, $\nabla u_n e^{\delta|u_n|} \rightharpoonup \nabla u e^{\delta|u|}$ weakly in $L^{p(x)}(\Omega;\mathbb{R}^{\mathbb{N}})$, then:

$$H_n \to 0 \tag{63}$$

Finally, $|\nabla\psi||\varphi z_n|$ converge strongly to zero in $L^{r(x)}(\Omega)$ for every $r(x) > 1$. by (20) the term $|g|e^{\delta|u_n|}$ is bounded in $L^{p'(x)}_{loc}(\Omega)$ then we have that:

$$L_n \to 0 \tag{64}$$

Putting all inequalities (56), (57), (58), (59), (60), (61), (62), (63) and (64) we can conclude:

$$A_{1,n} \to 0 \tag{65}$$

On the other hand we have

$$\int_{\{|u_n|>k\}} [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)]\nabla z_n \varphi'(z_n)e^{\delta|T_k(u_n)|}\psi = \int_{\{|u_n|>k\}} |\nabla T_k(u)|^{p(x)}\varphi'(k - T_k(u))e^{\delta k}\psi \to 0 \tag{66}$$

From (65) and (66) we can conclude that:

$$\int_{\Omega_0} [|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)] (\nabla T_k(u_n) - \nabla T_k(u)) \to 0 \tag{67}$$

Finally, using the Lemma 2 we have:

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{strongly in } L^{p(x)}(\Omega_0;\mathbb{R}^{\mathbb{N}}) \tag{68}$$

Step 3: End of the proof

Observing that:

$$\nabla u_n - \nabla u = \nabla T_k u_n - \nabla T_k u + \nabla G_k u_n - \nabla G_k u$$

Let Ω_0 be an open set compactly contained in Ω, using (54) and (68) we have:

$$\nabla u_n \to \nabla u \quad \text{strongly in } L^{p(x)}(\Omega_0;\mathbb{R}^{\mathbb{N}}) \tag{69}$$

To obtain (51) we have to pass to the limit in the distributional formulation of problem (5) using (69). Finally, statement (52) follows easily from Proposition 4 and (69), using Fatou's Lemma.

5 Boundedness of solutions

In this section we will gave some regularity on the solution of the problem (1) using an adaptation of a classical technique due to Stampacchia. To do this we need the following lemma (see [15]):

Lemma 4 *Let ϕ be a non-negative, non-increasing function defined on the halfline $[k_0, \infty)$. Suppose that there exist positive constants A, μ,β, with $\beta > 1$, such that*

$$\phi(h) \le \frac{A}{(h-k)^\mu}\phi(k)^\beta$$

for every $h > k \ge k_0$. Then $\phi(k) = 0$ for every $k \ge k_1$, where

$$k_1 = k_0 + A^{1/\mu}2^{\beta/(\beta-1)}\phi(k_0)^{(\beta-1)/\mu}$$

The result that we are going to prove is the following:

Theorem 2 *Suppose that* (3) *holds. Then every solution u of* (1)*; which is specified in* (4) *is essentially bounded, and*

$$\|u_n\|_{L^\infty(\Omega)} \le C \tag{70}$$

The proof relies on the combined use of the well-known technique by Stampacchia (see [15]) and suitable exponential test functions, as in [16].

Proof: Since (42) we can obtain an estimate for $\int_\Omega |u|^{p(x)-1}\varphi(G_k(u))$ then for some constant $k_0 = k(\lambda)$ sufficiently large we have

$$meas(A_{k_0}) < 1 \tag{71}$$

where

$$A_k = \{x \in \Omega : |u| > k\}$$

as before we can take the test function $\varphi(G_k(u))$ then we have:

$$\begin{aligned}
&\int_{A_k} |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)) + \alpha_0 \int_{A_k} |u|^{p(x)-1}|\varphi(G_k(u))|,\\
&\leq d\int_{A_k} |\nabla G_k(u)|^{p(x)}|\varphi(G_k(u))| + \int_{A_k\cap\{|f|>1\}} |f||\varphi(G_k(u))|\\
&+ \int_{A_k\cap\{|f|\leq 1\}} |\varphi(G_k(u))| + \int_{A_k} |g||\nabla G_k(u)|\varphi'(G_k(u))
\end{aligned} \tag{72}$$

As in the proof of Proposition 4 one has:

$$\begin{aligned}
&\int_{A_k} |g||\nabla G_k(u)|\varphi'(G_k(u))\\
&\leq \frac{1}{4}\int_{A_k} |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)) + C_{22}\int_{A_k} |g|^{p'(x)}\varphi'(G_k(u))
\end{aligned}$$

and if $\lambda \geq 4d$ and $k \geq k_0(\lambda)$(large enough) where

$$\alpha_0 k_0^{p_- -1} \geq 4 \tag{73}$$

then

$$\begin{aligned}
&\frac{1}{2}\int_{A_k} |\nabla G_k(u)|^{p(x)}\varphi'(G_k(u)) + \frac{3\alpha_0}{4}\int_{A_k} |u|^{p(x)-1}|\varphi(G_k(u))|,\\
&\leq \int_{(A_k\setminus A_{k+1})\cap\{|f|>1\}} |f||\varphi(G_k(u))|\\
&+ \int_{A_{k+1}\cap\{|f|>1\}} |f||\varphi(G_k(u))| + C_{22}\varphi'(1)\int_{A_k\setminus A_{k+1}} |g|^{p'(x)}\\
&+ C_{22}\int_{A_{k+1}} |g|^{p'(x)}\varphi'(G_k(u))
\end{aligned} \tag{74}$$

using Hölder inequality we have:

$$\begin{aligned}
&\int_{(A_k\setminus A_{k+1})\cap\{|f|>1\}} |f||\varphi(G_k(u))|\\
&\leq \varphi(1)\Big(\frac{1}{q_-}+\frac{1}{q'_-}\Big)\|f\|_{L^{q(x)}(\{|f|>1\})}(meas(A_k))^{\frac{1}{q'_+}}
\end{aligned}$$

by Hölder's inequality and interpolation we obtain:

$$\begin{aligned}
&\int_{A_{k+1}\cap\{|f|>1\}} |f||\varphi(G_k(u))|\\
&\leq \|f\|_{L^{q_-}(A_{k+1}\cap\{|f|>1\})}\|\varphi(G_k(u))\|^{\frac{N}{p_- q_-}}_{L^{p*_-/p_-}(A_{k+1})}\|\varphi(G_k(u))\|^{1-\frac{N}{p_- q_-}}_{L^1(A_{k+1})}
\end{aligned}$$

while (28), (71) and using Young's and sobolev's inequalities we can deduce that:

$$\begin{aligned}
&\int_{A_{k+1}\cap\{|f|>1\}} |f||\varphi(G_k(u))|\\
&\leq \frac{1}{8}\|\nabla\psi(G_k(u))\|^{p_-}_{L^{p(x)}(A_k)}\\
&+ C_{23}\|f\|^{\frac{p_- q_-}{p_- q_- - N}}_{L^{q_-}(\{|f|>1\})}\|\varphi(G_k(u))\|_{L^1(A_k)}\\
&\leq \frac{1}{8}\int_{A_k} |\nabla\psi(G_k(u))|^{p(x)} + 1dx\\
&+ C_{23}\|f\|^{\frac{p_- q_-}{p_- q_- - N}}_{L^{q_-}(\{|f|>1\})}\|\varphi(G_k(u))\|_{L^1(A_k)}\\
&\leq \frac{1}{8}\int_{A_k} |\nabla(G_k(u))|^{p(x)}\varphi'(G_k(u))dx + \frac{1}{8}meas(A_k)\\
&+ C_{23}\|f\|^{\frac{p_- q_-}{p_- q_- - N}}_{L^{q_-}(\{|f|>1\})}\|\varphi(G_k(u))\|_{L^1(A_k)}
\end{aligned}$$

Therfore, choosing k_0 such that:

$$C_{23}\|f\|^{\frac{p_- q_-}{p_- q_- - N}}_{L^{q_-}(\{|f|>1\})} \leq \frac{\alpha_0 k_0^{p_- -1}}{4} \tag{75}$$

the second integral in the right-hand side of (74) can be absorbed by the left-hand side.
In view of Hölder's inequality and (71) and (3) we have:

$$\begin{aligned}
&C_{22}\varphi'(1)\int_{A_k\setminus A_{k+1}} |g|^{p'(x)}\\
&\leq C_{23}\Big(\int_{A_k} |g|^{p'(x)}\Big)^{\eta}(meas(A_k))^{1-\frac{p'_+}{r_-}}\\
&\leq C_{24}(\|g\|_{L^{r(x)}(A_k)})^{\delta''}(meas(A_k))^{1-\frac{p'_+}{r_-}}
\end{aligned}$$

where

$$\eta = \begin{cases} \frac{p'_-}{r_+} & \text{if } \int_{A_k} |g|^{p'(x)} \leq 1,\\ \frac{p'_+}{r_-} & \text{if } \int_{A_k} |g|^{p'(x)} > 1.\end{cases}$$

$$\delta'' = \begin{cases} \frac{\eta}{r_-} & \text{if } \|g\|_{L^{r(x)}(A_k)} \geq 1,\\ \frac{\eta}{r_+} & \text{if } \|g\|_{L^{r(x)}(A_k)} < 1.\end{cases}$$

Finally, with similar calculations, using (39) we have:

$$\begin{aligned}
&C_{22}\int_{A_{k+1}} |g|^{p'(x)}\varphi'(G_k(u))\\
&\leq C_{24}\int_{A_{k+1}\cap\{|g|>1\}} |g|^{p'(x)}|\varphi(G_k(u))|\\
&+ C_{24}\int_{A_{k+1}\cap\{|g|\leq 1\}} |\varphi(G_k(u))|
\end{aligned}$$

If we choose k_0 such that:

$$\alpha_0 k_0^{p_- -1} > 4C_{24} \tag{76}$$

then

$$\begin{aligned}
&C_{22}\int_{A_{k+1}} |g|^{p'(x)}\varphi'(G_k(u))\\
&\leq C_{24}\int_{A_{k+1}\cap\{|g|>1\}} |g|^{p'_+}|\varphi(G_k(u))|\\
&+ \frac{\alpha_0}{4}\int_{A_k} |u|^{p(x)-1}|\varphi(G_k(u))|
\end{aligned}$$

by Hölder's inequality and interpolation we obtain:

$$C_{22}\int_{A_{k+1}}|g|^{p'(x)}\varphi'(G_k(u))$$
$$\leq C_{24}\|g\|^{p'_+}_{L^{r_-}(\Omega,\mathbb{R}^N)}\|\varphi(G_k(u))\|^{\frac{p'_+N}{p_-r_-}}_{L^{p*_-/p_-}(A_k)}\|\varphi(G_k(u))\|^{1-\frac{p'_+N}{p_-r_-}}_{L^1(A_k)}$$
$$+\frac{\alpha_0}{4}\int_{A_k}|u|^{p(x)-1}|\varphi(G_k(u))|$$

as before while (28), (71) and using Young's and sobolev's inequalities we can deduce that:

$$C_{22}\int_{A_{k+1}}|g|^{p'(x)}\varphi'(G_k(u))$$
$$\leq\frac{1}{8}\int_{A_k}|\nabla(G_k(u))|^{p(x)}\varphi'(G_k(u))dx+\frac{1}{8}meas(A_k)$$
$$+C_{25}\|g\|^{\frac{p'_+p_-r_-}{p_-r_--p'_+N}}_{L^{r_-}(\Omega,\mathbb{R}^N)}\|\varphi(G_k(u))\|_{L^1(A_k)}$$
$$+\frac{\alpha_0}{4}\int_{A_k}|u|^{p(x)-1}|\varphi(G_k(u))|$$

Therefore, by taking k_0 satistying (71), (73), (75), (76) and the further condition:

$$C_{25}\|g\|^{\frac{p'_+p_-r_-}{p_-r_--p'_+N}}_{L^{r_-}(\Omega,\mathbb{R}^N)}\leq\frac{\alpha_0k_0^{p_--1}}{4}\qquad(77)$$

one obtains, for every $k\geq k_0$:

$$\frac{1}{4}\int_{A_k}|\nabla G_k(u)|^{p(x)}\varphi'(G_k(u))$$
$$\leq\varphi(1)\left(\frac{1}{q_-}+\frac{1}{q'_-}\right)\|f\|_{L^{q(x)}(\{|f|>1\})}(meas(A_k))^{\frac{1}{q'_+}}+\frac{1}{4}meas(A_k)$$
$$+C_{24}(\|g\|_{L^{r(x)}(A_k)})^{\delta''}(meas(A_k))^{1-\frac{p'_+}{r_-}}$$
$$\leq C_{26}(meas(A_k))^m$$

where $m=\min(\frac{1}{q'_+},1-\frac{p'_+}{r_-})$ in view of (26) and Sobolev's inequality we can obtain:

$$\left(\int_{A_k}|\psi(G_k(u))|^{p*(x)}\right)^\beta\leq\|\psi(G_k(u))\|_{L^{p*(x)}(A_k)}$$
$$\leq C_{27}(meas(A_k))^{\frac{m}{\alpha}}$$

Where:

$$\alpha=\begin{cases}p_+ & \text{if } \|\nabla\psi(G_k(u))\|_{L^{p(x)}(A_k)}\leq 1,\\ p_- & \text{if } \|\nabla\psi(G_k(u))\|_{L^{p(x)}(A_k)}>1.\end{cases}$$
$$\beta=\begin{cases}\frac{N-p_-}{Np_-} & \text{if } \|\psi(G_k(u))\|_{L^{p*(x)}(A_k)}\leq 1,\\ \frac{N-p_+}{Np_+} & \text{if } \|\psi(G_k(u))\|_{L^{p*(x)}(A_k)}>1.\end{cases}$$

We now take $h-k>1$ and recall that there exists $C_{28}(\lambda,p_+,p_-)$such that $|\psi(s)|\geq C_{28}|s|$ for every $s\in\mathbb{R}$ so that

$$[C_{28}(h-k)]^{p*_-}meas(A_h)\leq\int_{A_h}|\psi(G_k(u))|^{p*(x)}$$
$$\leq\int_{A_k}|\psi(G_k(u))|^{p*(x)}$$
$$\leq C_{29}(meas(A_k))^{\frac{m}{\alpha\beta}}$$

Then it follows for every h and k (such that $h>k\geq k_0$) that

$$meas(A_h)\leq\frac{C_{30}}{[h-k]^{p*_-}}(meas(A_k))^{\frac{m}{\alpha\beta}}$$

Since by (3), $\frac{m}{\alpha\beta}>1$ Lemma 4 applied to the function $\phi(h)=meas(A_h)$ gives:

$$\|u_n\|_{L^\infty(\Omega)}\leq C$$

References

1. Y. Chen, S. Levine, M. Rao,, *Variable exponent, linear growth functionals in image restoration.*, SIAM J. Appl. Math., **66**, 1383-1406 (2006).

2. V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory.*, Math. USSR Izvestiya, **29**(1), 33-66 (1987).

3. E. Azroul, H. Hjiaj, A. Touzani, *Existence and regularity of entropy solutions for strongly nonlinear $p(x)$– elliptic equations*, Electronic Journal of Differential Equations, Vol. (**2013**), No. 68, pp. 1-27.

4. M. B. Benboubker,H. Chrayteh, M. El Moumni, H. Hjiaj; *Entropy and Renormalized Solutions for Nonlinear Elliptic Problem Involving Variable Exponent and Measure Data*, Acta Mathematica Sinica, English Series Jan., 2015, Vol. **31**, No. 1, pp. 151-169.

5. C. Yazough, E. Azroul, H. Redwane; *Existence of solutions for some nonlinear elliptic unilateral problems with measure data*, Electronic Journal of Qualitative Theory of Diferential Equations 2013, No. 43, 1-21;

6. Q. Zhang; *Existence of radial solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$*, J. Math. Anal. Appl. **315** (2006) 506-516.

7. A. Dall'Aglio D. Giachetti J.-P. Puel, *Nonlinear elliptic equations with natural growth in general domains*, Annali di Matematica **181**, 407-426 (2002).

8. V. Ferone, F. Murat; *Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small*, Nonlinear Analysis 42 (2000) 1309-1326.

9. Guowei Dai; *Infinitely many solutions for a $p(x)$-Laplacian equation in $\mathbb{R}^N$*, Nonlinear Analysis 71 (2009) 1133-1139;

10. X. L. Fan, D. Zhao; *On the generalised Orlicz-Sobolev Space $W^{k,p(x)}(\Omega)$*, J. Gansu Educ. College **12**(1) (1998), 1-6.

11. D. Zhao, W. J. Qiang, X. L. Fan; *On generalized Orlicz spaces $L^{p(x)}(\Omega)$*, J. Gansu Sci. 9(2), 1997, 1-7.

12. P. Harjulehto, P. Hästö; *Sobolev Inequalities for Variable Exponents Attaining the Values* 1 *and* n, Publ. Mat. 52 (**2008**), no. 2, 347-363.

13. L. Diening, P. Harjulehto, P. Hästö, M. Ržička; *Lebesgue and Sobolev Spaces with Variable Exponents*, vol. 2017 of Lecture Notes in Mathematics, Springer, Heidelberg, Germany, 2011.

14. J. L. Lions; *Quelques methodes de rêsolution des problèmes aux limites non linéaires*, Dunod et Gauthiers-Villars, Paris 1969.

15. G. Stampacchia; *Equations elliptiques du second ordre à coefficients discontinus*, Séminaire de Mathématiques Suprieures. No. 16, Montréal, Que.: Les Presses de l´Université de Montréal (1966)

16. L. Boccardo, F. Murat, J.-P. Puel,; *L^∞ estimate for some nonlinear elliptic partial differential equations and application to an existence result*,SIAM J. Math. Anal. (**2**) 23, 326-333 (1992)

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Nonresonance between the first two Eigencurves of Laplacian for a Nonautonomous Neumann Problem

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ABSTRACT
We consider the following Neumann elliptic problem

$$\begin{cases} -\Delta u = \alpha\, m_1(x)\, u + m_2(x)\, g(u) + h(x) & in\, \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & on\, \partial\Omega. \end{cases}$$

By means of Leray-Schauder degree and under some assumptions on the asymptotic behavior of the potential of the nonlinearity g, we prove an existence result for our equation for every given $h \in L^\infty(\Omega)$.

1 Introduction

Let Ω be a bounded domain of $I\!R^N$ $(N \geq 1)$, with $C^{1,1}$ boundary and let ν be the outward unit normal vector on $\partial\Omega$.
D. Del Santo and P. Omari, have studied in [1] the Dirichlet problem

$$\begin{cases} -\Delta u = g(u) + h(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

They have proved the existence of nontrivial weak solutions for this problem for every given $h \in L^p(\Omega)$ under some assumptions on the function g. In the case of Neumann elliptic problem J.-P. Gossez and P. Omari, have considered in [2] the following problem

$$\begin{cases} -\Delta u = g(u) + h(x) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

They have shown the existence of weak solutions for this problem for every given $h \in L^\infty(\Omega)$ under some conditions on function g. A.Dakkak and A. Anane studied in [3] the existence of weak solutions for the problem

$$\begin{cases} -\Delta u = \lambda_2 m(x) u + g(u) + h(x) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\lambda_2 = \lambda_2(m)$ is the second eigenvalue of $-\Delta$ with weight m, with $m \in M^+(\Omega) = \{m \in L^\infty(\Omega) : \text{meas}(\{x \in \Omega : m(x) > 0\}) \neq 0\}$.

We investigate in the present work the following Neumann elliptic problem

$$(\mathcal{P}) \begin{cases} -\Delta u = \alpha\, m_1(x) u + m_2(x) g(u) + h(x) & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

where $-\Delta$ is the Laplacian operator. The functions $m_1, m_2 \in M^+(\Omega)$, $h \in L^\infty(\Omega)$, $g : I\!R \to I\!R$ is a continuous function and α is a real parameter such that $\alpha \geq \lambda_2(m_1)$ or $\alpha \leq \lambda\_2(m_1)$, with $\lambda\_2(m_1) = -\lambda_2(-m_1)$. By a solution of $(\mathcal{P})$ we mean a function $u \in H^1(\Omega) \cap L^\infty(\Omega)$, such that

$$\int_\Omega \nabla u \nabla w = \int_\Omega (\alpha\, m_1 u + m_2 g(u) + h) w$$

for every $w \in H^1(\Omega)$.

This paper is organized as follows. In section 2, we recall some results that we will use later. Section 3 is concerned with the existence of principal eigencurve of the Laplacian operator with Neumann boundary conditions. In section 4, we show a theorem of nonresonance between the first and second eigenvalue (see theorem 2). In section 5, we prove the nonresonance between the first two eigencurves for problem $(\mathcal{P})$.

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2 Preliminary

Let us briefly recall some properties of the spectrum of $-\Delta$ with weight and with Neumann boundary condition to be used later. Let be Ω a smooth bounded domain in $I\!R^N$ $(N \geq 1)$ and let $m \in M^+(\Omega)$. the eigenvalue problem is

$$\begin{cases} -\Delta u = \lambda\, m(x) u & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

this spectrum contains a sequence of nonnegative eigenvalues $(\lambda_n)_{n>0}$ given by

$$\frac{1}{\lambda_n} = \frac{1}{\lambda_n(m)} = \sup_{K\in\Gamma_n} \min_{u\in K} \frac{\int_\Omega m u^2}{\int_\Omega |\nabla u|^2}, \tag{2}$$

where $\Gamma_n = \{K \subset S : \text{K is symetric, compact and } \gamma(K) \geq n\}$, S is the unit sphere of $H^1(\Omega)$ and γ is the genus function. This formulation can be found in [4], the sequence $(\lambda_n)_{n>0}$ verify:
i) $\lambda_n \to +\infty$ as $n \to +\infty$
ii) If m change its sign in Ω and $\int_\Omega m dx < 0$, then the first eigenvalue defined by

$$\lambda_1(m) = \inf\left\{\int_\Omega |\nabla u|^2, u \in H^1(\Omega) / \int_\Omega m u^2 dx = 1\right\} \tag{3}$$

it is known that $\lambda_1(m)$ is > 0, simple and the associated eigenfunction φ_1 can be chosen such that $\varphi_1 > 0$ in Ω and $\|\varphi_1\|_{H^1} = 1$ hold. Moreover $\lambda_1(m)$ is isolated in the spectrum, which allows to define the second positive eigenvalue $\lambda_2(m)$ as

$$\lambda_2(m) = \min\{\lambda \in I\!R : \lambda \text{ is eigenvalue and } \quad \lambda > \lambda_1(m)\} \tag{4}$$

it is also known that any eigenfunction associated to a positive eigenvalue different from $\lambda_1(m)$ changes sign in Ω.
iii) $\lambda_1(m)$ is strictly monotone decreasing with respect to m (i.e. $m \lneqq m'$ implies $\lambda_1(m) > \lambda_1(m')$).
Throughout this work, the functions m_1 and m_2 satisfies the following assumptions:

$$(\mathbf{A}) \qquad m_1, m_2 \in M^+(\Omega) \text{ and } ess\inf_\Omega m_2 > 0.$$

Proposition 1 *([3]). Let $m, m' \in M^+(\Omega)$.*

1. *If $m \leq m'$, then $\lambda_2(m) \geq \lambda_2(m')$.*

2. *$\lambda_2 : m \to \lambda_2(m)$ is continuous in $(M^+(\Omega), \|.\|_\infty)$.*

Proposition 2 *([3]). Let $(m_k)_k$ be a sequence in $M^+(\Omega)$ such that $m_k \to m$ in $L^\infty(\Omega)$. then $\lim_{k\to\infty} \lambda_2(m_k) = +\infty$ if and only if $m \leq 0$ almost everywhere in Ω.*

3 Existence of the second eigencurve of the $-\Delta$ with weighs in the Neumann case

The second eigencurve of the $-\Delta$ with weighs is defined as a set $\mathcal{C}_2$ of those $(\alpha, \beta) \in I\!R^2$ such that the following Neumann problem

$$\begin{cases} -\Delta u = \alpha\, m_1(x) u + \beta m_2(x) u & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

has a nontrivial solution $u \in H^1(\Omega)$ (i.e. $\mathcal{C}_2 = \left\{(\alpha,\beta) \in I\!R^2; \lambda_2(\alpha\, m_1 + \beta m_2) = 1\right\}$), where m_1 and m_2 satisfies the condition $(\mathbf{A})$. For more details see [5, 6]. The purpose in this section is to study the following problem: For $\beta < 0$, we prove the existence and the uniqueness of reel $\alpha_2^+(\beta)$ such that $(\alpha_2^+(\beta), \beta) \in \mathcal{C}_2$). Given $m \in M^+(\Omega)$, we denote by $\Omega_m^+ = \{x \in \Omega; m(x) > 0\}$ and $\Omega_m^- = \{x \in \Omega; m(x) < 0\}$.

Remark 1 *Let $(\alpha, \beta) \in \mathcal{C}_2$.*
1. *If $\alpha > \lambda_2(m_1)$, then we have $\beta < 0$.*
2. *If meas$(\Omega_m^-) > 0$ and $\alpha < \lambda\_2(m_1)$, we have $\beta < 0$.*

Indeed, assume by contradiction if $\alpha > \lambda_2(m_1)$ and $\beta \geq 0$, then

$$\alpha\, m_1 \leq \alpha\, m_1 + \beta\, m_2,$$

using the monotony property of λ_2, we obtain

$$\lambda_2(\alpha\, m_1 + \beta\, m_2) \leq \lambda_2(\alpha\, m_1) = \frac{\lambda_2(m_1)}{\alpha} < 1,$$

since, $(\alpha, \beta) \in \mathcal{C}_2$, we have $\lambda_2(\alpha\, m_1 + \beta\, m_2) = 1$, thus necessarily $\beta < 0$. The proof of the second assertion is similar.

Theorem 1 *Let m_1, m_2 satisfy $(\boldsymbol{A})$, then we have:*

i) *For all $\beta < 0$, there exists $\alpha_2^+(\beta) > \lambda_2(m_1)$ such that $(\alpha_2^+(\beta), \beta) \in \mathcal{C}_2$.*

ii) *If meas$(\Omega_{m_1}^-) > 0$, then for all $\beta < 0$, there exists $\alpha_2^-(\beta) < \lambda\_2(m_1)$ such that $(\alpha_2^-(\beta), \beta) \in \mathcal{C}_2$.*

Proof To prove **i)**, we consider $\beta < 0$ and we define $\alpha_2^+(\beta)$ as follows

$$\frac{1}{\alpha_2^+(\beta)} = \sup_{K\in\Gamma_2} \inf_{u\in K} \frac{\int_\Omega m_1\, u^2}{\int_\Omega |\nabla u|^2 - \beta \int_\Omega m_1\, u^2} \tag{5}$$

by definition of $\alpha_2^+(\beta)$ and, using the fact that for any eigenfunction associated to $\lambda_2(m_1)$ changes sign in Ω, we obtain that there exists eigenfunction u which change sign in Ω such that

$$\int_\Omega \nabla u.\nabla v - \beta \int_\Omega m_2\, u\, v = \int_\Omega \alpha_2^+(\beta)\, m_1\, v, \tag{6}$$

for all $v \in H^1(\Omega)$, we deduce also that, if $w \in H^1(\Omega)$ is eigenfunction of operator $-\Delta(.) - \beta\, m_2(.)$ which change singe in Ω with the corresponding eigenvalue $\lambda > 0$, then $\lambda \geq \alpha_2^+(\beta)$. In view of the (6), we have

$$\int_\Omega \nabla u.\nabla v = \int_\Omega (\alpha_2^+(\beta)\, m_1 + \beta m_2\, u) v \quad \text{for all}\, v \in H^1(\Omega)$$

it follows that the real 1 is eigenvalue of $-\Delta$ with weight $(\alpha_2^+(\beta)+\beta m_2)$, since the corresponding eigenfunction u change singe in Ω, we conclude that

$$\lambda_2(\alpha_2^+(\beta)m_1+\beta m_2)\leq 1. \tag{7}$$

On the other hand, using (5) for all $K\in\Gamma_2$, there exists $u_K\in K$ such that

$$\min_{u\in K}\frac{\int_\Omega m_1 u^2}{\int_\Omega|\nabla u|^2-\beta\int_\Omega m_1 u^2} = \frac{\int_\Omega m_1 u_K^2}{\int_\Omega|\nabla u_K|^2-\beta\int_\Omega m_2 u_K^2}\leq\frac{1}{\alpha_2^+(\beta)},$$

it follows that

$$\frac{\int_\Omega(\alpha_2^+(\beta)m_1+\beta m_2)u_K^2}{\int_\Omega|\nabla u_K|^2}\leq 1.$$

So that

$$\min_{u\in K}\frac{\int_\Omega(\alpha_2^+(\beta)m_1+\beta m_2)u^2}{\int_\Omega|\nabla u|^2} \leq\frac{\int_\Omega(\alpha_2^+(\beta)m_1+\beta m_2)u_K^2}{\int_\Omega|\nabla u_K|^2}\leq 1 \text{ for all } K\in\Gamma_2,$$

this implies

$$\sup_{K\in\Gamma_2}\min_{u\in K}\frac{\int_\Omega(\alpha_2^+(\beta)m_1+\beta m_2)u^2}{\int_\Omega|\nabla u|^2}\leq 1.$$

Since

$$\frac{1}{\lambda_2(\alpha_2^+(\beta)m_1+\beta m_2)}=\sup_{K\in\Gamma_2}\min_{u\in K}\frac{\int_\Omega(\alpha_2^+(\beta)m_1+\beta m_2)u^2}{\int_\Omega|\nabla u|^2}$$

we deduce that

$$\lambda_2(\alpha_2^+(\beta)m_1+\beta m_2)\geq 1. \tag{8}$$

By combining (7) and (8), we obtain

$$\lambda_2(\alpha_2^+(\beta)m_1+\beta m_2)=1.$$

Let $\gamma>0$ such that $\lambda_2(\gamma m_1+\beta m_2)=1$, there exists eigenfunction ω change singe in Ω and

$$\int_\Omega\nabla u.\nabla\omega=\int_\Omega(\gamma m_1+\beta m_2 u)\omega\quad\forall\,\omega\in H^1(\Omega)$$

hence

$$\int_\Omega\nabla u.\nabla\omega-\int_\Omega\beta m_2 u\,\omega=\gamma\int_\Omega m_1\omega\quad\forall\,\omega\in H^1(\Omega) \tag{9}$$

from (9), we obtain that γ is eigenvalue of the operator $-\Delta(.)-\beta m_2(.)$ with weight m_1, since the eigenfunction ω change singe, we conclude that

$$\gamma\geq\alpha_2^+(\beta).$$

Assume by contradiction that $\gamma>\alpha_2^+(\beta)$, then

$$\frac{1}{\gamma}<\frac{1}{\alpha_2^+(\beta)}=\sup_{K\in\Gamma_2}\min_{u\in K}\frac{\int_\Omega m_1 u^2}{\int_\Omega|\nabla u|^2-\beta\int_\Omega m_2 u^2},$$

by the inequality above we deduce that there exists $K_0\in\Gamma_2$ such that

$$\frac{1}{\gamma}<\min_{u\in K_0}\frac{\int_\Omega m_1 u^2}{\int_\Omega|\nabla u|^2-\beta\int_\Omega m_2 u^2},$$

since K_0 is compact, we conclude that, there exists $u_0\in K_0$

$$\frac{1}{\gamma}<\frac{\int_\Omega m_1 u_0^2}{\int_\Omega|\nabla u_0|^2-\beta\int_\Omega m_2 u_0^2},$$

hence

$$1<\min_{u\in K_0}\frac{\int_\Omega(\gamma m_1+\beta m_2)u_0^2}{\int_\Omega|\nabla u_0|^2}$$

it follows that

$$1<\sup_{K\in\Gamma_2}\min_{u\in K_0}\frac{\int_\Omega(\gamma m_1+\beta m_2)u_0^2}{\int_\Omega|\nabla u_0|^2}=\frac{1}{\lambda_2(\gamma m_1+\beta m_2)}=1,$$

which gives a contradiction, thus we have $\gamma=\alpha_2^+(\beta)$.

4 Nonresonance between the first and second eigenvalue

In this section we are interesting to the study of the existence results for the following Neumann problem

$$(\mathcal{P}_2)\begin{cases}-\Delta u=\lambda_2 m_1(x)u+m_2(x)g(u)+h(x) & \text{in } \Omega\\ \frac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega,\end{cases}$$

where $\lambda_2=\lambda_2(m_1)$ is the second eigenvalue of $-\Delta$ with weight m_1 under the Neumann boundary condition.

Lemma 1 *Let $m_1,m_2\in M^+(\Omega)$. Assume that **(A)** is verified, then there exists a unique real $c>0$ such that*

$$\lambda_1(\lambda_2 m_1-c\,m_2)=1 \tag{10}$$

Proof Put $a=\frac{ess\inf_\Omega\lambda_2 m_1}{ess\inf_\Omega m_2}$ and $b=\frac{ess\sup_\Omega\lambda_2 m_1}{ess\inf_\Omega m_2}$ since m is a nonconstant function, then we have $a<b$. So for $t\in[a,b[$ we consider the weight $m_t=\lambda_2 m_1-t\,m_2$ and the corresponding increasing and continuous function:

$$f:[a,b[\to I\!R \qquad t\mapsto\lambda_1(\lambda_2 m_1-t\,m_2) \tag{11}$$

which satisfy $f(a)=0$ and $\lim_{t\to b}f(t)=+\infty$. According to be strict monotony property of λ_1 with weights we observe that f is strictly increasing on $[a',b[$ where $a'=\max\{t:f(t)=0\}$.
Therefore, $f(a')=0\leq f(0)=\lambda_1(\lambda_2 m_1)<\lambda_2(\lambda_2 m_1)=1$, so the conclusion follows from the intermediate values theorem.

Theorem 2 *Let $m_1, m_2 \in M^+(\Omega)$. Assume that the weights m_1 and m_2 satisfy $(\boldsymbol{A})$ and the function g satisfy the following hypotheses*

$$-c \leq \liminf_{s\to\pm\infty} \frac{g(s)}{s} \leq \limsup_{s\to\pm\infty} \frac{g(s)}{s} \leq 0 \qquad (12)$$

$$-c < \limsup_{|s|\to\infty} \frac{2G(s)}{s^2}; \liminf_{s\to+\infty\, or -\infty} \frac{2G(s)}{s^2} < 0 \qquad (13)$$

where c is given in (10), then problem $(\mathcal{P}_2)$ admits at least one solution for any $h \in L^\infty(\Omega)$.

For the proof of theorem 2, we observe that the main trick introduce in [7] can be adapted in our situation. Furthermore the proof needs some technical lemmas, the two next lemmas concern an a-priori estimates on the possible solutions of the following homotopic problem.

$$\begin{cases} -\Delta u = ((1-\mu)\theta + \mu\lambda_2)m_1 u + \mu m_2 g(u) + \mu h & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \qquad (14)$$

where $\mu \in [0,1]$ and $\theta \in]\lambda_1, \lambda_2[$ to be variable and $\lambda_1 = \lambda_1(m_1)$.

Lemma 2 *Suppose that $(\boldsymbol{A})$ and (12) hold and assume that for some $\theta \in]\lambda_1, \lambda_2[$ there exists $\mu_{n,\theta} \in [0,1]$ and $u_{n,\theta}$ be a solution of (14), for all n. Then we have*

1) $(u_{n,\theta})_n$ is a sequence of $L^\infty(\Omega)$ and if $\|u_{n,\theta}\|_\infty \to +\infty$ when $n \to +\infty$, then

$$v_{n,\theta} = \frac{u_{n,\theta}}{\|u_{n,\theta}\|_\infty} \to v_\theta \text{ stongly in } C^1(\Omega), \qquad (15)$$

for some subsequence.
2) Assume that the following hypothesis holds

$$(H) \quad \exists \theta \in]\lambda_1, \lambda_2[/ \limsup_{n\to\infty} \mu_{n,\theta} = 1$$

then one of the following assertions i) or ii) holds, where

i) $v_\theta = \psi$, ψ is a normed ($\|\psi\|_\infty = 1$) eigenfunction associated to $\lambda_2(m_1)$ and

$$\int_\Omega \frac{|g(u_{n,\theta})|}{\|(u_{n,\theta}\|} dx \longrightarrow 0 \text{ when } n \to +\infty. \qquad (16)$$

Furthermore, there exists $\eta_1 > 0$, $\eta_2 > 0$ such that

$$\eta_1 < \frac{\max(u_{n,\theta})}{-\min(u_{n,\theta})} < \eta_2 \text{ for } n \text{ large enough.} \qquad (17)$$

ii) $v_\theta = \pm\varphi$, φ is a normed ($\|\varphi\|_\infty = 1$) eigenfunction associated to $\lambda_1(\lambda_2 m_1 - c m_2) = 1$ and

$$\int_\Omega \frac{|g(u_{n,\theta}) + c u_{n,\theta}|}{\|(u_{n,\theta}\|} dx \longrightarrow 0 \text{ when } n \to +\infty. \qquad (18)$$

Furthermore, $u_{n,\theta}$ not changes sign for n large enough.

*3) If **(H)** is false, then there exists a sequence $(\theta_k)_k \subset]\lambda_1, \lambda_2[$ and a strictly increasing sequence $(n_k)_k \subset \mathbb{N}$ such that*

a) $\lim_{k\to+\infty} \theta_k = \lambda_2$, $\lim_{k\to\infty} \mu_{n_k,\theta_k} = 1$ and $\lim_{k\to\infty} \|w_k\| = +\infty$ where $w_k = u_{n_k,\theta_k}$.

b) $\frac{w_k}{\|w_k\|} \longrightarrow \pm\varphi$ strongly in $C^1(\overline{\Omega})$ and

$$\int_\Omega \frac{|g(w_k) + c w_k|}{\|w_k\|} dx \longrightarrow 0 \text{ when } n \to +\infty.$$

Proof: 1) From the Anane's L^∞-estimation [8] and the Tolksdorf's-regularity [9] we can see that $(u_{n,\theta})_n \subset \mathcal{C}^{1,\alpha}(\overline{\Omega})$, since the embedding $\mathcal{C}^{1,\alpha}(\overline{\Omega}) \hookrightarrow L^\infty(\Omega)$ is continuous for some $\alpha \in]0,1[$ independent on n, furthermore $v_{n,\theta} = \frac{u_{n,\theta}}{\|u_{n,\theta}\|_\infty}$ remains a bounded sequence in $\mathcal{C}^{1,\alpha}(\overline{\Omega})$.
By using the following compact embedding $\mathcal{C}^{1,\alpha}(\overline{\Omega}) \hookrightarrow\hookrightarrow \mathcal{C}^1(\overline{\Omega})$, then there exists a subsequence still denoted $(v_{n,\theta})_n$ such that

$$v_{n,\theta} \to v_\theta \text{ stongly in } \mathcal{C}^1(\overline{\Omega}) \text{ and } \|v_\theta\|_\infty = 1. \qquad (19)$$

2) According to the function g satisfy the hypothesis (12), we deduce that for all $s \in \mathbb{R}$, we can write

$$g(s) = q(s)s + r(s) \qquad (20)$$

where $-c \leq q(s) \leq 0$ and $\frac{r(s)}{s} \longrightarrow 0$ uniformly, when $|s| \to +\infty$. Since $u_{n,\theta}$ is a solution of $(\mathcal{P}_{2,\theta,\mu_n})$, we get

$$\begin{aligned} \int_\Omega \nabla u_{n,\theta} \nabla w dx = \quad & \int_\Omega [(1-\mu_n)\theta + \mu_n\lambda_2)m_1 u_{n,\theta} \\ & + \mu_n m_2 g(u_{n,\theta}) + \mu_n h] w dx \end{aligned} \qquad (21)$$

for all $w \in H^1(\Omega)$.
On the other hand, since $(u_{n,\theta})_n \subset L^\infty(\Omega)$ and q is a continuous function, it follows that $q(u_n)$ is bonded in $L^\infty(\Omega)$, then for a subsequence we get

$$q(u_{n,\theta}) \rightharpoonup q_\theta \text{ in } L^\infty(\Omega) \text{ weak} - *,$$

and

$$\frac{|r(u_{n,\theta})|}{\|u_{n,\theta}\|_\infty} \to 0 \text{ strongly in } L^\infty(\Omega)$$

where $-c \leq q_\theta(x) \leq 0$ a.e. in Ω.
Dividing by $\|u_{n,\theta}\|_\infty$ and passing to the limit as $n \to \infty$ in (20), we get

$$\int_\Omega \nabla v_\theta \nabla w dx = \int_\Omega (\lambda_2 m_1 + q_\theta m_2) v_\theta w dx \ \forall w \in H^1(\Omega). \qquad (22)$$

Since $v_\theta \neq 0$, then 1 is an eigenvalue of Laplacain with weight $m_{q_\theta} = \lambda_2 m_1 + q_\theta m_2$.
By using the monotony property of λ_2 with respect to the weight, we obtain

$$\lambda_2(m_{q_\theta}) \geq \lambda_2(\lambda_2 m_1) = 1$$

hence $\lambda_2(m_{q_\theta}) = 1$ or $\lambda_2(m_{q_\theta}) > 1$.
First case: $\lambda_2(m_{q_\theta}) = 1$.
So, we have

$$q_\theta = 0 \quad \text{and} \quad v_\theta = \psi.$$

Moreover, let us denotes by F_{λ_2} be the eigenspace associated to $\lambda_2 = \lambda_2(m_1)$, since F_{λ_2} is a vector space of finite dimension then, we can take

$$\eta_1 < \min_{v \in F_{\lambda_2} \cap S} \frac{\max(v)}{-\min(v)} \quad \text{and} \quad \eta_2 > \max_{v \in F_{\lambda_2} \cap S} \frac{\max(v)}{-\min(v)}$$

where $S = \left\{v \in F_{\lambda_2} / \|v\|_\infty = 1\right\}$.

It is clear to see that $\frac{\max(v_{n,\theta})}{-\min(v_{n,\theta})} \to \frac{\max(\psi)}{-\min(\psi)}$ when $n \to +\infty$.

It follows that for n large enough

$$\eta_1 < \frac{\max(v_{n,\theta})}{-\min(v_{n,\theta})} < \eta_2.$$

According to (20), we get

$$\begin{aligned} \int_\Omega \frac{|m_2(x) g(u_{n,\theta})|}{\|u_{n,\theta}\|_\infty} dx &\leq \|m_2\|_\infty \int_\Omega -q(u_{n,\theta})|v_{n,\theta}| \\ &+ \|m_2\|_\infty \int_\Omega \frac{|r(u_{n\theta})|}{\|u_{n\theta}\|_\infty} dx \end{aligned}$$

by passage to the limit in the above inequality, we find (16).

Second case: $\lambda_2(m_{q_\theta}) > 1$.

So, we get $\lambda_1(m_{q_\theta}) = 1$ and by using the strict monotony property of λ_1, we can see that

$$q_\theta = -c \quad \text{and} \quad v_\theta = \varphi.$$

The rest of the proof follows directly as in first case.

3) We take $(\theta_k)_k \subset]\lambda_1, \lambda_2[$ such that $\lim_{k \to +\infty} \theta_k = \lambda_2$. Let us denotes by $\overline{\mu}_k = \lim_{n \to +\infty} \mu_{k,\theta_k}$, as in 2) it is clear to see that

$$\begin{cases} -\Delta u = [((1-\overline{\mu}_k)\theta_k + \overline{\mu}_k \lambda_2) m_1 + \overline{\mu}_k q_{\theta_k} m_2] v_{\theta_k} & \text{in } \Omega \\ \frac{\partial v_{\theta_k}}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

we recall that $0 \leq \overline{\mu}_k < 1$, $-c \leq q_{\theta_k} \leq 0$ and $q(u_{n,\theta_k}) \longrightarrow^* q_{\theta_k}$ in $L^\infty(\Omega)$ when $n \to +\infty$, where $g(s) = q(s)s + r(s)$, $-c \leq q(s) \leq 0$ and $\frac{r(s)}{r} \to 0$ when $|s| \to +\infty$.

Let us consider the following weight

$$\overline{M}_k(x) = ((1-\overline{\mu}_k)\theta_k + \overline{\mu}_k \lambda_2) m_1(x) + \overline{\mu}_k q_{\theta_k} m_2(x)$$

we have

$$\overline{M}_k(x) \leq ((1-\overline{\mu}_k)\theta_k + \overline{\mu}_k \lambda_2) m_1(x)$$

because $q_{\theta_k} \leq 0$.

Then

$$\begin{aligned} \lambda_2(\overline{M}_k(x)) &\geq \lambda_2(((1-\overline{\mu}_k)\theta_k + \overline{\mu}_k \lambda_2) m_1(x)) \\ &\geq \frac{\lambda_2}{(1-\overline{\mu}_k)\theta_k + \overline{\mu}_k \lambda_2} > 1. \end{aligned}$$

Thus it is clear to see that

$$\lambda_2(\overline{M}_k(x)) = 1. \tag{23}$$

Let $\overline{\mu}_k \to \overline{\mu}$ and $q(u_{n,\theta_k}) \longrightarrow^* \overline{q}_0$ in $L^\infty(\Omega)$ when $k \to +\infty$, where $-c \leq \overline{q}_0 \leq 0$. Then by letting k tends to infinity in (23), it follows that

$$\lambda_1(\lambda_2 m_1 + \overline{\mu}\,\overline{q}_0 m_2) = 1. \tag{24}$$

In view of lemma 1, we get

$$\overline{\mu} = 1,\ \overline{q}_0 = -c \quad \text{and } u_{\theta_k} \to \pm\varphi \text{ strongly in } C^1(\Omega) \text{ when } k \to +\infty.$$

On the other hand, using (20), we have

$$\begin{aligned} \int_\Omega \frac{|g(w_k) + c w_k|}{\|w_k\|_\infty} dx &= \int_\Omega \frac{|q(w_k) w_k + c w_k + r(w_k)|}{\|w_k\|_\infty} dx \\ &\leq \int_\Omega |q(w_k) + c| \frac{w_k|}{\|w_k\|_\infty} dx \\ &+ \int_\Omega \frac{|r(w_k)|}{\|w_k\|_\infty}. \end{aligned}$$

Since $\frac{r(s)}{s} \to 0$ when $|s| \to \infty$, then for all $\varepsilon > 0$ there exists $n_{\varepsilon,k}$ such that for all $n \geq n_{\varepsilon,k}$

$$\int_\Omega \frac{|g(w_k) + c w_k|}{\|w_k\|_\infty} dx \leq \int_\Omega |\overline{q}_{\theta_k} + c||v_{\theta_k}| dx + \varepsilon.$$

If we take $\varepsilon = \frac{1}{k}, n = n_k = n_{\varepsilon,k}$ and we replace in the last formula, then we can see that the second member of this last inequality goes to 0 when m goes to $+\infty$. Finally, we conclude by the fact that n_k will be adjusted such that $\|u_{n_k,\theta_k}\|_\infty > k$ and $\|v_{n_k,\theta_k} - v_{\theta_k}\|_{C^1(\overline{\Omega})} \leq \frac{1}{k}$.

Lemma 3 *Let us consider the assumptions and notations of lemma 2. We take $a \in \Omega$ and $\eta > 0$ such that $B(a,\eta) \subset \Omega$.*

*1) Assume that the hypothesis (**H**) holds, so, if $\|u_{n,\theta}\|_\infty \to +\infty$ when $n \to +\infty$ then $\limsup_{n \to +\infty} \mu_{n,\theta} = 1$ and*

i) If $v_\theta = \psi$ then

$$\lim_{n\to\infty} \int_0^1 \frac{|g(u_{n,\theta}(\sigma_x(t)))||\nabla u_{n,\theta}(\sigma_x(t))||x-a|}{\|u_{n,\theta}\|_\infty} dt = 0 \quad a.e.\ x \in \partial B(a,\eta) \tag{25}$$

where $\sigma_x(t) = a + t(x-a)$.

ii) If $v_\theta = \pm\varphi$ then

$$\lim_{n\to\infty} \int_0^1 \frac{|g(u_{n,\theta}(\sigma_x(t))) + c u_{n,\theta}(\sigma_x(t)||\nabla u_{n,\theta}(\sigma_x(t))||x-a|}{\|u_{n,\theta}\|_\infty} dt = 0 \quad a.e.\ x \in \partial B(a,\eta). \tag{26}$$

*2) Assume that (**H**) is false then we have*

$$\lim_{n\to\infty} \int_0^1 \frac{|g(w_k(\sigma_x(t))) + c w_k(\sigma_x(t)||\nabla w_k(\sigma_x(t))||x-a|}{\|w_k\|_\infty} dt = 0 \quad a.e.\ x \in \partial B(a,\eta). \tag{27}$$

Proof: We only show the relation (25) since the proof of the (26) and (27) one proceeds in the same way. Firstly we have

$$\int_{B(a,\eta)} \frac{|g(u_{n,\theta})|}{\|u_{n,\theta}\|_\infty} dx \leq \int_\Omega \frac{|g(u_{n,\theta})|}{\|u_{n,\theta}\|_\infty} dx.$$

According to (16), we obtain

$$\lim_{n\to\infty} \int_{B(a,\eta)} \frac{|g(u_{n,\theta})|}{\|u_{n,\theta}\|_\infty} dx = 0.$$

Which gives by passing in spherical coordinates

$$\lim_{n\to\infty}\int_{[0,\pi]^n}\int_0^{2\pi}\int_0^{\eta} t^{N-1}\frac{|g(u_{n,\theta}(a+t\omega))|}{\| u_{n,\theta}\|_\infty} \prod_{j=1}^{N-2}(\sin\theta_j)^{N-1-j}d\theta_j d\theta_{N-1}dt = 0 \tag{28}$$

where, $\omega = \frac{x-a}{\eta} \in \partial B(0,1)$.
The above equality imply

$$\frac{|g(u_{n,\theta}(\sigma_x(\tau)))|}{\|u_{n,\theta}\|_\infty} \to 0 \quad \text{when } n\to\infty$$
$$\text{a.e. } x\in\partial B(a,\eta) \text{ and a.e. } \tau\in[0,1].$$

By using (12) we can see that g satisfy the following growth condition:

$|g(s)| \le a|s| + b$, for some positive reals a, b and $\forall s \in I\!R$

hence $\left(\frac{|g(u_{n,\theta}(\sigma_x(.)))|}{\|u_{n,\theta}\|_\infty}\right)_n$ and $\left(\frac{|\nabla(u_{n,\theta}(\sigma_x(.)))|}{\|u_{n,\theta}\|_\infty}\right)_n$ are bounded in $L^\infty([0,1])$.
By using the Lebesgue dominated convergence theorem, we conclude this proof.

Lemma 4 *1) If G satisfy $-c < \limsup_{s\to+\infty}\frac{2G(s)}{s^2} \le 0$. then for all $r\in]0,1[$, there exists $\rho_r > -c(1-r^2)$ and a sequence of positive real numbers (S_n) such that*

$$\lim_{n\to+\infty} S_n = +\infty \text{ and } \lim_{n\to+\infty}\frac{2G(S_n)-2G(rS_n)}{S_n^2} = \rho_r.$$

2) If G satisfy $-c < \limsup_{s\to+\infty}\frac{2G(s)}{s^2} \le 0$ and $-c < \liminf_{s\to+\infty}\frac{2G(s)}{s^2} < 0$, then for all $r\in]0,1[$, there exists $\rho_r > -c(1-r^2)$ and a sequence of positive real numbers $(S_n^{'})$ such that

$$\lim_{n\to+\infty} S_n^{'} = +\infty, \quad \lim_{n\to+\infty}\frac{2G(S_n^{'})}{S_n^{'2}} = \rho$$
$$\text{and } \lim_{n\to+\infty}\frac{2G(S_n^{'})-2G(rS_n^{'})}{S_n^{'2}} = \rho_r.$$

The same conclusion is obtained when we replace $+\infty$ par$-\infty$, in this case we should note $\rho^{'}$ and $\rho_r^{'}$ in place of ρ and ρ_r respectively.

Proof: We pose $L_1 = \liminf_{s\to+\infty}\frac{2G(s)}{s^2}$ and $L_2 = \limsup_{s\to+\infty}\frac{2G(s)}{s^2}$.

We distinguish the following two cases:
Case1: If $L_1 = L_2$, it is enough to take $\rho = L_2$ and $\rho_r = (1-r^2)L_2$.
Case2: If $L_1 < L_2$, then we choose $\rho\in]L_1, L_2[$ neighbor of L_2 in such a way that:

$$\rho - r^2L_2 > -c(1-r^2).$$

According to the definition of L_1 and L_2, we conclude that, there existence a sequence (S_n) such that $\lim_{n\to+\infty} S_n = +\infty$ and $\frac{2G(S_n)-2G(rS_n)}{S_n^2} = \rho$. We put

$$\liminf_{n\to+\infty}\frac{2G(S_n)-2G(rS_n)}{S_n^2} = \rho - r^2\limsup_{n\to+\infty}\frac{2G(rS_n)}{rS_n^2}.$$

Since $\limsup_{n\to+\infty}\frac{2G(rS_n)}{(rS_n)^2} \le L_2$, then we have:

$$\rho_r \ge \rho - r^2L_2 > -c(1-r^2).$$

5 Proof of Theorem 2

Our purpose now, consists in building in $C(\overline{\Omega})$ an open bounded set $\mathcal{O}$ such that, there exist $\theta\in]\lambda_1,\lambda_2[$ such that, no solution of (14) with $\mu\in[0,1[$ occurs on the boundary $\partial\mathcal{O}$. Homotopy invariance of the degree then yields the conclusion. The set $\mathcal{O}$ will have the following form

$$\mathcal{O} = \mathcal{O}_{S,T} = \{u\in\mathcal{C}(\overline{\Omega});\quad T < u < S\},$$

where, S and T satisfy $T < 0 < S$.
Firstly, according to, (13) we will assume the following hypothesis holds

$$-c < \limsup_{|s|\to\infty}\frac{2G(s)}{s^2} \text{ and } \liminf_{s\to+\infty}\frac{2G(s)}{s^2} < 0 \quad (13_+).$$

An analogous proof will be adapted to the hypothesis (13_-) where (13) represents (13_+) or (13_-). Assume by contradiction, that for all $\theta\in]\lambda_1,\lambda_2[$ and for all S, T $(T < 0 < S)$, there exists $\mu = \mu_{\theta,S,T}\in[0,1[$ and $u = u_{\theta,S,T}\in\partial\mathcal{O}$ such that u is a solution of (14) which gives

$$\max(u) = S \text{ or } \min(u) = T. \tag{29}$$

According to (13_+), then owing to lemma 4, there exists two sequence (T_n) and (S_n) such that

$$\lim_{n\to+\infty} T_n = -\infty,\ \lim_{n\to+\infty}\frac{2G(T_n)-2G(rT_n)}{T_n^2} = \rho_r^{\prime} \tag{30}$$

$$\lim_{n\to+\infty} S_n = +\infty,\ \lim_{n\to+\infty}\frac{2G(S_n)}{S_n^2},\ \lim_{n\to+\infty}\frac{2G(S_n)-2G(rS_n)}{S_n^2} = \rho_r \tag{31}$$

where $r = \frac{\min\varphi}{\max\varphi}$ and ρ, ρ_r and $\rho_r^{\prime}$ provides from lemma 4.
So, we remark that, without loss of generality we can assume that

$$\frac{S_n}{-T_n} \le \eta_1. \tag{32}$$

We use the following notations:

$$T = T_n,\ S = S_n,\ \mu_{n,\theta} = \mu_{\theta,T,S} \text{ and } u_{n,\theta} = u_{\theta,T,S}.$$

According to (29) and (31), we obtain

$$\|u_{n,\theta}\|_\infty \to +\infty \text{ when } n\to+\infty.$$

We will distinguish two cases.
First case: We assume that the hypothesis **(H)** is satisfied (c.f. lemma 2).
According to lemma 2 we get

$$v_{n,\theta} = \frac{u_{n,\theta}}{\|u_{n,\theta}\|_\infty} \to v_\theta \text{ where } v_\theta = \psi \text{ or } v_\theta = \pm\varphi.$$

i) If $v_\theta = \psi$, we assert that for every n large enough we have
$$\max(u_{n,\theta}) = S_n.$$
Indeed, if not we have
$$\max(u_{n,\theta}) < S_n \text{ and } \min(u_{n,\theta}) = T_n$$
thus,
$$\frac{\max(u_{n,\theta})}{-\min(u_{n,\theta})} < \frac{S_n}{-S_n} \leq \eta_1.$$
which gives a contradiction from the definition of η_1 given in lemma 2. On the other hand, we have $u_{n,\theta}$ changes sign on Ω, so there exists $x_n \in \overline{\Omega}$, $y_n \in \Omega$ such that
$$u_{n,\theta}(x_n) = S_n \text{ and } u_{n,\theta}(y_n) = 0,$$
we write
$$\begin{aligned}\frac{2G(S_n)}{S_n^2} &= \frac{2G(u_{n,\theta}(x_n)) - 2G(u_{n,\theta}(x_n))}{\|u_{n,\theta}\|_\infty^2 \max(v_{n,\theta})^2} \\ &= \frac{2}{\max(v_{n,\theta})^2} \int_{\mathcal{C}_n} \frac{d(G \circ u_{n,\theta})}{\|u_{n,\theta}\|_\infty^2}\end{aligned}$$
where
$$d(G \circ u_{n,\theta})(\mathcal{C}_n) = g(u_{n,\theta}(\mathcal{C}_n))\nabla u_{n,\theta}(\mathcal{C}_n).\mathcal{C}_n', \text{ a.e.}$$
and $\mathcal{C}_n$ is a $\mathcal{C}^1$ with morsels line which connects extremity x_n and y_n.
According to lemma 3, we have
$$\lim_{n\to+\infty} \int_{\mathcal{C}_n} \frac{d(G \circ u_{n,\theta})}{\|u_{n,\theta}\|_\infty^2}.$$
Since $\max(v_{n,\theta}) \to \max(\psi)$ when $n \to +\infty$, we deduce that
$$\rho = \lim_{n\to+\infty} \frac{2G(S_n)}{S_n^2} = 0,$$
which is a contradiction since $\rho \in]-c, 0[$.
ii) If $v_\theta = \pm\varphi$: then, for n large enough $u_{n,\theta}$ not changes sign on Ω.
so,
$$\max(u_{n,\theta}) = S_n \text{ or } \min(u_{n,\theta}) = T_n.$$
Assume that $\max(u_{n,\theta}) = S_n$ (the same gait will be used for the case $\min(u_{n,\theta}) = T_n$), so it is clear to see that $v_{n,\theta} = +\varphi$ and $\min(u_{n,\theta}) > 0$ for n large enough.
We put
$$\overline{g}(s) = g(s) + cs, \; \overline{G}(s) = \int_0^s \overline{g}(s)ds.$$
Let x_n, $y_n \in \overline{\Omega}$ such that $u_{n,\theta}(x_n) = S_n$ and $u_{n,\theta}(y_n) = \min(u_{n,\theta})$.
We write
$$\begin{aligned}\frac{\overline{G}(S_n) - \overline{G}(r_n S_n)}{S_n^2} &= \frac{2\overline{G}(u_{n,\theta}(x_n)) - 2\overline{G}(u_{n,\theta}(x_n))}{\|u_{n,\theta}\|_\infty^2} \\ &= \int_{\mathcal{C}_n} \frac{d(\overline{G} \circ u_{n,\theta})}{\|u_{n,\theta}\|_\infty^2}\end{aligned}$$
where $r_n = \frac{\min(u_{n,\theta})}{\max(u_{n,\theta})} \to r = \frac{\min(\varphi)}{\max(\varphi)}$.
Using the Lemma 3, we deduce that
$$\lim_{n\to+\infty} \int_{\mathcal{C}_n} \frac{d(\overline{G} \circ u_{n,\theta})}{\|u_{n,\theta}\|_\infty^2} = 0. \tag{33}$$
On the other hand, it is easy to verify that
$$0 = \lim_{n\to+\infty} \frac{\overline{G}(S_n) - \overline{G}(r_n S_n)}{S_n^2} = \frac{\rho_r + c(1-r^2)}{2} > 0$$
which gives a contradiction.
Second case: Assume that the hypothesis **(H)** is not verified, so for all $\theta \in]\lambda_1, \lambda_2[$ such that $\limsup_{n\to+\infty} \mu_{n,\theta} < 1$.
We take a sequence (θ_k) such that
$$\lim_{k\to+\infty} \theta_k = \lambda_2,$$
and we consider the subsequences
$$(T_{n_k})_k, (S_{n_k})_k \text{ and } w_k = u_{n_k,\theta_k}.$$
Similarly, as in the second point of the previous case we obtain a contradiction. This completes the proof of theorem 2.

6 Nonresonance between the first two Eigencurves

In this section we will prove an existence result for problem $(\mathcal{P})$. We need more restrictive hypotheses on the nonlinearities g and G.

$$(\mathbf{A}_g) \qquad \beta_1 \leq \liminf_{s\to\pm\infty} \frac{g(s)}{s} \leq \limsup_{s\to\pm\infty} \frac{g(s)}{s} \leq \beta_2$$

$$(\mathbf{A}_G^{\pm}) \quad \beta_1 < \limsup_{|s|\to\infty} \frac{2G(s)}{s^2}; \; \liminf_{s\to+\infty\, or\, -\infty} \frac{2G(s)}{s^2} < \beta_2$$

where $(\beta_1, \beta_2) \in I\!R^2$ with $\beta_2 - \beta_1 = c$ and c is given in (10).

Theorem 3 *Let m_1, $m_2 \in M^+(\Omega)$. Assume that $(\boldsymbol{A})$, $(\boldsymbol{A}_g)$ and $(\boldsymbol{A}_G^{\pm})$ holds. Moreover, if $(\alpha, \beta_1) \in \mathcal{C}_1$ and $(\alpha, \beta_2) \in \mathcal{C}_2$, where $\alpha \geq \lambda_2(m_1)$ or $\alpha \leq \lambda\_2(m_1)$. Then the problem $(\mathcal{P})$ has at least one nontrivial weak solution $u \in H^1(\Omega)$ for any given $h \in L^\infty(\Omega)$.*

Proof: The problem $(\mathcal{P})$ can be written in the following equivalent form
$$(\mathcal{P}_e) \quad \begin{cases} -\Delta u = \widetilde{m}|u|^{p-2}u + m_2\widetilde{g}(u) + h & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$
where
$$\widetilde{g}(s) = g(s) - \beta_2 s,$$
and
$$\widetilde{m} = \alpha\, m_1 + \beta_2\, m_2.$$
Since $(\alpha, \beta_2) \in \mathcal{C}_2$, then 1 is the second eigenvalue of laplacian operator with weight $\widetilde{m}$ relating to Neumann boundary conditions. In view of theorem 2, there exists at least one weak solution $u \in H^1(\Omega)$ of

the problem $(\mathcal{P}_e)$ for all $h \in L^\infty(\Omega)$ if the function $\widetilde{g}$ and his potential $\widetilde{G}$ satisfy the two conditions (12) and (13). Indeed, in view of $(\mathbf{A}_g)$

$$\beta_1 - \beta_2 \leq \lim_{|s|\to\infty}\inf\left(\frac{g(s)}{s} - \beta_2\right) \leq 0$$

thus

$$-c \leq \lim_{|s|\to\infty}\inf\left(\frac{\widetilde{g}(s)}{s}\right) \leq 0.$$

Consequently, $\widetilde{g}$ satisfy (12).
On the other hand, using $(\mathbf{A}_G^{\pm})$, we have

$$\beta_1 - \beta_2 < \limsup_{|s|\to\infty}\left(\frac{2G(s)}{s^2} - \beta_2\right); \liminf_{s\to+\infty\, or -\infty}\left(\frac{2G(s)}{s^2} - \beta_2\right) < 0$$

hence

$$-c < \limsup_{|s|\to\infty}\left(\frac{2\widetilde{G}(s)}{s^2}\right); \liminf_{s\to+\infty\, or -\infty}\left(\frac{2\widetilde{G}(s)}{s^2}\right) < 0$$

which means that $\widetilde{G}$ satisfy (13).
Finally, since the problem $(\mathcal{P}_e)$ is a equivalent to the problem $(\mathcal{P})$. Then the problem $(\mathcal{P})$ has at least one nontrivial weak solution $u \in H^1(\Omega)$ for any given $h \in L^\infty(\Omega)$.

References

1. D. Del Santo et P. Omari, Nonresonance conditions on the potential for a semilinear elliptic problem. J. Diff. Equ. pp. 120-138, (1994).

2. J.-P. Gossez and P. Omari, A necessary and sufficient condition of nonresonance for a semilinear Neumann problem, Proc. Amer. Math. Soc. Vol. 114, No. 2 (Feb., 1992), pp. 433-442.

3. A. Dakkak, A. Anane, Nonresonance Conditions on the Potential for a Neumann Problem, proceedings of the international conference held in Fez, partial differential equations pp. 85-102.

4. A. Dakkak, Etude sur le spectre et la résonance pour des problèmes elliptiques de Neumann, Thèse de 3eme cycle, Faculté des sciences, Oujda, Maroc, (1995).

5. A. Dakkak , M. Hadda, Eigencurves of the p-Laplacian with weights and their asymptotic behavior EJDE, Vol 2007(2007), No. 35, pp. 1–7.

6. A. Dakkak and M. Moussaoui On the second eigencurve for the p-laplacian operator with weight, Bol. Soc. Paran. Mat., (3s.) v. 35 1 (2017): 281289.

7. J.-P. Gossez and P. Omari, On a semilinear elliptic Neumann problem with asymmetric nonlinearities. Transactions of Amer. Math. Soc. Volume 347, Number 7, pp. 2553-2562, July 1995.

8. A. Anane, Etude des valeurs propres et de la résonance pour l'operateur p-Laplacien , these de Doctorat, Universite Libre de Bruxelles, (1988).

9. P.Tolksdorf, Regularity for more general class of quasilinear elliptic equation J.Diff. Eq. 8, (1983), pp. 773–817.

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Existence Results for Nonlinear Anisotropic Elliptic Equation

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ABSTRACT

In this work, we shall be concerned with the existence of weak solutions of anisotropic elliptic operators $Au+\sum_{i=1}^{N} g_i(x,u,\nabla u)+\sum_{i=1}^{N} H_i(x,\nabla u) = f-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}k_i$, where the right hand side f belongs to $L^{p'_\infty}(\Omega)$ and k_i belongs to $L^{p'_i}(\Omega)$ for $i=1,\dots,N$ and A is a Leray-Lions operator. The critical growth condition on g_i is the respect to ∇u and no growth condition with respect to u, while the function H_i grows as $|\nabla u|^{p_i-1}$.

1 Introduction

In this paper we study the existence of weak solutions to anisotropic elliptic equations with homogeneous Dirichlet boundary conditions of the type

$$\begin{cases} Au+\sum_{i=1}^{N} g_i(x,u,\nabla u)+\sum_{i=1}^{N} H_i(x,\nabla u) \\ \qquad\qquad = f-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}k_i & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N\geq 2)$ with Lipschitz continuous boundary. The operator $Au = -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,u,\nabla u)$ is a Leray-Lions operator such that the functions a_i, g_i and H_i are the Carathodory functions satisfying the following conditions for all $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N, \xi'\in\mathbb{R}^N$ and a.e. in Ω:

$$\sum_{i=1}^{N} a_i(x,s,\xi)\xi_i \geq \lambda\sum_{i=1}^{N}|\xi_i|^{p_i},$$

$$|a_i(x,s,\xi)| \leq \gamma[|s|^{\frac{p_\infty}{p'_i}}+|\xi_i|^{p_i-1}],$$

$$(a_i(x,s,\xi)-a_i(x,s,\xi'))(\xi_i-\xi'_i)>0 \quad \text{for } \xi_i\neq\xi'_i,$$

$$g_i(x,s,\xi)s\geq 0,$$

$$|g_i(x,s,\xi)|\leq L(|s|)|\xi_i|^{p_i}\ \forall i=1,\dots,N,$$

$$|H_i(x,\xi)|\leq b_i|\xi_i|^{p_i-1},$$

where λ,γ,b_i are some positive constants, for $i=1,\dots,N$ and $L:\mathbb{R}^+\to\mathbb{R}^+$is a continuous and non decreasing function. The right hand side f and k_i for $i=1,\dots,N$ are functions belonging to $L^{p'_\infty}(\Omega)$ and $L^{p'_i}(\Omega)$ where $p'_i=\frac{p_i}{p_i-1}, p'_\infty=\frac{p_\infty}{p_\infty-1}$ with $p_\infty=\max\{\overline{p}*,p_+\}$ where $p_+=\max\{p_1,\dots,p_N\}$, $\overline{p}=\frac{1}{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}}$ and $\overline{p}*=\frac{N\overline{p}}{N-\overline{p}}$.

Since the growth and the coercivity conditions of each a_i for all $i=1,\dots,N$ depend on p_i, we have need to use the anisotropic Sobolev space. We mention some papers on anisotropic Sobolev spaces (see e.g.[1]-[5]).

If $p_i=p$ for all $i=1,\dots,N$, we refer some works such as by Guibé in [6], by Monetti and Randazzo in [7] and by Y. Akdim, A. Benkirane and M. El Moumni in [8].

In [3], L.Boccardo, T. Gallouet and P. Marcellini have studied the problem (1) when $a_i(x,u,\nabla u) = \left|\frac{\partial u}{\partial x_i}\right|^{p_i-1}\frac{\partial u}{\partial x_i}$, $g_i=0$, $H_i=0$, $k_i=0$ and $f=\mu$ is Radon's measure. In [5], F. Li has proved the existence and regularity of weak solutions of the problem (1) with $g_i=0$, $H_i=0$, $k_i=0$ for all $i=1,\dots,N$ and f belongs to $L^m(\Omega)$ with $m>1$. In [9], R. Di Nardo and F. Feo have proved the existence of weak solution of the problem (1) when $g_i=0$ for all $i=1,\dots,N$. In [10], we have proved the existence and uniqueness of weak solution of the problem (1) but when $Au=-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,\nabla u)$ (a_i depending only on x and ∇u).

In this work, we prove the existence of weak solu-

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tions of the problem (1), based on techniques related to that of Di castro in [11] and to the recent work's Di Nardo and F. Feo in [9].

2 Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz continuous boundary and let $1 < p_1, ..., p_N < \infty$ be N real numbers, $p^+ = \max\{p_1, ..., p_N\}, p^- = \min\{p_1, ..., p_N\}$ and $\overrightarrow{p} = (p_1, ..., p_N)$. The anisotropic Sobolev space (see [12])

$$W^{1,\overrightarrow{p}}(\Omega) = \left\{u \in W^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), i = 1, 2, ..., N\right\}$$

is a Banach space with respect to norm

$$\|u\|_{W^{1,\overrightarrow{p}}(\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega)}.$$

The space $W_0^{1,\overrightarrow{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to this norm. We recall a Poincarr-type inequality. Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ then there exists a constant C_p such that (see[13])

$$\|u\|_{L^{p_i}(\Omega)} \leq C_p \|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega)} \text{ for } i = 1, ..., N. \tag{1}$$

Moreover a Sobolev-type inequality holds. Let us denote by $\overline{p}$ the harmonic mean of these numbers, i.e. $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$. Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, then there exists (see [12]) a constant C_s such that

$$\|u\|_{L^q(\Omega)} \leq C_s \prod_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}. \tag{2}$$

Where $q = \overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}$ if $\overline{p} < N$ or $q \in [1, +\infty[$ if $\overline{p} \geq N$. We recall the arithmetic mean: Let $a_1, ..., a_N$ be positive numbers, it holds

$$\prod_{i=1}^{N} a_i^{\frac{1}{N}} \leq \frac{1}{N}\sum_{i=1}^{N} a_i. \tag{3}$$

Which implies by (2)

$$\|u\|_{L^q(\Omega)} \leq \frac{C_s}{N}\sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{L^{p_i}(\Omega)}. \tag{4}$$

When $\overline{p} < N$ hold, inequality (4) implies the continuous embedding of the space $W_0^{1,\overrightarrow{p}}(\Omega)$ into $L^q(\Omega)$ for every $q \in [1, \overline{p}^*]$. On the other hand the continuity of the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p^+}(\Omega)$ relies on inequality (1). Let us put $p_\infty := \max\{\overline{p}*, p^+\}$

Proposition 1 *For $q \in [1, p_\infty]$ there is a continuous embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^q(\Omega)$. If $q < p_\infty$ the embedding is compact.*

3 Assumptions and Definition

We consider the following class of nonlinear anisotropic elliptic homogenous Dirichlet problems

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + \sum_{i=1}^{N} g_i(x, u, \nabla u) + \\ \sum_{i=1}^{N} H_i(x, \nabla u) = f - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} k_i \text{ in } \Omega, \\ \quad u = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz continuous boundary $\partial\Omega$, $1 < p_1, ..., p_N < \infty$. We assume that a_i: $\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, g_i: $\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and H_i: $\Omega \times \mathbb{R}^N \to \mathbb{R}$ are Carathodory functions such that for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N, \xi' \in \mathbb{R}^N$ and a. e. in Ω :

$$\sum_{i=1}^{N} a_i(x, s, \xi)\xi_i \geq \lambda \sum_{i=1}^{N} |\xi_i|^{p_i}, \tag{5}$$

$$|a_i(x, s, \xi)| \leq \gamma[|s|^{\frac{p_\infty}{p_i'}} + |\xi_i|^{p_i - 1}], \tag{6}$$

$$(a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0 \quad \text{for } \xi_i \neq \xi_i', \tag{7}$$

$$g_i(x, s, \xi)s \geq 0, \tag{8}$$

$$|g_i(x, s, \xi)| \leq L(|s|)|\xi_i|^{p_i} \; \forall i = 1, ..., N, \tag{9}$$

$$|H_i(x, \xi)| \leq b_i |\xi_i|^{p_i - 1}, \tag{10}$$

where λ, γ, b_i are some positive constants, for $i = 1, ..., N$ and $L : \mathbb{R}^+ \to \mathbb{R}^+$is a continuous and non decreasing function. Moreover, we suppose that

$$f \in L^{p_\infty'}(\Omega), \tag{11}$$

$$k_i \in L^{p_i'}(\Omega) \text{ for } \; i = 1, ..., N. \tag{12}$$

Definition 1 *A function $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ is a weak solution of the problem (1) if $\sum_{i=1}^{N} g_i(x, u, \nabla u) \in L^1(\Omega)$ and u satisfies*

$$\sum_{i=1}^{N} \int_\Omega \left[a_i(x, u, \nabla u)\frac{\partial \varphi}{\partial x_i} + g_i(x, u, \nabla u)\varphi + H_i(x, \nabla u)\varphi\right]$$

$$= \int_\Omega \left[f\varphi + \sum_{i=1}^{N} k_i \frac{\partial \varphi}{\partial x_i}\right]$$

$\forall\, \varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega).$

4 Main results

In this section we prove the existence of at least a weak solution of the problem (1). We consider the approximate problems.

4.1 Approximate problems and a priori estimates

Let

$$g_i^n(x,u,\nabla u) = \frac{g_i(x,u,\nabla u)}{1+\frac{1}{n}|g_i(x,u,\nabla u)|}$$

and

$$H_i^n(x,\nabla u) = \frac{H_i(x,\nabla u)}{1+\frac{1}{n}|H_i(x,\nabla u)|}.$$

By Leray-Lions (see e.g. [14]), there exists at least a weak solution $u_n \in W_0^{1,\vec{p}}(\Omega)$ of the following approximate problem

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x,u_n,\nabla u_n) + \sum_{i=1}^N g_i^n(x,u_n,\nabla u_n) \\ \qquad + \sum_{i=1}^N H_i^n(x,\nabla u_n) = f - \sum_{i=1}^N \frac{\partial}{\partial x_i} k_i \quad \text{in } \Omega \\ u_n = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{13}$$

Lemma 1 *See ([9], lemma 4.2) Let $A \in \mathbb{R}^+$ and $u \in W_0^{1,\vec{p}}(\Omega)$, then there exists t measurable subsets $\Omega_1,...,\Omega_t$ of Ω and t functions $u_1,...,u_t$ such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, $|\Omega_t| \leq A$ and $|\Omega_s| = A$ for $s \in \{1,...,t-1\}$, $\{x \in \Omega : |\frac{\partial u_s}{\partial x_i}| \neq 0$ for $i = 1,...,N\} \subset \Omega_s$, $\frac{\partial u}{\partial x_i} = \frac{\partial u_s}{\partial x_i}$ a. e. in Ω_s, $\frac{\partial(u_1+...+u_s)}{\partial x_i} u_s = (\frac{\partial u}{\partial x_i}) u_s$, $u_1 + ... + u_s = u$ in Ω and $sign(u) = sign(u_s)$ if $u_s \neq 0$ for $s \in \{1,...,t\}$ and $i \in \{1,...,N\}$.*

Proposition 2 *Assume that $\overline{p} < N$, (5)-(12) hold and let $u_n \in W_0^{1,\vec{p}}(\Omega)$ be a solution to problem (13) then, we have*

$$\sum_{i=1}^N \int_\Omega |\frac{\partial u_n}{\partial x_i}|^{p_i} \leq C, \tag{14}$$

for some positive constant C depending on N, Ω, λ, γ, p_i, b_i, $\|f\|_{L^{p'_\infty}(\Omega)}$, $\|g_i\|_{L^{p'_i}(\Omega)}$ for $i = 1,...,N$.

Proof: Let A be a positive real number, that will be chosen later, Referring to lemma 1. Let us fix $s \in \{1,...,t\}$ and let us use $T_k(u_s)$ as test function in problem 13, using (5), (8), Young's and Hölder's inequalities and proposition 1 we obtain

$$\sum_{i=1}^N \int_{\{u_s \leq k\}} |\frac{\partial u_s}{\partial x_i}|^{p_i} \tag{15}$$

$$\leq C_1\Big(\|f\|_{L^{p'_\infty}(\Omega)} d_s^{\frac{1}{N}} + \sum_{i=1}^N \int_\Omega |H_i(x,\nabla u)||u_s| + \sum_{i=1}^N \|k_i\|^{p'_i}_{L^{p'_i}(\Omega)} \Big).$$

The dominated convergence theorem implies that

$$\sum_{i=1}^N \int_\Omega |\frac{\partial u_s}{\partial x_i}|^{p_i} \leq C_1\Big(\|f\|_{L^{p'_\infty}(\Omega)} d_s^{\frac{1}{N}} +$$

$$\sum_{i=1}^N \int_\Omega |H_i(x,\nabla u)||u_s| + \sum_{i=1}^N \|k_i\|^{p'_i}_{L^{p'_i}(\Omega)} \Big),$$

for some constant $C_1 > 0$, where $d_s = \prod_{i=1}^N \Big(\int_\Omega |\frac{\partial u_s}{\partial x_i}|^{p_i} \Big)^{\frac{1}{p_i}}$ here and in what follows the constants depend on the data but not on the function u.

Using condition (10), Hölder's and Young's inequalities, lemma 1 and proposition 1 we get

$$\sum_{i=1}^N \int_\Omega |H_i(x,\nabla u)||u_s| \leq \sum_{i=1}^N \int_\Omega b_i |\frac{\partial u}{\partial x_i}|^{p_i-1}|u_s|$$

$$\leq C_2 \sum_{i=1}^N \Big[\sum_{\sigma=1}^s \int_{\Omega_\sigma} |\frac{\partial u_\sigma}{\partial x_i}|^{p_i-1}|u_s| + \int_{\Omega \setminus \cup_{\sigma=1}^s \Omega_\sigma} |\frac{\partial u}{\partial x_i}|^{p_i-1}|u_s| \Big]$$

$$\leq C_2 \sum_{i=1}^N \Big[\sum_{\sigma=1}^s \int_{\Omega_\sigma} |\frac{\partial u_\sigma}{\partial x_i}|^{p_i-1}|u_s| \Big]$$

$$\leq C_2 \sum_{i=1}^N \sum_{\sigma=1}^s \Big(\int_{\Omega_\sigma} |\frac{\partial u_\sigma}{\partial x_i}|^{p_i} \Big)^{\frac{1}{p'_i}} \Big(\int_\Omega |u_s|^{p_\infty} \Big)^{\frac{1}{p_\infty}} |\Omega_\sigma|^{\frac{1}{p_i} - \frac{1}{p_\infty}}$$

$$\leq C_2 \sum_{i=1}^N \sum_{\sigma=1}^s \Big[\frac{1}{p'_i} \int_{\Omega_\sigma} |\frac{\partial u_\sigma}{\partial x_i}|^{p_i} + \frac{1}{p_i} \Big(\int_\Omega |u_s|^{p_\infty} \Big)^{\frac{p_i}{p_\infty}} \Big] |\Omega_\sigma|^{\frac{1}{p_i} - \frac{1}{p_\infty}}$$

since $\Big(\|u_s\|_{L^{\overline{p}^*}(\Omega)} \leq C_s \prod_{j=1}^N \|\frac{\partial u_s}{\partial x_j}\|^{\frac{1}{N}}_{L^{p_j}(\Omega)}$ and $|\Omega_\sigma| \leq A \Big)$, hence

$$\sum_{i=1}^N \int_\Omega |H_i(x,\nabla u)||u_s| \tag{16}$$

$$\leq C_2 \sum_{i=1}^N A^{\frac{1}{p_i} - \frac{1}{p_\infty}} \sum_{\sigma=1}^s \Big[\frac{1}{p'_i} \int_{\Omega_\sigma} |\frac{\partial u_\sigma}{\partial x_i}|^{p_i} + \frac{CC_s}{p_i} \prod_{j=1}^N \|\frac{\partial u_s}{\partial x_j}\|^{\frac{p_i}{N}}_{L^{p_j}(\Omega)} \Big]$$

$$\leq C_3 \sum_{i=1}^N A^{\frac{1}{p_i} - \frac{1}{p_\infty}} \Big[\int_{\Omega_s} |\frac{\partial u_s}{\partial x_i}|^{p_i} + \sum_{\sigma=1}^{s-1} \int_{\Omega_\sigma} |\frac{\partial u_\sigma}{\partial x_i}|^{p_i} + d_s^{\frac{p_i}{N}} \Big]$$

for some constant $C_3 > 0$. Putting (16) in (15) we obtain

$$\sum_{i=1}^N \int_\Omega |\frac{\partial u_s}{\partial x_i}|^{p_i} \leq C_1 \Big\{ \|f\|_{L^{p'_\infty}(\Omega)} d_s^{\frac{1}{N}} + \sum_{i=1}^N \|k_i\|^{p'_i}_{L^{p'_i}(\Omega)} +$$

$$C_3 \sum_{i=1}^N A^{\frac{1}{p_i} - \frac{1}{p_\infty}} \Big[\int_{\Omega_s} |\frac{\partial u_s}{\partial x_i}|^{p_i} + \sum_{\sigma=1}^{s-1} \int_{\Omega_\sigma} |\frac{\partial u_\sigma}{\partial x_i}|^{p_i} + d_s^{\frac{p_i}{N}} \Big] \Big\}. \tag{17}$$

If A is such that

$$1 - C_1 C_3 \sum_{i=1}^N A^{\frac{1}{p_i} - \frac{1}{p_\infty}} > 0, \tag{18}$$

inequality (17) becomes

$$\sum_{i=1}^{N}\int_{\Omega}|\frac{\partial u_s}{\partial x_i}|^{p_i} \le C_4\Big\{\|f\|_{L^{p'_\infty}(\Omega)} d_s^{\frac{1}{N}} + \sum_{i=1}^{N}\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)} +$$

$$\sum_{\sigma=1}^{s-1}\sum_{i=1}^{N} A^{\frac{1}{p_i}-\frac{1}{p_\infty}}\Big(\sum_{j=1}^{N}\int_{\Omega_s}|\frac{\partial u_s}{\partial x_j}|^{p_j}\Big)+\sum_{i=1}^{N}A^{\frac{1}{p_i}-\frac{1}{p_\infty}} d_s^{\frac{p_i}{N}}\Big\} \quad (19)$$

for some constant $C_4 > 0$ and for $s = 1$, we get

$$\int_{\Omega}|\frac{\partial u_1}{\partial x_i}|^{p_i} \le \sum_{i=1}^{N}\int_{\Omega}|\frac{\partial u_1}{\partial x_i}|^{p_i}$$

$$\le C_4\Big\{\|f\|_{L^{p'_\infty}(\Omega)} d_1^{\frac{1}{N}} + \sum_{i=1}^{N}\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)} + \sum_{i=1}^{N}A^{\frac{1}{p_i}-\frac{1}{p_\infty}} d_1^{\frac{p_i}{N}}\Big\}. \quad (20)$$

Let us choose A such that (18) and $1 - C_4\sum_{i=1}^{N} A^{\frac{N}{p_i}(\frac{1}{p_i}-\frac{1}{p_\infty})} > 0$ hold, (see [9]).
For example, we can take

$$A < \min\Big\{1, (\frac{1}{2C_3})^{\frac{1}{\min_{i=1...N}\{\frac{1}{p_i}-\frac{1}{p_\infty}\}}}; (\frac{1}{2C_3})^{\frac{1}{\min_{i=1...N}\{\frac{N}{p_i}(\frac{1}{p_i}-\frac{1}{p_\infty})\}}}\Big\}.$$

By this choice, we obtain

$$d_1 = \prod_{i=1}^{N}\Big(\int_{\Omega}|\frac{\partial u_1}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p_i}}$$

$$\le C_5\Big[\Big(\|f\|^{\frac{N}{\overline{p}}}_{L^{p'_\infty}(\Omega)} + \|\nu_2\|^{\frac{N}{\overline{p}}}_{L^{p'_\infty}(\Omega)}\Big)d_1^{\frac{1}{\overline{p}}} + \sum_{i=1}^{N}\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)}\Big].$$

Then there exists a constant $C_6 > 0$ such that $d_1 \le C_6$ and by (20), we obtain

$$\sum_{i=1}^{N}\int_{\Omega}|\frac{\partial u_1}{\partial x_i}|^{p_i} \le C_7, \quad (21)$$

for some constant $C_7 > 0$. Moreover using (21) in (19) and iterating on s, we have

$$\sum_{i=1}^{N}\int_{\Omega}|\frac{\partial u_s}{\partial x_i}|^{p_i}$$

$$\le C_8\Big[\|f\|_{L^{p'_\infty}(\Omega)} d_s^{\frac{1}{N}} + \sum_{i=1}^{N}\|k_i\|^{p'_i}_{L^{p'_i}(\Omega)} + 1 + \sum_{i=1}^{N}A^{\frac{1}{p_i}-\frac{1}{p_\infty}} d_s^{\frac{p_i}{N}}\Big]$$

then arguing as before, we obtain $\sum_{i=1}^{N}\int_{\Omega}|\frac{\partial u_s}{\partial x_i}|^{p_i} \le C_9$, for some constant $C_9 > 0$, then

$\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \le k\sum_{i=1}^{N}\Big(\sum_{s=1}^{t}\int_{\Omega}|\frac{\partial u_s}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p_i}} \le C_{10}$, for some positive $k > 0$.

Since u_n is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$ and the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p_-}(\Omega)$ is compact, we obtain the following results.

Corollaire 1 *If u_n is a weak solution of problem (13), then there exists a subsequence $(u_n)_n$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,\overrightarrow{p}}(\Omega)$, strongly in $L^{p_-}(\Omega)$ and a. e. in Ω.*

4.2 Strong convergence of $T_k(u_n)$

Lemma 2 *Assume that $u_n \rightharpoonup u$ weakly in $W_0^{1,\overrightarrow{p}}(\Omega)$ and a. e. in Ω and $\sum_{i=1}^{N}\int_{\Omega}[a_i(x,u_n,\nabla u_n)-a_i(x,u_n,\nabla u)](\frac{\partial u_n}{\partial x_i}-\frac{\partial u}{\partial x_i}) \to 0$ then,*

$$u_n \to u \text{ strongly in } W_0^{1,\overrightarrow{p}}(\Omega).$$

Proof: The proof follows as in Lemma 5 of [15] taking into account the anisotropy of operator. □

Proposition 3 *Let u_n be a solution to the approximate problem (13), then*

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,\overrightarrow{p}}(\Omega)$$

.

Proof: Let us fix k and let δ be a real number such that $\delta \ge (\frac{L(k)}{2\lambda})^2$. Let us define $z_n = T_k(u_n) - T_k(u)$ and $\varphi(s) = se^{\delta s^2}$, it is easy to check that for all $s \in \mathbb{R}$ one has

$$\varphi'(s) - \frac{L(k)}{\lambda}|\varphi(s)| \ge \frac{1}{2}. \quad (22)$$

Using $\varphi(z_n)$ as test function in (13), we get

$$\sum_{i=1}^{N}\int_{\Omega}a_i(x,u_n,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i}$$

$$+\sum_{i=1}^{N}\int_{\Omega}g_i^n(x,u_n,\nabla u_n)\varphi(z_n) + \sum_{i=1}^{N}\int_{\Omega}H_i^n(x,\nabla u_n)\varphi(z_n) =$$

$$\int_{\Omega}f\varphi(z_n) + \sum_{i=1}^{N}\int_{\Omega}k_i(x)\frac{\partial\varphi(z_n)}{\partial x_i}. \quad (23)$$

Since $\varphi(z_n) \rightharpoonup 0$ weakly in $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^{p_\infty}(\Omega)$ and $f \in L^{p'_\infty}(\Omega)$ then $\int_{\Omega}f\varphi(z_n) \to 0$ as $n \to +\infty$. Since $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,\overrightarrow{p}}(\Omega)$, $k_i \in L^{p'_i}(\Omega)$ and $(\varphi'(z_n))_n$ is bounded then $\sum_{i=1}^{N}\int_{\Omega}k_i(x)\frac{\partial\varphi(z_n)}{\partial x_i} \to 0$ as $n \to +\infty$. On the other hand, we have

$$|\sum_{i=1}^{N}\int_{\Omega}H_i^n(x,\nabla u_n)\varphi(z_n)|$$

$$\le \sum_{i=1}^{N}\int_{\Omega}|\frac{\partial u_n}{\partial x_i}|^{p_i-1}|b_i\varphi(z_n)|$$

$$\le \sum_{i=1}^{N}\Big(\int_{\Omega}|b_i\varphi(z_n)|^{p_i}\Big)^{\frac{1}{p_i}}\Big(\int_{\Omega}|\frac{\partial u_n}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p'_i}}$$

$$\le C\sum_{i=1}^{N}\Big(\int_{\Omega}|b_i\varphi(z_n)|^{p_i}\Big)^{\frac{1}{p_i}}.$$

Since $b_i\varphi(z_n) \to 0$ a.e. in Ω and $|b_i\varphi(z_n)| \le |b_i| \times 2ke^{4k^2\delta} \in L^{p_i}(\Omega)$, then by dominated convergence theorem $b_i\varphi(z_n) \to 0$ strongly in $L^{p_i}(\Omega)$, then $\Big|\sum_{i=1}^{N}\int_{\Omega}H_i^n(x,\nabla u_n)\varphi(z_n)dx\Big| \to 0$ as $n \to +\infty$. Denote by $\varepsilon_1(n), \varepsilon_2(n), ...$ various sequences of real numbers which converge to zero when n tends to $+\infty$. Using (8), we obtain that $g_i^n(x,u_n,\nabla u_n)\varphi(z_n) \ge 0$ on the set $\{|u_n| > k\}$. Then we have

$$\sum_{i=1}^{N}\int_{\Omega}a_i(x,u_n,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i}+$$

$$\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}g_i^n(x,u_n,\nabla u_n)\varphi(z_n)dx \le \varepsilon_1(n) \quad (24)$$

On the other hand, we have
$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i}$$
$$=\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)$$
$$=\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)$$
$$-\sum_{i=1}^{N}\int_{\{u_n>k\}} a_i(x,u_n,\nabla u_n)\frac{\partial T_k(u)}{\partial x_i}\varphi'(z_n).$$
The sequence $\Big(a_i(x,u_n,\nabla u_n)\varphi'(z_n)\Big)_n$ is bounded in $L^{p_i'}(\Omega)$, then since $\frac{\partial T_k(u)}{\partial x_i}\chi_{\{|u_n|>k\}}\to 0$ strongly in $L^{p_i}(\Omega)$, one has
$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i}dx$$
$$=\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)$$
$$+\varepsilon_2(n)$$
which we can write
$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i}dx$$
$$=\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$$
$$\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)dx$$
$$+\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)dx$$
$$+\varepsilon_2(n).$$
Since $u_n\to u$ a.e. in Ω, we have $a_i(x,T_k(u_n),\nabla T_k(u))\varphi'(z_n)$ converges to $a_i(x,T_k(u),\nabla T_k(u))$ a. e. in Ω. Let E be measurable subset of Ω, by the growth condition (6), we get
$$\int_E |a_i(x,T_k(u_n),\nabla T_k(u))\varphi'(z_n)|^{p_i'}dx$$
$$\le 2^{p_i'-1}(1+8\delta k^2)^{p_i'}e^{4\delta k^2 p_i'}\Big(k^{p_\infty}|E|+\int_E|\frac{\partial T_k(u)}{\partial x_i}|^{p_i}\Big).$$
Then the sequence $\Big(a_i(x,T_k(u_n),\nabla T_k(u))\varphi'(z_n)\Big)_n$ is equi-integrable and by Vitali's theorem one has $a_i(x,T_k(u_n),\nabla T_k(u))\varphi'(z_n)$ converges to $a_i(x,T_k(u),\nabla T_k(u))$ strongly in $L^{p_i'}(\Omega)$. Since $\frac{\partial T_k(u_n)}{\partial x_i}\rightharpoonup\frac{\partial T_k(u)}{\partial x_i}$ weakly in $L^{p_i}(\Omega)$, then
$$\lim_{n\to+\infty}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))$$
$\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)=0$. It follows that
$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)\frac{\partial\varphi(z_n)}{\partial x_i}$$
$$=\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$$
$$\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi'(z_n)+\varepsilon_3(n). \quad (25)$$
On the other hand, by virtue of (5) and (9)
$$|\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} g_i^n(x,u_n,\nabla u_n)\varphi(z_n)dx|$$
$$\le L(k)\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}|\frac{\partial u_n}{\partial x_i}|^{p_i}|\varphi(z_n)|dx$$
$$\le L(k)\sum_{i=1}^{N}\int_{\Omega}|\frac{\partial T_k u_n}{\partial x_i}|^{p_i}|\varphi(z_n)|dx$$
$$\le\frac{L(k)}{\lambda}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n)).\frac{\partial T_k(u_n)}{\partial x_i}|\varphi(z_n)|dx$$
$$\le\frac{L(k)}{\lambda}\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-$$
$$a_i(x,T_k(u_n),\nabla T_k(u))\Big)\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)|\varphi(z_n)|dx+$$
$$\frac{L(k)}{\lambda}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n)).\frac{\partial T_k(u)}{\partial x_i}|\varphi(z_n)|dx+$$
$$\frac{L(k)}{\lambda}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))$$
$$.\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)|\varphi(z_n)|dx.$$
Similarly as above, it's easy to see that by (6), corollary 1 and Vitali's theorem one has $a_i(x,T_k(u_n),\nabla T_k(u))\to a_i(x,T_k(u),\nabla T_k(u))$ strongly in $L^{p_i'}(\Omega)$. Writing
$$\Big|\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u)).\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi(z_n)dx\Big|$$
$$\le\varphi(2k)\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u))|.\Big|\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big|$$
and taking into account that $\frac{\partial T_k(u_n)}{\partial x_i}\rightharpoonup\frac{\partial T_k(u)}{\partial x_i}$ weakly in $L^{p_i}(\Omega)$, we obtain
$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u)).\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\varphi(z_n)dx$$
tends to zero as $n\to+\infty$. Thanks to (6) and (14), the sequence $\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))\Big)_n$ is bounded in $L^{p_i'}(\Omega)$, so that there exists $l_k^i\in L^{p_i'}(\Omega)$ such that $a_i(x,T_k(u_n),\nabla T_k(u_n))\rightharpoonup l_k^i$ weakly in $L^{p_i'}(\Omega)$. We have
$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\frac{\partial T_k(u)}{\partial x_i}\varphi(z_n)dx=$$
$$\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-l_k^i\Big)\frac{\partial T_k(u)}{\partial x_i}\varphi(z_n)dx$$
$$+\sum_{i=1}^{N}\int_{\Omega} l_k^i\frac{\partial T_k(u)}{\partial x_i}\varphi(z_n)dx.$$
Since $\varphi(z_n)\to 0$ weak* in $L^\infty(\Omega)$, we conclude that $\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\frac{\partial T_k(u)}{\partial x_i}\varphi(z_n)dx\to 0$ as $n\to+\infty$. Hence, we get

$$|\sum_{i=1}^{n}\int_{\{|u_n|\le k\}} g_i^n(x,u_n,\nabla u_n)\varphi(z_n)dx|$$

$$\le \frac{L(k)}{\lambda}\sum_{i=1}^{N}\int_\Omega \Big(a_i(x,T_k(u_n),\nabla T_k(u_n)) \tag{26}$$

$-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)|\varphi(z_n)|dx$
$+\varepsilon_5(n)$.

Combining (24), (25) and (26), we obtain

$$\sum_{i=1}^{N}\int_\Omega \Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$$
$$\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\Big(\varphi'(z_n)-\frac{L(k)}{\lambda}|\varphi(z_n)|\Big)dx \le \varepsilon_6(n).$$

Which gives by using (22)

$$0\le \sum_{i=1}^{N}\int_\Omega \Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$$
$$\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)\le 2\varepsilon_6(n),$$

then lemma 2 gives

$$T_k(u_n)\to T_k(u) \text{ strongly in } W_0^{1,\vec{p}}(\Omega). \tag{27}$$

4.3 Existence results

Theorem 1 *Assume that $\overline{p}<N$ and (5)-(12) hold. Then there exists at least a weak solution of the problem (1).*

Proof: By (4.2) the sequence $(\frac{\partial u_n}{\partial x_i})_n$ is bounded in $L^{p_i}(\Omega)$, so we have that $\frac{\partial u_n}{\partial x_i}\rightharpoonup \frac{\partial u}{\partial x_i}$ weakly in $L^{p_i}(\Omega)$ for $i=1,\dots,N$ and $u_n\to u$ strongly in $L^{p_-}(\Omega)$. By (27) there exists a subsequence, which we still denote by u_n such that $\frac{\partial u_n}{\partial x_i}\to\frac{\partial u}{\partial x_i}$ a. e. in Ω for $i=1,\dots,N$, then for $i=1,\dots,N$, we have

$$\begin{cases} a_i(x,u_n,\nabla u_n)\to a_i(x,u,\nabla u) \text{ a. e. in } \Omega,\\ g_i^n(x,u_n,\nabla u_n)\to g_i(x,u,\nabla u) \text{ a. e. in } \Omega,\\ H_i^n(x,\nabla u_n)\to H_i(x,\nabla u) \text{ a. e. in } \Omega.\end{cases}$$

Moreover by (6) and (10), we have $\int_\Omega |a_i(x,u_n,\nabla u_n)|^{p_i'}\le C\Big[\int_\Omega |u_n|^{p_\infty}+\int_\Omega |\frac{\partial u_n}{\partial x_i}|^{p_i}\Big]$ and $\int_\Omega |H_i^n(x,\nabla u_n)|^{p_i'}\le C\int_\Omega |\frac{\partial u_n}{\partial x_i}|^{p_i}$ by (14), $\Big(a_i(x,u_n,\nabla u_n)\Big)_n$ and $\Big(H_i(x,\nabla u_n)\Big)_n$ are bounded in $L^{p_i'}(\Omega)$ then $a_i(x,u_n,\nabla u_n)\rightharpoonup a_i(x,u,\nabla u)$ weakly in $L^{p_i'}(\Omega)$ and $H_i(x,\nabla u_n)\rightharpoonup H_i(x,\nabla u)$ weakly in $L^{p_i'}(\Omega)$. Now we prove that $g_i^n(x,u_n,\nabla u_n)$ is uniformly equi-integrable for $i=1,\dots,N$. For any measurable E of Ω and for any $k\in\mathbb{R}^+$, we have

$$\int_E |g_i^n(x,u_n,\nabla u_n)|$$
$$=\int_{E\cap\{|u_n|\le k\}}|g_i^n(x,u_n,\nabla u_n)|+\int_{E\cap\{|u_n|>k\}}|g_i^n(x,u_n,\nabla u_n)|$$
$$\le \int_{E\cap\{|u_n|\le k\}}L(k)|\frac{\partial T_k(u_n)}{\partial x_i}|^{p_i}+\int_{E\cap\{|u_n|>k\}}|g_i^n(x,u_n,\nabla u_n)|,$$

for fixed k and for $i=1,\dots,N$. For the first term we recall $\frac{\partial T_k(u_n)}{\partial x_i}$ strongly converges to $\frac{\partial T_k(u)}{\partial x_i}$ in $L^{p_i}(\Omega)$ for $i=1,\dots,N$. Taking $T_k(u_n)$ as test function in (4.1), then

$$\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\frac{\partial T_k(u_n)}{\partial x_i}+\sum_{i=1}^{N}\int_\Omega g_i^n(x,u_n,\nabla u_n)T_k(u_n)+$$
$$\sum_{i=1}^{N}\int_\Omega H_i^n(x,\nabla u_n)T_k(u_n)=\int_\Omega fT_k(u_n)+\sum_{i=1}^{N}\int_\Omega k_i\frac{\partial T_k(u_n)}{\partial x_i},$$

which implies that

$$\sum_{i=1}^{N}\int_\Omega g_i^n(x,u_n,\nabla u_n)T_k(u_n)\le\Big(\int_\Omega |f|^{p_\infty'}\Big)^{\frac{1}{p_\infty'}}\Big(\int_\Omega |u_n|^{p_\infty}\Big)^{\frac{1}{p_\infty}}+$$
$$\sum_{i=1}^{N}\Big(\int_\Omega |k_i|^{p_i'}\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega |\frac{\partial u_n}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p_i}}+$$
$$\sum_{i=1}^{N}\int_\Omega b_i|\frac{\partial u_n}{\partial x_i}|^{p_i-1}|T_k(u_n)|$$
$$\le\Big(\int_\Omega |f|^{p_\infty'}\Big)^{\frac{1}{p_\infty'}}\Big(\int_\Omega |u_n|^{p_\infty}\Big)^{\frac{1}{p_\infty}}+$$
$$\sum_{i=1}^{N}\Big(\int_\Omega |k_i|^{p_i'}\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega |\frac{\partial u_n}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p_i}}+$$
$$\sum_{i=1}^{N}\Big(\int_\Omega b_i^{r_i}\Big)^{\frac{1}{r_i}}\Big(\int_\Omega |\frac{\partial u_n}{\partial x_i}|^{p_i}\Big)^{\frac{1}{p_i'}}\Big(\int_\Omega |u_n|^{p_\infty}\Big)^{\frac{1}{p_\infty}}$$
$$\le C_1,$$

where r_i is such that $\frac{1}{r_i}=\frac{1}{p_i}-\frac{1}{p_\infty}$. Let E a measurable subset of Ω and for any $m\in\mathbb{R}^+$, we have

$$\int_E |g_i^n(x,u_n,\nabla u_n)|dx$$
$$=\int_{E\cap\{|u_n|\le k\}}|g_i^n(x,u_n,\nabla u_n)|dx+\int_{E\cap\{|u_n|>k\}}|g_i^n(x,u_n,\nabla u_n)|dx$$
$$\le\int_{E\cap\{|u_n|\le k\}}L(k)\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i}dx+\int_{E\cap\{|u_n|>k\}}|g_i^n(x,u_n,\nabla u_n)|dx$$
$$\le\int_{E\cap\{|u_n|\le k\}}L(k)\Big|\frac{\partial T_k(u_n)}{\partial x_i}\Big|^{p_i}dx+\frac{1}{k}\int_{\{|u_n|>k\}}T_k(u_n)g_i^n(x,u_n,\nabla u_n)dx$$

we have $\frac{\partial T_k(u_n)}{\partial x_i}\to\frac{\partial T_k(u)}{\partial x_i}$ strongly in $L^{p_i}(\Omega)$ and $\int_{\{|u_n|>k\}}T_k(u_n)g_i^n(x,u_n,\nabla u_n)dx\le C_1$

then g_i^n is uniformly equi-integrable for any i, since $g_i^n(x,u_n,\nabla u_n)\to g_i(x,u,\nabla u)$ a. e. in Ω, we get $g_i^n(x,u_n,\nabla u_n)\to g_i(x,u,\nabla u)$ in $L^1(\Omega)$. That allow us to pass to the limit in the approximate problem.

Remark 1 *The condition (6) can be substituted by $|a_i(x,s,\xi)|\le\gamma\Big[j_i+|s|^{\frac{p_\infty}{p_i'}}+|\xi_i|^{p_i-1}\Big]$, where j_i is a positive function in $L^{p_i'}(\Omega)$ for $i=1,\dots,N$, and the condition (9) can be substituted by $|g_i(x,s,\xi)|\le L(|s|)\Big(C_i+|\xi_i|^{p_i}\Big)$ where C_i is a positive function in $L^1(\Omega)$ for $i=1,\dots,N$.*

References

1. A. Alvino, M.F. Betta, A. Mercaldo, "Comparison principle for some classes of nonlinear elliptic equations.", J. Differential Equations., **12**, 3279–3290, 2010. http://dx.doi.org/10.1016/j.jde.2010.07.030
2. S. Antontsev, M. Chipot, "Anisotropic equations: uniqueness and existence results.", Differential Integral Equations **21** (5), 401–419, 2008. https://projecteuclid.org/euclid.die/1356038624
3. L. Boccardo, T. Gallouét, P. Marcellini," Anisotropic equations in L^1.", Differential Integral Equations., **9**(1), 209-212, 1996.
4. G, Bottaro, M. Marina," Problema di Dirichlet per equazioni ellittiche di tipo variazionale su insiemi non limitati.", Boll. Un. Mat. Ital, **8**, 46-56, 1973.
5. F. Li, "Anisotropic elliptic equations in L^m.", J. Convex Anal, **8** (2), 417-422, 2001.
6. O. Guibé, A. Mercaldo," Uniqueness results for non coercive nonlinear elliptic equations with two lower order terms.", Commun. Pure Appl. Anal. **7** (1), 163-192, 2008.
7. V. Monetti, L. Randazzo," Existence results for nonlinear elliptic equations with p-growyh in the gradient.", Ricerche di Matematica **1**,. 163-181, 2000.
8. Y. Akdim, A. Benkirane, M. El Moumni, " Existence results for nonlinear elliptic problems with lower order terms.", International Journal of Evolution Equations (IJEE)., **4**, 1–20, 2014.
9. R. Di Nardo, F. Feo, "Existence and uniqueness For nonlinear anisotropic elliptic equations.", Arch. Math.**102** 141-153, 2014.
10. Y. Akdim, A. Salmani, " Existence And Uniqueness Results for nonlinear Anisotropic Elliptic Equations.", Journal of Nonlinear Evolution Equations and Applications (JNEEA)., **6**, 95–111, 2016.
11. A. Di Castro,"Existence and regularity results for anisotropic elliptic problems.", Adv. Nonlinear Stud, **9**, 367-393, 2009.
12. M.Troisi," Teoremi di inclusione per spazi di Sobolev non isotropi.", recerche Mat.**18**, 3-24, 1969.
13. I. Fragala, F. Gazzola, B. Kawohl," Existence and nonexistence results for anisotropic quasi- linear elliptic equations.", Ann. Inst. H. Poincar Anal. Non Linéaire, **21**(5), 715-734, 2004.
14. J. Leray, J. L. Lions," Quelques rsultats de Visik sur les problémes elliptiques nonlinéaires par les méthodes de Minty-Browder.", Bull. Soc. Math. France **93** 97-107, 1965.
15. L. Boccardo, F. Murat, J. P. Puel," Existence of bounded solution for non linear elliptic unilateral problems.", Ann. Mat. pura appl., **152**, 183-196, 1988.

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Boundary gradient exact enlarged controllability of semilinear parabolic problems

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A B S T R A C T

The aim of this paper is to study the boundary enlarged gradient controllability problem governed by parabolic evolution equations. The purpose is to find and compute the control u which steers the gradient state from an initial gradient one ∇y_0 to a gradient vector supposed to be unknown between two defined bounds b_1 and b_2, only on a subregion Γ of the boundary $\partial\Omega$ of the system evolution domain Ω. The obtained results have been proved via two approaches, The subdifferential and Lagrangian multiplier approach.

1 Introduction

The concept of controllability has been widely developed since the sixties [1,2]. Later, the notion of regional controllability was introduced by El Jai [3,4], and interesting results have been obtained, in particular, the possibility to control a state only on a subregion ω of Ω. These results have been extended to the case where ω is a part of the boundary $\partial\Omega$ of the evolution domain Ω [5,6]. Then the concept of regional gradient controllability and regional enlarged controllability were introduced and developed for linear and semilinear systems [7–11]. Here instead of steering the system to a desired gradient, we are interested in steering its gradient between two prescribed functions given only on a boundary subregion $\Gamma \subset \partial\Omega$ of system evolution domain .

Many reasons are motivating this problem: first, the mathematical models are obtained from measurements or from approximation techniques and they are very often affected by perturbations. And the solution of such system is approximately known. Second, in many real problems the target required to be between two bounds. Moreover, there are many applications of gradient modeling. For example, controlling the concentration regulation of a substrate at the upper bottom of a bioreactor between two levels (see Figure 1).

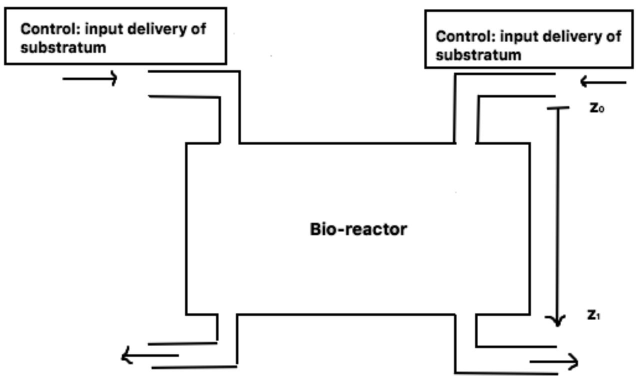

Figure 1: Regulation of the concentration flux of the substratum at the upper bottom of the bio-reactor

Motivated by the arguments above, in this paper, our goal is to study the regional boundary enlarged controllability of the gradient. For that we consider a semilinear system of parabolic equations where the control is exerted in an intern subregion ω of the system evolution domain Ω.

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Thus, let Ω be an open bounded set of $\mathbb{R}^n (n \geq 1)$ with regular boundary $\partial\Omega$. We consider the Banach spaces $L^2(\Omega)$ and $H_0^1(\Omega)$ with their corresponding norms. For a given $T > 0$, we denote $Q_T = \Omega \times]0,T[$, $\Sigma_T = \partial\Omega \times]0,T[$ and we consider the following problem :

$$\text{Minimize} \quad J(u) = \frac{1}{2}\|u\|_U^2 \tag{1}$$

$$\text{subject to:} \begin{cases} \dfrac{\partial y(x,t)}{\partial t} - \mathcal{A}y(x,t) - \mathcal{N}y(x,t) = Bu(t) & \text{in } Q_T \\ y(x,0) = y_0(x) & \text{in } \Omega \\ \dfrac{\partial y(\xi,t)}{\partial \nu_{\mathcal{A}}} = 0 & \text{on } \Sigma_T \end{cases} \tag{2}$$

where:

- χ_Γ the restriction operator defined by:

$$\begin{array}{rl} \chi_\Gamma : \left(H^{1/2}(\partial\Omega)\right)^n & \longrightarrow \left(H^{1/2}(\Gamma)\right)^n \\ y & \longmapsto \chi_\Gamma y = y|_\Gamma, \end{array}$$

  while $\Gamma \in \partial\Omega$ is of Lebesgue positive measure and χ_Γ^* denotes its adjoint.

- ∇ is nabla operator given by the formula:

$$\begin{array}{rl} \nabla : H_0^1(\Omega) \cap H^2(\Omega) & \longrightarrow \left(L^2(\Omega)\right)^n \\ y & \longmapsto \nabla y = \left(\dfrac{\partial y}{\partial x_1}, \ldots, \dfrac{\partial y}{\partial x_n}\right), \end{array}$$

- $\gamma : \left(L^2(\Omega)\right)^n \to \left(H^{1/2}(\partial\Omega)\right)^n$ is the extension of trace operator of order zero which is linear, continuous and surjective,

- $y_u(T)$ is the mild solution of (2) at the time T ($y_u(T) \in H_0^1(\Omega)$ [12]), and u is a the control function in the control space $U = L^2(0,T;\mathbb{R}^m)$ (where m is the number of actuators),

- $\mathcal{A}$ is linear, second order operator with dense domain such that the coefficients do not depend on t and generates a C_0-semigroup $(S(t))_{t\geq 0}$ on $L^2(\Omega)$ which we consider compact,

- $\mathcal{N} : L^2(0,T;L^2(\Omega)) \to L^2(\Omega)$ is a non linear k-Lipschitz operator [13],

- $B \in \mathcal{L}\left(\mathbb{R}^m, L^2(\Omega)\right)$ and y_0 is the initial datum in $L^2(\Omega)$.

The problem (1-2) is well-posed and has a unique solution.

The remainder contents of this paper are structured as follows. Some preliminary results are introduced in the next section. In section 3 we give the definitions of the gradient enlarged controllability on the boundary. Two approaches steering the system (2) from the initial gradient vector to a target gradient function between two bounds with minimum energy control is presented in section 4.

2 Preliminary results

In this section, we introduce some preliminary results to be used there after.

Let D be an open subset of $L^2(\Omega)$, we consider the system in (2), where $y_0 \in D$ and $y : [0,T] \to D$.
A continuous function $u : [0,T] \to D$ is said to be a classical solution of (2) if $u(0) = y_0(x)$, u is differentiable on $[0,T]$. and $u(t) \in D$ for $t \in [0,T]$ and y satisfies (2) on $[0,T]$.
It is well known that if y is a classical solution of (2), then it satisfies the following:

$$y_u(t) = S(t)y_0 + \int_0^t S(t-s)\mathcal{N}y(s)ds + \int_0^t S(t-s)Bu(s)ds. \tag{3}$$

Definition 1 (14) *A continuous function y from $[0,T]$ into D is called a mild solution of (2) if y satisfies the integral equation (3) on $[0,T]$.*

Let $\Gamma \subseteq \partial\Omega$ a part of the boundary with $y(x,t)$ satisfies (3), then the regional enlarged controllability on the boundary problem is concerned whether there exists a control u to steer the system (2) from the initial gradient vector ∇y_0 to a gradient vector between two functions in $\left(H^{1/2}(\Gamma)\right)^n$.

By [15], the adjoint of the gradient operator on a connected, open bounded subset Ω with a Lipschitz continuous boundary $\partial\Omega$ is the minus of the divergence operator. Then $\nabla^* : (L^2(\Omega))^n \to H^{-1}(\Omega)$ is given by

$$\xi \to \nabla^*\xi := v, \tag{4}$$

and v solves the following Dirichlet problem

$$\begin{cases} v = -div(\xi) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

We recall the following definitions

Definition 2 (16) *The system (2) is exactly (resp. weakly) gradient controllable on Γ if for every desired gradient $g_d \in \left(H^{1/2}(\Gamma)\right)^n$ (resp. for every $\epsilon > 0$), there exists a control $u \in U$ such that $\chi_\Gamma\gamma\nabla y_u(T) = g_d$ (resp. $\|\chi_\Gamma\gamma\nabla y_u(T) - g_d\| \leq \epsilon$)*

We recall that an actuator is conventionally defined by a couple (D,f), where D is a nonempty closed part of Ω, and it represents the geometric support of the actuator. And $f \in L^2(D)$ defines the spatial distribution of the action on the support D. For more details about the notion of actuators we refer the readers to [3,17].

We need also to recall this important result:

Theorem 1 *Let $(X,(\cdot,\cdot)_X,\|\cdot\|_X)$, $(Y,(\cdot,\cdot)_Y,\|\cdot\|_Y)$ be two Hilbert spaces and let $A \subseteq X$, $B \subseteq Y$ be non-empty, closed, convex subsets. Assume that a real functional $L : A \times B \to \mathbb{R}$ satisfies the following conditions*

$\forall \mu \in B$, $v \to L(v,\mu)$ is convex and lower semicontinuous;

$\forall v \in A$, $\mu \to L(v,\mu)$, is concave and upper semicontinuous.

Moreover,

$$A \text{ is bounded or } \lim_{\substack{\|v\|_X \to \infty \\ v \in A}} L(v, \mu_0) = +\infty \ \forall \mu_0 \in B$$

$$B \text{ is bounded or } \lim_{\substack{\|\mu\|_Y \to \infty \\ \mu \in B}} \inf_{v \in A} L(v, \mu) = -\infty$$

Then, the functional L has at least one saddle point.

Demonstration: For the proof of this theorem see [18].

3 Gradient enlarged controllability on the boundary

In this section, we give the definition of the concept of the gradient enlarged controllability on the boundary. To do so, we need to introduce the closed sub-vectorial space G of $H_0^1(\Omega)$. Hence, we have the following definition:

Definition 3 *Given $T > 0$. We say that there is gradient enlarged controllability on Γ, if, for every y_0 (in a suitable functional space), we can find a control u such that*

$$\chi_\Gamma \gamma \nabla y_u(T) \in G$$

Remark 1
- *This notion depends of course on the choice of the functional space G.*
- *If $G = \{0\}$, we retrieve the classical notion of the exact controllability.*
- *if G is the space skimmed by $y_u(T)$, the notion is empty.*

For the particular case, we will study the boundary gradient enlarged controllability in $[a_i(\cdot), b_i(\cdot)]$ $\forall i \in \{1, \ldots, n\}$. For that let's consider $(a(\cdot))_i$ and $(b(\cdot))_i$ be two given functions in $\left(H^{1/2}(\Gamma)\right)^n$ such that $(a(\cdot))_i \leq (b(\cdot))_i$ for $i = 1, \ldots, n$ a.e on Γ, and we set:

$$\Theta = [a_i(\cdot), b_i(\cdot)] = \left\{(y_1, \ldots, y_n) \in \left(H^{1/2}(\Gamma)\right)^n \mid a_i(\cdot) \leq y_i(\cdot) \leq b_i(\cdot) \quad \text{a.e on } \Gamma\right\}$$

$$\text{for all } i \in \{1, \ldots, n\}$$

So the definition of the boundary gradient enlarged controllability in Θ is the following:

Definition 4 *We say that (2) is Θ-gradient controllable on Γ at time T if there exist a command $u \in U$ such that:*

$$\chi_\Gamma \gamma \nabla y_u(T) \in \Theta\cdot$$

Let us define the following operators:

- The Duhamel operator H from U to $H_0^1(\Omega) \cap H^2(\Omega)$ such that for $u \in U$:

$$Hu = \int_0^T S(T-s)Bu(s)ds,$$

- and the operator G_Θ given by:

$$\begin{aligned} G_\Theta : L^2(0,T; H_0^1(\Omega)) &\longrightarrow H_0^1(\Omega) \cap H^2(\Omega) \\ y(\cdot) &\longmapsto \int_0^T S(T-\tau)\mathcal{N}y(\tau)d\tau\cdot \end{aligned} \tag{6}$$

Finally, the solution of the system (2) at the time T could be written as follow:

$$y_u(T) = S(T)y_0 + G_\Theta y + Hu\cdot$$

And we have the following proposition:

Proposition 1 *We say that system (2) is Θ-gradient controllable on Γ at time T if and only if:*

$$\left(Im\, \chi_\Gamma \gamma \nabla G_\Theta + Im\, \chi_\Gamma \gamma \nabla H\right) \cap \Theta \neq \emptyset\cdot$$

Demonstration: We suppose that the system (2) is Θ-gradient controllable on Γ at time T which is equivalent to write: $\chi_\Gamma \gamma \nabla y_u(T) \in \Theta$.
Hence we can write:

$$\begin{aligned} \chi_\Gamma \gamma \nabla y_u(T) &= \chi_\Gamma \gamma \nabla\left(S(T)y_0 + G_\Theta y + Hu\right) \\ &= \chi_\Gamma \gamma \nabla S(T)y_0 + \chi_\Gamma \gamma \nabla G_\Theta y + \chi_\Gamma \gamma \nabla Hu. \end{aligned}$$

And we have $\nabla S(T)y_0 = 0$, furthermore, let's denote by: $z_1 = \chi_\Gamma \gamma \nabla G_\Theta y$, $z_2 = \chi_\Gamma \gamma \nabla Hu$ and $z = \chi_\Gamma \gamma \nabla y_u(T)$. Which leads to: $z_1 \in Im\left(\chi_\Gamma \gamma \nabla G_\Theta\right)$ and $z_2 \in Im\left(\chi_\Gamma \gamma \nabla H\right)$. Thus, $z \in Im\left(\chi_\Gamma \gamma \nabla G_\Theta\right) + Im\left(\chi_\Gamma \gamma \nabla H\right)$ and $z \in \Theta$ which gives:

$$\left(Im\, \chi_\Gamma \gamma \nabla G_\Theta + Im\, \chi_\Gamma \gamma \nabla H\right) \cap \Theta \neq \emptyset\cdot$$

Conversely, we suppose that the following expression is verified:

$$\left(Im\, \chi_\Gamma \gamma \nabla G_\Theta + Im\, \chi_\Gamma \gamma \nabla H\right) \cap \Theta \neq \emptyset,$$

then, there exists $z \in \Theta$ such that $z \in \left(Im\, \chi_\Gamma \gamma \nabla G_\Theta + Im\, \chi_\Gamma \gamma \nabla H\right)$. So $z = z_1 + z_2$ where $z_1 = \chi_\Gamma \gamma \nabla G_\Theta y$ with $y \in L^2(0,T; H_0^1(\Omega))$ and $\exists\, u \in U$ such that $z_2 = \chi_\Gamma \gamma \nabla Hu$. Then $z = \chi_\Gamma \gamma \nabla G_\Theta y + \chi_\Gamma \gamma \nabla Hu$.
While $\chi_\Gamma \gamma \nabla S(T)y_0 = 0$, one can write:

$$\begin{aligned} z &= \chi_\Gamma \gamma \nabla S(T)y_0 + \chi_\Gamma \gamma \nabla G_\Theta y + \chi_\Gamma \gamma \nabla Hu \\ &= \chi_\Gamma \gamma \nabla\left(S(T)y_0 + G_\Theta y + Hu\right) \\ &= \chi_\Gamma \gamma \nabla y_u(T). \end{aligned}$$

Hence, $z = \chi_\Gamma \gamma \nabla y_u(T) \in \Theta$, and we prove the equivalence.

Remark 2
1. *The above definition means that we are interested in the transfer of the system (2) to an unknown state just in Θ.*
2. *If $\Theta = \{0\}$ or $a_i(\cdot) = b_i(\cdot)$ $\forall i \in \{0, 1\}$ we retrieve the regional exact controllability. So, for $a_i(\cdot) \neq b_i(\cdot)$ the Θ-gradient controllability on Γ constitutes an extension of the regional controllability.*

We can also characterize the enlarged controllability by using the notion of strategic actuators. And we can say:

Definition 5 *The actuator (D, f) is said to be Θ-strategic on Γ if the excited system is Θ-gradient controllable on Γ.*

4 Computation of the control

In order to compute the control subject to our problem (1-2), we will use two approaches. The first one relies on the subdifferential techniques and convex analysis and the second one on the Lagrangian multiplier approach.

4.1 Subdifferential approach

In this sub-section we are using the subdifferential techniques to compute the control steering the system from an initial gradient state to a final one between two functions on the boundary [19,20]. For that, we consider the optimization problem in (1).

$$\begin{cases} \inf J(u) = \frac{1}{2}\|u\|^2 \\ u \in U_{ad}, \end{cases} \tag{7}$$

where $U_{ad} = \left\{ u \in U \,|\, \chi_\Gamma \gamma \nabla y_u(T) \in \Theta \right\}$.

- Let f be a nontrivial, lower semi-continuous, proper and convex function from a Hilbert space W to $\widetilde{R} =]-\infty, +\infty[$. We denote $\mathcal{F}(W)$ the set of the functions f and $\|\cdot\|$ is the Hilbert norm of W.
- For $f \in \mathcal{F}(W)$, the polar function f^* of f is given by

$$f^*(v^*) = \sup_{v \in dom(f)} \{\langle v^*, v\rangle - f(v)\} \quad \forall v^* \in W$$

  where $dom(f) = \{v \in W \mid f(v) < +\infty\}$.
- For $v_0 \in dom(f)$, the set:

$$\partial f(v_0) = \left\{ v^* \in W \,|\, f(v) \geq f(v_0) + \langle v^*, v - v_0\rangle \ \forall v \in W \right\}$$

  denotes the subdifferential of f at v_0, then we have $v_1 \in \partial f(v^*)$ if and only if

$$f(v^*) + f^*(v_1) = \langle v^*, v_1 \rangle$$

With all these notations, the problem (7) is equivalent to:

$$\begin{cases} \inf\left(\sigma + \Psi_{U_{ad}}\right)(u) \\ u \in U_{ad}, \end{cases} \tag{8}$$

where:

- $\Psi_{U_{ad}}(u) = \begin{cases} 0 & \text{if } u \in U_{ad} \\ +\infty & \text{otherwise} \end{cases}$ denotes the indicator functional of U_{ad} (U_{ad} a non empty subset of U).
- $\sigma(u) = \frac{1}{2}\|u\|^2$.

Hence the solution of (8) is characterized by the following result.

Theorem 2 *u^* is a solution of (7) if and only of if the system (2) is Θ-gradient controllable on Γ and:*

$$u^* \in U_{ad} \quad and \quad \Psi^*_{U_{ad}}(-u^*) = -\|u^*\|^2. \tag{9}$$

Demonstration: The system (2) is Θ-gradient controllable on Γ, then $U_{ad} \neq \emptyset$. Using Fermat's rule, we have u^* is a minimum of (7) if and only if $0 \in \partial\left(\sigma + \Psi_{U_{ad}}\right)(u^*)$.
We prove that U_{ad} is convex, for that we consider u and v two elements of U_{ad}. So $\chi_\Gamma \gamma \nabla y_u(T) \in \Theta$ and $\chi_\Gamma \gamma \nabla y_{tu+(1-t)v}(T) \in \Theta$ for $t \in [0,1]$ which prove the convexity.
And we have $\sigma \in \mathcal{F}(U)$ and since U_{ad} is closed, convex and non empty, then $\Psi_{U_{ad}} \in \mathcal{F}(U)$
Moreover, $dom\,\sigma \cap dom\,\Psi_{U_{ad}} \neq \emptyset$ because the system (2) is Θ-gradient controllable on Γ.
Furthermore, $\partial\left(\sigma + \Psi_{U_{ad}}\right)(u^*) = \partial\sigma(u^*) + \partial\Psi_{U_{ad}}(u^*)$ (σ is continuous). It follows that u^* is a solution of (7) if and only if

$$0 \in \partial\sigma(u^*) + \partial\Psi_{U_{ad}}(u^*).$$

Besides we have σ is Frechet-Differentiable, hence $\partial\sigma(u^*) = \{u^*\}$. Furthermore, u^* is the solution of (7) if and only if $-u^* \in \partial\Psi_{U_{ad}}(u^*)$ which is equivalent to $\Psi_{U_{ad}}(u^*) + \Psi^*_{U_{ad}}(-u^*) = -\|u^*\|^2$. And which gives $u^* \in U_{ad}$ and $\Psi^*_{U_{ad}}(-u^*) = -\|u^*\|^2$.

4.2 Lagrangian approach

We consider the problem (1) when the system (2) is excited by one actuator (D, f). The following result gives a useful characterization of the solution of the problem.

Theorem 3 *If the system (2) is Θ-gradient controllable on Γ then the solution of (1-2) is given by:*

$$u^* = -\left(\chi_\Gamma \gamma \nabla H\right)^* \lambda^*, \tag{10}$$

where $\lambda^* \in \left(H^{1/2}(\Gamma)\right)^n$ satisfies:

$$\begin{cases} z^* = P_\Theta(\rho\lambda^* + z^*) \\ R_\Theta \lambda^* + z^* = \chi_\Gamma \gamma \nabla\left[S(T)y_0 + G_\Theta y\right], \end{cases} \tag{11}$$

while $P_\Theta : \left(H^{1/2}(\Gamma)\right)^n \to \Theta$ denotes the projection operator, $\rho > 0$ and $R_\Theta = \left(\chi_\Gamma \gamma \nabla H\right)\left(\chi_\Gamma \gamma \nabla H\right)^*$.

Demonstration: If the system (2) is Θ-gradient controllable on Γ then $U_{ad} \neq \emptyset$ and the problem (1-2) has a unique solution.
The minimization problem (1) is equivalent to the following saddle point problem:

$$\begin{cases} \inf \frac{1}{2}\|u\|^2 \\ (u, z) \in Z, \end{cases} \tag{12}$$

where $Z = \left\{ (u,z) \in U_{ad} \times \Theta \mid \chi_\Gamma \gamma \nabla y_u(T) - z = 0 \right\}$.
To study this constraints, we will use a Lagrangian functional and steer the problem (12) to a saddle point problem.

We associate to the problem (12) the Lagrangian functional defined by:

$$\begin{aligned}&\forall(u,z,\lambda)\in U_{ad}\times\Theta\times\left(H^{1/2}(\Gamma)\right)^n\\&L(u,z,\lambda)=\frac{1}{2}\|u\|^2+\left\langle\lambda,\chi_\Gamma\gamma\nabla y_u(T)-z\right\rangle\cdot\end{aligned}\tag{13}$$

▹ We prove that L admits a saddle point:

The set $U_{ad}\times\Theta$ is non empty, closed and convex. The functional L satisfies these conditions:

• $(u,z)\mapsto L(u,z,\lambda)$ is convex and lower semi-continuous for all $\lambda\in\left(H^{1/2}(\Gamma)\right)^n$.
• $\lambda\mapsto L(u,z,\lambda)$ is concave and upper semi-continuous for all $(u,z)\in U_{ad}\times\Theta$.
Moreover, there exists $\lambda_0\in\left(H^{1/2}(\Gamma)\right)^n$ such that:

$$\lim_{\|(u,z)\|\to+\infty}L(u,z,\lambda_0)=+\infty,\tag{14}$$

and there exists $(u_0,z_0)\in U_{ad}\times\Theta$ such that

$$\lim_{\|\lambda\|\to+\infty}L(u_0,z_0,\lambda)=-\infty\cdot\tag{15}$$

Then, the functional L admits a saddle point. For more details we refer to [21].

▹ We prove then that u^* is a solution of (1) and it is the minimum one:

Let (u^*,z^*,λ^*) be a saddle point of L. Hence, we have:

$$\begin{aligned}&L(u^*,z^*,\lambda)\le L(u^*,z^*,\lambda^*)\le L(u,z,\lambda^*)\\&\forall\,(u,z,\lambda)\in U_{ad}\times\Theta\times\left(H^{1/2}(\Gamma)\right)^n\end{aligned}\tag{16}$$

From the first inequality of (16) we have:

$$\begin{aligned}&\left\langle\lambda,\chi_\Gamma\gamma\nabla y_{u^*}(T)-z^*\right\rangle\le\left\langle\lambda^*,\chi_\Gamma\gamma\nabla y_{u^*}(T)-z^*\right\rangle\\&\forall\,\lambda\in\left(H^{1/2}(\Gamma)\right)^n,\end{aligned}$$

which implies $\chi_\Gamma\gamma\nabla y_{u^*}(T)=z^*$ and hence $\chi_\Gamma\gamma\nabla y_{u^*}(T)\in\Theta$.
The second inequality of (16) means that for all $u\in U_{ad}$ and $z\in\Theta$, we have:

$$\begin{aligned}\frac{1}{2}\|u^*\|^2&+\left\langle\lambda^*,\chi_\Gamma\gamma\nabla y_{u^*}(T)-z^*\right\rangle\le\frac{1}{2}\|u\|^2\\&+\left\langle\lambda^*,\chi_\Gamma\gamma\nabla y_u(T)-z\right\rangle\quad\forall\,(u,z)\in U_{ad}\times\Theta\cdot\end{aligned}$$

Since $\chi_\Gamma\gamma\nabla y_{u^*}(T)=z^*$, it follows that:

$$\frac{1}{2}\|u^*\|^2\le\frac{1}{2}\|u\|^2+\left\langle\lambda^*,\chi_\Gamma\gamma\nabla y_u(T)-z\right\rangle\quad\forall\,(u,z)\in U_{ad}\times\Theta.$$

Taking $z=\chi_\Gamma\gamma\nabla y_u(T)\in\Theta$, we obtain:

$$\frac{1}{2}\|u^*\|^2\le\frac{1}{2}\|u\|^2,$$

which implies that u^* is of minimum energy.

▹ (u^*,z^*) is an optimal solution of (12), then there exists a Lagrange multiplier $\lambda^*\in\left(H^{1/2}(\Gamma)\right)^n$ such that the following optimality conditions hold:

$$\langle u^*,u-u^*\rangle+\left\langle\lambda^*,\chi_\Gamma\gamma\nabla H(u-u^*)\right\rangle=0\qquad\forall\,u\in U_{ad},\tag{17}$$

$$-\langle\lambda^*,z-z^*\rangle\ge0\qquad\forall\,z\in\Theta,\tag{18}$$

$$\left\langle\lambda-\lambda^*,\chi_\Gamma\gamma\nabla y_{u^*}(T)-z^*\right\rangle=0\qquad\forall\,\lambda\in\left(H^{1/2}(\Gamma)\right)^n\cdot\tag{19}$$

For more details about the saddle point and its theory, we refer to [22–24].
From (17) we deduce (10).
The equation (19) is equivalent to:

$$\chi_\Gamma\gamma\nabla y_{u^*}(T)=z^*$$

And since $y_{u^*}(T)=S(T)y_0+G_\Theta y(\cdot)+Hu^*$, we have:

$$\chi_\Gamma\gamma\nabla\left[S(T)y_0+G_\Theta y(\cdot)+Hu^*\right]-z^*=0\cdot$$

Then $\chi_\Gamma\gamma\nabla\left[S(T)y_0+G_\Theta y(\cdot)\right]+\chi_\Gamma\gamma\nabla Hu^*=z^*$. And with (10), we have:

$$\chi_\Gamma\gamma\nabla\left[S(T)y_0+G_\Theta y(\cdot)\right]-\left(\chi_\Gamma\gamma\nabla H\right)\left(\chi_\Gamma\gamma\nabla H\right)^*\lambda^*=z^*,$$

with $R_\Theta=\left(\chi_\Gamma\gamma\nabla H\right)\left(\chi_\Gamma\gamma\nabla H\right)^*$, we obtain the first equation of (11).
And from inequality (18), we obtain:

$$\langle(\rho\lambda^*+z^*)-z^*,z-z^*\rangle_{\left(H^{1/2}(\Gamma)\right)^n}\le0\qquad\forall\,z\in\Theta\,,\ \rho>0,$$

which is equivalent to the second equation of (11).

5 Conclusion

This paper is concerned with the regional gradient controllability, which is motivated by many real applications where the objective is to explore the minimum energy control to steer the system under consideration from the initial gradient vector ∇y_0 to any gradient vector in an interested subregion of the whole domain. The presented results here can provide some insight into the control theory analysis of such systems. They can also be extended to complex fractional order distributed parameter systems and various open questions are still under consideration.

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References

1. R. F. Curtain, H. Zwart, An Introduction to Infinite Dimensional Linear Systems Theory, Texts in Applied Mathematics, **21**, Springer-Verlag, New York, 1995.
2. R. F. Curtain, A. J. Pritchard, Infinite Dimensional Linear Systems Theory, Springer Lecture Notes in Control and Informations, Springer-Verlag, Berlin Heidelberg, 1978.
3. A. El Jai, A. J. Pritchard, Sensors and Actuators in the Analysis of Distributed Systems, Wiley, New York, 1988.

4. A. El Jai, M. C. Simon, E. Zerrik, A. J. Pritchard, ”Regional controllability of distributed parameter systems.”, International Journal of Control, **62**(6), 1351-1365, 1995.

5. E. Zerrik, R. Larhrissi, ”Regional target control of the wave equation.”, Int. J. Syst. Sci., **32**, 105-110, 2001.

6. E. Zerrik, R. Larhrissi, ”Regional Boundary Controllability of Hyperbolic Systems. Numerical Approach.”, Journal of Dynamical and Control Systems, **8**, 293-311, 2002.

7. E. Zerrik, F. Ghafrani, ”Minimum energy control subject to output constraints numerical approach.”, IEE Proc.-Control Theory Appl., **149**(1), 105-110, 2002.

8. A. Boutoulout, H. Bourray, F. Z. El Alaoui, L. Ezzahri, ”Constrained Controllability for Distributed Hyperbolic Systems.”, Math. Sci. Lett., **3**(3), 207-214, 2014.

9. T. Karite, A. Boutoulout, ”Regional constrained controllability for parabolic semilinear systems.”, International Journal of Pure and Applied Mathematics, **113**(1), 113-129, 2017.

10. T. Karite, A. Boutoulout, F. Z. El Alaoui, ”Regional Gradient Exact Enlarge Controllability of the Semilinear Heat Equation.”, General Letters in Mathematics, **2**(2), 84-93, 2017.

11. T. Karite, A. Boutoulout, ”Regional boundary controllability of semi-linear parabolic systems with state constraints.”, Int. J. Dynamical Systems and Differential Equations, **8**(1/2), 150-159 , 2018

12. J. L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications, Dunod, Paris, **1**, 1968.

13. E. Zeidler, Applied functional analysis : Applications to mathematical physics, Springer-Verlag, New York, 1995.

14. S. B. Agase, V. Rchavendre, ”Existence of mild solutions of semilinear differential equations in Banach spaces.”, Indian J. pure appl. Math., **21**(9), 813-821, 1990.

15. M. Kurula, H. Zwart, ”The Duality Between the Gradient and Divergence Operators on Bounded Lipschitz Domains.”, Department of Applied Mathematics, University of Twente, October 2012.

16. A. Kamal, A. Boutoulout, S. A. Ould Beinane, ”Regional Gradient Controllability of Semi-Linear Parabolic Systems.”, International Review of Automatic Control, **6**(5), 2013.

17. E. Zerrik, A. El Jai, A. Boutoulout, ”Actuators and regional boundary controllability of parabolic system.”, Int. J. Syst. Sci. **31**(1), 73-82, 2000.

18. I. Ekeland, R. Temam, Convex analysis and variational problems, (Studies in mathematics and its applications; **1**), North-Holland Publ. Comp., 1976.

19. J. P. Aubin, S. Wilson, Optima and Equilibria: An Introduction to Nonlinear Analysis, Springer-Verlag, Berlin Heidelberg, 2002.

20. A. G. Kusraev, S. S. Kutateladze, Subdifferentials: Theory and Applications, Springer-Verlag, Netherlands, 1995.

21. A. Matei, ”Weak solvability via Lagrange multipliers for two frictional contact models.”, Annals of the university of Bucharest (mathematical series), **4**, (LXII), 179-191, 2013.

22. F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, 1991.

23. M. Fortin, R. Glowinski, Augmented Lagrangian Methods: Applications to the numerical solution of boundary-value problems, North-Holland, Amsterdam, **15**, 1983.

24. R. T. Rockafellar, ”Lagrange multipliers and optimality”, SIAM Review, **35**(2), 183-238, 1993.

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L^∞-Estimates for Nonlinear Degenerate Elliptic Problems with p-growth in the Gradient

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ABSTRACT

In this work, we will prove the existence of bounded solutions for the nonlinear elliptic equations

$$-\,div(a(x,u,\nabla u)) = g(x,u,\nabla u) - \,div f,$$

in the setting of the weighted Sobolev space $W^{1,p}(\Omega,w)$ where a, g are Carathéodory functions which satisfy some conditions and f satisfies suitable summability assumption.

1 Introduction

Let Ω be a regular bounded domain of $\mathbb{R}^N$, $N > 1$ and let us consider the problem:

$$\begin{cases} -\operatorname{div}\Big(a(x,u,\nabla u)\Big) = g(x,u,\nabla u) - \operatorname{div} f \text{ in } \mathcal{D}'(\Omega), \\ u \in W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega), \end{cases} \tag{1}$$

where $-\operatorname{div}(a(x,u,\nabla u))$ is a Leray-Lions operator acting from $W_0^{1,p}(\Omega,w)$ into its dual $W^{-1,p'}(\Omega,w^{1-p'})$ with $p > 1$ and $\frac{1}{p}+\frac{1}{p'} = 1$, g is a nonlinearity which satisfies the growth condition, but it does not satisfy any sign condition. And f satisfies suitable summability assumption.

In [1], the authors proved the existence results in the setting of weighted Sobolev spaces for quasilinear degenerated elliptic problems associated with the following equation $-\operatorname{div}\Big(a(x,u,\nabla u)\Big) + g(x,u,\nabla u) = f$, where g satisfies the sign condition.

In [2], Benkirane and Bennouna studied L^∞ estimates of the solutions in $W_0^{1,p}(\Omega,w)$ of the problem $-\operatorname{div} a(x,u,\nabla u) - \operatorname{div} \phi(u) + g(x,u) = f$ with a non-uniform elliptic condition, and g satisfies the sign condition.

In [3], the authors proved the existence of bounded solutions of the problem $-\operatorname{div} a(x,u,\nabla u) = g - \operatorname{div} f$, whose principal part has a degenerate coercivity, in the setting of weighted Sobolev spaces $W_0^{1,p}(\Omega,w)$.

The equations like (1) have been studied by many authors in the non-degenerate case (i.e. with $w(x) \equiv 1$) (see, e.g., [4] and the references therein).

The aim of this article is to establish a bounded solution for the problem (1) based on rearrangement properties. The results of this work can be considered as an extension of the results in [4] to the weighted case.

In order to perform L^∞-Estimates, the paper is organized in the following way. In section 1, we presented the introduction of the current work. In Section 2 we will state some basic knowledge of Sobolev spaces with weight and properties of the relative rearrangement. Finally in Section 3, we will introduce the essential assumptions, and we will prove our main result.

2 Preliminary results

2.1 Sobolev spaces with weight

In order to discuss the problem (1), we need some theories on $W^{1,p}(\Omega,w)$ which is called Sobolev spaces with weight.

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Firstly we state some basic properties of spaces $W^{1,p}(\Omega, w)$ which will be used later (for details, see [5]). Let Ω be an open subset of $\mathbb{R}^N$ with $N \geq 2$, and $1 \leq p < \infty$ a real number.

Let $w = w(x)$ be a weighted function which is measurable and positive function a.e. in Ω. Define $L^p(\Omega, w) = \{u \ measurable : uw^{\frac{1}{p}} \in L^p(\Omega)\}$. We shall denote by $W^{1,p}(\Omega, w)$ the function space which consists of all real functions $u \in L^p(\Omega)$ such that their weak derivatives $\frac{\partial u}{\partial x_i}$, for all $i = 1, \ldots, N$ (in the sense of distributions) satisfy $\frac{\partial u}{\partial x_i} \in L^p(\Omega, w)$, for all $i = 1, \ldots, N$. Endowed with the norm

$$|||u|||_{p,w} = \left(\int_\Omega |u|^p dx + \int_\Omega |\nabla u|^p w(x) dx \right)^{\frac{1}{p}}, \quad (2)$$

$W^{1,p}(\Omega, w)$ is a Banach space. Further more, we suppose that

$$w \in L^1_{loc}(\Omega), \quad (3)$$

$$w^{-\frac{1}{p-1}} \in L^1(\Omega), \quad (4)$$

Due to condition (3), $C_0^\infty(\Omega)$ is a subset of $W^{1,p}(\Omega, w)$.

Since we are dealing with compactness methods to get solutions of nonlinear elliptic equations, a compact imbedding is necessary. This leads us to suppose that the weight function w also satisfies

$$w^{-q} \in L^1(\Omega), \text{ where } 1 + \frac{1}{q} < p \text{ and } q > \frac{N}{p}. \quad (5)$$

Condition (5) ensures that the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega) \quad (6)$$

is compact.

Therefore, we denote by $W_0^{1,p}(\Omega, w)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{p,w} = \left(\int_\Omega |\nabla u|^p w(x) dx \right)^{\frac{1}{p}}.$$

We remark that condition (4) implies that $W^{1,p}(\Omega, w)$ as well as $W_0^{1,p}(\Omega, w)$ are reflexive Banach spaces if $1 < p < \infty$.

Let us give the following lemmas which will be needed later.

Lemma 2.1 *(See [1]) Assume that* (6) *holds. Let $F : \mathbb{R} \to \mathbb{R}$ be a uniformly Lipschitz function such that $F(0) = 0$. Then, F maps $W_0^{1,p}(\Omega, w)$ into itself. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial (Fou)}{\partial x_i} = \begin{cases} F'(u) \frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.2 *(See [1]) Let $u \in L^r(\Omega, w)$ and let $u_n \in L^r(\Omega, w)$, with $\|u_n\|_{L^r(\Omega,w)} \leq c$, $1 < r < \infty$. If $u_n \to u$ a.e. in Ω, then $u_n \rightharpoonup u$ in $L^r(\Omega, w)$, where $\rightharpoonup$ denotes weak convergence.*

2.2 Properties of the relative rearrangement

In this paragraph, we recall some standard notations and properties about decreasing rearrangements which will be used throughout this paper.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $v : \Omega \to \mathbb{R}$ be a measurable function. If one denotes by $|E|$ the Lebesgue measure of a set E, one can define the distribution function $\mu_v(t)$ of v as:

$$\mu_v(t) = |\{x \in \Omega : |v(x)| > t\}|, \ t \geq 0.$$

The decreasing rearrangement v^* of v is defined as the generalized inverse function of μ_v:

$$v^*(s) = \inf \{t \geq 0 : \mu_v(t) \leq s\}, \ s \in [0, |\Omega|].$$

We recall that v and v^* are equimeasurable, i.e.,

$$\mu_v(t) = \mu_{v^*}(t), \ t \in \mathbb{R}^+.$$

This implies that for any Borel function ψ, it holds that

$$\int_\Omega \psi(v(x)) dx = \int_0^{|\Omega|} \psi(v^*(s)) ds.$$

In particular,

$$\|v^*\|_{L^p(0,|\Omega|)} = \|v\|_{L^p(\Omega)}, \ 1 \leq p < \infty. \quad (7)$$

and, if $v \in L^\infty(\Omega)$,

$$v^*(0) = ess \sup_\Omega |v|.$$

The theory of rearrangements is well known, and its exhaustive treatments can be found for example in [6, 7,8]. Now we recall two notions which allow us to define a "generalized" concept of rearrangement of a function f with respect to a given function v.

Definition 2.1 *(See [9]). Let $f \in L^1(\Omega)$ and $v \in L^1(\Omega)$. We will say that a function $\overline{f}_v \in L^1(0, |\Omega|)$ is a pseudo-rearrangement of f with respect to v if there exists a family $\{D(s)\}_{s \in (0,|\Omega|)}$ of subsets of Ω satisfying the properties:*

(i) $|D(s)| = s$,

(ii) $s_1 < s_2 \Rightarrow D(s_1) \subset D(s_2)$,

(iii) $D(s) = \{x \in \Omega : v(x) > t\}$ *if* $s = \mu_v(t)$,

such that

$$\overline{f}_v(s) = \frac{d}{ds} \int_{D(s)} f(x), \ in \ \mathcal{D}'(\Omega).$$

Definition 2.2 *(See [10]). Let $f \in L^1(\Omega)$ and $v \in L^1(\Omega)$. The following limit exists:*

$$\lim_{\lambda \searrow 0} \frac{(v + \lambda f)^* - v^*}{\lambda} = f_v^*,$$

where the convergence is in $L^p(\Omega)$-weak, if $f \in L^p(\Omega)$, $1 \leq p < \infty$, and in $L^\infty(\Omega)-weak^*$, if $f \in L^\infty(\Omega)$. The function

f_v^* is called the relative rearrangement of f with respect to v. Moreover, one has

$$f_v^*(s) = \frac{dG}{ds}, \text{ in } \mathcal{D}'(\Omega),$$

where

$$G(s) = \int_{\{v>v^*(s)\}} f(x)dx + \int_0^{s-|\{v>v^*(s)\}|} (f|_{\{v=v^*(s)\}})(\sigma)d\sigma.$$

The two notions are equivalent in some precise sense (see [6]). For this reason we will denote both $\overline{f}_v$ and f_v^* by F_v. We only recall a few results which hold for both the pseudo- and the relative rearrangements.

If f and v are non-negative and $v \in W_0^{1,1}(\Omega)$, it is possible to prove the following properties:

$$-\frac{d}{dt}\int_{\{v>t\}} f(x)dx = F_v(\mu_v(t))(-\mu_v'(t)), \text{ for a.e. } t > 0; \quad (8)$$

$$\|F_v\|_{L^p(0,|\Omega|)} \le \|f\|_{L^p(\Omega)}, \quad 1 \le p < \infty. \quad (9)$$

The proofs of (8) and (9) can be found in [9] (for pseudo-rearrangements) and in [11, 12] (for relative rearrangements). We finally recall the following chain of inequalities which holds for any non-negative $v \in W_0^{1,p}(\Omega)$:

$$\begin{aligned} NC_N^{1/N}\mu_v(t)^{1-1/N} &\le -\frac{d}{dt}\int_{\{v>t\}} |\nabla v| dx \\ &\le (-\mu_v'(t))^{1/p'}\left(-\frac{d}{dt}\int_{\{v>t\}} |\nabla v|^p dx\right)^{1/p}, \end{aligned} \quad (10)$$

where C_N denotes the measure of the unit ball in $\mathbb{R}^N$. It is a consequence of the Fleming-Rishel formula [13], the isoperimetric inequality [14] and the Hölder's inequality.

3 Assumptions and main results

Let us now give the precise hypotheses on the problem (1), we assume that the following assumptions: Ω is a bounded open set of $\mathbb{R}^N$ ($N > 1$), $1 < p < +\infty$, Let w be a non-negative real valued measurable function defined on Ω which satisfies (3), (4) and (5). Let $a : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function, such that

$$a(x,s,\xi)\xi \ge \alpha w(x)|\xi|^p, \quad (11)$$

$$[a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0, \quad (12)$$

where α is a strictly positive constant and for all $(\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N$, with $\xi \ne \eta$.

$$|a(x,s,\xi)| \le \beta w(x)^{\frac{1}{p}}(d(x) + |s|^{p-1} + w(x)^{\frac{1}{p'}}|\xi|^{p-1}), \quad (13)$$

for a.e. $x \in \Omega$, all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, some positive function $d(x) \in L^{p'}(\Omega)$, $1 < p \le N$, and $\beta > 0$.

Furthermore, let $g(x,s,\xi) : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which satisfies, for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the following condition

$$|g(x,s,\xi)| \le b_1(x) + b_2(x)w(x)|\xi|^p, \quad (14)$$

where $b_1(x) \in L^m(\Omega)$, $\frac{1}{m} < \frac{p}{N} - \frac{1}{q}$, $m > 1$ and $b_1(x) \ge 0$ a.e. $|b_2(x)| \le \lambda$ a.e. in Ω where λ is a strictly positive constant.

Finally, the right hand side we assume that

$$f w^{-\frac{1}{p}} \in (L^{mp'}(\Omega))^N. \quad (15)$$

Now, we give the definition of weak solutions of problem (1).

Definition 3.1 *We say that a function $u \in W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega)$ is a weak solution of problem* (1) *if*

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla v\,dx = \int_\Omega g(x,u,\nabla u)v\,dx + \int_\Omega f\cdot\nabla v\,dx, \quad (16)$$

for all $v \in W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega)$.

Our main results are collected in the following existence result:

Theorem 3.1 *Suppose that the assumptions* (3)–(5) *and* (11)–(15) *hold true. Then there exists at least one weak solution $u \in W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega)$ of problem* (1).

And in the following theorem

Theorem 3.2 *Let u be a solution of* (1) *and let us assume that* (3)–(5) *and* (11)–(15) *hold true. If b_1 and f satisfy the inequality*

$$\begin{aligned} &\left(NC_N^{1/N}\right)^{-p'}\left(\frac{p'}{\alpha}\|b_1\|_{L^m(\Omega)} + \frac{\lambda p'}{\alpha^{p'+1}}\|f\|^{\frac{p}{p-1}}_{(L^{mp'}(\Omega,w^{\frac{-1}{p}}))^N}\right)^{\frac{p'}{p}} \times \\ &\times \|w^{-q}\|^{\frac{1}{q(p-1)}}_{L^1(\Omega)}\left(\frac{N\gamma}{q}\times\frac{q(p-1)-1}{p\gamma-N}\right)^{1-\frac{1}{q(p-1)}}|\Omega|^{\frac{p\gamma-N}{N\gamma(p-1)}} \\ &+ \frac{p'}{\alpha^{p'/p}NC_N^{1/N}}\|w^{-q}\|^{\frac{1}{qp}}_{L^1(\Omega)}\|f\|^{\frac{1}{p-1}}_{(L^{mp'}(\Omega,w^{\frac{-1}{p}}))^N} \times \\ &\times\left(\frac{N(p\gamma-1)}{p\gamma-N}\right)^{1-\frac{1}{p\gamma}}|\Omega|^{\frac{p\gamma-N}{Np\gamma}} \\ &\le \frac{\alpha(p-1)}{\lambda p'}, \end{aligned} \quad (17)$$

where $1/\gamma = 1/m + 1/q$.
Then there exists a constant $M > 0$, which depends only on N, p, p', q, $|\Omega|$, $\|f\|_{(L^{mp'}(\Omega,w^{\frac{-1}{p}}))^N}$, $\|w^{-q}\|_{L^1(\Omega)}$ and $\|b_1\|_{L^m(\Omega)}$, such that

$$\|u\|_{L^\infty(\Omega)} \le M. \quad (18)$$

Lemma 3.1 *Let u be a solution of* (1) *and let us assume that* (3)–(5) *and* (11)–(15) *hold true. Define*

$$\varphi = \frac{e^{k|u|}-1}{k}, \qquad k = \frac{\lambda p'}{\alpha(p-1)}. \quad (19)$$

Then the decreasing rearrangement of φ satisfies the following differential inequality:

$$
\begin{aligned}
&(-\varphi^*(s))' \\
&\leq \frac{\left[\left(w(x)^{-\frac{1}{p-1}}\right)^*_{\varphi}\right]^{1/p'}(s)[(-\varphi^*(s))']^{1/p}}{NC_N^{1/N}s^{1-1/N}}\times \\
&\quad\times\left(\int_0^s \psi^*(\tau)(k\varphi^*(\tau)+1)^{p-1}d\tau\right)^{1/p} \\
&+\frac{k\varphi^*(s)+1}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\left[\left(w(x)^{-\frac{1}{p-1}}\right)^*_{\varphi}\right]^{1/p'}(s)\Big(F_{\varphi}(s)\Big)^{1/p}
\end{aligned}
\tag{20}
$$

Proof. Let us define two real functions $\phi_1(z)$, $\phi_2(z)$, $z\in\mathbb{R}$, as follows:

$$
\begin{cases}
\phi_1(z)=e^{k(p-1)|z|}\,\mathrm{sign}(z),\\
\phi_2(z)=\frac{(e^{kz}-1)}{k},
\end{cases}
\tag{21}
$$

where $k=\frac{\lambda p'}{\alpha(p-1)}$, we observe that $\phi_2(0)=0$ and for $z\neq 0$, $\phi_1'(z)>0$, $\phi_2'(z)>0$,

$$
\phi_1(z)\phi_2'(|z|)\,\mathrm{sign}(z)=|\phi_2'(|z|)|^p,
\tag{22}
$$

$$
\phi_1'(z)-\frac{\lambda p'}{\alpha}|\phi_1(z)|=0.
\tag{23}
$$

Furthermore, for $t>0$, $h>0$, let us put

$$
S_{t,h}(z)=\begin{cases}
\mathrm{sign}(z) & \text{if } |z|>t+h,\\
((|z|-t)/h)\,\mathrm{sign}(z) & \text{if } t<|z|\leq t+h,\\
0 & \text{if } |z|\leq t.
\end{cases}
\tag{24}
$$

We use in (1) the test function $v\in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)$ defined by

$$
v=\phi_1(u)S_{t,h}(\varphi)=\phi_1(u)S_{t,h}(\phi_2(|u|)),
\tag{25}
$$

where $\varphi=\frac{e^{k|u|}-1}{k}$. Using (24) we have

$$
\begin{aligned}
&\frac{1}{h}\int_{\{t<\varphi\leq t+h\}} a(x,u,\nabla u)\nabla u\,\phi_1(u)\phi_2'(|u|)\mathrm{sign}\,(u)\,dx \\
&-\int_{\{\varphi>t\}} g(x,u,\nabla u)\phi_1(u)S_{t,h}(\varphi)\,dx \\
&+\int_{\{\varphi>t\}} a(x,u,\nabla u)\nabla u\,\phi_1'(u)S_{t,h}(\varphi)\,dx \\
&=\int_{\{\varphi>t\}}\sum_{i=1}^N f_i\frac{\partial u}{\partial x_i}\phi_1'(u)S_{t,h}(\varphi)\,dx \\
&+\frac{1}{h}\int_{\{t<\varphi\leq t+h\}}\sum_{i=1}^N f_i\frac{\partial u}{\partial x_i}\phi_1(u)\phi_2'(|u|)\mathrm{sign}\,(u)\,dx.
\end{aligned}
\tag{26}
$$

Taking into account (22) and Young's inequality, it follows that

$$
\begin{aligned}
&\frac{1}{h}\int_{\{t<\varphi\leq t+h\}} a(x,u,\nabla u)\nabla u\,\phi_1(u)\phi_2'(|u|)\mathrm{sign}(u)\,dx \\
&-\int_{\{\varphi>t\}} g(x,u,\nabla u)\phi_1(u)S_{t,h}(\varphi)\,dx \\
&+\int_{\{\varphi>t\}} a(x,u,\nabla u)\nabla u\,\phi_1'(u)S_{t,h}(\varphi)\,dx \\
&\leq\frac{\alpha^{-p'/p}}{p'}\int_{\{\varphi>t\}}|f|^{p'}w^{\frac{-p'}{p}}\phi_1'(u)S_{t,h}(\varphi)dx \\
&+\frac{\alpha}{p}\int_{\{\varphi>t\}} w(x)|\nabla u|^p\phi_1'(u)S_{t,h}(\varphi)\,dx \\
&+\frac{\alpha^{-p'/p}}{p'h}\int_{\{t<\varphi\leq t+h\}}|f|^{p'}w^{\frac{-p'}{p}}|\phi_2'(|u|)|^p\,dx \\
&+\frac{\alpha}{ph}\int_{\{t<\varphi\leq t+h\}} w(x)|\nabla u|^p|\phi_2'(|u|)|^p\,dx,
\end{aligned}
$$

using (14), and the ellipticity condition (11), we obtain

$$
\begin{aligned}
&\frac{\alpha}{h}\int_{\{t<\varphi\leq t+h\}} w(x)|\nabla u|^p|\phi_2'(|u|)|^p\,dx \\
&\leq\int_{\{\varphi>t\}}\Big(\big(b_1(x)+b_2(x)w(x)|\nabla u|^p\big)\phi_1(u) \\
&-\alpha w(x)|\nabla u|^p\phi_1'(u)\Big)S_{t,h}(\varphi)\,dx \\
&+\frac{\alpha^{-p'/p}}{p'}\int_{\{\varphi>t\}}|f|^{p'}w^{\frac{-p'}{p}}\phi_1'(u)S_{t,h}(\varphi)dx \\
&+\frac{\alpha}{p}\int_{\{\varphi>t\}} w(x)|\nabla u|^p\phi_1'(u)S_{t,h}(\varphi)\,dx \\
&+\frac{\alpha^{-p'/p}}{p'h}\int_{\{t<\varphi\leq t+h\}}|f|^{p'}w^{\frac{-p'}{p}}|\phi_2'(|u|)|^p\,dx \\
&+\frac{\alpha}{ph}\int_{\{t<\varphi\leq t+h\}} w(x)|\nabla u|^p|\phi_2'(|u|)|^p\,dx,
\end{aligned}
$$

By(23), it follows that

$$
\begin{aligned}
&\frac{1}{h}\int_{\{t<\varphi\leq t+h\}} w(x)|\nabla u|^p|\phi_2'(|u|)|^p\,dx \\
&\leq\int_{\{\varphi>t\}}\left(\frac{\lambda p'}{\alpha}|\phi_1(u)|-\phi_1'(u)\right)w(x)|\nabla u|^pS_{t,h}(\varphi)\,dx \\
&+\int_{\{\varphi>t\}}\left(\frac{p'}{\alpha}b_1(x)+\frac{\lambda p'}{\alpha^{p'+1}}|f|^{p'}w^{\frac{-p'}{p}}\right)|\phi_1(u)|S_{t,h}(\varphi)\,dx \\
&+\frac{1}{\alpha^{p'}h}\int_{\{t<\varphi\leq t+h\}}|f|^{p'}w^{\frac{-p'}{p}}|\phi_2'(|u|)|^p\,dx.
\end{aligned}
$$

Using (23) and the definition of ϕ_1, ϕ_2 in (21), the above inequality gives:

$$
\begin{aligned}
&\frac{1}{h}\int_{\{t<\varphi\leq t+h\}} w(x)|\nabla\varphi|^p dx \\
&\qquad\leq\int_{\{\varphi>t\}}\psi(k\varphi+1)^{p-1}S_{t,h}(\varphi)dx \\
&\qquad+\frac{1}{\alpha^{p'}h}\int_{\{t<\varphi\leq t+h\}}|f|^{p'}w^{\frac{-p'}{p}}(k\varphi+1)^p dx
\end{aligned}
\tag{27}
$$

where $\psi=\frac{p'}{\alpha}b_1(x)+\frac{\lambda p'}{\alpha^{p'+1}}|f|^{p'}w^{\frac{-p'}{p}}$.

Letting h go to 0 in a standard way we get:

$$-\frac{d}{dt}\int_{\{\varphi>t\}} w(x)|\nabla\varphi|^p\,dx \le \int_{\{\varphi>t\}} \psi(k\varphi+1)^{p-1}\,dx + \frac{(kt+1)^p}{\alpha^{p'}}\Big(-\frac{d}{dt}\int_{\{\varphi>t\}} F(x)\,dx\Big),$$

with $F(x)=|f(x)|^{p'}w^{\frac{-p'}{p}}(x)$.
Using Hardy-Littlewood's inequality and the inequality (8). It follows that

$$-\frac{d}{dt}\int_{\{\varphi>t\}} w(x)|\nabla\varphi|^p\,dx \le \int_0^{\mu_\varphi(t)} \psi^*(s)(k\varphi^*(s)+1)^{p-1}\,ds + \frac{(kt+1)^p}{\alpha^{p'}}(-\mu'_\varphi(t))F_\varphi(\mu_\varphi(t)), \tag{28}$$

where F_φ is a pseudo-rearrangement (or the relative rearrangement) of $|f|^{p'}$ with respect to φ.

On the other hand, thanks to Hölder's inequality, we can easily check that

$$-\frac{d}{dt}\int_{\{\varphi>t\}} |\nabla u|\,dx \le \Big(-\frac{d}{dt}\int_{\{\varphi>t\}} w(x)|\nabla u|^p\,dx\Big)^{\frac{1}{p}} \times \Big(-\frac{d}{dt}\int_{\{\varphi>t\}} w(x)^{-\frac{1}{p-1}}\,dx\Big)^{1-\frac{1}{p}}. \tag{29}$$

Since $w(x)^{-\frac{1}{p-1}}\in L^1(\Omega)$, we write

$$-\frac{d}{dt}\int_{\{\varphi>t\}} w(x)^{-\frac{1}{p-1}}\,dx = \Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi(\mu_\varphi(t))\times(-\mu'_\varphi(t)), \tag{30}$$

for almost every $t>0$, where $\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi$ is the relative rearrangement of $w(x)^{-\frac{1}{p-1}}$ with respect to φ.
Using the Fleming–Rishel formula (see [8]), we can write

$$-\frac{d}{dt}\int_{\{\varphi>t\}} |\nabla u|\,dx \ge NC_N^{\frac{1}{N}}(\mu_\varphi(t))^{1-\frac{1}{N}} \tag{31}$$

for almost every $t>0$.
Combining (28), (29), (30) and (31), we obtain

$$\begin{aligned}
&NC_N^{1/N}\mu_\varphi(t)^{1-1/N}\\
&\le \Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(\mu_\varphi(t))(-\mu'_\varphi(t))^{1/p'}\times\\
&\qquad\times\Big(\int_0^{\mu_\varphi(t)}\psi^*(s)(k\varphi^*(s)+1)^{p-1}ds\Big)^{1/p}\\
&+\frac{kt+1}{\alpha^{p'/p}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(\mu_\varphi(t))(-\mu'_\varphi(t))\Big(F_\varphi(\mu_\varphi(t))\Big)^{1/p}
\end{aligned}$$

and then, using the definition of $\varphi^*(s)$, we have:

$$\begin{aligned}
&(-\varphi^*(s))'\\
&\le \frac{\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(s)[(-\varphi^*(s))']^{1/p}}{NC_N^{1/N}s^{1-1/N}}\times\\
&\quad\times\Big(\int_0^s\psi^*(\tau)(k\varphi^*(\tau)+1)^{p-1}d\tau\Big)^{1/p}\\
&+\frac{k\varphi^*(s)+1}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(s)\Big(F_\varphi(s)\Big)^{1/p}
\end{aligned}$$

that is (20).

Proof of Theorem 3.2. By using Young's inequality and (20) of Lemma 3.1 implies:

$$\begin{aligned}
(-\varphi^*(s))' &\le \frac{1}{p}(-\varphi^*(s))' + \frac{\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big](s)}{p'(NC_N^{1/N}s^{1-1/N})^{p'}}\times\\
&\quad\times\Big(\int_0^s\psi^*(\tau)(k\varphi^*(\tau)+1)^{p-1}d\tau\Big)^{p'/p}\\
&\quad+\frac{k\varphi^*(s)+1}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(s)\Big(F_\varphi(s)\Big)^{\frac{1}{p}}.
\end{aligned}$$

We deduce that

$$\begin{aligned}
(-\varphi^*(s))' &\le \frac{\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big](s)}{(NC_N^{1/N}s^{1-1/N})^{p'}}\times\\
&\quad\times\Big(\int_0^s\psi^*(\tau)(k\varphi^*(\tau)+1)^{p-1}d\tau\Big)^{p'/p}\\
&\quad+\frac{p'(k\varphi^*(s)+1)}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(s)\Big(F_\varphi(s)\Big)^{\frac{1}{p}}.
\end{aligned}$$

Integrating between 0 and $|\Omega|$, since $\varphi(|\Omega|)=0$, we have

$$\begin{aligned}
&\varphi^*(0)=\int_0^{|\Omega|}(-\varphi^*(s))'ds\\
&\le\int_0^{|\Omega|}\frac{1}{(NC_N^{1/N}s^{1-1/N})^{p'}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big](s)\Big(\int_0^s\psi^*(\tau)d\tau\Big)^{p'/p}ds\times\\
&\times(k\varphi^*(0)+1)^{(p-1)p'/p}\\
&+\int_0^{|\Omega|}\frac{p'}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(s)\Big(F_\varphi(s)\Big)^{\frac{1}{p}}ds\times\\
&\times(k\varphi^*(0)+1).
\end{aligned}$$

Since φ^* attains its maximum at 0, we can write

$$\|\varphi\|_{L^\infty(\Omega)}\le kA\|\varphi\|_{L^\infty(\Omega)}+A, \tag{32}$$

where $k=\frac{\lambda p'}{\alpha(p-1)}$ and

$$\begin{aligned}
A&=\int_0^{|\Omega|}\frac{1}{(NC_N^{1/N}s^{1-1/N})^{p'}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big](s)\Big(\int_0^s\psi^*(\tau)d\tau\Big)^{p'/p}ds\\
&\quad+\int_0^{|\Omega|}\frac{p'}{\alpha^{p'/p}NC_N^{1/N}s^{1-1/N}}\Big[\Big(w(x)^{-\frac{1}{p-1}}\Big)^*_\varphi\Big]^{1/p'}(s)\Big(F_\varphi(s)\Big)^{\frac{1}{p}}ds\\
&=I+J.
\end{aligned}$$

In order to estimate I, we use the Hölder's inequality and (7), obtaining

$$\begin{aligned}&\int_0^s \psi^*(\tau)d\tau \leq \|\psi\|_{L^m(\Omega)} s^{1-1/m}\\ &\leq (\tfrac{p'}{\alpha}\|b_1\|_{L^m(\Omega)} + \tfrac{\lambda p'}{\alpha^{p'+1}}\||f w^{\frac{-1}{p}}|^{p'}\|_{L^m(\Omega)}) s^{1-1/m} \\ &\leq (\tfrac{p'}{\alpha}\|b_1\|_{L^m(\Omega)} + \tfrac{\lambda p'}{\alpha^{p'+1}}\|f\|^{\frac{p}{p-1}}_{(L^{mp'}(\Omega, w^{\frac{-1}{p}}))^N}) s^{1-1/m},\end{aligned} \tag{33}$$

which we use to get

$$I \leq \left(NC_N^{1/N}\right)^{-p'} \left(\frac{p'}{\alpha}\|b_1\|_{L^m(\Omega)} + \frac{\lambda p'}{\alpha^{p'+1}}\|f\|^{\frac{p}{p-1}}_{(L^{mp'}(\Omega, w^{\frac{-1}{p}}))^N}\right)^{\frac{p'}{p}} \times \int_0^{|\Omega|} \left(w(x)^{-\frac{1}{p-1}}\right)^*_{\varphi}(s) s^{\frac{p'}{p}(1-\frac{1}{m})-p'(1-\frac{1}{N})} ds.$$

By (5) one has $q(p-1) > 1$, we use again Hölder's inequality to obtain

$$I \leq \left(NC_N^{1/N}\right)^{-p'} \left(\frac{p'}{\alpha}\|b_1\|_{L^m(\Omega)} + \frac{\lambda p'}{\alpha^{p'+1}}\|f\|^{\frac{p}{p-1}}_{(L^{mp'}(\Omega, w^{\frac{-1}{p}}))^N}\right)^{\frac{p'}{p}} \times \|w^{-q}\|^{\frac{1}{q(p-1)}}_{L^1(\Omega)} \left(\int_0^{|\Omega|} t^{(\frac{p'}{p}(1-\frac{1}{m})-p'(1-\frac{1}{N}))\frac{q(p-1)}{q(p-1)-1}} dt\right)^{1-\frac{1}{q(p-1)}}.$$

The assumptions on exponents (5) and (14) allow us to get

$$I \leq \left(NC_N^{1/N}\right)^{-p'} \left(\frac{p'}{\alpha}\|b_1\|_{L^m(\Omega)} + \frac{\lambda p'}{\alpha^{p'+1}}\|f\|^{\frac{p}{p-1}}_{(L^{mp'}(\Omega, w^{\frac{-1}{p}}))^N}\right)^{\frac{p'}{p}} \times \|w^{-q}\|^{\frac{1}{q(p-1)}}_{L^1(\Omega)} \left(\frac{N\gamma}{q} \times \frac{q(p-1)-1}{p\gamma - N}\right)^{1-\frac{1}{q(p-1)}} |\Omega|^{\frac{p\gamma-N}{N\gamma(p-1)}}. \tag{34}$$

We now turn to estimate J. Since $p\gamma > N > 1$, we can consider the Hölder conjugate exponent $\eta = \frac{p\gamma}{p\gamma - 1}$. The conjugate exponent η satisfies the identity

$$\frac{1}{qp} + \frac{1}{mp} + \frac{1}{\eta} = 1,$$

so that by Hölder's inequality we obtain

$$J \leq \frac{p'}{\alpha^{p'/p} NC_N^{1/N}} \|w^{-q}\|^{\frac{1}{qp}}_{L^1(\Omega)} \left(\int_\Omega |f|^{mp'} w^{\frac{-m}{p-1}} dx\right)^{\frac{1}{mp}} \times \left(\int_0^{|\Omega|} s^{\eta(\frac{1}{N}-1)} ds\right)^{\frac{1}{\eta}}.$$

Then we have

$$J \leq \frac{p'}{\alpha^{p'/p} NC_N^{1/N}} \|w^{-q}\|^{\frac{1}{qp}}_{L^1(\Omega)} \|f\|^{\frac{1}{p-1}}_{(L^{mp'}(\Omega, w^{\frac{-1}{p}}))^N} \times \left(\frac{N(p\gamma-1)}{p\gamma - N}\right)^{1-\frac{1}{p\gamma}} |\Omega|^{\frac{p\gamma-N}{Np\gamma}}. \tag{35}$$

Using (34) and (35) we can estimate the quantity A in (32), Obtaining that under assumption (17) the following inequality holds:

$$\frac{\lambda p'}{\alpha(p-1)} A < 1.$$

Then (32) implies (18).

Proof of Theorem 3.1.

Let us define for $\varepsilon > 0$ the approximation

$$g_\varepsilon(x, s, \xi) = \frac{g(x, s, \xi)}{1 + \varepsilon |g(x, s, \xi)|}.$$

On note that $|g_\varepsilon(x, s, \xi)| \leq |g(x, s, \xi)|$, and $|g_\varepsilon(x, s, \xi)| < \frac{1}{\varepsilon}$. we consider the approximate problem

$$\begin{cases} -\operatorname{div}\Big(a(x, u_\varepsilon, \nabla u_\varepsilon)\Big) = g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) - \operatorname{div} f \text{ in } \mathcal{D}'(\Omega), \\ u_\varepsilon \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega). \end{cases} \tag{36}$$

Since the operator $A - g_\varepsilon : W_0^{1,p}(\Omega, w) \to W^{-1,p'}(\Omega, w^{1-p'})$ is bounded, coercive, and pseudo-monotone operator, where $A(u) = -\operatorname{div}\Big(a(x, u, \nabla u)\Big)$, there exists at least one solution $u_\varepsilon \in W_0^{1,p}(\Omega, w)$ of the problems (36)(see [15], [16] and in the weighted case [1]).
Using Stampacchia's method [17], one can prove that any solution u_ε of (36) belongs to $L^\infty(\Omega)$ for fixed ε. Finally using the L^∞ estimate obtained in Theorem 3.2 and working as in [15] and [1], but with obvious modifications, we obtain Theorem 3.1.

4 References

1. Y. Akdim, E. Azroul, and A. Benkirane, "Existence results for quasilinear degenerated equations via strong convergence of truncations", - Rev. Mat. Complut. 17:2, 2004, 359–379.
2. A. Benkirane, J. Bennouna, "Existence of solutions for nonlinear elliptic degenerate equations", Nonlinear Anal.,(2003), **54**, 9–37.
3. J. Bennouna, M. Hammoumi and A. Youssfi, "on L^∞–regularity result for some degenerate nonlinear elliptic equations", Annales Academiae Scientiarum Fennicae. Mathematica.,(2014) **39**, 873–886.
4. V. Ferone, M. R. Posteraro and J. M. Rakotoson, "L^∞–Estimates for nonlniear elliptic problems with p-growth in the gradient",J. of lnequal. and Appl., (1999), **3**, 109–125.
5. P. Drabek, A. Kufner, and F. Nicolosi, "Non linear elliptic equations, singular and degenerate cases", University of West Bohemia, Pilsen, 1996.
6. V. Ferone, "Riordinamenti e applicazioni", Tesi di Dottorato, Universit di Napoli, (1990).
7. B. Kawohl, "Rearrangements and Convexity of Level Sets in P.D.E.", Lecture Notes in Math., No. **1150**, Springer, Berlin, New York, (1985).
8. J. Mossino, "Inégalités isopérimétriques et applications en physique", Collection Travaux en Cours, Hermann, Paris, (1984).
9. A. Alvino and G. Trombetti, "Sulle migliori costanti di maggiorazione per una classe di equazioni ellitfiche degeneri", Ricerche Mat., (1978), **27**, 413–428.
10. J. Mossino and J.M. Rakotoson, "Isoperimetric inequalities in parabolic equations", Ann. Scuola Norm. Sup Pisa, (1986), **13** , 51–73.
11. J.M. Rakotoson and R. Temam, "Relative rearrangement in quasilinear variational inequalities", Indiana Math. J., (1987), **36**, 757–810.

12. G. Talenti, ”Elliptic equations and rearrangements”, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (1976), (4) **3**, 697–718.

13. W. Fleming and R. Rishel, ”An integral formula for total gradient variation”, Arch.Math., (1960), **11**, 218–222.

14. E. De Giorgi, ”Su una teoria generale della misura (r-1)-dimensionale in uno spazio ad dimensioni”, Ann. Mat. Pura e Appl., (1954), **36**, 191–213.

15. L. Boccardo, F. Murat and J.P. Puel, ”Existence of bounded solutions for nonlinear elliptic unilateral problems”, Ann. Mat. Pura Appl.,(1988) **152**, 183–196.

16. J.L. Lions, ”Quelques mthodes de rsolution des problmes aux limites non linaires”, Dunod, Paris, 1969.

17. G. Stampacchia, ”Le probleme de Dirichlet pour les quations elliptiques du second ordre à coefficients discontinus”. Ann. Inst. Fourier (Grenoble), 15, (1965), 189-258.

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Doubly Nonlinear Parabolic Systems In Inhomogeneous Musielak-Orlicz-Sobolev Spcaes

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ABSTRACT

In this paper, we discuss the solvability of the nonlinear parabolic systems associated to the nonlinear parabolic equation: $\frac{\partial b_i(x,u_i)}{\partial t} - div(a(x,t,u_i,\nabla u_i)) - \phi_i(x,t,u_i)) + f_i(x,u_1,u_2) = 0$, *where the function* $b_i(x,u_i)$ *verifies some regularity conditions, the term* $\Big(a(x,t,u_i,\nabla u_i)\Big)$ *is a generalized Leray-Lions operator and* ϕ_i *is a Carathéodory function assumed to be continuous on* u_i *and satisfy only a growth condition. The source term* $f_i(t,u_1,u_2)$ *belongs to* $L^1(\Omega\times(0,T))$.

1 Introduction

Given a bounded-connected open set $\Omega$of $\mathbb{R}^N$ $(N=2)$, with Lipschitz boundary $\partial\Omega$, $Q_T = \Omega\times(0,T)$ is the generic cylinder of an arbitrary finite hight, $T<+\infty$. We prove the existence of a renormalized solutions for the nonlinear parabolic systems:

$$\frac{\partial b_i(x,u_i)}{\partial t} - \mathrm{div}(a(x,t,u_i,\nabla u_i)) - div(\Phi_i(x,t,u_i)) + f_i(x,u_1,u_2) = 0 \quad \text{in } Q_T, \tag{1.1}$$

$$u_i = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{1.2}$$

$$b_i(x,u_i)(t=0) = b_i(x,u_{i,0}) \quad \text{in } \Omega, \tag{1.3}$$

where i=1,2. Here the vector field $a:\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that $-\mathrm{div}(a(x,t,u_i,\nabla u_i))$ is a Leray-lions operator defined from the Inhomogeneous Musielak-Orlicz-Sobolev Spcaes $W_0^{1,x}L_\varphi(Q_T)$into its dual $W^{-1,x}L_\psi(Q_T)$. Let $b_i : \Omega\times\mathbb{R}\to\mathbb{R}$ a Carathéodory function such that for every $x\in\Omega$, $b_i(x,.)$ is a strictly increasing C^1-function, the divegential term $\Phi_i(x,t,u_i)$ is a Carathéodory function satisfy only a polynomial growth with respect to the anisotropic N-function φ (see (4.6)), the data $u_{0,i}$ is in $L^1(\Omega)$ such that $b_i(.,u_{0,i})$ in $L^1(\Omega)$ and the source f_i is a Carathéodory function satisfy the assumptions ((4.7)-(4.10)). When problem ((1.1)) is investigated, there is a difficulty due to the fact that the data $b_1(x,u_0^1(x))$ and $b_2(x,u_0^2(x))$ only belong to L^1 and the functions $a(x,t,u_i,\nabla u_i)$,$\Phi_i(x,t,u_i)$ and $f_i(x,u_1,u_2)$ do not belong to $(L^1_{loc}(Q_T))^N$ in general, so that proving existence of weak solution seems to be an arduous task, and we cannot use the Stocks formula in the a priori estimates of the nonlinearity ,$\Phi_i(x,t,u_i)$. In order to overcome this difficulty, we work with the framework of renormalized solutions (see Definition 3.1). One of the models of applications of these operators is the system of Boussinesq:

$$\frac{\partial u}{\partial t} + (u.\nabla))u - 2div(\mu(\theta)\varepsilon(u)) + \nabla p = F(\theta) \quad \text{in} \quad Q_T$$

$$\frac{\partial b(\theta)}{\partial t} + u.\nabla b(\theta) - \triangle\theta = 2\mu(\theta)|\varepsilon(u)|^2 \quad \text{in} \quad Q_T$$

$$u(t=0) = u_0,\ b(\theta)(t=0) = b(\theta_0) \quad \text{on} \quad \Omega$$

$$u = 0 \quad \theta = 0 \quad \text{on} \quad \partial\Omega\times(0,T)$$

Equation first equation is the motion conservation equation, the unknowns are the fields of displacement $u : Q_T \to \mathbb{R}^N$ and temperature $\theta : Q_T \to \mathbb{R}$, The field $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$ is the strain rate tensor.

It is our purpose, in this paper to generalize the result of ([1], [2], [3]) and we prove the existence of a renormalized solution of system (1.1).

The plan of the paper is as follows: In Section 2 we give the framework space, in Section 3 and 4 we

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give some useful Lemmas and basics assumptions. In Section 5 we give the definition of a renormalized solution of (1.1), and we establish (Theorem 5.1) the existence of such a solution.

2 Preliminaries

2.1 Musielak-Orlicz function

Let Ω be an open subset of $\mathbb{R}^N$ ($N \geq 2$), and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying conditions:

Φ_1:$\varphi(x,.)$ is an N-function for all $x \in \Omega$ (i.e. convex, non-decreasing, continuous, $\varphi(x,0) = 0$,$\varphi(x,0) > 0$ for $t > 0$, $\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t} = 0$ and $\lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t} = \infty$).

Φ_2:$\varphi(.,t)$ is a measurable function for all $t \geq 0$.

A function φ which satisfies the conditions Φ_1 and Φ_2 is called a Musielak-Orlicz function.

For a Musielak-Orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its non-negative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t$$

Let φ and γ be two Musielak-Orlicz functions, we say that φ dominate γ, and we write $\gamma \prec \varphi$, near infinity (resp.globally) if there exist two positive constants c and t_0 such that for a.e. $x \in \Omega$
$\gamma(x,t) \leq \varphi(x,ct)$ for all $t \geq t_0$ (resp. for all $t \geq 0$). We say that γ grows essentially less rapidly than φ at 0(resp. near infinity) and we write $\gamma << \varphi$, for every positive constant c, we have
$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0$
(resp.$\lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0\Big)$

Remark 2.1 *[4]. If $\gamma << \varphi$ near infinity,then $\forall\epsilon > 0$ there exist $k(\epsilon) > 0$ such that for almost all $x \in \Omega$, we have*

$$\gamma(x,t) \leq k(\epsilon)\varphi(x,\epsilon t) \quad \forall t \geq 0$$

2.2 Musielak-Orlicz space

For a Musielak-Orlicz function φ and a measurable function $u : \Omega \to \mathbb{R}$, we define the functionnal

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x,|u(x)|)dx.$$

The set $K_\varphi(\Omega) = \{u : \Omega \to \mathbb{R} \text{ mesurable} : \varrho_{\varphi,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class . The Musielak-Orlicz space $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$; that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \{u : \Omega \to \mathbb{R} \quad \text{mesurable} : \quad \varrho_{\varphi,\Omega}(\frac{u}{\lambda}) < \infty, \quad \text{for some} \quad \lambda > 0\}$$

For any Musielak-Orlicz function φ, we put $\psi(x,s) = \sup_{t\geq 0}(st - \varphi(x,s))$.

ψ is called the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sense of Young with respect to s. We say that a sequence of function $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that $\lim_{n\to\infty}\varrho_{\varphi,\Omega}(\frac{u_n - u}{\lambda}) = 0$

This implies convergence for $\sigma(\Pi L_\varphi, \Pi L_\psi)$[see [5]].

In the space $L_\varphi(\Omega)$, we define the following two norms

$$\|u\|_\varphi = \inf\left\{\lambda > 0 : \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx \leq 1\right\}$$

which is called the Luxemburg norm, and the so-called Orlicz norm by

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1}\int_\Omega |u(x)v(x)|dx$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent [8]. $K_\varphi(\Omega)$ is a convex subset of $L_\varphi(\Omega)$. The closure in $L_\varphi(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is by denoted $E_\varphi(\Omega)$. It is a separable space and $(E_\varphi(\Omega))^* = L_\varphi(\Omega)$. We have $E_\varphi(\Omega) = K_\varphi(\Omega)$, if and only if φ satisfies the Δ_2-condition for large values of t or for all values of t, according to whether Ω has finite measure or not.

We define

$$W^1L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : D^\alpha u \in L_\varphi(\Omega), \quad \forall\alpha \leq 1\}$$
$$W^1E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) : D^\alpha u \in E_\varphi(\Omega), \quad \forall\alpha \leq 1\}$$

where $\alpha = (\alpha_1, ..., \alpha_N)$,$|\alpha| = |\alpha_1| + ... + |\alpha_N|$ and : $D^\alpha u$ denote the distributional derivatives. The space $W^1L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space.

Let
$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq 1}\varrho_{\varphi,\Omega}(D^\alpha u)$ and $\|u\|^1_{\varphi,\Omega} = \inf\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1\}$ for $u \in W^1L_\varphi(\Omega)$.

These functionals are convex modular and a norm on $W^1L_\varphi(\Omega)$, respectively. Then pair $(W^1L_\varphi(\Omega), \|u\|^1_{\varphi,\Omega})$ is a Banach space if φ satisfies the following condition [6].

$$\text{There exists a constant} \quad c > 0 \quad \text{such that} \quad \inf_{x\in\Omega}\varphi(x,1) > c$$

The space $W^1L_\varphi(\Omega)$ is identified to a subspace of the product $\prod_{\alpha\leq 1}L_\varphi(\Omega) = \prod L_\varphi$ We denote by $\mathcal{D}(\Omega)$ the Schwartz space of infinitely smooth functions with compact support in Ω and by $\mathcal{D}(\overline{\Omega})$ the restriction of $\mathcal{D}(\mathbb{R})$ on Ω. The space $W^1_0L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^1L_\varphi(\Omega)$ and the space $W^1_0E_\varphi(\Omega)$ as the(norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1L_\varphi(\Omega)$. For two complementary Musielak-Orlicz functions φ and ψ, we have (See [7]).

- The Young inequality:

$$st \leq \varphi(x,s) + \psi(x,t) \text{ for all } s,t \geq 0 \text{ , } x \in \Omega.$$

- The Hölder inequality

$$\left|\int_\Omega u(x)v(x)dx\right| \leq \|u\|_{\varphi,\Omega}|||v|||_{\psi,\Omega} \text{ for all } u \in L_\varphi(\Omega), v \in L_\psi(\Omega)$$

We say that a sequence of functions u_n converges to u for the modular convergence in $W^1L_\varphi(\Omega)$ (respectively in $W_0^1L_\varphi(\Omega)$) if, for some $\lambda > 0$.

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0$$

The following spaces of distributions will also be used

$$W^{-1}L_\psi(\Omega) = \Big\{f \in \mathcal{D}'(\Omega) : f = \sum_{\alpha\le 1}(-1)^\alpha D^\alpha f_\alpha \quad \text{where} \quad f_\alpha \in L_\psi(\Omega)\Big\}$$

and

$$W^{-1}E_\psi(\Omega) = \Big\{f \in \mathcal{D}'(\Omega) : f = \sum_{\alpha\le 1}(-1)^\alpha D^\alpha f_\alpha \quad \text{where} \quad f_\alpha \in E_\psi(\Omega)\Big\}$$

2.3 Inhomogeneous Musielak-Orlicz-Sobolev spcaes:

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $Q = \Omega\times]0,T[$ with some given $T > 0$. let φ be a Musielak-Orlicz function.For each $\alpha \in N^N$, denote by D_x^α the distributional derivative on Q_T of order α with respect to the variable $x \in R^N$. The inhomogeneous Musielak-Orlicz-Sovolev spaces of order 1 are defined as follows

$$W^{1,x}L_\varphi(Q) = \Big\{u \in L_\varphi(Q) : \forall|\alpha| \le 1, \quad D_x^\alpha u \in L_\varphi(Q)\Big\}$$

$$W^{1,x}E_\varphi(Q) = \Big\{u \in E_\varphi(Q) : \forall|\alpha| \le 1, \quad D_x^\alpha u \in E_\varphi(Q)\Big\}$$

The last is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha|\le m} \|D_x^\alpha u\|_{\varphi,Q}$$

We can easily show that they form a complementary system when Ω is a Lipschitz domain. These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q)$ which has $(N+1)$ copies. We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$. If $u \in W^{1,x}L_\varphi(Q)$ then the function : $t \mapsto u(t) = u(t,.)$ is defined on $(0,T)$ with values in $W^1L_\varphi(\Omega)$. If, further, $u \in W^{1,x}E_\varphi(Q)$ then this function is a $W^1E_\varphi(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds: $W^{1,x}E_\varphi(Q) \subset L^1(0,T;W^1E_\varphi(\Omega))$. The space $W^{1,x}L_\varphi(Q)$ is not in general separable, if $u \in W^{1,x}L_\varphi(Q)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto u(t) = \|u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_\varphi(Q)$ is defined as the (norm) closure in $W^{1,x}E_\varphi(Q)$ of $\mathcal{D}(\Omega)$. We can easily show that when Ω is a Lipschitz domain then each element u of the closure of $\mathcal{D}(\Omega)$ with respect of the weak* topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is limit, in $W^{1,x}L_\varphi(Q)$, of some subsequence $(u_i) \in \mathcal{D}(\Omega)$ for the modular convergence, i.e. there exists $\lambda > 0$ such that for all $|\alpha| \le 1$,

$$\int_Q \varphi(x, (\frac{D_x^\alpha u_i - D_x^\alpha u}{\lambda})dxdt \to 0 \quad \text{as} \quad i \to \infty$$

, this implies that (u_i) converge to u in $W^{1,x}L_\varphi(Q)$ for the weak topology $\sigma(\Pi L_\varphi, \Pi L_\psi)$. Consequently

$$\mathcal{D}(Q)^{\sigma(\Pi L_\varphi, \Pi E_\psi)} = \mathcal{D}(Q)^{\sigma(\Pi L_\varphi, \Pi L_\psi)}$$

, this space will be denoted by $W_0^{1,x}L_\varphi(Q)$. Furthermore $W_0^{1,x}E_\varphi(Q) = W_0^{1,x}L_\varphi(Q) \cap \Pi E_\varphi$. We have the following complementary system F being the dual space of $W_0^{1,x}E_\varphi(Q)$. It is also, except for an isomorphism, the quotient of ΠL_ψ by the polar set $W_0^{1,x}E_\varphi(Q)^\perp$, and will be denoted by $F = W^{1,x}L_\psi(Q)$ and it is shown that

this space will be equipped with the usual quotient norm

where the inf is taken on all possible decompositions

The space F_0 is then given by $F_0 = W^{-1,x}E_\psi(Q)$.

Lemma 2.1 *[4]. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

- *There exists a constant $c > 0$ such that*

$$\inf_{x\in\Omega} \varphi(x,1) > c \tag{2.1}$$

- *$\exists A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \le \frac{1}{2}$, we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \le t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \textit{for all} \quad t \ge 1. \tag{2.2}$$

- $$\int_\Omega \varphi(y,1)dx < \infty \tag{2.3}$$

- $$\exists C > 0 \quad \textit{such that} \quad \psi(y,t) \le C \quad \textit{a.e.} \quad \textit{in} \quad \Omega \tag{2.4}$$

Under this assumptions $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L\psi(\Omega)$ on an element u of $W_0^1L_\varphi(\Omega)$ is well defined. It will be denoted by $< S, u >$.

2.4 Truncation Operator

T_k, $k > 0$, denotes the truncation function at level k defined on $\mathbb{R}$ by $T_k(r) = \max(-k, \min(k, r))$. The following abstract lemmas will be applied to the truncation operators.

Lemma 2.2 *[4]. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian,with $F(0) = 0$. Let φ be an Musielak-Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$(resp.$u \in W^1 E_\varphi(\Omega)$). Then $F(u) \in W^1 L_\varphi(\Omega)$(resp.$u \in W_0^1 E_\varphi(\Omega)$).Moreover, if the set of discontinuity points D of F' is finite,then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(x)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega;\, u(x) \notin D\} \\ 0 & \text{a.e. in } \{x \in \Omega;\, u(x) \in D\} \end{cases}$$

Lemma 2.3 *Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then, there exists a sequence $u_n \in \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_\varphi(\Omega).$$

Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \le (N+1)\|u\|_\infty$.

Let Ω be an open subset of $\mathbb{R}^N$ and let φ be a Musielak-Orlicz function satisfying :

$$\int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt = \infty \quad \text{a.e.} \quad x \in \Omega \tag{2.5}$$

and the conditions of Lemma 2.1. We may assume without loss of generality that

$$\int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt < \infty \quad \text{a.e.} \quad x \in \Omega \tag{2.6}$$

Define a function $\varphi^* : \Omega \times [0, \infty)$ by $\varphi^*(x, s) = \int_0^s \frac{\varphi_x^{-1}(t)}{t^{\frac{N+1}{N}}} dt$ $x \in \Omega$ and $s \in [0, \infty)$.
φ^*its called the Sobolev conjugate function of φ (see [1] for the case of Orlicz function).

Theorem 2.1 *Let Ω be a bounded Lipschitz domain and let φ be a Musielak-Orlicz function satisfying 2.5,2.6 and the conditions of Lemma 2.1. Then*

$$W_0^1 L_\varphi(\Omega) \hookrightarrow L_{\varphi^*}(\Omega)$$

where φ^* is the Sobolev conjugate function of φ. Moreover, if ϕ is any Musielak-Orlicz function increasing essentially more slowly than φ^* near infinity, then the imbedding

$$W_0^1 L_\varphi(\Omega) \hookrightarrow L_\phi(\Omega)$$

is compact

Corollary 2.1 *Under the same assumptions of theorem 5.1, we have*

$$W_0^1 L_\varphi(\Omega) \hookrightarrow\hookrightarrow L_\varphi(\Omega)$$

Lemma 2.4 *If a sequence un $u_n \in L_\varphi(\Omega)$ converges a.e. to u and if u_n remains bounded in $L_\varphi(\Omega)$, then $u \in L_\varphi(\Omega)$ and $u_n \rightharpoonup u$ for $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$.*

Lemma 2.5 *Let $u_n, u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \rightharpoonup u$ for $\sigma(L_\varphi(\Omega), L_\psi(\Omega))$.*

See ([8]).

3 Technical lemma

Lemma 3.1 *Under the assumptions of lemma 2.1, and by assuming that $\varphi(x, t)$ decreases with respect to one of coordinate of x, there exists a constant $c_1 > 0$ which depends only on Ω such that*

$$\int_\Omega \varphi(x, |u|) dx \le \int_\Omega \varphi(x, c_1|\nabla u|) dx \tag{3.1}$$

Theorem 3.1 *Let Ω be a bounded Lipschitz domain and let φ be a Musielak-Orlicz function satisfying the same conditions of Theorem 5.1. Then there exists a constant $\lambda > 0$ such that*

$$\|u\|_\varphi \le \lambda \|\nabla u\|_\varphi, \quad \forall \in W_0^1 L_\varphi(\Omega)$$

4 Essential assumptions

Let Ω be an open subset of $\mathbb{R}^N$ $(N \ge 2)$ satisfying the segment property,and let φ and γ be two Musielak-Orlicz functions such that φ and its complementary ψ satisfies conditions of Lemma 2.1 and $\gamma << \varphi$.

$$b_i : \Omega \times \mathbb{R} \to \mathbb{R}$$

is a Carathéodory function such that for every $x \in \Omega$, (4.1)
$b_i(x, .)$ is a strictly increasing $\mathcal{C}^1(\mathbb{R})$-function and $b_i \in L^\infty(\Omega \times \mathbb{R})$ with $b_i(x, 0) = 0$. Next for any $k > 0$, there exists a constant $\lambda_k^i > 0$ and functions $A_k^i \in L^\infty(\Omega)$ and $B_k^i \in L_\varphi(\Omega)$ such that:

$$\lambda_k^i \le \frac{\partial b_i(x,s)}{\partial s} \le A_k^i(x) \quad \text{and} \quad \left|\nabla_x\left(\frac{\partial b_i(x,s)}{\partial s}\right)\right| \le B_k^i(x)$$

$$\text{a.e. } x \in \Omega \text{ and } \forall\ |s| \le k. \tag{4.2}$$

$A : D(A) \subset W_0^1 L_\varphi(Q_T) \to W^{-1} L_\psi(Q_T)$ defined by $A(u) = -diva(x, t, u, \nabla u)$,where $a : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$,$\xi, \xi^* \in \mathbb{R}^N$,$\xi \ne \xi^*$

(A_1) : $|a(x, t, s, \xi)| \le \beta(c(x) + \psi_x^{-1}(\gamma(x, \nu_1|s|)) + \psi_x^{-1}(\varphi(x, \nu_2|\xi|)))$,

$$\beta > 0, \quad c(x) \in E_\psi(\Omega), \tag{4.3}$$

$$(A_2) : (a(x, t, s, \xi) - a(x, s, \xi^*)(\xi - \xi^*) > 0, \tag{4.4}$$

$$(A_3) : a(x, t, s, \xi).\xi \ge \alpha\varphi(x, |\xi|). \tag{4.5}$$

$\Phi(x, s, \xi) : \Omega \times IR \times IR^N \to IR^N$ is a Carathéodory function such that

$$|\Phi_i(x, t, s)| \le \psi_x^{-1}\varphi(x, |s|), \tag{4.6}$$

$f_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with

$$f_1(x, 0, s) = f_2(x, s, 0) = 0 \quad \text{a.e. } x \in \Omega, \forall s \in \mathbb{R}. \tag{4.7}$$

and for almost every $x \in \Omega$, for every $s_1, s_2 \in \mathbb{R}$,

$$sign(s_i) f_i(x, s_1, s_2) \geq 0. \tag{4.8}$$

The growth assumptions on f_i are as follows: For each $K > 0$, there exists $\sigma_K > 0$ and a function F_K in $L^1(\Omega)$ such that

$$|f_1(x, s_1, s_2)| \leq F_K(x) + \sigma_K \, |b_2(x, s_2)| \tag{4.9}$$

a.e. in Ω, for all s_1 such that $|s_1| \leq K$, for all $s_2 \in \mathbb{R}$. For each $K > 0$, there exists $\lambda_K > 0$ and a function G_K in $L^1(\Omega)$ such that

$$|f_2(x, s_1, s_2)| \leq G_K(x) + \lambda_K \, |b_1(x, s_1)|, \tag{4.10}$$

for almost every $x \in \Omega$, for every s_2 such that $|s_2| \leq K$, and for every $s_1 \in \mathbb{R}$.
Finally, we assume the following condition on the initial data $u_{i,0}$: for i=1,2.

$$u_{i,0} \text{ is a measurable function such that } b_i(., u_{i,0}) \in L^1(\Omega), \tag{4.11}$$

In this paper, for $K > 0$, we denote by $T_K : r \mapsto \min(K, max(r, -K))$ the truncation function at height K. For any measurable subset E of Q_T, we denote by $meas(E)$ the Lebesgue measure of E. For any measurable function v defined on Q and for any real number s, $\chi_{\{v<s\}}$ (respectively, $\chi_{\{v=s\}}, \chi_{\{v>s\}}$) denote the characteristic function of the set $\{(x,t) \in Q_T \; ; \; v(x,t) < s\}$ (respectively, $\{(x,t) \in Q_T; v(x,t) = s\}, \{(x,t) \in Q_T; v(x,t) > s\}$).

Definition 4.1 *A couple of functions* (u_1, u_2) *defined on* Q *is called a renormalized solution of* (4.1)-(4.11)*if for* $i = 1, 2$ *the function* u_i *satisfies*

$$T_K(u_i) \in W_0^{1,x} L_\varphi(Q_T) \quad and \quad b_i(x, u_i) \in L^\infty(0, T; L^1(\Omega)), \tag{4.12}$$

$$\int_{\{\, m \leq |u_i| \leq m+1\}} a(x,t,u_i,\nabla u_i) \nabla u_i \, dx\, dt \to 0 \quad as \; m \to +\infty, \tag{4.13}$$

For every function S *in* $W^{2,\infty}(\mathbb{R})$ *which is piecewise* C^1 *and such that* S' *has a compact support,we have*

$$\frac{\partial B_{i,S}(x,u_i)}{\partial t} - div(S'(u_i) a(x,t,u_i,\nabla u_i))$$
$$+ S''(u_i) a(x,t,u_i,\nabla u_i) \nabla u_i$$

$$+ div(S'(u_i)\phi_i(x,t,u_i)) - S''(u_i)\phi_i(x,t,u_i)\nabla u_i$$
$$+ f_i(x,u_1,u_2) S'(u_i) = 0, \tag{4.14}$$

$$B_{i,S}(x,u_i)(t=0) = B_{i,S}(x,u_{i,0}) \quad in \; \Omega, \tag{4.15}$$

where $B_{i,S}(r) = \int_0^r b_i'(x,s) S'(s)\, ds$.

Due to (4.12), each term in (4.14) has a meaning in $W^{-1,x}L_\psi(Q_T) + L^1(Q_T)$.
Indeed, if K such that $suppS \subset [-K, K]$, the following identifications are made in (4.14)

- $B_{i,S}(x,u_i) \in L^\infty(Q_T)$, since $|B_{i,S}(x,u_i)| \leq K \|A_K^i\|_{L^\infty(\Omega)} \|S'\|_{L^\infty(\mathbb{R})}$
- $S'(u_i) a(x,t,u_i,\nabla u_i)$ can be identified with $S'(u_i) a(x,t,T_K(u_i),\nabla T_K(u_i))$ a.e. in Q_T. Since indeed $|T_K(u_i)| \leq K$ a.e. in Q_T, . As a consequence of (4.3) , (4.12) and $S'(u_i) \in L^\infty(Q_T)$, it follows that
$$S'(u_i) a(x, T_K(u_i), \nabla T_K(u_i)) \in (L_\psi(Q_T))^N.$$
- $S'(u_i) a(x,t,u_i,\nabla u_i)\nabla u_i$ can be identified with $S'(u_i) a(x,t,T_K(u_i),\nabla T_K(u_i))\nabla T_K(u_i)$ a.e. in Q_T.with (4.2) and (4.12) it has
$$S'(u_i) a(x,t,T_K(u_i),\nabla T_K(u_i))\nabla T_K(u_i) \in L^1(Q_T)$$
- $S'(u_i)\Phi_i(u_i)$ and $S''(u_i)\Phi_i(u_i)\nabla u_i$ respectively identify with $S'(u_i)\Phi_i(T_K(u_i))$ and $S''(u_i)\Phi(T_K(u_i))\nabla T_K(u_i)$. In view of the properties of S and (4.6), the functions S', S'' and $\Phi \circ T_K$ are bounded on $\mathbb{R}$ so that (4.12) implies that $S'(u_i)\Phi_i(T_K(u_i)) \in (L^\infty(Q_T))^N$ and $S''(u_i)\Phi_i(T_K(u_i))\nabla T_K(u_i) \in (L_\psi(Q_T))^N$.
- $S'(u_i) f_i(x,u_1,u_2)$ identifies with $S'(u_i) f_1(x, T_K(u_1), u_2)$ a.e. in Q_T
(or $S'(u_i) f_2(x, u_1, T_K(u_2))$ a.e. in Q_T). Indeed, since $|T_K(u_i)| \leq K$ a.e. in Q_T, assumptions (4.9) and (4.10) and using (4.12) and of $S'(u_i) \in L^\infty(Q)$, one has
$$S'(u_1) f_1(x, T_K(u_1), u_2) \in L^1(Q_T)$$
$$\text{and} \quad S'(u_2) f_2(x, u_1, T_K(u_2)) \in L^1(Q_T).$$
As consequence, (4.14) takes place in $D'(Q_T)$ and that
$$\frac{\partial B_{i,S}(x,u_i)}{\partial t} \in W^{-1,x}L_\psi(Q_T) + L^1(Q_T). \tag{4.16}$$
Due to the properties of S and (4.2)
$$B_{i,S}(x,u_i) \in W_0^{1,x} L_\varphi(Q_T). \tag{4.17}$$
Moreover (4.16) and (4.17) implies that $B_{i,S}(x,u_i) \in C^0([0,T], L^1(\Omega))$ so that the initial condition (4.15) makes sense.

5 Existence result

We shall prove the following existence theorem

Theorem 5.1 *Assume that* (4.1)-(4.11) *hold true. There at least a renormalized solution* (u_1, u_2) *of Problem* (1.1).

We divide the prof in 5 steps.
Step 1: Approximate problem.
Let us introduce the following regularization of the data: for $n > 0$ and $i = 1, 2$

$$b_{i,n}(x,s) = b_i(x, T_n(s)) + \frac{1}{n}\, s \quad \forall s \in \mathbb{R}, \tag{5.1}$$

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \text{ a.e. in } \Omega, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N, \tag{5.2}$$

$$\Phi_{i,n}(x,t,s) = \Phi_{i,n}(x,t,T_n(s)) \quad \text{a.e.} \quad (x,t) \in Q_T, \quad \forall s \in IR. \tag{5.3}$$

$$f_{1,n}(x,s_1,s_2) = f_1(x,T_n(s_1),s_2) \quad \text{a.e. in } \Omega, \forall s_1,s_2 \in \mathbb{R}, \tag{5.4}$$

$$f_{2,n}(x,s_1,s_2) = f_2(x,s_1,T_n(s_2)) \quad \text{a.e. in } \Omega, \forall s_1,s_2 \in \mathbb{R}, \tag{5.5}$$

$$u_{i,0n} \in C_0^\infty(\Omega), b_{i,n}(x,u_{i,0n}) \to b_i(x,u_{i,0}) \quad \text{in} L^1(\Omega)$$

$$\text{as } n \text{ tends to } +\infty \tag{5.6}$$

Let us now consider the regularized problem $\frac{\partial b_{i,n}(x,u_{i,n})}{\partial t} - \operatorname{div}(a_n(x,u_{i,n},\nabla u_{i,n})) - div(\Phi_{i,n}(x,t,u_{i,n}))$

$$+ f_{i,n}(x,u_{1,n},u_{2,n}) = 0 \quad \text{in } Q_T, \tag{5.7}$$

$$u_{i,n} = 0 \quad \text{on } (0,T)\times\partial\Omega, \tag{5.8}$$

$$b_{i,n}(x,u_{i,n})(t=0) = b_{i,n}(x,u_{i,0n}) \quad \text{in } \Omega. \tag{5.9}$$

In view of (5.1), for $i = 1,2$, we have

$$\frac{\partial b_{i,n}(x,s)}{\partial s} \geq \frac{1}{n}, \quad |b_{i,n}(x,s)| \leq \max_{|s|\leq n} |b_i(x,s)| + 1 \quad \forall s \in \mathbb{R},$$

In view of (4.9)-(4.10), $f_{1,n}$ and $f_{2,n}$ satisfy: There exists $F_n \in L^1(\Omega), G_n \in L^1(\Omega)$ and $\sigma_n > 0, \lambda_n > 0$, such that

$|f_{1,n}(x,s_1,s_2)| \leq F_n(x) + \sigma_n \ \max_{|s|\leq n} |b_i(x,s)|$ a.e. in $x \in \Omega, \forall s_1,s_2 \in \mathbb{R}$,

$|f_{2,n}(x,s_1,s_2)| \leq G_n(x) + \lambda_n \ \max_{|s|\leq n} |b_i(x,s)|$ a.e. in $x \in \Omega, \forall s_1,s_2 \in \mathbb{R}$. As a consequence, proving the existence of a weak solution $u_{i,n} \in W_0^{1,x}L_\varphi(Q_T)$ of (5.7)-(5.9) is an easy task (see e.g. [9]).

Step2:A priori estimates.

Let $t \in (0,T)$ and using $T_k(u_{i,n})\chi_{(0,t)}$ as a test function in problem (5.7), we get:

$$\int_\Omega B^n_{i,k}(x,u_{i,n}(t))dx + \int_{Q_t} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_k(u_{i,n})dx\,dt$$

$$+ \int_{Q_t} \phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})dx\,dt \tag{5.10}$$

$$+ \int_{Q_t} f_{i,n} T_k(u_{i,n})\,dx\,dt \leq \int_\Omega B^n_{i,k}(x,u_{i,0n})dx,$$

where $B^n_{i,k}(x,r) = \int_0^r \frac{\partial b_{i,n}(x,s)}{\partial s} T_k(s)ds$.

Due to definition of $B^n_{i,k}$ we have:

$$\int_\Omega B^n_{i,k}(x,u_{i,n}(t))dx \geq \frac{\lambda_n}{2}\int_\Omega |T_k(u_{i,n})|^2\,dx, \quad \forall k > 0, \tag{5.11}$$

and

$$0 \leq \int_\Omega B^n_{i,k}(x,u_{i,0n})dx \leq k\int_\Omega |b_{i,n}(x,u_{i,0n})|dx \tag{5.12}$$

$$\leq k\|b_i(x,u_{i,0})\|_{L^1(\Omega)}, \quad \forall k > 0.$$

In view of (4.8), we have $\int_{Q_t} f_{i,n}T_k(u_{i,n})\,dx\,dt \geq 0$ Also, we obtain with Young inequality:

$$\int_{Q_t} \phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})dx\,dt$$

$$= \int_{\{|u_{i,n}|\leq k\}} \phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})dx\,dt$$

$$\leq \int_{\{|u_{i,n}|\leq k\}} \psi(x, \frac{1}{\alpha_0^i}\phi_{i,n}(x,t,u_{i,n}))dxdt$$

$$+ \int_{\{|u_{i,n}|\leq k\}} \varphi(x,\alpha_0^i\nabla T_k(u_{i,n}))dxdt$$

$$\leq \int_{\{|u_{i,n}|\leq k\}} \psi(x, \frac{1}{\alpha_0^i}\psi_x^{-1}\varphi(x,|k|))dxdt$$

$$+ \int_{\{|u_{i,n}|\leq k\}} \varphi(x,\alpha_0^i\nabla T_k(u_{i,n})dxdt$$

$$\leq \int_{\{|u_{i,n}|\leq k\}} \psi(x, \frac{1}{\alpha_0^i}\psi_x^{-1}\varphi(x,|k|))dxdt$$

$$+ \int_{\{|u_{i,n}|\leq k\}} \varphi(x,\alpha_0^i\nabla T_k(u_{i,n})dxdt$$

then

$$\int_{Q_t} \phi_{i,n}(x,t,T_k(u_{i,n}))\nabla T_k(u_{i,n})dx\,dt$$

$$\leq C_{i,k} + \alpha_0^i\int_{Q_t} \varphi(x,\nabla T_k(u_{i,n}))dx\,dt \tag{5.13}$$

We conclude that

$$\frac{\lambda}{2}\int_\Omega |T_k(u_{i,n})|^2\,dx + \alpha^i\int_{Q_t} \varphi(x,\nabla T_k(u_{i,n})\,dx\,dt$$

$$\leq \alpha_0^i\int_{Q_t} \varphi(x,\nabla T_k(u_{i,n}))\,dt\,dx + C_{i,k} + k\|b_i(x,u_{i,0n})\|_{L^1(\Omega)}$$

Then

$$\frac{\lambda}{2}\int_\Omega |T_k(u_{i,n})|^2\,dx + (\alpha^i - \alpha_0^i)\int_{Q_t} \varphi(x,\nabla T_k(u_{i,n}))\,dt\,dx \leq C_i.k$$

Choosing α_0^i such that

$$0 < \alpha_0^i < \min(1,\alpha^i)$$

we get

$$\int_{Q_t} \varphi(x,\nabla T_k(u_{i,n}))\,dx\,dt \leq C_i.k \tag{5.14}$$

Then, by (5.14), we conclude that $T_k(u_{i,n})$ is bounded in $W^{1,x}L_\varphi(Q_T)$ independently of n and for any $k \geq 0$, so there exists a subsequence still denoted by u_n such that

$$T_k(u_{i,n}) \to \psi_{i,k} \tag{5.15}$$

weakly in $W_0^{1,x}L_\varphi(Q_T)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$ strongly in $E_\varphi(Q_T)$ and a.e in Q_T.

Since Lemma(3.1)and (5.14), we get also,

$$\varphi(x,k)\ meas\Big\{\{|u_{i,n}| > k\} \cap B_R \times [0,T]\Big\}$$

$$\leq \int_0^T \int_{\{|u_{i,n}|>k\}\cap B_R} \varphi(x,T_k(u_{i,n}))dxdt$$

$$\leq \int_{Q_T} \varphi(x,T_k(u_{i,n}))dxdt$$

$\leq diamQ_T \int_{Q_T} \varphi(x, \nabla T_k(u_{i,n}))dxdt$

Then

$$meas\Big\{\{|u_{i,n}| > k\} \cap B_R \times [0,T]\Big\} \leq \frac{diamQ_T.C_i.k}{\varphi(x,k)}$$

Which implies that:
$\lim_{k\to+\infty} meas\Big\{\{|u_{i,n}| > k\} \cap B_R \times [0,T]\Big\} = 0.$ uniformly with respect to n.

Now we turn to prove the almost every convergence of $u_{i,n}$, $b_{i,n}(x,u_{i,n})$ and convergence of $a_{i,n}(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))$.

Proposition 5.1 *Let $u_{i,n}$ be a solution of the approximate problem, then:*

$$u_{i,n} \to u_i \quad a.e \text{ in } \quad Q_T. \tag{5.16}$$

$$b_{i,n}(x,u_{i,n}) \to b_i(x,u_i) \quad a.e \text{ in } \quad Q_T$$

$$b_i(x,u_i) \in L^\infty(0,T,L^1(\Omega)). \tag{5.17}$$

$$a_n(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n})) \rightharpoonup X_{i,k}$$

$$in \quad (L_\psi(Q_T))^N for \quad \sigma(\Pi L_\psi, \Pi E_\varphi) \tag{5.18}$$

$$for\ some\ X_{i,k} \in (L_\psi(Q_T))^N$$

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{m\leq|u_{i,n}|\leq m+1} a_i(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}dxdt = 0 \tag{5.19}$$

Proof of (5.16) **and** (5.17)**:**
Consider now a function non decreasing $g_k \in C^2(I\!R)$ such that $g_k(s) = s$ for $|s| \leq \frac{k}{2}$ and $g_k(s) = k$ for $|s| \geq k$. Multiplying the approximate equation by $g_k'(u_{i,n})$, we get

$$\frac{\partial B_{k,g}^{i,n}(x,u_{i,n})}{\partial t} - div\Big(a_n(x,t,u_{i,n},\nabla u_{i,n})g_k'(u_{i,n})\Big)$$
$$+ a_n(x,t,u_{i,n},\nabla u_{i,n})g_k''(u_{i,n})\nabla u_{i,n} \tag{5.20}$$
$$+ div\Big(\phi_{i,n}(x,t,u_{i,n})g_k'(u_{i,n})\Big) - g_k''(u_{i,n})\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}$$
$$+f_{i,n}g_k'(u_n) = 0 \qquad \text{in } D'(Q_T)$$

where $B_{k,g}^{i,n}(x,z) = \displaystyle\int_0^z \frac{\partial b_{i,n}(x,s)}{\partial s} g_k'(s)ds.$

Using (5.20),we can deduce that $g_k(u_{i,n})$ is bounded in $W_0^{1,x}L_\varphi(Q_T)$ and $\frac{\partial B_{k,g}^{i,n}(x,u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_\psi(Q_T)$ independently of n.

thanks to (4.6) and properties of g_k, it follows that

$$|\int_{Q_T} \phi_{i,n}(x,t,u_n)g_k'(u_{i,n})dxdt|$$

$$\leq \|g_k'\|_\infty \int_{Q_T} c_i(x,t)\psi^{-1}\varphi(x,T_k(u_{i,n}))dxdt$$

$$\leq \|g_k'\|_\infty\Big(\psi^{-1}\varphi(x,k)\Big)\int_{Q_T} c_i(x,t)dxdt\Big) \leq C_{i,k}^1$$

By (5.13), we get

$$|\int_{Q_T} g_k''(u_{i,n})\phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}dxdt|$$

$$\leq \|g_k'\|_\infty\Big(C_{i,k} + c_0^i\int_{Q_T}\psi(x,\nabla T_k(u_{i,n}))dx\,dt\Big) \leq C_{i,k}^2$$

where $C_{i,k}^1$ and $C_{i,k}^2$ constants independently of n.

we conclude that $\frac{\partial g_k(u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_\psi(Q_T)$ for $k < n$. which implies that $g_k(u_{i,n})$ is compact in $L^1(Q_T)$.Due to the choice of g_k, we conclude that for each k,the sequence $T_k(u_{i,n})$ converges almost everywhere in Q_T,which implies that the sequence $u_{i,n}$ converge almost everywhere to some measurable function u_i in Q_T.

Then by the same argument in [9], we have

$$u_{i,n} \to u_i \text{ a.e. } Q_T, \tag{5.21}$$

where u_i is a measurable function defined on Q_T. and

$$b_{i,n}(x,u_{i,n}) \to b_i(x,u_i) \quad \text{a.e. in} \quad Q_T$$

by (5.15) and (5.21) we have

$$T_k(u_{i,n}) \to T_k(u_i) \tag{5.22}$$

weakly in $W_0^{1,x}L_\varphi(Q_T)$ for $\sigma(\Pi L_\varphi,\Pi E_\psi)$ strongly in $E_\varphi(Q_T)$ and a.e in Q_T.

We now show that $b_i(x,u_i) \in L^\infty(0,T;L^1(\Omega))$. Indeed using $\frac{1}{\varepsilon}T_\varepsilon(u_{i,n})$ as a test function in (5.7),

$$\frac{1}{\varepsilon}\int_\Omega b_{i,n}^\varepsilon(x,u_{i,n})(t)\,dx + \frac{1}{\varepsilon}\int_{Q_T} a_n(x,u_{i,n},\nabla u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt$$
$$-\frac{1}{\varepsilon}\int_{Q_T}\Phi_{i,n}(x,t,u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt + \frac{1}{\varepsilon}\int_{Q_T} f_{i,n}(x,u_{1,n},u_{2,n})T_\varepsilon(u_{i,n})$$
$$= \frac{1}{\varepsilon}\int_\Omega b_{i,n}^\varepsilon(x,u_{i,0n})\,dx, \tag{5.23}$$

for almost any t in $(0,T)$. Where, $b_{i,n}^\varepsilon(r) = \int_0^r b_{i,n}'(s)T_\varepsilon(s)\,ds$. Since a_n satisfies (4.5) and $f_{i,n}$ satisfies (4.8), we get

$$\int_\Omega b_{i,n}^\varepsilon(x,u_{i,n})(t)\,dx \leq \int_{Q_T}\Phi_{i,n}(x,t,u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt$$
$$+\int_\Omega b_{i,n}^\varepsilon(x,u_{i,0n})\,dx, \tag{5.24}$$

By Young inequality and (4.6), we get

$$\int_{Q_T}\Phi_{i,n}(x,t,u_{i,n})\nabla T_\varepsilon(u_{i,n})\,dx\,dt \leq \int_{|u_{i,n}|\leq\varepsilon}\psi(x,\Phi_{i,n}(x,t,u_{i,n}))\,dx\,dt$$
$$+\int_{|u_{i,n}|\leq\varepsilon}\varphi(x,\nabla T_\varepsilon(u_{i,n}))\,dx\,dt$$
$$\leq \varepsilon\psi(x,\frac{\alpha}{\lambda+1}\psi^{-1}\varphi(x,1)).meas(Q_T)+\int_{|u_{i,n}|\leq\varepsilon}(\varphi(x,\nabla T_\varepsilon(u_{i,n}))\,dx\,dt \tag{5.25}$$

Using the Lebesgue's Theorem and $\varphi(x, \nabla T_\varepsilon(u_{i,n})) \in W_0^{1,x}L(Q_T)$ in second term of the left hand side of the (5.25) and Letting $\varepsilon \to 0$ in (5.24)we obtain

$$\int_\Omega |b_{i,n}(x, u_{i,n})(t)|dx \leq \|b_{i,n}(x, u_{i,0n})\|_{L^1(\Omega)} \quad (5.26)$$

for almost $t \in (0, T)$. thanks to (5.6) , (5.16), and passing to the limit-inf in (5.26), we obtain $b_i(x, u_i) \in L^\infty(0, T; L^1(\Omega))$. **Proof of** (5.18) **:**
Following the same way in([10]),we deduce that $a_n(x, t, T_k(u_{i,n}), \nabla T_k(u_{i,n}))$ is a bounded sequence in $(L_\psi(Q_T))^N$,and we obtain (5.18).
Proof of (5.19) **:**
Multiplying the approximating equation (5.7) by the test function $\theta_m(u_{i,n}) = T_{m+1}(u_{i,n}) - T_m(u_{i,n})$

$$\int_\Omega B_{i,m}(x, u_{i,n}(T))dx + \int_{Q_T} a_n(x, t, u_{i,n}, \nabla u_{i,n})\nabla\theta_m(u_{i,n})dx\,dt$$
$$+ \int_{Q_T} \phi_{i,n}(x, t, u_{i,n})\nabla\theta_m(u_{i,n})dx\,dt \quad (5.27)$$
$$+ \int_{Q_T} f_{i,n}\theta_m(u_{i,n})\,dx\,dt \leq \int_\Omega B_{i,m}(x, u_{i,0n})dx,$$

where $B_{i,m}(x, r) = \displaystyle\int_0^r \theta_m(s)\frac{\partial b_{i,n}(x, s)}{\partial s}ds.$
By (4.6),we have

$$\int_{Q_T} \phi_{i,n}(x, t, u_{i,n})\nabla\theta_m(u_{i,n})dx\,dt$$
$$\leq \int_{m\leq|u_{i,n}|\leq m+1} \psi(x, \frac{\beta}{\epsilon}\psi_x^{-1}\varphi(x, |u_{i,n}|))dx\,dt$$
$$+\epsilon \int_{m\leq|u_{i,n}|\leq m+1} \varphi(x, \nabla\theta_m(u_{i,n}))dx\,dt$$

Also $\int_{Q_T} f_{i,n}\theta_m(u_{i,n})dxdt \geq 0$ in view of (4.8).Then, The same argument in step 2 , we obtain,

$$\int_{Q_T} \varphi(x, \nabla u_{i,n})dxdt$$
$$\leq C_i\Big(\int_{m\leq|u_{i,n}|\leq m+1} \psi(x, \frac{\beta}{\epsilon}\psi^{-1}\varphi(x, |u_{i,n}|))dx\,dt$$
$$+ \int_\Omega B_{i,m}(x, u_{i,0n})dx\Big)$$

Where $C_i = \frac{1}{\alpha^i - \epsilon}$ where $0 < \epsilon < \alpha^i$.
passing to limit as $n \to +\infty$,since the pointwise convergence of $u_{i,n}$ and strongly convergence in $L^1(Q_T)$ of $B_{i,m}(x, u_{i,0n})$ we get

$$\lim_{n\to+\infty} \int_{Q_T} \varphi(x, \nabla u_{i,n})dxdt$$
$$\leq C_i\Big(\int_{m\leq|u_i|\leq m+1} \psi(x, \frac{\beta}{\epsilon}\psi_x^{-1}\varphi(x, |u_i|))dx\,dt$$
$$+ \int_\Omega B_{i,m}(x, u_{i,0})dx\Big)$$

By using Lebesgue's theorem and passing to limit as $m \to +\infty$, in the all term of the right-hand side, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty} \int_{m\leq|u_i|\leq m+1} \varphi(x, \nabla u_{i,n})dxdt = 0 \quad (5.28)$$

and the other hand, we have

$$\lim_{m\to+\infty}\lim_{n\to+\infty} \int_{Q_T} \phi_{i,n}(x, t, u_{i,n})\nabla\theta_m(u_{i,n})dxdt$$
$$\leq \lim_{m\to+\infty}\lim_{n\to+\infty} \int_{m\leq|u_i|\leq m+1} \varphi(x, \nabla\theta_m(u_{i,n}))dxdt$$
$$+ \lim_{m\to+\infty}\lim_{n\to+\infty} \int_{m\leq|u_{i,n}|\leq m+1} \psi(x, \phi_{i,n}(x, t, u_{i,n}))dxdt$$

Using the pointwise convergence of $u_{i,n}$ and by Lebesgue's theorem,in the second term of the right side ,we get

$$\lim_{n\to+\infty} \int_{m\leq|u_{i,n}|\leq m+1} \psi(x, \phi_{i,n}(x, t, u_{i,n}))dxdt$$
$$= \int_{m\leq|u_i|\leq m+1} \psi(x, \phi_i(x, t, u_i))dxdt,$$

and also ,by Lebesgue's theorem

$$\lim_{m\to+\infty} \int_{m\leq|u_i|\leq m+1} \psi(x, \phi_i(x, t, u_i))dxdt = 0 \quad (5.29)$$

we obtain with (5.28) and (5.29),

$$\lim_{m\to+\infty}\lim_{n\to+\infty} \int_{Q_T} \phi_{i,n}(x, t, u_{i,n})\nabla\theta_m(u_{i,n})dxdt = 0$$

then passing to the limit in (5.27), we get the (5.19).
Step 3: Let $v_{i,j} \in \mathcal{D}(Q_T)$ be a sequence such that $v_{i,j} \to u_i$ in $W_0^{1,x}L_\varphi(Q_T)$ for the modular convergence.
This specific time regularization of $T_k(v_{i,j})$ (for fixed $k \geq 0$) is defined as follows.
Let $(\alpha_{i,0}^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$\alpha_{i,0}^\mu \in L^\infty(\Omega) \cap W_0^1 L_\varphi(\Omega) \quad \text{for all} \mu > 0 \quad (5.30)$$
$$\|\alpha_{i,0}^\mu\|_{L^\infty(\Omega)} \leq k \text{ for all} \mu > 0.$$

and $\alpha_{i,0}^\mu$ converges to $T_k(u_{i,0})$ a.e. in Ω
and $\frac{1}{\mu}\|\alpha_{i,0}^\mu\|_{\varphi,\Omega}$ converges to 0 $\mu \to +\infty$.
For $k \geq 0$ and $\mu > 0$, let us consider the unique solution $(T_k(v_{i,j}))_\mu \in L^\infty(Q) \cap W_0^{1,x}L_\varphi(Q_T)$ of the monotone problem:

$$\frac{\partial(T_k(v_{i,j}))_\mu}{\partial t} + \mu((T_k(v_{i,j}))_\mu - T_k(v_{i,j})) = 0 \text{ in } D'(\Omega), \quad (5.31)$$
$$(T_k(v_{i,j}))_\mu(t = 0) = \alpha_{i,0}^\mu \text{ in } \Omega. \quad (5.32)$$

Remark that due to

$$\frac{\partial(T_k(v_{i,j}))_\mu}{\partial t} \in W_0^{1,x}L_\varphi(Q_T) \quad (5.33)$$

We just recall that,

$$(T_k(v_{i,j}))_\mu \to T_k(u_i) \quad \text{a.e. in} \quad Q_T, \quad \text{weakly}* \quad \text{in } L^\infty(Q_T), \tag{5.34}$$

$$(T_k(v_{i,j}))_\mu \to (T_k(u_i))_\mu \quad \text{in} \quad W_0^{1,x}L_\varphi(Q_T) \tag{5.35}$$

for the modular convergence as $j \to +\infty$.

$$(T_k(u_i))_\mu \to T_k(u_i) \quad \text{in} \quad W_0^{1,x}L_\varphi(Q_T) \tag{5.36}$$

for the modular convergence as $\mu \to +\infty$.

$$\|(T_k(v_{i,j}))_\mu\|_{L^\infty(Q_T)} \leq max(\|(T_k(u_i))\|_{L^\infty(Q_T)}, \|\alpha_0^\mu\|_{L^\infty(\Omega)}) \leq k \tag{5.37}$$

$\forall\ \mu > 0$,$\forall\ k > 0$. Now, we introduce a sequence of increasing $C^\infty(\mathbb{R})$-functions S_m such that, for any $m \geq 1$

$$S_m(r) = r \text{ for } |r| \leq m, \quad \text{supp}(S'_m) \subset [-(m+1), (m+1)], \tag{5.38}$$

$$\|S''_m\|_{L^\infty(\mathbb{R})} \leq 1.$$

Through setting, for fixed $K \geq 0$,

$$W^n_{i,j,\mu} = T_K(u_{i,n}) - T_K(v_{i,j})_\mu \quad \text{and} \quad W^n_{i,\mu} = T_K(u_{i,n}) - T_K(u_i)_\mu \tag{5.39}$$

we obtain upon integration,

$$\begin{aligned}
&\int_{Q_T} \Big\langle \frac{\partial b_{i,S_m}(u_{i,n})}{\partial t}, W^n_{i,j,\mu} \Big\rangle dx\,dt \\
&+ \int_{Q_T} S'_m(u_{i,n}) a_n(x, u_i^n, \nabla u_{i,n}) \nabla W^n_{i,j,\mu}\, dx\,dt \\
&+ \int_{Q_T} S''_m(u_{i,n}) W^n_{i,j,\mu} a_n(x, u_{i,n}, \nabla u_{i,n}) \nabla u_{i,n}\, dx\,dt \\
&+ \int_{Q_T} \Phi_{i,n}(x,t,u_{i,n}) S'_m(u_{i,n}) \nabla W^n_{i,j,\mu}\, dx\,dt \\
&+ \int_{Q_T} S''_m(u_{i,n}) W^n_{i,j,\mu} \Phi_{i,n}(x,t,u_{i,n}) \nabla u_{i,n}\, dx\,dt \\
&+ \int_{Q_T} f_{i,n}(x, u_{1,n}, u_{2,n}) S'_m(u_{i,n}) W^n_{i,j,\mu}\, dx\,dt = 0
\end{aligned} \tag{5.40}$$

Next we pass to the limit as n tends to $+\infty$, j tends to $+\infty$, μ tends to $+\infty$ and then m tends to $+\infty$, the real number $K \geq 0$ being kept fixed. In order to perform this task we prove below the following results for fixed $K \geq 0$:

$$\liminf_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \int_{Q_T} \Big\langle \frac{\partial b_{i,S_m}(u_{i,n})}{\partial t}, S'_m(u_{i,n}) W^n_{i,j,\mu} \Big\rangle \geq 0, \tag{5.41}$$

$$\lim_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \int_{Q_T} S'_n(u_{i,n}) \Phi_{i,n}(x,t,u_{i,n}) \nabla W^n_{i,j,\mu} = 0, \tag{5.42}$$

$$\lim_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \int_{Q_T} S''_m(u_{i,n}) W^n_{i,\mu} \Phi_{i,n}(x,t,u_{i,n}) \nabla u_{i,n} = 0, \tag{5.43}$$

$$\lim_{m\to+\infty} \overline{\lim_{\mu\to+\infty}} \lim_{j\to+\infty} \overline{\lim_{n\to+\infty}} \Big| \int_{Q_T} S''_m(u_{i,n}) W^n_{i,j,\mu} a_n(x,t,u_{i,n},\nabla u_{i,n}) \nabla u_{i,n} \Big| = 0 \tag{5.44}$$

$$\lim_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \int_{Q_T} f_{i,n}(x, u_{1,n}, u_{2,n}) S'_m(u_{i,n}) W^n_{i,j,\mu} = 0. \tag{5.45}$$

$$\limsup_{n\to+\infty} \int_{Q_T} a(x,t,u_{i,n},\nabla T_K(u_{i,n})) \nabla T_K(u_{i,n})\, dx\,dt \tag{5.46}$$

$$\leq \int_{Q_T} X_{i,K} \nabla T_K(u_i)\, dx\,dt. \tag{5.47}$$

$$\int_{Q_T} [a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n})) - a(x,t,T_k(u_{i,n}),\nabla T_k(u_i))]$$

$$[\nabla T_k(u_{i,n}) - \nabla T_k(u_i)]\, dx\,dt \to 0. \tag{5.48}$$

Proof of (5.41):

Lemma 5.1

$$\int_{Q_T} \Big\langle \frac{\partial b_{i,n}(x,u_{i,n})}{\partial t}, S'_m(u_{i,n}) W^n_{i,j,\mu} \Big\rangle dx\,dt \geq \epsilon(n,j,\mu,m), \tag{5.49}$$

See [23]. **Proof of** (5.42):
If we take $n > m+1$, we get

$$\phi_{i,n}(x,t,u_{i,n}) S'_m(u_{i,n}) = \phi_i(x,t,T_{m+1}(u_{i,n})) S'_m(u_{i,n})$$

Using (4.6), we have:

$$\psi(\phi_{i,n}(x,t,T_{m+1}(u_{i,n}) S'_m(u_{i,n})) \leq (m+1) \psi(\phi_i(x,t,T_{m+1}(u_{i,n})))$$

$$\leq (m+1) \psi(\|c(x,t)\|_{L^\infty(Q_T)} \psi^{-1} M(m+1))$$

Then $\phi_{i,n}(x,t,u_n) S_m(u_{i,n})$ is bounded in $L_\psi(Q_T)$, thus, by using the pointwise convergence of $u_{i,n}$ and Lebesgue's theorem we obtain $\phi_{i,n}(x,t,u_{i,n}) S_m(u_{i,n}) \to \phi_i(x,t,u_i) S_m(u_i)$ with the modular convergence as $n \to +\infty$, then $\phi_{i,n}(x,t,u_{i,n}) S_m(u_{i,n}) \to \phi(x,t,u_i) S_m(u_i)$ for $\sigma(\prod L_\psi, \prod L_\varphi)$.
In the other hand $\nabla W^n_{i,j,\mu} = \nabla T_k(u_{i,n}) - \nabla(T_k(v_{i,j}))_\mu$ for converge to $\nabla T_k(u_i) - \nabla(T_k(v_{i,j}))_\mu$ weakly in $(L_\varphi(Q_T))^N$,then

$$\int_{Q_T} \phi_{i,n}(x,t,u_{i,n}) S_m(u_{i,n}) \nabla W^n_{i,j,\mu}\, dx\,dt$$

$$\to \int_{Q_T} \phi_i(x,t,u_i) S_m(u_i) \nabla W_{i,j,\mu}\, dx\,dt$$

as $n \to +\infty$.
By using the modular convergence of $W_{i,j,\mu}$ as $j \to +\infty$ and letting μ tends to infinity, we get (5.42).
Proof of (5.43):
For $n > m+1 > k$, we have $\nabla u_{i,n} S''_m(u_{i,n}) = \nabla T_{m+1}(u_{i,n})$ a.e. in Q_T. By the almost every where convergence of $u_{i,n}$ we have $W^n_{i,j,\mu} \to W_{i,j,\mu}$ in $L^\infty(Q_T)$ weak-* and since the sequence $(\phi_{i,n}(x,t,T_{m+1}(u_{i,n})))_n$ converge strongly in $E_\psi(Q_T)$ then

$$\phi_{i,n}(x,t,T_{m+1}(u_{i,n}))\; W^n_{i,j,\mu} \to \phi_i(x,t,T_{m+1}(u_i))\; W_{i,j,\mu}$$

converge strongly in $E_\psi(Q_T)$ as $n \to +\infty$.By virtue of $\nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u_i)$ weakly in $(L_\varphi(Q_T))^N$ as $n \to +\infty$ we have

$$\int_{m\le|u_{i,n}|\le m+1} \phi_{i,n}(x,t,T_{m+1}(u_{i,n}))\nabla u_{i,n} S''_m(u_{i,n}) W^n_{i,j,\mu}\,dx\,dt$$

$$\to \int_{m\le|u_i|\le m+1} \phi(x,t,u_i))\nabla u_i W_{i,j,\mu}\,dx\,dt$$

as $n \to +\infty$
with the modular convergence of $W_{i,j,\mu}$ as $j \to +\infty$ and letting $\mu \to +\infty$ we get (5.43).
Proof of (5.44):
For any $m \ge 1$ fixed, we have

$$\Big|\int_{Q_T} S''_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} W^n_{i,j,\mu}\,dx\,dt\Big|$$

$$\le \|S''_m\|_{L^\infty(\mathbb{R})}\|W^n_{i,j,\mu}\|_{L^\infty(Q_T)} \int_{\{m\le|u_{i,n}|\le m+1\}} a_n(x,t,u_{i,n},\nabla u_{i,n}) \times \nabla u_{i,n}\,dx\,dt,$$

for any $m \ge 1$, and any $\mu > 0$. In view (5.37) and (5.38), we can obtain

$$\limsup_{n\to+\infty}\Big|\int_{Q_T} S''_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} W^n_{i,j,\mu}\,dx\,dt\Big| \le 2K \limsup_{n\to+\infty}\int_{\{m\le|u_{i,n}|\le m+1\}} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt, \tag{5.50}$$

for any $m \ge 1$. Using (5.19) we pass to the limit as $m \to +\infty$ in (5.50) and we obtain (5.44).
Proof of (5.45):
For fixed $n \ge 1$ and $n > m+1$, we have

$$f_{1,n}(x,u_{1,n},u_{2,n})S'_m(u_{1,n}) = f_1(x,T_{m+1}(u_{1,n}),T_n(u_{2,n}))S'_m(u_{1,n}),$$

$$f_{2,n}(x,u_{1,n},u_{2,n})S'_m(u_{2,n}) = f_2(x,T_n(u_{1,n}),T_{m+1}(u_{2,n}))S'_m(u_{2,n}),$$

In view (4.9),(4.10),(5.22) and Lebesgue's the theorem allow us to get, for

$$\lim_{n\to+\infty}\int_{Q_T} f_{i,n}(x,u_{1,n},u_{2,n})S'_m(u_{i,n})W^n_{i,j,\mu}\,dx\,dt = \int_{Q_T} f_i(x,u_1,u_2)S'_m(u_i)W_{i,j,\mu}\,dx\,dt$$

Using (5.35), we follow a similar way we get as $j \to +\infty$,

$$\lim_{j\to+\infty}\int_{Q_T} f_i(x,u_1,u_2)S'_m(u_i)W_{i,j,\mu}\,dx\,dt = \int_{Q_T} f_i(x,u_1,u_2)S'_m(u_i)(T_K(u_i)-T_K(u_i)_\mu)\,dx\,dt$$

we fixed $m > 1$, and using (5.36), we have

$$\lim_{\mu\to+\infty}\int_{Q_T} f_i(x,u_1,u_2)S'_m(u_i)(T_K(u_i)-T_K(u_i)_\mu)\,dx\,dt = 0$$

Then we conclude the proof of (5.45).
Proof of (5.46):
If we pass to the lim-sup when n ,j and μ tends to $+\infty$ and then to the limit as m tends to $+\infty$ in (5.40). We obtain using (5.41)-(5.45), for any $K \ge 0$,

$$\lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{j\to+\infty}\limsup_{n\to+\infty}\int_{Q_T} S'_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) \Big(\nabla T_K(u_{i,n}) - \nabla T_K(v_{i,j})_\mu\Big)\,dx\,dt \le 0.$$

Since

$$S'_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_K(u_{i,n}) = a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_K(u_{i,n})$$

for $n > K$ and $K \le m$. Then, for $K \le m$,

$$\limsup_{n\to+\infty}\int_{Q_T} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_K(u_{i,n})\,dx\,dt \le \lim_{m\to+\infty}\limsup_{\mu\to+\infty}\limsup_{j\to+\infty}\limsup_{n\to+\infty}\int_{Q_T} S'_m(u_{i,n}) a_n(x,u_{i,n},\nabla u_{i,n})\nabla T_K(v_{i,j})_\mu\,dx\,dt. \tag{5.51}$$

Thanks to (5.38), we have in The right hand side of (5.51), for $n > m+1$,

$$S'_m(u_{i,n}) a_n(x,t,u_{i,n},\nabla u_{i,n}) = S'_m(u_{i,n}) a\Big(x,t,T_{m+1}(u_{i,n}),\nabla T_{m+1}(u_{i,n})\Big) \text{ a.e. in } Q_T.$$

Using (5.18), and fixing $m \ge 1$, we get

$$S'_m(u_{i,n}) a_n(u_{i,n},\nabla u_{i,n}) \rightharpoonup S'_m(u_i) X_{i,m+1} \text{ weakly in } (L_\psi(Q_T))^N.$$

when $n \to +\infty$.
We pass to limit as $j \to +\infty$ and $\mu \to +\infty$, and using (5.35)-(5.36)

$$\limsup_{\mu\to+\infty}\limsup_{j\to+\infty}\limsup_{n\to+\infty}\int_{Q_T} S'_m(u_{i,n})) a_n(x,t,u_{i,n},\nabla u_{i,n}))\nabla T_K(v_{i,j})_\mu\,dx\,dt = \int_{Q_T} S'_m(u_i) X_{i,m+1}\nabla T_K(u_i)\,dx\,dt = \int_{Q_T} X_{i,m+1}\nabla T_K(u_i)\,dx\,dt \tag{5.52}$$

where $K \leq m$, since $S'_m(r) = 1$ for $|r| \leq m$.
On the other hand, for $K \leq m$, we have

$$a\Big(x,t,T_{m+1}(u_{i,n}),\nabla T_{m+1}(u_{i,n})\Big)\chi_{\{|u_{i,n}|<K\}}$$
$$= a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\chi_{\{|u_{i,n}|<K\}},$$

a.e. in Q_T. Passing to the limit as $n \to +\infty$, we obtain

$$X_{i,m+1}\chi_{\{|u_i|<K\}} = X_{i,K}\chi_{\{|u_i|<K\}} \quad \text{a.e. in } Q_T - \{|u_i| = K\} \text{ for } K \leq n. \tag{5.53}$$

Then

$$X_{m+1}\nabla T_K(u_i) = X_K\nabla T_K(u_i) \quad \text{a.e. in } Q_T. \tag{5.54}$$

Then we obtain (5.46).
Proof of (5.48):
Let $K \geq 0$ be fixed. Using (4.5) we have

$$\int_{Q_T}\Big[a(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})) - a(x,t,T_K(u_{i,n}),\nabla T_K(u_i))\Big]$$
$$\Big[\nabla T_K(u_{i,n}) - \nabla T_K(u_i)\Big]dx\,dt \geq 0, \tag{5.55}$$

In view (1.1) and (5.22), we get

$$a(x,t,T_K(u_{i,n}),\nabla T_K(u_i)) \to a(x,t,T_K(u_i),\nabla T_K(u_i)) \quad \text{a.e. in } Q_T,$$

as $n \to +\infty$, and by (4.2) and Lebesgue's theorem, we obtain

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_i)\Big) \to a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big) \tag{5.56}$$

strongly in $(L_\psi(Q_T))^N$. Using (5.46), (5.22), (5.18) and (5.56), we can pass to the lim-sup as $n \to +\infty$ in (5.55) to obtain (5.48).
To finish this step, we prove this Lemma:

Lemma 5.2 *For $i = 1,2$ and fixed $K \geq 0$, we have*

$$X_{i,K} = a\Big(xt,,T_K(u_i),\nabla T_K(u_i)\Big) \quad \text{a.e. in } Q. \tag{5.57}$$

Also, as $n \to +\infty$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n}) \rightharpoonup a\Big(x,t,T_K(u_i),DT_K(u_i)\Big)\nabla T_K(u_i), \tag{5.58}$$

weakly in $L^1(Q_T)$.

Proof of (5.57):
It's easy to see that

$$a_n(x,t,T_K(u_{i,n}),\xi) = a(x,t,T_K(u_{i,n}),\xi) = a_K(x,t,T_K(u_{i,n}),\xi)$$

a.e. in Q_T
for any $K \geq 0$, any $n > K$ and any $\xi \in \mathbb{R}^N$.
In view of (5.18), (5.48) and (5.56) we obtain

$$\lim_{n\to+\infty}\int_{Q_T} a_K\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n})\,dx\,dt$$
$$= \int_{Q_T} X_{i,K}\nabla T_K(u_i)\,dx\,dt. \tag{5.59}$$

Since (1.1), (4.4) and (5.22), imply that the function $a_K(x,s,\xi)$ is continuous and bounded with respect to s. Then we conclude that (5.57).
Proof of (5.58):
Using (4.5) and (5.48), for any $K \geq 0$ and any $T' < T$, we have

$$\Big[a(x,t,T_K(u_{i,n},\nabla T_K(u_{i,n})) - a(x,t,T_K(u_{i,n}),\nabla T_K(u))\Big]$$
$$\times\Big[\nabla T_K(u_{i,n}) - \nabla T_K(u_i)\Big] \to 0 \tag{5.60}$$

strongly in $L^1(Q_{T'})$ as $n \to +\infty$.
On the other hand with (5.22), (5.18), (5.56) and (5.57), we get

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_i)$$
$$\rightharpoonup a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$

weakly in $L^1(Q_T)$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_i)\Big)\nabla T_K(u_{i,n})$$
$$\rightharpoonup a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$

weakly in $L^1(Q_T)$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$
$$\to a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i),$$

strongly in $L^1(Q)$, as $n \to +\infty$.
It's results from (5.60), for any $K \geq 0$ and any $T' < T$,

$$a\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n})$$
$$\rightharpoonup a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i) \tag{5.61}$$

weakly in $L^1(Q_{T'})$ as $n \to +\infty$.then for $T' = T$, we have (5.58). Finally we should prove that u_i satisfies (4.13).
Step 4:Pass to the limit.
we first show that u satisfies (4.13)

$$\int_{m\leq|u_{i,n}|\leq m+1\}} a(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt$$
$$= \int_{Q_T} a_n(x,t,u_{i,n},\nabla u_{i,n})\Big[\nabla T_{m+1}(u_{i,n}) - \nabla T_m(u_{i,n})\Big]dx\,dt$$
$$= \int_{Q_T} a_n\Big(x,t,T_{m+1}(u_{i,n}),\nabla T_{m+1}(u_{i,n})\Big)\nabla T_{m+1}(u_{i,n})\,dx\,dt$$
$$- \int_{Q_T} a_n\Big(x,t,T_m(u_{i,n}),\nabla T_m(u_{i,n})\Big)\nabla T_m(u_{i,n})\,dx\,dt$$

for $n > m+1$. According to (5.58), one can pass to the limit as $n \to +\infty$; for fixed $m \geq 0$ to obtain

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{m\leq|u_{i,n}|\leq m+1\}} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\,dx\,dt\\
&=\int_Q a\Big(x,t,T_{m+1}(u_i),\nabla T_{m+1}(u_i)\Big)\nabla T_{m+1}(u_i)\,dx\,dt\\
&\quad-\int_Q a\Big(x,t,T_m(u_i),\nabla T_m(u_i)\Big)\nabla T_m(u_i)\,dx\,dt\\
&=\int_{m\leq|u_i|\leq m+1\}} a(x,t,u_i,\nabla u_i)\nabla u_i\,dx\,dt
\end{aligned} \tag{5.62}$$

Pass to limit as m tends to $+\infty$ in (5.62) and using (5.19) show that u_i satisfies (4.13).

Now we shown that u_i to satisfy (4.14)and (4.15). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let K be a positive real number such that $\operatorname{supp} S' \subset [-K,K]$. the Pointwise multiplication of the approximate equation (1.1) by $S'(u_{i,n})$ leads to

$$\begin{aligned}
&\frac{\partial B^n_{i,S}(u_{i,n})}{\partial t}-\operatorname{div}\Big(S'(u_{i,n})a_n(x,u_{i,n},\nabla u_{i,n})\Big)\\
&+S''(u_{i,n})a_n(x,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}\\
&-\operatorname{div}\Big(S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\Big)\\
&+S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}\\
&=f_{i,n}(x,u_{1,n},u_{1,n})S'(u_{i,n})
\end{aligned} \tag{5.63}$$

in $D'(Q_T)$, for $i = 1,2$.
Now we pass to the limit in each term of (5.63).
Limit of $\frac{\partial B^n_{i,S}(u_{i,n})}{\partial t}$: Since $B^n_{i,S}(u_{i,n})$ converges to $B_{i,S}(u_i)$ a.e. in Q_T and in $L^\infty(Q_T)$ weak $\star$ and S is bounded and continuous. Then $\frac{\partial B^n_{i,S}(u_{i,n})}{\partial t}$ converges to $\frac{\partial b_{i,S}(u_i)}{\partial t}$ in $D'(Q_T)$ as n tends to $+\infty$.
Limit of $div\Big(S'(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\Big)$: Since $\operatorname{supp} S' \subset [-K,K]$, for $n > K$, we have

$$S'(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})$$

$$= S'(u_{i,n})a_n\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)$$

a.e. in Q_T. Using the pointwise convergence of $u_{i,n}$, (5.38),(5.18) and (5.57), imply that

$$S'(u_{i,n})a_n\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)$$

$$\rightharpoonup S'(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)$$

weakly in $(L_\psi(Q_T))^N$, for $\sigma(\Pi L_\psi,\Pi E_\varphi)$ as $n \to +\infty$, since $S'(u_i) = 0$ for $|u_i| \geq K$ a.e. in Q_T. And

$$S'(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big) = S'(u_i)a(x,t,u_i,\nabla u_i)$$

a.e. in Q_T.

Limit of $S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}$. Since $\operatorname{supp} S'' \subset [-K,K]$, for $n > K$, we have

$$S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}$$

$$= S''(u_{i,n})a_n\Big(x,t,T_K(u_{i,n}),\nabla T_K(u_{i,n})\Big)\nabla T_K(u_{i,n}) \quad \text{a.e. in } Q_T.$$

The pointwise convergence of $S''(u_{i,n})$ to $S''(u_i)$ as $n \to +\infty$, (5.38) and (5.58) we have

$$S''(u_{i,n})a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n}$$

$$\rightharpoonup S''(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$

weakly in $L^1(Q_T)$, as $n \to +\infty$, and

$$S''(u_i)a\Big(x,t,T_K(u_i),\nabla T_K(u_i)\Big)\nabla T_K(u_i)$$

$$= S''(u_i)a(x,t,u_i,\nabla u_i)\nabla u_i \quad \text{a.e.in } Q_T.$$

Limit of $S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})$: We have

$$S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})$$

$$= S'(u_{i,n})\Phi_{i,n}(x,t,T_K(u_{i,n}))$$

a.e.in Q_T, Since $\operatorname{supp} S' \subset [-K,K]$.Using (4.5), (5.24) and (5.16), it's easy to see that $S'(u_{i,n})\Phi_{i,n}(x,t,u_{i,n}) \rightharpoonup S'(u_i)\Phi_i(x,t,T_K(u_i))$ weakly for $\sigma(\Pi L_\psi,\Pi L_\varphi)$ as $n \to +\infty$. And $S'(u_i)\Phi_i(x,t,T_K(u_i)) = S'(u_i)\Phi_i(x,t,u_i)$ a.e. in Q_T.
Limit of $S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}$: Since $S' \in W^{1,\infty}(\mathbb{R})$ with $\operatorname{supp} S' \subset [-K,K]$, we have $S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} = \Phi_{i,n}(x,t,T_K(u_{i,n}))\nabla S'(T_K(u_{i,n}))$ a.e. in Q_T. The weakly convergence of truncation allows us to prove that

$$S''(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} \rightharpoonup \Phi_i(x,t,u_i)\nabla S'(u_i),$$

strongly in $L^1(Q_T)$.

Limit of $f_{i,n}(x,u_{1,n},u_{2,n})S'(u_{i,n})$: Using (4.9), (4.10), (5.4) and (5.5), we have $f_{i,n}(x,u_{1,n},u_{2,n})S'(u_{i,n}) \to f_i(x,u_1,u_2)S'(u_i)$ strongly in $L^1(Q_T)$, as $n \to +\infty$.
It remains to show that for i=1,2 $B_S(x,u_i)$ satisfies the initial condition (4.15).
To this end, firstly remark that, in view of the definition of S'_φ, we have $B_\varphi(x,u_{i,n})$ is bounded in $L^\infty(Q_T)$.
Secondly, by (5.41) we show that $\dfrac{\partial B_\varphi(x,u_{i,n})}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_\psi(Q_T))$. As a consequence, an Aubin's type Lemma (see e.g., [11], Corollary 4) implies that $B_\varphi(x,u_{i,n})$ lies in a compact set of $C^0([0,T];L^1(\Omega))$.
It follows that, on one hand,$B_\varphi(x,u_i,n)(t=0)$ converges to $B_\varphi(x,u_i)(t=0)$ strongly in $L^1(\Omega)$. On the order hand, the smoothness of B_φ imply that $B_\varphi(x,u_{i,n})(t=0)$ converges to $B_\varphi(x,u_i)(t=0)$ strongly in $L^1(\Omega)$, we conclude that $B_\varphi(x,u_{i,n})(t=0) = B_\varphi(x,u_{i,0n})$ converges to $B_\varphi(x,u_i)(t=0)$ strongly in $L^1(\Omega)$, we obtain $B_\varphi(x,u_i)(t=0) = B_\varphi(x,u_{i,0})$ a.e. in Ω and for all $M > 0$, now letting M to $+\infty$, we conclude that $b(x,u_i)(t=0) = b(x,u_{i,0})$ a.e. in Ω.

As a conclusion, the proof of Theorem (5.1) is complete.

References

1. Adams RA. " Sobolev spaces "New York (NY):Academic Press; (1975).
2. R. Di Nardo, F. Féo and O. Guibé. "Existence result for nonlinear parabolic equations with lower order terms".*Anal. Appl*. Singap. 9, no. 2 (2011), 161–186.
3. Redwane, H. , "Existence of a solution for a class of nonlinear parabolic systems". Electron. J. Qual. Theory Differ. Equ., 24, , 18pp.(2007)
4. Benkirane A., Bennouna J.:" Existence and uniqueness of solution of unilateral problems with L^1-data in Orlicz spaces". Ital. J. Pure Appl. Math. 16, 87-102 (2004).
5. Kellou M.A., Benkirane A., Douiri S.M.," An inequality of type Poincarée in Musielak spaces and applications to some nonlinear elliptic problems with L^1-data". Complex Variables and Elliptic Equations 60, pp. 1217-1242, (2015).
6. A. Aberqi , J. Bennouna 2and M. Hammoumi, " Existence Result for Nonlinear Degenerated Parabolic Systems". Nonlinear Dynamics and Systems Theory, 17 (3) (2017) 217–229.
7. Gossez J.P. :" A strongly nonlinear elliptic problem in Orlicz Sobolev spaces". Proc. Am. Math. Soc. Symp. Pure Math. ,45,455-462 (1986).
8. Azroul, E., Redwane, H., M. Rhoudaf, " Existence of a renormalized solution for a class of nonlinear parabolic equations in Orlicz spaces", Port. Math. 66 (1) (2009), 29-63.
9. Aharouch L., Bennouna J.: " Existence and uniqueness of solutions of unilateral problems in Orlicz spaces". Nonlinear Anal. 72, 3553-3565 (2010)
10. Musielak J., :" Modular spaces and Orlicz spaces". Lecture Notes in Math. (1983).
11. Redwane, H. , "Existence of a solution for a class of nonlinear parabolic systems". Electron. J. Qual. Theory Differ. Equ., 24, , 18pp.(2007)
12. Simon, J., " Compact set in the space $L^p(0,T,B)$". Ann. Mat. Pura, Appl. 146 , pp. 65-96 (1987).