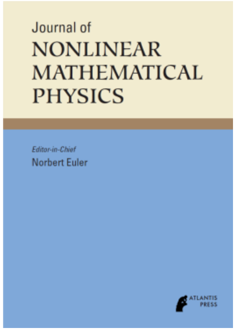

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BERNOULLI NUMBERS AND SOLITONS — REVISITED

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In the present paper we propose a new proof of the Grosset–Veselov formula connecting one-soliton solution of the Korteweg–de Vries equation to the Bernoulli numbers. The approach involves Eulerian numbers and Riccati's differential equation.

Keywords: Eulerian numbers; Riccati's equation; Bernoulli numbers; KdV equation; soliton.

Mathematics Subject Classification: 11B68, 35Q51

1. Introduction

By B_n $(n = 0, 1, 2, \ldots)$ we denote the nth Bernoulli number. The Bernoulli numbers have the following generating function $B(\xi)$ (see [4])

$$B(\xi) = B_0 + B_1\xi + B_2\frac{\xi^2}{2!} + \cdots = \frac{\xi}{e^{\xi} - 1}, \quad |\xi| < 2\pi. \tag{1.1}$$

It is well known that B_n vanishes for odd $n \geq 3$. The numbers are rational and they appear in relations such that

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n+1}\frac{2^{2n-1}\pi^{2n}}{(2n)!}B_{2n} \quad n = 1, 2, \ldots.$$

The first few nonzero Bernoulli numbers are as follows

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42},$$
$$B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}.$$

The Korteweg–de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1.2}$$

is well known and widely present in the literature. Miura, Gardner and Kruskal, in a seminal paper [8], proved that KdV equation has infinitely many conservation laws. The equation is famous for its family of solutions known as solitons, the simplest of which is one-dimensional soliton solution

$$u = -\frac{2}{\cosh^2(x-4t)},$$

corresponding to the initial profile $u(x,0) = -2/\cosh^2 x$. Fairlie and Veselov [3] proved, by using the conservation laws, that KdV equation is directly related to the Faulhaber polynomials and the Bernoulli polynomials (see [1,7]). Grosset and Veselov [5] demonstrated the formula

$$B_{2m} = \frac{(-1)^{m-1}}{2^{2m+1}} \int_{-\infty}^{+\infty} \left(\frac{d^{m-1}}{dx^{m-1}} \frac{1}{\cosh^2 x} \right)^2 dx, \tag{1.3}$$

in two ways, using the previously cited results and then adapting an idea due to Logan described in the book [4]. Boyadzhiev [2] gave the alternative proof of (1.3), based on the Fourier transform theory.

In the present article we would like to indicate some formulas, relating to KdV equation, which explain the appearance of Bernoulli numbers in this theory and imply (1.3).

In order to do this we will need Eulerian numbers (see [4]). The Eulerian number $\left\langle {n \atop k} \right\rangle$ is defined as the number of permutations of the set $\{1, 2, \dots, n\}$ having k permutation ascents. Let $\{a_1, a_2, \dots, a_n\}$ be a permutation of the set $\{1, 2, \dots, n\}$. Then $\{a_j, a_{j+1}\}$ is an ascent of the permutation if $a_j < a_{j+1}$. For example for $n = 3$ the permutation $\{1, 2, 3\}$ has two ascents, namely $\{1, 2\}$ and $\{2, 3\}$, and $\{3, 2, 1\}$ has no ascents. Each of the other four permutations of the set has exactly one ascent. Thus $\left\langle {3 \atop 0} \right\rangle = 1$, $\left\langle {3 \atop 1} \right\rangle = 4$, and $\left\langle {3 \atop 2} \right\rangle = 1$. It is well known that Eulerian numbers satisfy the following relations:

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k-1} \right\rangle,$$
$$\left\langle {n+1 \atop k} \right\rangle = (k+1)\left\langle {n \atop k} \right\rangle + (n-k+1)\left\langle {n \atop k-1} \right\rangle. \tag{1.4}$$

2. Two Theorems

We will reformulate, to the case of complex holomorphic functions, some results of the paper [9]. Consider a holomorphic function $z = z(t)$, defined in a domain $t \in D \subset C$ which fulfils Riccati's differential equation with constant coefficients

$$z'(t) = az^2 + bz + c \tag{2.1}$$

where a, b, c are complex numbers, $a \neq 0$, $b^2 - 4ac \neq 0$. Examples of such functions and equations are:

(1) $z(t) = \tan t, \; z'(t) = z^2 + 1$,
(2) $z(t) = \tanh t, \; z'(t) = -z^2 + 1$,
(3) $z(t) = 1/(1 + e^t), \; z'(t) = z^2 - z$,

(4) $z(t) = 1/(1+e^{-t})$, $z'(t) = -z^2 + z$, (more generally the logistic function $z(t) = q/(1+pe^{-rt})$, $z'(t) = \frac{r}{q}(q-z)z$ with $p > 1, q > 0, r > 0$).

Let $az^2 + bz + c = a(z-\alpha)(z-\beta)$.

Theorem 2.1. *If a function $z(t)$ fulfils Eq.* (2.1)

$$z'(t) = a(z-\alpha)(z-\beta) \tag{2.2}$$

then the nth derivative of $z(t)$ can be expressed by the following formula

$$\begin{aligned} z^{(n)}(t) &= a^n\left(\left\langle {n \atop 0} \right\rangle (z-\alpha)(z-\beta)^n + \left\langle {n \atop 1} \right\rangle (z-\alpha)^2(z-\beta)^{n-1} \right. \\ &\qquad \left. + \left\langle {n \atop 2} \right\rangle (z-\alpha)^3(z-\beta)^{n-2} + \cdots + \left\langle {n \atop n-1} \right\rangle (z-\alpha)^n(z-\beta) \right) \\ &= a^n \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle (z-\alpha)^{k+1}(z-\beta)^{n-k} \end{aligned} \tag{2.3}$$

where $n = 2, 3, \ldots$.

For convenience of the reader we give a proof of the theorem.

Proof. The proof is based on mathematical induction. By (2.2) we get

$$z''(t) = a[(z-\alpha)+(z-\beta)]z'(t) = a^2[(z-\alpha)(z-\beta)^2 + (z-\alpha)^2(z-\beta)]$$

which establishes (2.3) for $n = 2$. Let us assume that for an integer $n \geq 2$ formula (2.3) holds. Using recurrence formula (1.4), in the last step of the following calculation, we get

$$\begin{aligned} z^{(n+1)}(t) &= a^n \frac{d}{dt} \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle (z-\alpha)^{k+1}(z-\beta)^{n-k} \\ &= a^{n+1} \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \left[(k+1)(z-\alpha)^{k+1}(z-\beta)^{n-k+1} + (n-k)(z-\alpha)^{k+2}(z-\beta)^{n-k}\right] \\ &= a^{n+1} \left[\left\langle {n \atop 0} \right\rangle (z-\alpha)(z-\beta)^{n+1} + \sum_{k=1}^{n-1} \left((k+1) \left\langle {n \atop k} \right\rangle + (n-k+1) \left\langle {n \atop k-1} \right\rangle \right) \right. \\ &\qquad \left. \times (z-\alpha)^{k+1}(z-\beta)^{n-k+1} + \left\langle {n \atop n-1} \right\rangle (z-\alpha)^{n+1}(z-\beta) \right] \\ &= a^{n+1} \sum_{k=0}^{n} \left\langle {n+1 \atop k} \right\rangle (z-\alpha)^{k+1}(z-\beta)^{n-k+1}, \end{aligned}$$

which ends the proof. ∎

Let us denote the polynomial (2.3) of z (of degree $n+1$) by $P_{n+1}(z)$. Therefore for $n = 2, 3, \ldots$ we have

$$P_n(z) = \sum_{k=0}^{n-2} \left\langle {n-1 \atop k} \right\rangle (z-\alpha)^{k+1}(z-\beta)^{n-k-1}. \tag{2.4}$$

For $n = 1$, we put $P_1(z) = z - \alpha$. Since $P_n(z)$ is the entire function in C we can consider the integral $\int_\alpha^\beta P_n(z)dz$ over any curve (piecewise smooth) joining the points α and β.

Theorem 2.2. *For $n = 1, 2, \ldots$ it holds*

$$\int_\alpha^\beta P_n(z)dz = -(\beta-\alpha)^{n+1}B_n. \tag{2.5}$$

Proof. Substituting in integral on the left-hand side of (2.5) $z = z(w)$, $z - \alpha = (\beta - \alpha)w$ or equivalently $z - \beta = (\beta - \alpha)(w - 1)$ $(z(0) = \alpha,\ z(1) = \beta)$ we see that it suffices to prove formula (2.5) in the particular case when $\alpha = 0$, $\beta = 1$, where the respective Riccati's equation $z'(t) = z^2 - z$ has a solution $z = f(t) = 1/(1 + \exp(t))$. We will prove that in case of

$$P_n(z) = \sum_{k=0}^{n-2} \left\langle {n-1 \atop k} \right\rangle z^{k+1}(z-1)^{n-k-1}, \quad P_1(z) = z, \tag{2.6}$$

one gets

$$\int_0^1 P_n(z)dz = -B_n. \tag{2.7}$$

Let us observe that the generating function, for the polynomials (2.6),

$$g(z,\xi) = P_1(z) + P_2(z)\xi + P_3(z)\frac{\xi^2}{2!}\cdots$$

is the Taylor expansion of $f(t+\xi)$ at the point $t = \log((1-z)/z)$ (i.e. $z = 1/(1 + \exp(t))$. Thus

$$g(z,\xi) = f(t+\xi) = \frac{1}{1+e^{t+\xi}} = \frac{1}{1+e^t e^\xi} = \frac{1}{1+\frac{1-z}{z}e^\xi} = \frac{z}{z+(1-z)e^\xi}. \tag{2.8}$$

Since the generating function (2.8) is a rational function of the variable z we find easily its antiderivative

$$\frac{1}{1-e^\xi}\left[z - \frac{e^\xi}{1-e^\xi}\log(z+(1-z)e^\xi)\right],$$

and then use it to compute $\int_0^1 g(z,\xi)dz$. We get

$$\int_0^1 g(z,\xi)dz = \frac{1-e^\xi+\xi e^\xi}{(1-e^\xi)^2} \tag{2.9}$$

and check that the right-hand side of (2.9) is equal to $-B'(\xi)$ (where $B(\xi)$ is defined by (1.1)). By comparing the coefficients of $\xi^{n-1}/(n-1)!$ we get (2.7) and the theorem is proved. $\square$

Corollary. *Putting in* (2.5) $z = z(t)$ (*a solution of Riccati's equation* (2.2)) *and using* (2.3) *we get*

$$\int_\gamma z^{(n-1)}(t)z'(t)dt = -a^{n-1}(\beta-\alpha)^{n+1}B_n, \tag{2.10}$$

over a curve γ, *such that its image* $z(\gamma)$ *is the curve joining the points* α *and* β.

3. Applications to KdV Equation

When looking for soliton solutions of KdV equation (1.2) of the form $u = f(x - ct)$, we see that the function $f(x)$ must fulfill the ordinary differential equation

$$-cf'(x) - 6ff'(x) + f'''(x) = 0$$

i.e. for a constant A

$$\frac{d^2 f}{dx^2} = 3f^2 + cf + A. \tag{3.1}$$

Multiplying both sides of (3.1) by $\frac{df}{dx}$ one gets

$$\frac{1}{2}\frac{d}{dx}\left(\frac{df}{dx}\right)^2 = (3f^2 + cf + A)\frac{df}{dx},$$

and then

$$\left(\frac{df}{dx}\right)^2 = 2f^3 + cf^2 + 2Af + B \tag{3.2}$$

for a constant B. Denote by $F(x)$ a primitive function of $f(x)$. It is easy to check that in case $A = B = 0$ the function $F(x)$, under assumption that $dF/dx = h(F)$, must fulfill the following equation

$$\frac{dF}{dx} = \frac{1}{2}F^2 + c_1 F + \frac{1}{2}c_1^2 - \frac{1}{2}c \tag{3.3}$$

for some constant c_1.

We see that equation (3.3) is a particular case of the Riccati equation (2.1), and we can apply, for its solution $F(x)$, the formula (2.10). We have

$$\int_\gamma F^{(n-1)}(x)F'(x)dx = -\frac{(\beta-\alpha)^{n+1}}{2^{n-1}}B_n. \tag{3.4}$$

For example if $c_1 = 0,\ c = 1$, Eq. (3.3) takes the form

$$\frac{dF}{dx} = \frac{1}{2}(F^2 - 1),$$

and one of its solutions is $F(x) = -\tanh(x/2)$. As the curve γ we may take the real axis, which is the preimage of the interval $(-1, 1)$ under the function F. Thus we get

$(\beta = 1, \alpha = -1)$

$$-\int_{-\infty}^{\infty} (-\tanh(x/2))^{(n-1)}(-\tanh(x/2))' dx = -4B_n.$$

Substituting, in the above formula, $t = x/2$, $n = 2m$ and then integrating by parts $m-1$ times we get (1.3), the main result of the paper [5].

4. Concluding Remarks

Hoffman [6] considers more general case of a function $f(x)$ which fulfills the equation $f'(x) = P(f(x))$, where P is a polynomial. By computing successive derivatives of $f(x)$ he obtains, on the right-hand side of this equation, a sequence of polynomials (derivative polynomials). Next he finds, for them, a generating function and a recurrence formula and uses the results to get exact formulas for some integrals and series. The formulas derived by Hoffman could be applied to find e.g. the generating function of derivative polynomials of the function $f(x) = \tanh(x)$. Then proceeding similarly as in paper [5] one could get the Grosset–Veselov formula (1.3).

Formula (2.3) can be seen as a closed form formula for the derivative polynomials of a function satisfying the Riccati equation with constant coefficients. The two Theorems 2.1 and 2.2 can be used not only in KdV equation theory, but in each case of such function. The results of the paper might be applied e.g. to the logistic function (see item 4 at the beginning of the second section), which has a great importance in physics (the Fermi–Dirac statistics), medicine (modeling of growth of tumors), economics (production function, population growth). Theorem 2.1 allows us to calculate, at any point, all derivatives of the logistic function.

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