Fordy-Kulish model and spinor Bose-Einstein condensate

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Abstract

A three-component nonlinear Schrodinger-type model which describes spinor Bose-Einstein condensate (BEC) is considered. This model is integrable by the inverse scattering method and using Zakharov-Shabat dressing method we obtain three types of soliton solutions. The multi-component nonlinear Schrödinger type models related to symmetric spaces $\mathbf{C.I} \simeq \mathrm{Sp}(4)/\mathrm{U}(2)$ is studied.

1   Introduction

The dynamics of spinor BEC is described by a three-component Gross-Pitaevskii (GP) system of equations. In the one-dimensional approximation the GP system goes into the following nonlinear Schrödinger (MNLS) equation in (1D) x-space [1]:

$$
\begin{aligned}
&i\partial_t \Phi_1 + \partial_x^2 \Phi_1 + 2(|\Phi_1|^2 + 2|\Phi_0|^2)\Phi_1 + 2\Phi_{-1}^* \Phi_0^2 = 0, \\
&i\partial_t \Phi_0 + \partial_x^2 \Phi_0 + 2(|\Phi_{-1}|^2 + |\Phi_0|^2 + |\Phi_1|^2)\Phi_0 + 2\Phi_0^* \Phi_1 \Phi_{-1} = 0, \\
&i\partial_t \Phi_{-1} + \partial_x^2 \Phi_{-1} + 2(|\Phi_{-1}|^2 + 2|\Phi_0|^2)\Phi_{-1} + 2\Phi_1^* \Phi_0^2 = 0.
\end{aligned}
\tag{1}
$$

We consider BEC of alcali atoms in the $F = 1$ hyperfine state, elongated in x direction and confined in the transverse directions y, z by purely optical means. Thus the assembly of atoms in the $F = 1$ hyperfine state can be described by a normalized spinor wave vector

$$
\mathbf{\Phi}(x,t) = (\Phi_1(x,t), \Phi_0(x,t), \Phi_{-1}(x,t))^T
\tag{2}
$$

whose components are labelled by the values of $m_F = 1, 0, -1$. The above model is integrable by means of inverse scattering transform method [1]. It also allows an exact description of the dynamics and interaction of bright solitons with spin degrees of freedom. Matter-wave solitons are expected to be useful in atom laser, atom interferometry and coherent atom transport. It could contribute to the realization of quantum information processing or computation, as a part of new field of atom optics. Lax pairs and geometric interpretation of the model (1) are given in [4]. Darboux transformation for this special integrable model is developed in [5].

The aim of present paper is to show that the system (1) is related to symmetric space **C.I** $\simeq$ Sp(4)/U(2) (in the Cartan classification [8]) with canonical $\mathbb{Z}_2$-reduction and has natural Lie algebraic interpretation. The model allows also a special class of soliton solutions. We will show that they can be obtained by a suitable modification of the generalization of the so-called "dressing method", proposed in [9].

2 Solving the model: algebraic and analytic aspects.

The model (1) belongs to the class of multicomponent NLS equations that can be solved by the inverse scattering method [7, 6]. It is a particular case of the MNLS related to the **C.I** type symmetric space Sp(4)/U(2) [4]. These MNLS systems allow Lax representation with the generalized Zakharov–Shabat system as the Lax operator:

$$L\psi(x,t,\lambda) \equiv i\frac{d\psi}{dx} + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0. \tag{3}$$

where J and $Q(x,t)$ are 4×4 matrices: $J = \text{diag}\,(1,1,-1,-1)$ and $Q(x,t)$ is a block-off-diagonal matrix:

$$Q(x,t) = \begin{pmatrix} 0 & \mathbf{q}(x,t) \\ \mathbf{p}(x,t) & 0 \end{pmatrix}, \quad \mathbf{q}(x,t) = \begin{pmatrix} \Phi_0(x,t) & -\Phi_1(x,t) \\ \Phi_{-1}(x,t) & -\Phi_0(x,t) \end{pmatrix},$$
$$\mathbf{p}(x,t) = \begin{pmatrix} \Phi_0^*(x,t) & \Phi_{-1}^*(x,t) \\ -\Phi_1^*(x,t) & -\Phi_0^*(x,t) \end{pmatrix}. \tag{4}$$

Solving the direct and the inverse scattering problem for L uses the Jost solutions $\phi = (\phi^+, \phi^-)$ and $\psi = (\psi^-, \psi^+)$ of (3) which are defined by, see [14] and the references therein:

$$\lim_{x\to-\infty} \phi(x,t,\lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to\infty} \psi(x,t,\lambda)e^{i\lambda Jx} = \mathbb{1} \tag{5}$$

These definitions are compatible with the class of smooth potentials $Q(x,t)$ vanishing sufficiently rapidly at $x \to \pm\infty$. It can be shown that ϕ^+ and ψ^+ (resp. ϕ^- and ψ^-) composed by 4 rows and 2 columns are analytic in the upper (resp. lower) half plane of λ. The scattering matrix associated to (3) is defined as

$$T(t,\lambda) = (\psi(x,t,\lambda))^{-1}\phi(x,t,\lambda) = \begin{pmatrix} a^+(t,\lambda) & -b^-(t,\lambda) \\ b^+(t,\lambda) & a^-(t,\lambda) \end{pmatrix},$$
$$(T(t,\lambda))^{-1} = \begin{pmatrix} c^-(t,\lambda) & d^-(t,\lambda) \\ -d^+(t,\lambda) & c^+(t,\lambda) \end{pmatrix}, \tag{6}$$

where $a^\pm(t,\lambda)$ and $b^\pm(t,\lambda)$ are 2×2 block matrices. The blocks $a^\pm$, $b^\pm$, $c^\pm$ and $d^\pm$ satisfy a number of relations [11, 12]; for example

$$a^+(\lambda)c^-(\lambda) + b^-(\lambda)d^+(\lambda) = \mathbb{1}, \qquad a^+(\lambda)d^-(\lambda) - b^-(\lambda)c^+(\lambda) = 0, \tag{7}$$

etc. The fundamental analytic solutions (FAS) $\chi^\pm(x,t,\lambda)$ of $L(\lambda)$ are analytic functions of λ for $\text{Im}\,\lambda \gtrless 0$ and are related to the Jost solutions by:

$$\chi^\pm(x,t,\lambda) = \phi(x,t,\lambda)S_J^\pm(t,\lambda) = \psi(x,t,\lambda)T_J^\mp(t,\lambda). \tag{8}$$

Here $S_J^\pm$, $T_J^\pm$ upper- and lower- block-triangular matrices:

$$S_J^+(t,\lambda) = \begin{pmatrix} \mathbb{1} & d^-(t,\lambda) \\ 0 & c^+(t,\lambda) \end{pmatrix}, \quad S_J^-(t,\lambda) = \begin{pmatrix} c^-(t,\lambda) & 0 \\ -d^+(t,\lambda) & \mathbb{1} \end{pmatrix},$$

$$T_J^+(t,\lambda) = \begin{pmatrix} \mathbb{1} & -b^-(t,\lambda) \\ 0 & a^-(t,\lambda) \end{pmatrix}, \quad T_J^-(t,\lambda) = \begin{pmatrix} a^+(t,\lambda) & 0 \\ b^+(t,\lambda) & \mathbb{1} \end{pmatrix},$$

satisfying $T_J^\pm(t,\lambda)\hat{S}_J^\pm(t,\lambda) = T(t,\lambda)$ and can be viewed as the factors of a generalized Gauss decompositions of $T(t,\lambda)$ [13]. If $Q(x,t)$ evolves according to (1) then the scattering matrix and its elements satisfy the following linear evolution equations:

$$i\frac{db^\pm}{dt} + 2\lambda^2 b^\pm(t,\lambda) = 0, \qquad i\frac{da^\pm}{dt} = 0, \tag{9}$$

so the block-matrices $a^\pm(\lambda)$ can be considered as generating functionals of the integrals of motion. The fact that all 4 matrix elements of $a^+(\lambda)$ for $\lambda \in \mathbb{C}_+$ (resp. of $a^-(\lambda)$ for $\lambda \in \mathbb{C}_-$) generate integrals of motion reflect the superintegrability of the model and are due to the degeneracy of the dispersion law of (1).

The system (1) can be written in a Hamiltonian form by introducing the Poisson brackets:

$$\{q_j(x), p_k(y)\} = 2i\delta_{kj}\delta(x-y), \qquad \{q_{12}(x), p_{12}(y)\} = i\delta(x-y), \tag{10}$$

and the Hamiltonian $H = H_{\text{kin}} + H_{\text{int}}$:

$$H_{\text{kin}} = \int_{-\infty}^{\infty} dx \left(\frac{\partial\Phi_0}{\partial x}\frac{\partial\Phi_0^*}{\partial x} + \frac{1}{2}\left(\frac{\partial\Phi_1}{\partial x}\frac{\partial\Phi_1^*}{\partial x} + \frac{\partial\Phi_{-1}}{\partial x}\frac{\partial\Phi_{-1}^*}{\partial x} \right) \right), \tag{11}$$

$$H_{\text{int}} = -\int_{-\infty}^{\infty} dx \left((|\Phi_0|^2 + |\Phi_1|^2)^2 + (|\Phi_0|^2 + |\Phi_{-1}|^2)^2 \right)$$
$$-\int_{-\infty}^{\infty} dx \left(|\Phi_0\Phi_{-1}^* + \Phi_1\Phi_0^*|^2 \right).$$

As mentioned above, one can use any of the matrix elements of $a^\pm(\lambda)$ as generating functional of integrals of motion of our model. Generically such integrals would have non-local densities and will not be in involution.

The classical R-matrix approach [6, 4] is an effective method to determine the generating functionals of local integrals of motion which are in involution. From it there follows that such integrals are generated by expanding $\ln m_k^\pm(\lambda)$ over the inverse powers of λ, see [13]. Here $m_k^\pm(\lambda)$ are the principal minors of $T(\lambda)$; in our case

$$m_1^+(\lambda) = a_{11}^+(\lambda), \qquad m_2^+(\lambda) = \det a^+(\lambda),$$
$$m_1^-(\lambda) = a_{22}^-(\lambda), \qquad m_2^-(\lambda) = \det a^-(\lambda). \tag{12}$$

If we consider

$$\ln m_k^+(\lambda) = \sum_{s=1}^{\infty} \lambda^{-k} I_s^{(k)},$$

then one can prove that the densities of $I_s^{(k)}$ are local in $Q(x,t)$. The fact that [13]:

$$\{m_k^\pm(\lambda), m_j^\pm(\mu)\} = 0, \qquad \text{for } k,j = 1,2,$$

and for all $\lambda, \mu \in \mathbb{C}_\pm$ allow one to conclude that $\{I_s^{(k)}, I_p^{(j)}\} = 0$ for all $k, j = 1, 2$ and $s, p \geq 1$.

In particular, the Hamiltonian of our model is proportional to $I_3^{(2)}$, i.e. $H = 8iI_3^{(2)}$.

3   Soliton solutions for the spinor BEC: The $so(5)$ connection

The soliton solutions of the $\mathfrak{sp}(4)$ MNLS (1) were derived by using the dressing method [3]. They can be considered as particular cases of the soliton solutions of the generic MNLS eqs., derived through the matrix version of the Gel'fand-Levitan-Marchenko equation, see [2, 1, 3]. Here we extend further these results and combining the ideas of [3, 15] we specify three types of solitons for the model (1).

We start our analysis with the well-known isomorphism between the algebras $sp(4, \mathbb{C})$ and $so(5, \mathbb{C})$ [8]. Since the Lax representation is of pure algebraical nature it is natural to expect that our model (1) can be treated also by an equivalent Lax operator L' whose potential $Q'(x, t)$ and J' take values in $so(5)$. A consequence of the above-mentioned isomorphism is that the typical representation of $sp(4)$ used above is equivalent to the spinor representation of $so(5)$.

So we first remind some of our results in [10, 11, 12], where we have constructed the fundamental analytic solutions, the dressing factors, the soliton solutions etc. for a class of Lax operators (including L'), related to the simple Lie algebra $\mathfrak{g}$, in the typical representation of $\mathfrak{g}$. So we first have to specify (if necessary) $\mathfrak{g} \simeq so(5)$ and then reformulate the corresponding results for the spinor representation of $so(5)$.

The main goal of the dressing method is, starting from a solution $\chi_0^\pm(x, t, \lambda)$ of $L_0(\lambda)$ with potential $Q_{(0)}(x, t)$ to construct a new singular solution $\chi_1^\pm(x, t, \lambda)$ with singularities located at prescribed positions $\lambda_1^\pm$; the reduction $\mathbf{p} = \mathbf{q}^\dagger$ used in eq. (4) ensures that $\lambda_1^- = (\lambda_1^+)^*$. The new solutions $\chi_1^\pm(x, t, \lambda)$ will correspond to a potential $Q_{(1)}(x, t)$ of $L(\lambda)$ (3) with two discrete eigenvalues $\lambda_1^\pm$. It is related to the regular one by a dressing factor $u(x, \lambda)$

$$\chi_1^\pm(x, t, \lambda) = u(x, \lambda)\chi_0^\pm(x, t, \lambda)u_-^{-1}(\lambda). \qquad u_-(\lambda) = \lim_{x \to -\infty} u(x, \lambda) \qquad (13)$$

Note that $u_-(\lambda)$ is a diagonal matrix. The dressing factor $u(x, \lambda)$ must satisfy the equation

$$i\frac{du}{dx} + Q_{(1)}(x)u - uQ_{(0)}(x) - \lambda[J, u(x, \lambda)] = 0, \qquad (14)$$

and the normalization condition $\lim_{\lambda \to \infty} u(x, \lambda) = \mathbb{1}$. Besides $\chi_i^\pm(x, \lambda)$, $i = 0, 1$ and $u(x, \lambda)$ must belong to the corresponding Lie group $Sp(4, \mathbb{C})$; in addition $u(x, \lambda)$ by construction has poles and/or zeroes at $\lambda_1^\pm$.

The construction of $u(x, \lambda)$ is based on an appropriate anzats specifying explicitly the form of its λ-dependence.

$$u(x, \lambda) = \mathbb{1} + (c_1(\lambda) - 1) P_1(x, t) + \left(\frac{1}{c_1(\lambda)} - 1\right) \overline{P}_1(x, t),$$

$$c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-}, \qquad (15)$$

where the projectors $P_1(x,t)$ and $\overline{P}_1(x,t)$ are are of rank 1 and are related by $\overline{P}_1(x) = SP_1^T(x)S^{-1}$. They must satisfy $\overline{P}_1(x,t)P_1(x,t) = P_1(x,t)\overline{P}_1(x,t) = 0$. By S we have denoted the special matrix which enters in the definition of the orthogonal algebra, i.e. $X \in so(5)$ if $X + SX^TS^{-1} = 0$. In the typical representation of $so(5)$ we have $S = \sum_{k=1}^{5}(-1)^{k+1}E_{k,6-k}$ where $(E_{ij})_{km} = \delta_{ik}\delta_{jm}$. The construction of $P_1(x,t)$ and $\overline{P}_1(x,t)$ using the 'polarization' vectors is outlined in [11] and we skip it. The new potential is obtained from

$$Q_{(1)}(x,t) - Q_{(0)}(x,t) = (\lambda_1^+ - \lambda_1^-)[J, P_1(x,t) - \overline{P}_1(x,t)].$$

Here we show that the λ-dependence of $u(x,\lambda)$ may depend [10] on the choice of the representation of $so(5,\mathbb{C}) \simeq sp(4,\mathbb{C}))$. For $so(5)$ it was shown[10, 11, 12, 15] that there are three types of solitons:

- the first type of soliton solutions are generated by dressing factors of the form (15). For generic choice of the polarization vectors $P_1(x,t) - \overline{P}_1(x,t) \in so(5)$.

- the second type of soliton solutions are generated analogously with dressing factor (15), but due to a specific choice of the polarization vectors $P_1(x,t) - \overline{P}_1(x,t) \in so(3) \subset so(5)$.

- the third type of soliton solutions are generated again by (15) but now the corresponding projectors $P_1(x,t)$ and $\overline{P}_1(x,t)$ have rank 2.

Each of these types of soliton solutions have their counterpart relevant to our model on $sp(4)$. To the first type of soliton solutions there correspond dressing factor and potential $Q_{(1)}(x,t)$ of the form[3]:

$$\tilde{u}(x,\lambda) = \sqrt{c_1(\lambda)}\pi_1(x) + \frac{1}{\sqrt{c_1(\lambda)}}\overline{\pi}_1(x),$$

$$Q_{(1)}(x,t) - Q_{(0)}(x,t) = \frac{1}{2}[J, \pi_1(x,t) - \overline{\pi}_1(x,t)], \tag{16}$$

where $\pi_1(x)$ and $\overline{\pi}_1(x)$ are *rank 2* projectors, such that

$$\overline{\pi}_1(x)\pi_1(x) = \pi_1(x), \quad \overline{\pi}_1(x) = 0, \qquad \overline{\pi}_1(x) + \pi_1(x) = \mathbb{1}. \tag{17}$$

This last property ensures the non-degeneracy of $u(x,\lambda)$. Note that now the dressing factor is not a rational function of λ but for the dressed FAS $\chi(x,\lambda)$ eq. (13) we get:

$$\chi_1^{\pm}(x,t,\lambda) = \left(\pi_1(x,t) + \frac{1}{c_1(\lambda)}\overline{\pi}_1(x,t)\right)\chi_0^{\pm}(x,t,\lambda)\left(\pi_1^- + c_1(\lambda)\overline{\pi}_1^-\right),$$

$$\pi_1^- = \lim_{x\to-\infty}\pi_1(x,t), \tag{18}$$

i.e., the fractional powers of $c_1(\lambda)$ disappear.

The second type of solitons with rank 2 projector $P_1(x)$ after recalculating to the spinor representation formally keeps the same form (15) with $P_1(x)$ replaced by $A_1(x)$ which has rank 1 but generically is not a projector, see [3].

The third type of solitons is similar to the second one but with additional constraints on the factor $A_1(x)$ so that $A_1(x) - \overline{A}_1(x) \in sp(2) \subset sp(4)$.

Consider the purely solitonic case when $Q_{(0)} = 0$. From now on we introduce the following notations $\lambda_1^{\pm} = \mu_1 \pm i\nu_1$ and

$$A = -2i((\lambda_1^+)^2 - (\lambda_1^-)^2)t - i(\lambda_1^+ - \lambda_1^-)x,$$
$$B = -2((\lambda_1^+)^2 + (\lambda_1^-)^2)t - (\lambda_1^+ + \lambda_1^-)x. \tag{19}$$

Here $A(x,t)$ and $B(x,t)$ are x and t dependent real valued functions. Making use of the explicit form of the projectors $P_{\pm 1}(x)$ valid for the typical representations of $\mathbf{B}_2$ we obtain[11]:

$$\Phi_{(1)}(x,t) = \frac{4(\lambda_1^+ - \lambda_1^-)}{\langle m|n \rangle} \left(n_{0,1}m_{0,2}e^A + n_{0,\bar{2}}m_{0,\bar{1}}e^{-A} \right) e^{iB} \tag{20}$$

$$\Phi_{(0)}(x,t) = i\frac{2\sqrt{2}(\lambda_1^+ - \lambda_1^-)}{\langle m|n \rangle} \left(n_{0,1}m_{0,3}e^A - n_{0,\bar{3}}m_{0,\bar{1}}e^{-A} \right) e^{iB} \tag{21}$$

$$\Phi_{(-1)}(x,t) = -\frac{4(\lambda_1^+ - \lambda_1^-)}{\langle m|n \rangle} \left(n_{0,1}m_{0,\bar{2}}e^A + n_{0,2}m_{0,\bar{1}}e^{-A} \right) e^{iB} \tag{22}$$

where the denominator in the above formula is given by:

$$\langle m|n \rangle = m_{0,1}n_{0,1}(e^{2A}) + m_{0,\bar{1}}n_{0,\bar{1}}(e^{-2A})$$
$$+ m_{0,2}n_{0,2} + m_{0,\bar{2}}n_{0,\bar{2}} + m_{0,3}n_{0,3}. \tag{23}$$

and $m_{0,k}$, $n_{0,k}$ are the components of the polarization vectors.

Choosing appropriately the elements of the polarization vectors $|n_0\rangle$ and $|m_0\rangle$, one can show that the conjecture that the Zakharov-Shabat dressing procedure and the Gel'fand-Levitan Marchenko formalism lead to comparable soliton solutions is true. It is not a problem to multiply the polarization vectors $|n_0\rangle$ and $|m_0\rangle$ by an appropriate scalar and thus to adjust the two solutions. Such a multiplication easily goes through the whole scheme outlined above. The involution $Q_{(1)}^\dagger = Q_{(1)}$ that the potential of the Lax operator (3) associated with the system (1) is subject to results in the following relations between the elements of the "polarization" vectors $|n_0\rangle$ and $\langle m_0|$, namely $n_{0,k} = m_{0,k}^*$. Utilizing the above and a proper change of field components, we can relate the solution

$$\mathbf{q}(x,t) = 4\nu_1 \frac{C^\dagger e^A + \sigma_2 C^t \sigma_2 \det\{C^\dagger\}e^{-A}}{e^{2A} + W + |\det\{C\}|^2 e^{-2A}} e^{iB}, \tag{24}$$

where $W = (2|c_{12}|^2 + |c_1|^2 + |c_2|^2)$ and the "polarization" matrix can be cast into the form

$$C = \begin{pmatrix} c_{12} & c_1 \\ c_2 & -c_{12} \end{pmatrix} \tag{25}$$

In the special case when $W = 1$ and $\det\{C\} = 0$ we obtain

$$\mathbf{q}(x,t) = \frac{2\nu_1 e^{iB}}{\cosh A} C^\dagger \tag{26}$$

Thus we confirm the result obtained in[1], aquaired with the help of GLM formalism and the solution (20), derived within the generalized Zakharov-Shabat dressing procedure, provided we make sure that the extra condition on the vector $|m\rangle$:

$$-2m_{0,1}m_{0,\bar{1}} + 2m_{0,2}m_{0,\bar{2}} = (m_{0,3})^2, \tag{27}$$

and analogous one for $|n\rangle$ holds true. Setting

$$m_{0,1} = 1, \quad m_{0,\bar{1}} = -(c_{12}^*)^2 - c_1^* c_2^*,$$
$$m_{0,2} = ic_1^*, \quad m_{0,\bar{2}} = ic_2^*, \quad m_{0,3} = m_{0,\bar{3}} = -\sqrt{2}c_{12}^*$$

we establish the equivalence between the two solutions.

4   Conclusions

We have derived the soliton solutions of the three-component system of NLS type on the symmetric space $\mathrm{Sp}(4)/\mathrm{U}(2)$ which is related to spinor Bose-Einstein condensate model (with $F = 1$). Furthermore, we have described briefly the Hamiltonian properties of the model and the integrals of motion. Using the classical r-matrix approach, we showed that the integrals of motion, that belong to the principal series are in involution.

The reduction of the multi-component nonlinear Schrödinger (NLS) equations on symmetric space $\mathbf{C.I} \simeq \mathrm{Sp}(2p)/\mathrm{U}(p)$ for $p = 2$ is related to spinor model of Bose-Einstein condensate. Other interesting reductions of MNLS type equations were reported in [11] and a systematic study of the problem is on the way. Recently the authors of [16] develop a perturbation theory for bright solitons of the $F = 1$ integrable spinor BEC model. Both rank-one and rank-two soliton solutions are obtained using Riemann-Hilbert method and are compared with known results.

Acknowledgments

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