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Managing Default Risk

Anna Zabai*

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Abstract

This paper argues that sovereigns can reduce the likelihood of debt crises through debt management: for any given financing need arising from the budget, they can adjust the timing and size of market recourse in response to borrowing costs. I model debt management by letting the sovereign adopt a funding rule that can respond to interest rates, and characterise the optimal rule accounting for the sovereign's large-agent status. I then decompose default risk into a solvency component driven by fundamentals and a coordination component driven by self-fulfilling expectations, and use numerical exercises to show that debt management operates on both.

Keywords: debt crises, default risk, debt management, coordination issues, self-fulfilling expectations.

JEL Classification: D84, H63.

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1 Introduction

High levels of public debt in advanced economies have recently raised the spectre of sovereign default and revived the debate about coordination problems and self-fulfilling beliefs. Indeed, policymakers' views and actions during the 2009-2013 euro area debt crisis appeared to reflect the dual nature of default risk (Blanchard, 2011; Draghi, 2012).[1] On the one hand, default could happen because fundamentals make debt unsustainable (solvency risk). On the other hand, default could also happen because investors expect it to happen (coordination risk).

Policymakers have a number of options to manage default risk. Ex-ante, fiscal consolidations that reduce the level of debt make sovereigns less vulnerable to both solvency and coordination risk. Ex-post, sovereigns can apply for a bailout from a lender of last resort (LoLR). Neither solution is without drawbacks, however. Fiscal consolidations require either spending cuts or tax hikes, and they are typically difficult to implement in countries where fundamentals are already bad. Moreover, to the extent that consolidation is a requirement for bailout applications to the LoLR, obtaining this type of funding may not be a viable option. Even without consolidation prerequisites, bailout applications may be put off by reputational penalties.[2]

This paper contributes to the sovereign default literature by arguing that sovereigns can also rely on debt management, namely, for any given financing need arising from the budget, they can adjust the timing and size of recourse to the markets in response to borrowing costs. The intuition is as follows. Since borrowing costs tend to be large when economic fundamentals are bad, debt management can reduce solvency risk by decreasing

[1] Sentiments featured prominently in the outlooks and recommendations of policymakers across the world. The Economic Counsellor of the International Monetary Fund declared that "the world economy is pregnant with multiple equilibria" (Blanchard, 2011). The President of the European Central Bank justified the introduction of the new Outright Monetary Transactions with a "need to help the large parts of the Euro area who find themselves in a bad equilibrium", and openly decried "pessimistic self-fulfilling expectations that threaten the union as a whole" (Draghi, 2012).

[2] Although the recent debate on default risk in advanced (especially euro area) economies has focused on fiscal austerity and bailouts by a LoLR, these are not the only options available to countries seeking to reduce default risk. Economic growth, debt restructuring, surprise inflation (another form of default) and financial repression have all been used at some point (see Reinhart and Rogoff, 2015).

the pro-cyclicality of the debt burden, in a similar way as state-contingent debt.[3] Unlike state-contingent debt, however, debt management can be implemented with *non-contingent bonds*, and does not require that fundamentals be observable or verifiable. Moreover, debt management can also reduce coordination risk. By letting their demand for funding fall as interest rates rise, sovereigns can weaken the link between prices and the burden of servicing the debt, thus reining in expectations and reducing borrowing costs. In sum, debt management can help address both components of default risk.

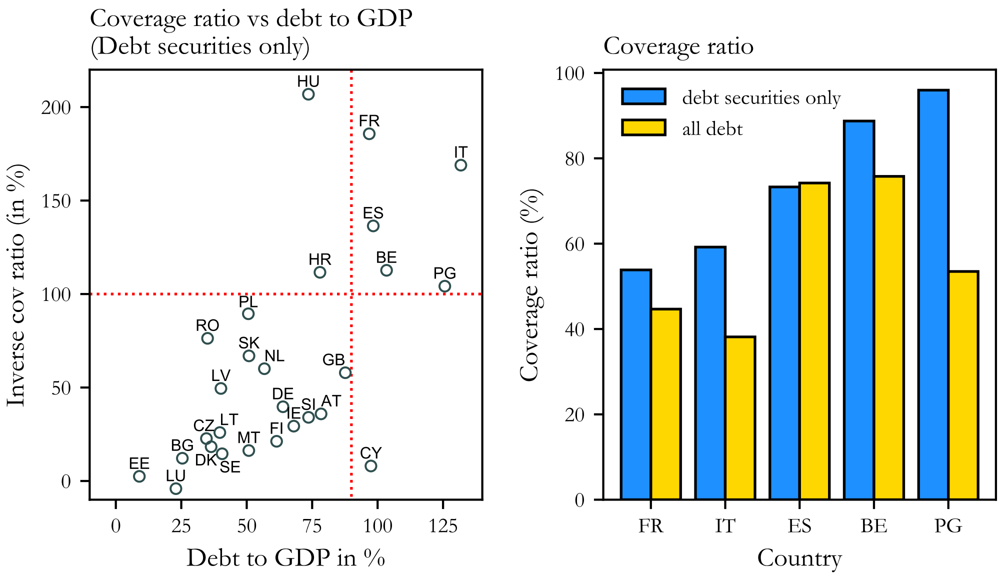
One may legitimately wonder whether sovereign debt management is feasible. After all, the *gross* government financing need (the sum of the primary deficit and maturing debt) is to a large degree predetermined.[4] But debt managers do not necessarily have to take gross financing needs as given and borrow accordingly. Because governments own financial assets, they can adjust new borrowing by liquidating some of the existing holdings. Adopting a liquidation rule that varies with interest rates is enough to ensure that the *net* government financing need (that is, new borrowing) responds to borrowing costs.

This feasibility argument hinges on the size of government holdings of financial assets. In the euro area, liquidity reserves are large relative to sovereign gross financing needs, so the margin of adjustment for debt managers is not trivial (Figure 1). Most euro area countries had large enough holdings of financial assets to cover their entire gross financing needs for the year 2017 (left panel). In the same year, all the five most heavily indebted euro area economies – including Italy, Portugal and Spain – would have been able to meet at least a third of their funding needs with asset liquidations (right panel). And selling assets is not a merely theoretical possibility. At the height of the sovereign debt crisis, Treasury departments in Italy, Spain and Ireland each cancelled a number of bond auctions, suggesting that sovereigns do rely on liquidations, at least in times of crisis.

[3] For a recent summary of the literature on state-contingent sovereign debt see Shiller, Ostry, Benford, and Joy (2018).

[4] To the extent that adjustments may be possible, they would happen through changes in the government budget, which is the outcome of political negotiations outside the remit of a debt management office.

FIGURE 1: Gross versus net government financing needs in the euro area (2017).



The coverage ratio in year t is defined as the ratio of gross government financial assets in year t (the sum of currency and deposits, debt securities, and loans) over the gross government financing need (GGFN) in year t. In turn, the GGFN is the sum of the primary deficit in year t and debt maturing in year t. The blue bars on the right panel take maturing debt to be equal to maturing debt securities only, while the yellow bars correspond to a broader measure of maturing debt that includes deposits and loans. *Sources:* Eurostat, ECB Government Financial Statistics (GFS) Database. Data is not available for Greece.

Framework. I assess the usefulness of debt management in a two-period model populated by a Sovereign and foreign financiers. The Sovereign borrows from financiers by issuing a one-period, non-contingent discount bond whose price is endogenously determined. The bond is risky because the Sovereign has limited commitment and can default on its liabilities.

The default decision is endogenous to the interest rate, in the Calvo (1988) tradition. To choose the optimal haircut, the sovereign trades off the benefit of a smaller debt burden against output losses incurred upon default (see, inter alia, Arellano, 2008). Therefore, the higher the interest rate, the stronger the incentive to default. This complementarity creates the potential for self-fulfilling expectations and price multiplicity.

I model debt management by letting the sovereign adopt a *funding rule* that can respond to interest rates. Under perfect foresight, and in the absence of some exogenous equilibrium selection device such as a sunspot, if the sovereign were to choose an optimal funding rule it would have to choose one prescribing that the level of bond issuance does not respond to interest rates. These rules deliver price uniqueness by eliminating complementarity, as in the Eaton and Gersovitz (1981) tradition.

To allow for the possibility that issuance may respond to interest rates – consistent with the euro area record of cancelled auctions during the recent sovereign debt crisis – while allowing coordination risk to be endogenous to the choice of borrowing, I characterise the optimal funding rule in a setting with uncertainty. Provided that bond issuance does not respond too strongly to funding costs, so complementarity is not too pronounced, the sovereign can still ensure price uniqueness.

The default cost (fundamentals) is a natural candidate for the source of uncertainty, for two reasons. First, it parameterises the strength of the sovereign's commitment problem. Second, it is likely more difficult to gauge than other relevant fundamental variables, like future output.

Results. I use this framework to show that by managing the debt (using an interest-elastic funding rule), a sovereign can experience lower funding costs and default risk than it would have obtained if it were restricted to not liquidating any assets (using an interest-inelastic rule). Next, I decompose default risk into a solvency component and a coordination component, and I use numerical exercises to confirm the intuition that debt management operates on both. The simulations also underscore how the optimal funding rule responds more aggressively to interest rates if the sovereign internalises coordination risk in choosing its demand for funding (coordination risk management).

Theoretical contribution. As a large agent, the Sovereign has to internalise how its choice of funding rule affects the non-competitive rational expectations equilibrium interest rate map, much like the informed traders in Kyle (1989) explicitly take into account the effect of their trades on the equilibrium price. This makes the optimal policy problem hard to solve.

The solution strategy begins by solving an ex-post problem. I suppose the sovereign has indeed announced some funding rule. Conditional on a particular realisation of the interest rate, it must decide whether to adhere to the funding level prescribed by said rule or not. Accordingly, the sovereign chooses an ex-post optimal borrowing level, taking as given a market-clearing price map that depends on new borrowing. This pointwise optimisation returns an optimal funding requirement and an optimal market-clearing interest rate, both as a function of the (perceived) fundamentals of the economy. Since the optimal interest rate map is invertible, a solution to the ex-ante problem can be constructed by composition from the solution to the ex-post problem (Proposition 2).

Relationship to the literature. Because I model coordination issues as manifesting in interest rates, this paper is in the Calvo tradition. As summarised by Aguiar, Chatterjee, Cole, and Stangebye (2017), there are two approaches to self-fulling expectations in the sovereign default literature. The first tradition was introduced by Calvo (1988). It emphasises how coordination issues driven by complementarity between current market-clearing prices and future default decisions feed back into current prices.[5] By letting the future default decision be endogenous to current interest rates, these papers – including mine – do not adopt the convention typically followed in empirical literature beginning with Eaton and Gersovitz (1981). The second approach, pioneered by Giavazzi and Pagano (1989) and subsequently developed by Cole and Kehoe (2000), focuses on contemporaneous links

[5] Recent contributions in this tradition are Lorenzoni and Werning (2013), Corsetti and Dedola (2016) and Ayres, Navarro, Nicolini, and Teles (2018).

between market prices and default decisions.[6] In this strand of the literature, coordination risk manifests as rollover risk, and thus debt crises are similar to bank runs.

The solution strategy developed here could be applied to other settings in which a large agent has to optimally choose actions while internalising their impact on market prices. A prime candidate is the global games literature with endogenous prices (e.g. Werning and Angeletos (2006), Tarashev (2007)), where this paper is most closely related to Tsyvinski, Mukherji, and Hellwig (2006). They present a model of currency crises to show that when public information is endogenous, equilibrium multiplicity may be restored in a global-games framework provided complementarity is sufficiently strong. As in this paper, coordination issues manifest in the asset price.In the Appendix, I show how to characterise the optimal funding rule in an environment with dispersed information.

Layout. Section 2 introduces the model. Section 3 presents the perfect foresight benchmark. Section 4 defines and characterises the equilibrium under uncertainty. In Section 5, I show how debt management improves debt sustainability. Section 6 concludes.

2 The setup

Consider a two-period economy. In each period, the economy produces a consumption good. The economy is inhabited by a risk-averse, benevolent government (from now on, the Sovereign) with access to international financial markets. Financial markets are populated by a measure-one continuum of risk-neutral (foreign) financiers indexed by $i \in [0, 1]$. The Sovereign can borrow from financiers by issuing a one-period, non-contingent discount bond. International debt contracts are not enforceable, so the Sovereign can default. Defaulting is costly.

Timing. To capture the fact that a government may adjust the size and timing of market recourse in response to borrowing costs by liquidating financial assets (debt management), I assume that the Sovereign announces

[6] Early contributions to this tradition include Alesina, Prati, and Tabellini (1990). More recent contributions are Aguiar, Amador, Farhi, and Gopinath (2015) and Aguiar, Chatterjee, Cole, and Stangebye (2017).

a *funding rule*, $b(r)$, at the beginning of the first period. The rule specifies the Sovereign's funding requirement (in terms of current consumption) for any level of the interest rate that can arise in equilibrium, $r \in \mathcal{E}_b$, where $\mathcal{E}_b$ denotes the set of possible market clearing rates.[7] As a government can typically only cover a fraction of its gross financing need by liquidating financial assets (see Figure 1), I assume that borrowing is bounded below, $b \geq \underline{b} > 0$. Finally, the rule could be interest-inelastic, $b(r) \equiv b$.

Financiers formulate beliefs about the economy's fundamentals, denoted by θ, and they then post lending functions that respond the interest rate (I discuss the fundamentals in greater detail shortly). Once financiers have submitted lending schedules, the bonds are priced. In turn, this pins down the value of the debt obligation, which is uncertain before the market-clearing price is realised, and the sovereign issues bonds. In the second period, there is no new borrowing. The Sovereign may or may not default depending on the fundamentals and the debt burden.

By allowing the Sovereign to choose a funding rule denominated in units of current consumption, this paper departs from the standard in the literature on debt crises. The literature has largely followed the Eaton and Gersovitz (1981) convention by assuming that the Sovereign commits to a level of bond issuance, denominated in terms of future consumption (e.g. Aguiar and Gopinath, 2006; Arellano, 2008; Yue, 2010). As a result, the face value of debt is not endogenous to interest rates, while the amount of funds raised from investors is. Here, instead, I follow the Calvo (1988) convention, whereby the Sovereign adjusts bond issuance to meet its financing needs, given the realisation of the bond price. Debt issuance is thus endogenous to interest rates.[8]

That said, because the Sovereign is allowed to choose a financing rule, the distinction between Calvo and Eaton Gersovitz timing becomes immaterial. Adopting a rule such that the funding requirement can respond to the price of debt, $b(r)$, implies adopting to an issuance rule that responds

[7] I index an equilibrium object by b to emphasise that it is associated with a particular funding rule b. For example, R_b is the equilibrium interest rate that is associated with the rule b. This convention is also used by Albagli, Hellwig, and Tsyvinski (2013).

[8] The Calvo approach to modelling the role of expectations in debt crises has re-emerged in the aftermath of the euro area debt crisis. See, inter alia, Lorenzoni and Werning (2013), Corsetti and Dedola (2016) and Ayres, Navarro, Nicolini, and Teles (2018).

to the interest rate, $B(r) = rb(r)$. Vice-versa, adopting an issuance rule that can respond to the interest rate, $B(r)$, implies adopting a rule for the funding need, $b(r) = B(r)/r$.

The Sovereign. The Sovereign's objective is:

$$w_b(r,\theta) = \frac{c_b(r)^{1-\rho}}{1-\rho} + \beta\frac{C_b(r,\theta)^{1-\rho}}{1-\rho}, \tag{1}$$

with $\rho > 0$ to ensure that there be a borrowing motive. This objective can be rationalised as the equilibrium level of social welfare that would arise in a microfounded model (see Appendix). The first term on the right-hand side of (1) represents utility from current consumption, $c_b(r) = y + b(r)$, the sum of current output y and borrowing b. The second term on the right-hand side of (1) represents utility from future consumption, weighted by $\beta > 0$. Future consumption is equal to:

$$C_b(r,\theta) = Y - rb(r) + Y(\kappa rb(r) - \theta)\mathbb{1}\{\theta \leq \kappa rb(r)\}, \tag{2}$$

where Y is future output and $rb(r)$ the future debt burden (bond issuance).

Future utility reflects the possibility of default, as captured by the indicator function $\mathbb{1}\{\theta \leq \kappa rb\}$. Taking the fundamentals as given, the Sovereign defaults at some haircut, $\eta \in (0,1)$, if θ is below a threshold,

$$\theta \leq \kappa rb(r), \tag{3}$$

where $\kappa > 0$. [9] Absent default, future consumption is $Y - rb$. In case of default, it is $Y - rb + Y(\kappa rb - \theta)$.

The fundamentals, $\theta \in (0,1)$, capture the severity of the Sovereign's commitment problem, and can be interpreted as a default cost. The threshold default rule described above arises endogenously as the optimal default rule in the underlying microfounded model. The default choice involves trading off the benefit of smaller payments to foreigners, ηrb, against a proportional output loss, θY. The threshold κrb then returns the value of the fundamentals that equates the cost and benefit of default, with $\kappa \equiv \eta/Y$.

Financiers. Foreign financiers are risk-neutral, with preferences given by $\mathbb{E}[c_i^* + \beta^* C_i^*]$, where c_i^* denotes the consumption of financier i in the

[9] The restriction $\eta < 1$ ensures that there be an equilibrium with positive borrowing and default in the perfect foresight benchmark of Section 3.

first period and C_i^* the consumption of financier i in the second period. Here, $\beta^* > 0$ denotes the relative weight of future utility. The expectation operator is conditional on all relevant information available in the first period. In the first period, financiers have an endowment, $\omega > 0$. I restrict the endowment as follows.

A1. $\omega > \left(Y - y\left(\frac{\beta}{\beta^*}\right)^{\frac{1}{\rho}}\right) / \left(\frac{1}{\beta^*} + \left(\frac{\beta}{\beta^*}\right)^{\frac{1}{\rho}}\right) \equiv b^c$.

A2. $\omega < \beta^* (1 - \eta) Y$.

Assumption 1 ensures that the commitment level of borrowing, b^c, can be sustained in equilibrium. Assumption 2 guarantees that future consumption be positive for all fundamentals.

Financiers can lend but they cannot borrow, $0 \leq l_i$, where l_i denotes the amount of resources transferred to the Sovereign. The budget constraint for financier i is $c_i^* + l_i \leq \omega$ in the first period, and $C_i^* \leq (1 - \eta \mathbb{1}\{\theta \leq \kappa r b\}) r l_i$ in the second period.

Market clearing. Market clearing requires that $\int_0^1 l_i di = b$.

3 A Calvo benchmark

In this Section, I assume perfect foresight about the fundamentals θ. I use this "Calvo benchmark" to illustrate how complementarity engenders self-fulling expectations and price multiplicity, and how the Sovereign may use debt management to improve debt sustainability.

Consider the model in Section 2, and assume that the funding need is exogenously given at some level, $b \geq \underline{b}$ (a scalar). Financiers choose lending for any value of the interest rate. Before making lending decisions, they observe b and θ. They take the threshold default rule, (3), as given. Financiers are identical so they all choose the same lending schedule to solve the following problem:

$$\max_{l_b} \mathbb{E}\left[-l_b + (1 - \eta \mathbb{1}\{\theta \leq \kappa r b\}) r l_b\right] \text{ s.t. } l_b \in [0, \omega]. \tag{4}$$

Problem (4) returns optimal lending, l_b, as a function of the fundamentals, θ, and the interest rate, r, $l_b(\theta, r)$. Market clearing then requires that $l_b(\theta, r) = b$. I can now define and characterise an equilibrium.

Definition 1. *An equilibrium is a lending map,* $l_b(\theta, r)$*, that solves the problem of financiers,* (4)*, and an interest rate map,* $R_b(\theta)$*, that returns the set of solutions to the market-clearing condition,* $R_b(\theta) = \{r : l_b(\theta, r) = b\}$*.*

Risk neutral financiers are only willing to lend provided the interest rate is at least as high as a reservation rate, $1/\beta^*(1 - \eta \Pr\{\theta \leq \kappa r b\}) \in \{\underline{r}, \bar{r}\}$. Here, $\underline{r} \equiv 1/\beta^*$ is the reservation rate of an investor who assigns probability 0 to default and $\bar{r} \equiv 1/\beta^*(1-\eta)$ is the reservation rate of an investor who assigns probability 1 instead. Let $\pi(r)$ denote the unique solution to $r = 1/\beta^*(1 - \eta\pi)$, that is, the default probability that makes a financier indifferent between lending and not lending given some r. Optimal lending is given by $l_b = 0$ if $\pi < \Pr\{\theta \leq \kappa r b\}$; $l_b = l \in [0, \omega]$ if $\pi = \Pr\{\theta \leq \kappa r b\}$; and $l_b = \omega$ if $\pi > \Pr\{\theta \leq \kappa r b\}$. I assume that an indifferent financier lends.

In an interior equilibrium, the market-clearing interest rate, R_b, is equal to the reservation rate of investors.[10] It thus satisfies the indifference condition:

$$R_b(\theta) = \{r : \pi(r) = \Pr_\theta\{\theta \leq \kappa r b\}\}. \tag{5}$$

The indifference condition (5) highlights how the endogeneity of the debt obligation to interest rates results in strategic complementarity between foreign investors and the Sovereign. Suppose that financiers become more pessimistic about default risk, so they require a higher interest rate to finance the Sovereign, which raises the market-clearing rate. Since the threshold $\kappa r b$ rises with interest rates, the Sovereign is indeed more likely to default, validating the initial change in expectations.

As a result, the market-clearing price is not uniquely determined for all fundamentals. By (5), the market-clearing price must make a financier indifferent between lending and not lending, and therefore it must be equal to the corresponding reservation rate, $1/\beta^*(1 - \eta \Pr\{\theta \leq \kappa r b\})$. Since the probability of default is either 0 or 1, the set of possible equilibrium prices is $\{\underline{r}, \bar{r}\}$. By (3), $\underline{r}$ can be sustained in equilibrium if and only if $\theta > \kappa \underline{r} b$. Similarly, $\bar{r}$ is an equilibrium rate if and only if $\theta \leq \kappa \bar{r} b$. Since $\bar{r} > \underline{r}$, multiplicity ensues. For $\theta \leq \kappa \underline{r} b \equiv \underline{\theta}_b$, there exist a unique equilibrium

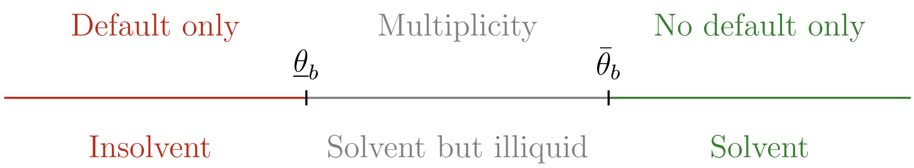
[10] Because of the lower bound on funding demand, $b \geq \underline{b} > 0$, lending must be positive in equilibrium, $l_b > 0$. As well as interior equilibria, there could exist equilibria with $l_b = \omega$.

price, $\bar{r}$. For $\theta > \kappa \bar{r} b \equiv \bar{\theta}_b$, the unique equilibrium interest rate is $\underline{r}$. For intermediate values of the fundamentals, $\theta \in \left(\underline{\theta}_b, \bar{\theta}_b\right]$, both $\underline{r}$ and $\bar{r}$ are market-clearing prices. I collect these results in the Proposition below.

Proposition 1. *For $\theta \leq \underline{\theta}_b$, there exist a unique equilibrium price, $R_b(\theta) = \bar{r}$. For $\theta > \bar{\theta}_b$, there exists a unique equilibrium price, $R_b(\theta) = \underline{r}$. For intermediate values of the fundamentals, $\theta \in \left(\underline{\theta}_b, \bar{\theta}_b\right]$, there are multiple market-clearing prices, $R_b(\theta) = \{\underline{r}, \bar{r}\}$.*

The space of fundamentals is thus partitioned into three regions (Figure 2). For $\theta > \bar{\theta}_b$, the Sovereign does not default (solvency region). For $\theta \leq \underline{\theta}_b$, the Sovereign defaults irrespective of the interest rate (insolvency region). For $\theta \in \left(\underline{\theta}_b, \bar{\theta}_b\right]$, instead, the Sovereign defaults only if the market-clearing interest rate embeds expectations of default, $R_b = \bar{r}$ (illiquidity region). Because these price-driven defaults could have been avoided if financiers had coordinated on a different set of (optimistic) expectations, default risk is coordination risk for $\theta \in \left(\underline{\theta}_b, \bar{\theta}_b\right]$.

FIGURE 2: Partition of the space of fundamentals for given b.



3.1 Debt management

So far, I have assumed that the funding need is exogenously given at some b. Next, I argue that debt management can be used to affect both the coordination and the solvency component of default risk. Namely, a Sovereign can choose the interest elasticity of funding demand to eliminate coordination risk, and then select the level of bond issuance that induces the optimal level of solvency risk.

Consider coordination risk first. Given that equilibrium multiplicity arises because of complementarity, a funding rule that eliminates complementarity delivers a unique equilibrium.[11] Suppose the Sovereign announced a funding rule such that bond issuance does not respond to r, $b(r) = B/r$, for some $B \geq \underline{rb}$. Then the default threshold would be constant, and thus the market-clearing interest rate would be $\bar{r}$ for $\theta \leq \kappa B$ and $\underline{r}$ otherwise. These rules endogenously generate the Eaton-Gersovitz convention in a setting with Calvo timing: the Sovereign promises to issue exactly B bonds, regardless of the price. Conditional on having chosen such a rule, all default risk is solvency risk. And since defaults happens for $\theta \leq \kappa B$, by choosing issuance, B, the Sovereign could pin down solvency risk for any θ.

In the absence of some exogenous equilibrium selection device such as a sunspot, only rules of the kind B/r ensure price uniqueness under perfect foresight. Hence, if the sovereign were to choose an optimal rule – a rule that selects a welfare-maximising equilibrium – its interest elasticity would be constrained to be -1 by the requirement to ensure uniqueness, and issuance would not respond to interest rates.

To allow for the possibility that (i) the interest-elasticity of the optimal funding rule may not be driven exclusively by the need to lessen complementarity and (ii) that bond issuance may respond to interest rates (consistent with the euro area record of cancelled/postponed auctions during the recent debt crisis), while still allowing coordination to be endogenous to the choice of borrowing, I characterise the optimal rule in a more realistic setting with uncertainty. Under uncertainty, complementarity is necessary but not sufficient for multiplicity, and funding rules such that bond issuance responds to interest rates become admissible.

[11] Adopting an interest-elastic borrowing rule is not the only option to deal with multiplicity. One could use an exogenous equilibrium selection device, like a sunspot à la Cole and Kehoe (2000). The sunspot suffers from two drawbacks, however. First, coordination risk would be exogenous, and would therefore not be affected by the Sovereign's choice of funding. Second, the occurrence of a default crisis would have the unappealing feature of being driven by a non-fundamental, non-microfounded variable.

4 The model with uncertainty

Consider again the model in Section 2. Assume that nature draws the fundamentals, θ, from a Gaussian distribution with mean $\theta_0 \in (0,1)$ and precision $\alpha_0 > 0$. The Sovereign and foreign financiers do not observe the fundamentals in the first period. They have a common prior about θ equal to its unconditional distribution. In addition, financiers observe a Gaussian public signal about θ, $z = \theta + s/\sqrt{\alpha_z}$, where s a standard normal shock independent of θ, and $\alpha_z > 0$ denotes the signal precision. The public signal z allows to model the occurrence of default as a function of the fundamentals, θ.

At the beginning of the first period, nature draws the fundamentals, θ, and the shock, s, from their respective distributions, pinning down the realisation of z. Then the Sovereign announces a funding rule, b, specifying its funding requirement for each possible value of the equilibrium interest rate. The sovereign does not observe the realisation of the public signal prior to announcing the funding rule.[12] Financiers observe the realisation of z and submit lending schedules, l_i. The funding market clears. Once the market-clearing price has been pinned down, the Sovereign issues bonds. The Sovereign issues exactly as many bonds as prescribed by the funding rule given the realised market clearing rate, $rb(r)$, no more and no less.[13]

Financiers choose lending for any r. Before making lending decisions, they know the funding rule, b, and they observe the public signal z. They take the threshold default, (3), as given. Each financier thus solves the following problem:

$$\max_{l_b} \mathbb{E}_\theta \left[-l_b + (1 - \eta \mathbb{1}\{\theta \le \kappa r b(r)\}) r l_b | z\right] \text{ s.t. } l_b \in [0, \omega], \text{ for any } r. \quad (6)$$

Solving Problem (6) returns optimal lending as a function of z and r, $l_b(z, r)$. I can now define an equilibrium.

Definition 2. *An equilibrium is a lending map, $l_b(z, r)$, that solves the*

[12] Because I focus on equilibria such that the interest rate mapping is an invertible function of the public signal, $R_b(z)$, this is without loss of generality. Given invertibility of the map R_b, letting the Sovereign choose $b(z, r)$ is equivalent to letting it choose a function of r only, $b\left(R_b^{-1}(r), r\right)$.

[13] As the optimal rule is time consistent (see Section 4.2.3), it is not necessary to impose commitment to a funding rule.

problem of financiers, (6)*, and an interest rate map,* $R_b(z)$*, that returns the set of solutions to the market-clearing condition,* $R_b(z) = \{r : l_b(z, r) = b\}$.

Since the equilibrium definition encompasses the interest rate map, it pins down the joint probability of the equilibrium interest rate, R_b, and the state variable, z. Because θ and s are drawn from independent Gaussian distributions, the public signal z is also a Gaussian, with mean θ_0 and precision $\frac{\alpha_0 \alpha_z}{\alpha_0+\alpha_z}$. As a result, $\Pr_z\{R_b \leq r\} = \int_{\{z: R_b(z) \leq r\}} d\Phi\left(\sqrt{\frac{\alpha_0\alpha_z}{\alpha_0+\alpha_z}}(z-\theta_0)\right)$, where $\Phi(x)$ denotes the cdf of a standard normal random variable.

Restrictions. I only study equilibria in which the variable z is a sufficient statistic for the interest rate, that is, equilibria in which the interest rate map, R_b, is an invertible function. This requirement rules out multiplicity.

While the set of market-clearing rates need not be single-valued for all z, price uniqueness can be obtained by appropriately restricting the set of admissible funding rules to anchor expectations, like in the Calvo benchmark, as I argue next.

4.1 Equilibrium characterisation

In this Section I show that if the Sovereign's choice is appropriately restricted to rules that do not feature too much complementarity (admissible rules), then the Sovereign can sustain an equilibrium such that the market-clearing interest rate map, R_b, is an invertible function, consistent with the restriction imposed above.

As in the perfect foresight benchmark, risk neutral financiers are only willing to lend provided the interest rate is at least as high as their reservation rate, $1/\beta^*(1 - \eta \Pr\{\theta \leq \kappa r b | z\})$. Here, the probability of default is computed according to the posterior distribution of θ given z, which is a Gaussian with mean $(1-\psi)\theta_0 + \psi z$ and precision $\alpha \equiv \alpha_0 + \alpha_z$, $\psi \equiv \alpha_z/\alpha$. Optimal lending is thus given by $l_b = 0$ if $\pi < \Pr\{\theta \leq \kappa r b|z\}$; $l_b = l \in [0, \omega]$ if $\pi = \Pr\{\theta \leq \kappa r b|z\}$; and $l_b = \omega$ if $\pi > \Pr\{\theta \leq \kappa r b|z\}$. I assume that an indifferent financier lends.

In an interior equilibrium, financiers are indifferent between lending and not lending. The market-clearing rate map is then implicitly defined by the indifference condition of financiers as:

$$R_b(z) = \left\{ r : \pi(r) = \Phi\left(\sqrt{\alpha}\left(\kappa r b(r) - (1-\psi)\theta_0 - \psi z\right)\right)\right\}. \tag{7}$$

Since the probability of default lies in $(0, 1)$, the set of interest rate that can arise in equilibrium, $\mathcal{E}_b$, is a subset of $(\underline{r}, \bar{r})$.

The market-clearing interest rate map defined by (7) is not necessarily single-valued (i.e. a function rather than a correspondence), because changes in r have potentially opposing effects on the indifference condition of financiers, $\pi - \Phi\left(\sqrt{\alpha}\left(\kappa r b - (1-\psi)\theta_0 - \psi z\right)\right) = 0$. There are two channels. First, higher r makes lending to the Sovereign more attractive for financiers, increasing their willingness to lend (return effect). The return effect is captured by π, which is increasing in r. Second, changes in r affect the default threshold, $\kappa r b$ (default effect). If the funding rule features complementarity for some r, so $\kappa r b$ is locally increasing, the default effect is positive. Default becomes more likely, thereby dampening investors' willingness to lend and pushing for multiplicity. The reverse is true if the rule does not display complementarity.

Price determinacy thus depends on the degree of complementarity associated with a funding rule. If complementarity is weak enough that the default effect does not outweigh the return effect for any r, then the interest rate map R_b is a function rather than a correspondence. Unlike in the perfect foresight benchmark, complementarity is necessary but not sufficient for multiplicity.

4.2 The best equilibrium

4.2.1 The ex-ante problem

In this Section, I construct the funding rule that selects the best equilibrium. I start by defining the set of admissible funding rules. A funding rule b is admissible if its corresponding market-clearing interest rate map, R_b, is an invertible function.

Definition 3. *Let $\mathcal{A}$ denote the class of admissible funding rules. A funding rule, b, is admissible, $b \in \mathcal{A}$, if (i) it is continuous on $\mathcal{E}_b$; (ii) $b \geq \underline{b}$; and (iii) the map $R_b : \mathbb{R} \to \mathcal{E}_b$, defined by (7) is an invertible function.*

It can be verified that admissible functions sustain decreasing market-clearing interest rate maps. Supposing $b \in \mathcal{A}$ to be differentiable, by (7) one obtains that dR_b/dz is proportional to $-\psi\sqrt{\alpha}\phi < 0$ for all z, where $\phi(x)$ denotes the standard Gaussian pdf. Moreover, because complementarity is not a sufficient condition for multiplicity like in the perfect foresight benchmark, the set of admissible funding rules is not limited to Eaton Gersovitz rules of the kind $b(r) = B/r$.

The Sovereign is benevolent, and chooses an admissible rule to select an equilibrium that maximises expected social welfare. The Sovereign's problem in the first period is:

$$\max_{b\in\mathcal{A}} \quad \mathbb{E}_\theta\left[\mathbb{E}_z\left[w_b\left(R_b\left(z\right),\theta\right)\middle|\,\theta\right]\right], \tag{8}$$

where social welfare w is given by (1), the set $\mathcal{A}$ is described by Definition 3 and the equilibrium interest rate map, R_b, is given by (7). The inner expectation in (8) is taken with respect to the distribution of the public signal z conditional on the fundamentals θ, which is a Gaussian with mean θ and precision α_z. The outer expectation is taken with respect to the prior distribution of θ, which is also Gaussian with mean θ_0 and precision α_0. I refer to (8) as the ex-ante problem.

In choosing the optimal funding rule b, the Sovereign has to take into account its large-agent status, which means it has to internalise the impact of its choice of b on the market-clearing interest rate function, R_b. Therefore, problem (8) cannot be solved pointwise, that is, r-by-r. I tackle this issue next.

4.2.2 The ex-post problem

A time-consistent solution to problem (8) can be constructed by solving a related pointwise maximisation. Consider the following alternative setup. The Sovereign observes z, and formulates posterior beliefs about the fundamentals, θ. It then chooses a funding level, a scalar $b \in [\underline{b}, \omega]$, to maximise expected social welfare:

$$\max_{b\in[\underline{b},\omega]} \mathbb{E}_\theta\left[w\left(b, R\left(b,z\right),\theta\right)|z\right], \tag{9}$$

where the expectation is taken with respect to the posterior distribution of θ conditional on z, a Gaussian with mean $(1-\psi)\,\theta_0 + \psi z$ and precision $\alpha \equiv \alpha_0 + \alpha_z$.[14] Here, w is given by

$$w\,(b, r, \theta) = \frac{(y+b)^{1-\rho}}{1-\rho} + \beta \frac{(Y - rb + Y\,(\kappa r b - \theta)\,\mathbb{1}\,\{\theta \leq \kappa r b\})^{1-\rho}}{1-\rho},$$

and R is a market-clearing interest rate map implicitly given by:

$$R\,(b, z) = \left\{ r : \pi\,(r) = \Phi\left(\sqrt{\alpha}\,(\kappa r b - (1-\psi)\,\theta_0 - \psi z)\right)\right\}. \qquad (10)$$

Problem (9) can be interpreted as the "ex-post" problem faced by a Sovereign that, having committed to some financing rule b, upon observing z must decide whether or not to stick to the funding level prescribed by the rule given the interest rate associated with z, $b\,(R_b\,(z))$. I thus refer to R as the ex-post market-clearing interest rate map.

In order to study problem (9), I choose the information parameters $(\theta_0, \alpha_0, \alpha_z)$ to ensure that the Sovereign assign an arbitrarily small prior and posterior probability to the region $\theta \leq 0$ (see Appendix). In addition, I make the following new assumptions.

A3. $\frac{1}{\bar{r}^2 \beta^* \eta} > \kappa \omega \phi\,(0)\,\sqrt{\alpha}$.

A4. $\beta^*\,(1 - \eta \Phi\,(1)) > \kappa \omega \sqrt{\alpha}$ and $\rho > \eta^2$.

Assumption 3 is a sufficient condition for uniqueness of the market-clearing interest rate. It guarantees that the return effect dominate the default effect, ruling out the possibility that (10) may have multiple solutions for some b. In this sense, it can be interpreted as an upper bound on complementarity. Assumption 4 is a sufficient condition for the Sovereign's objective to be strictly concave. It must be imposed because of the Gaussian information structure, which implies that problem (9) is not a priori convex.

The ex-post interest rate map R defined by (10) is increasing in b and decreasing in z. It is increasing in b because a larger funding need raises the debt burden, in turn pushing up the probability of default for all z. It is

[14] Even though the Sovereign is allowed to choose $b = \omega$, strict concavity of the objective in problem (9) implies that optimal borrowing is interior, $b_e^* < \omega$.

decreasing in z because a larger z represents better perceived fundamentals, and therefore lowers default risk for all b by raising the posterior mean, $(1-\psi)\,\theta_0+\psi z$. The following Lemma formalises.

Lemma 1. *The ex-post market-clearing interest rate map defined by (10) is continuously differentiable in* b *and* z*, with* $\partial R/\partial b>0$ *and* $\partial R/\partial z<0$*. The bond issuance function induced by* R*, the map* $R\,(b,z)\,b$*, is also continuously differentiable in both its arguments, with* $\partial\,(Rb)\,/\partial b>0$ *and* $\partial\,(Rb)\,/\partial z<0$*.*

4.2.3 Optimal funding

Solving problem (9) returns two objects: the optimal funding need as a (continuously differentiable) function of the signal z, $b_e^*\,(z)\in(\underline{b},\omega)$, as well as the corresponding ex-post market-clearing rate, $R^*\,(z)=R\,(b_e^*\,(z)\,,z)$, also as a (continuously differentiable) function of z. I now characterise the dependency of each object on z.

The ex-post optimal borrowing function, b_e^*, may not be monotonic, because of countervailing effects of higher z on the marginal cost of borrowing. Consider problem (9). The marginal benefit of borrowing is higher current consumption, $(y+b)^{\;-\rho}>0$, and it does not depend on z. The marginal cost is lower future consumption because of a higher debt burden,

$$\beta\frac{\partial\,(Rb)}{\partial b}\mathbb{E}_\theta\left[(1-h)\,C^{-\rho}|z\right]>0. \tag{11}$$

Differentiating (11) with respect to z returns:

$$\beta\frac{\partial^2\,(Rb)}{\partial z\partial b}\mathbb{E}_\theta\left[(1-h)\,C^{-\rho}|z\right]+\beta\frac{\partial\,(Rb)}{\partial b}\,(1-\eta)\,Y\psi\mathbb{E}_\theta\left[\rho C^{-\rho-1}|z,\theta\leq\kappa Rb\right]$$
$$+\,\beta\left(\frac{\partial\,(Rb)}{\partial b}\right)^2\mathbb{E}_\theta\left[(1-h)^2\,\rho\,(C)^{-\rho-1}\,|z\right]. \tag{12}$$

The first term on the first line of (12) returns the impact of higher z on the sensitivity of the debt burden, Rb, to changes in borrowing, b. Because the sign of $\frac{\partial^2(Rb)}{\partial z\partial b}$ is not constant for all (b,z), the sign of this term is ambiguous (see Appendix). The second term on the first line of (12) returns the impact of a higher posterior mean on the expected value of future consumption conditional on default, $Y\,(1-\theta)-(1-\eta)\,Rb$. A higher z raises the probability of a larger output loss in case of default. As a result, future consumption in the default state falls, raising its marginal value and increasing the marginal cost of borrowing. This pushes for a lower b_e^*. The term on the second line of (12) captures the impact of changes in bond issuance, Rb. Higher z lowers the future debt burden, $\frac{\partial (Rb)}{\partial z} = \frac{\partial R}{\partial z} b < 0$, thereby increasing future consumption for all values of θ, and thus lowering its expected value. As a result, the marginal cost of borrowing tends to decrease. This effect pushes for a higher b_e^*.

Intuitively, borrowing costs ought to fall as "perceived" fundamentals improve, that is, as z rises. A necessary condition for the optimal ex-post market-clearing interest rate map, R^*, to be decreasing in z is that $\frac{db_e^*}{dz} < -\frac{\partial R}{\partial z} / \frac{\partial R}{\partial b}$. Since Lemma 1 implies that $-\frac{\partial R}{\partial z} / \frac{\partial R}{\partial b} > 0$, monotonicity of R^* requires that b_e^* do not increase too much as perceived fundamentals improve. The next Lemma establishes that this is indeed the case.

Lemma 2. *The ex-post optimal market-clearing interest rate function, $R^*(z) \equiv R(b_e^*(z), z)$, is strictly decreasing.*

Since the ex-post optimal market-clearing rate, R^*, is a decreasing function of z, it is invertible. Hence, it is possible to construct a corresponding funding rule, $b^*(r) \equiv b_e^*\left((R^*)^{-1}(r)\right)$. The next Proposition shows that the funding rule thus constructed is a solution to the ex-ante problem, (8).

Proposition 2. *Let $b_e^* : \mathbb{R} \to (\underline{b}, \omega)$ denote the solution of the ex-post problem, (9), and $R^* : \mathbb{R} \to S \subseteq (\underline{r}, \bar{r})$ the corresponding ex-post market-clearing interest rate, implicitly defined by (10). The function R^* is monotonically decreasing and thus invertible, and the funding rule $b^* : S \to (\underline{b}, \omega)$, constructed as $b^*(r) \equiv b_e^*\left((R^*)^{-1}(r)\right)$, solves problem (8).*

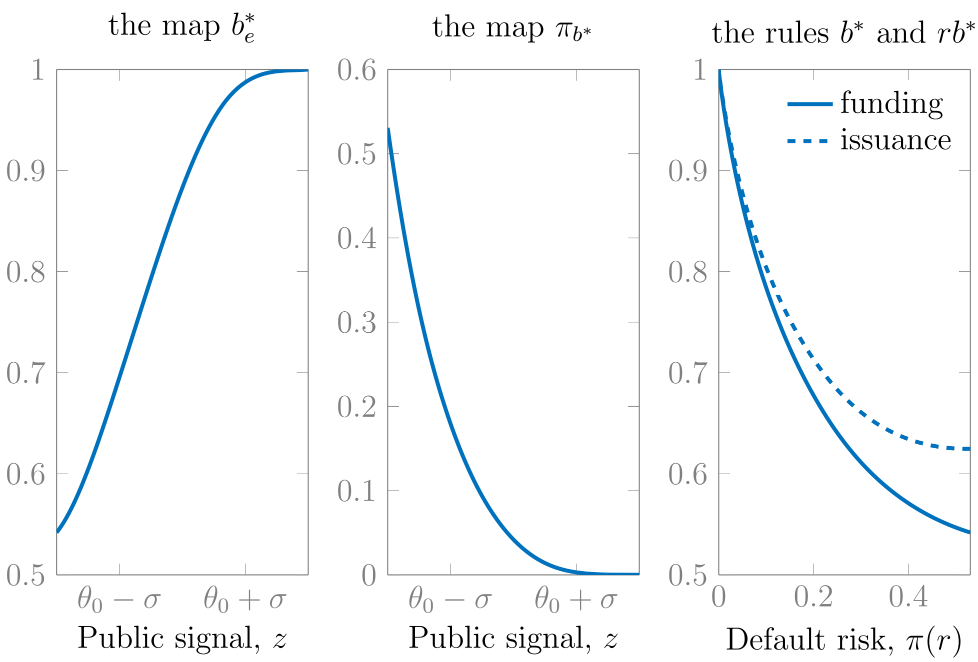
Proposition 2 provides an algorithm for solving the ex-ante problem, (8). First, solve the ex-post problem, (9), z-by-z to obtain the maps b_e^* and R^*. Second, get the rule b^* by inverting R^* and plugging into b_e^*.

The solution algorithm returns a time consistent rule. Suppose the Sovereign announced b^*. Conditional on some interest rate realisation r, would the Sovereign still be willing to issue exactly $rb^*(r)$ worth of bonds? Yes. Given r, the Sovereign would infer the price signal $(R^*)^{-1}(r)$. If it wanted to change issuance given $(R^*)^{-1}(r)$, it would have to solve problem (9) taking the equilibrium interest rate schedule given by (10) as given.

The optimal issuance level would then be $R^*\left(\left(R^*\right)^{-1}(r)\right) b_e^*\left(\left(R^*\right)^{-1}(r)\right)$, which is equal to $rb^*(r)$ by construction of the rule b^*.

Figure 3 shows an example. The ex-post optimal funding map, b_e^*, is decreasing (left panel). In this parameterisation, higher z tends to reduce the marginal cost of borrowing. Consistent with Lemma 2, the optimal interest rate function is decreasing (middle panel). Bond issuance is not constant across r (right panel, dashed curve), suggesting that Eaton Gersovitz rules are not optimal in the setting considered here. The optimal funding rule under uncertainty preserves some degree of complementarity, giving rise to some coordination risk.

FIGURE 3: Optimal demand for funding and interest rates, an example.



The optimal demand for funds and bond issuance are reported as a share of their commitment values (left and right). The optimal interest rate function (centre) is reported as the probability that makes a financier indifferent between lending and not lending, π_{b^*}. Here (and in all remaining figures), $\sigma \equiv \sqrt{1/\alpha_0 + 1/\alpha_z}$ denotes the unconditional standard deviation of the public signal, z. Figure (and all remaining figures) drawn for $y = 1$, $Y = 1.78$, $\rho = 1$, $\eta = 0.25$, $\beta = 1$, $\beta^* = 1$, $\omega = 0.4$, $\theta_0 = 0.08$, $1/\sqrt{\alpha_0} = 0.02$, $1/\sqrt{\alpha_z} = 0.03$.

5 Debt and risk management

In this Section, I establish that by managing the debt, a Sovereign can reduce its funding costs and default risk. I also put forward a decomposition of interest rates and default risk, which I use to show that debt management works on both solvency and coordination risk.

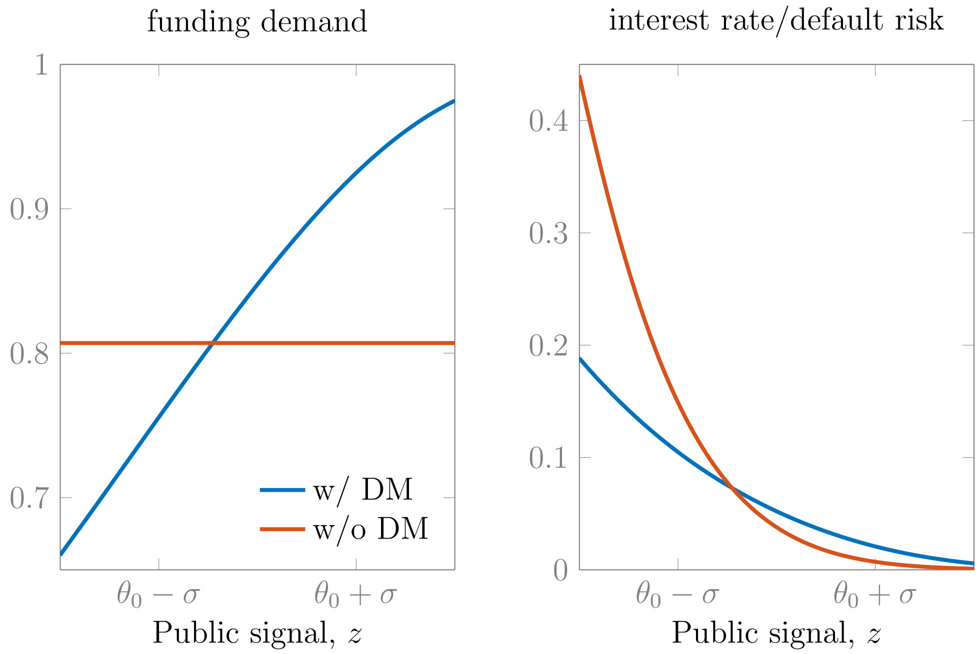
5.1 Debt management, funding costs and default risk

By managing the debt, a Sovereign can reduce its funding costs and default risk. In Figure 4, I compare the equilibrium outcomes faced by an unconstrained Sovereign (that optimally chooses an interest-elastic rule, as shown in Figure 3) to those faced by a Sovereign constrained to choosing interest-inelastic funding demand rules. The constrained Sovereign is the model counterpart of the case in which a Treasury department takes the funding need determined by the budget as given. In bad states (that is, low z), the unconstrained Sovereign borrows less than the constrained one (left panel). At these values of z, the ex-post interest rate map, (10), is very responsive to changes in b. As a result, the unconstrained Sovereign experiences lower funding costs and default risk (right panel). In good states (high z), instead, the ex-post market-clearing interest rate map is not as responsive to changes in borrowing, b. As a result, the unconstrained Sovereign pays only a slightly higher price for its borrowing, despite a significantly larger demand for funding.

5.2 Solvency and coordination risk

In order to determine whether the reduction in funding costs and default risk associated with debt management stems from solvency or coordination risk, I introduce a formal decomposition of interest rates/default risk based on the perfect foresight benchmark of Section 3. Any interest rate that arises in equilibrium given some funding rule b, $r \in \mathcal{E}_b$, has to satisfy the indifference condition $\pi(r) = \Pr_\theta\{\theta \leq \kappa r b(r) | R_b^{-1}(r)\}$. The posterior default probability that makes financiers indifferent between lending and

Figure 4: Debt management, funding costs and default risk.



Blue curves correspond to the case in which the Sovereign is not constrained (debt management). Orange curves correspond to the case in which the Sovereign is constrained to choosing interest-inelastic funding rules (no debt management).

not lending can be decomposed as:

$$\Pr\left\{\theta \leq \kappa r b\left(r\right)|R_b^{-1}\left(r\right)\right\} = \Sigma_b\left(r\right) + \Psi_b\left(r\right) \cdot \Lambda_b\left(r\right), \tag{13}$$

where $\Sigma_b\left(r\right) \equiv \Pr_\theta\{\theta \leq \underline{\theta}_b(r)|R_b^{-1}\left(r\right)\}$, $\Psi_b\left(r\right) \equiv \frac{\Pr_\theta\{\theta\in\left(\underline{\theta}_b(r),\theta_b^*(r)\right)|R_b^{-1}(r)\}}{\Pr_\theta\{\theta\in\left(\underline{\theta}_b(r),\bar{\theta}_b(r)\right)|R_b^{-1}(r)\}}$ and $\Lambda_b\left(r\right) \equiv \Pr_\theta\{\theta \in \left(\underline{\theta}_b(r), \bar{\theta}_b\left(r\right)\right)|R_b^{-1}\left(r\right)\}$. Here, $\underline{\theta}_b(r) \equiv \kappa \underline{r} b\left(r\right)$ and $\bar{\theta}_b\left(r\right) \equiv \kappa \bar{r} b\left(r\right)$, in analogy with the perfect foresight benchmark.

I refer to Σ_b as the solvency component of default risk (solvency risk), and to its complement, $\Psi_b\Lambda_b$ as the coordination component (coordination risk). To rationalise the labels, suppose that, in the perfect foresight benchmark, the Sovereign borrowed exactly the amount prescribed by the rule b given r, $b\left(r\right)$. For $\theta \leq \underline{\theta}_b\left(r\right)$, it would default regardless of the interest rate. In the same benchmark, defaults would be driven by self-fulfilling beliefs for $\theta \in \left(\underline{\theta}_b\left(r\right), \bar{\theta}_b\left(r\right)\right]$. In principle, investors could use a sunspot to coordinate on the default equilibrium with some exogenous probability,

Ψ.[15] The term Ψ_b in (13) can thus be interpreted as a "sunspot equivalent" capturing the likelihood of those defaults that would not have happened if there was perfect foresight and investors had coordinated on the no-default equilibrium (coordination failures).[16] Finally, as Λ_b returns the probability of what would be the multiplicity region in the benchmark, I refer to it as the probability of being solvent but illiquid (liquidity risk).

Debt management significantly lowers solvency risk in bad states of the world, and it increases it in good states, although not to the same extent (Figure 5, left panel). The rationale is that in bad (good) states the unconstrained Sovereign borrows less (more) than its constrained counterpart, and the probability of being in the solvency region is more (less) sensitive to changes in b when the state is bad. The solvency risk differential accounts for the majority of the risk differential. At the same time, debt management also reduces coordination risk in bad states of the world (centre panel). Significantly, the coordination risk differential stems from a lower risk of coordination failures (right panel) rather than from lower liquidity risk. Integrating over z returns that these results also hold in expectation (Figure 6).

5.3 Coordination risk and optimal debt management

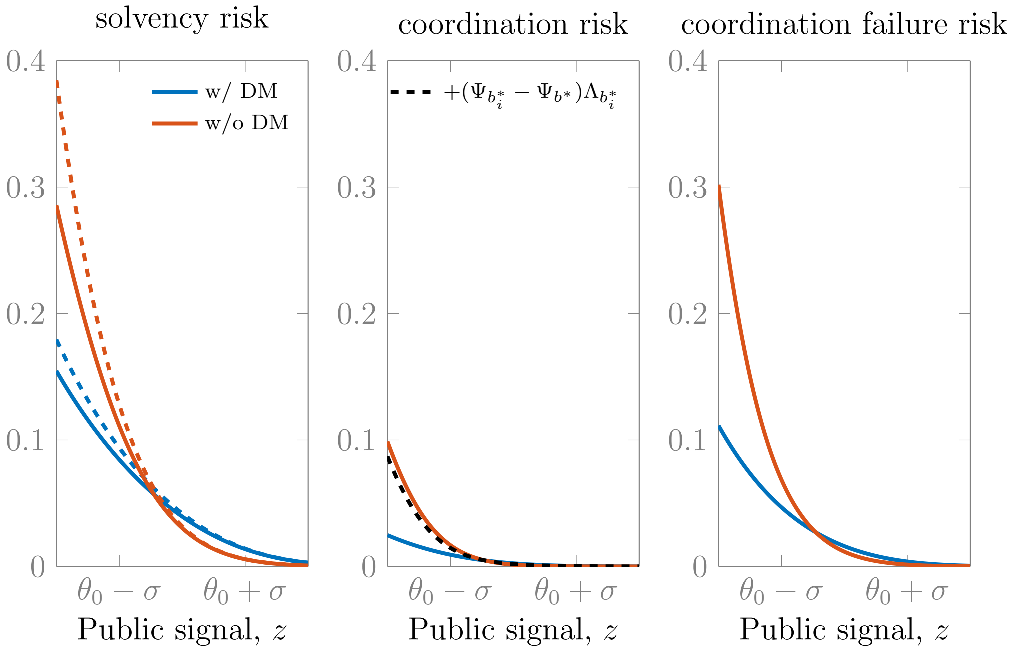
Next, I show that a sovereign that internalises coordination risk in the optimal choice of funding selects a funding rule that responds more aggressively to interest rates.

In order to study the extent to which optimal debt management can be ascribed to the management of coordination risk, I compare the optimal funding rule to the optimal rule selected by a Sovereign who behaves as if coordination risk did not matter at all for interest rates. This "naive" sovereign solves the ex-post problem, (9), as if the ex-post market-clearing interest rate map were given by $1/\beta^* \left(1 - \Pr\left\{\theta \leq \kappa \underline{r} b | z\right\}\right)$ rather than by (10). In this sense, it does not manage coordination risk.

[15] Footnote 11 explains why I did not pursue the sunspot route.

[16] The sunspot equivalent, Ψ_b, is closely linked to the "liquidity crisis index" of Guimaraes and Morris (2007), $\hat{\theta}_b(r) = \frac{\theta_b^*(r) - \underline{\theta}_b(r)}{\bar{\theta}_b(r) - \underline{\theta}(r)}$, which measures the share of defaults that would have been avoidable in the perfect foresight benchmark if investors had coordinated on the no-default equilibrium.

FIGURE 5: Debt management and the interest rate/risk decomposition given z.



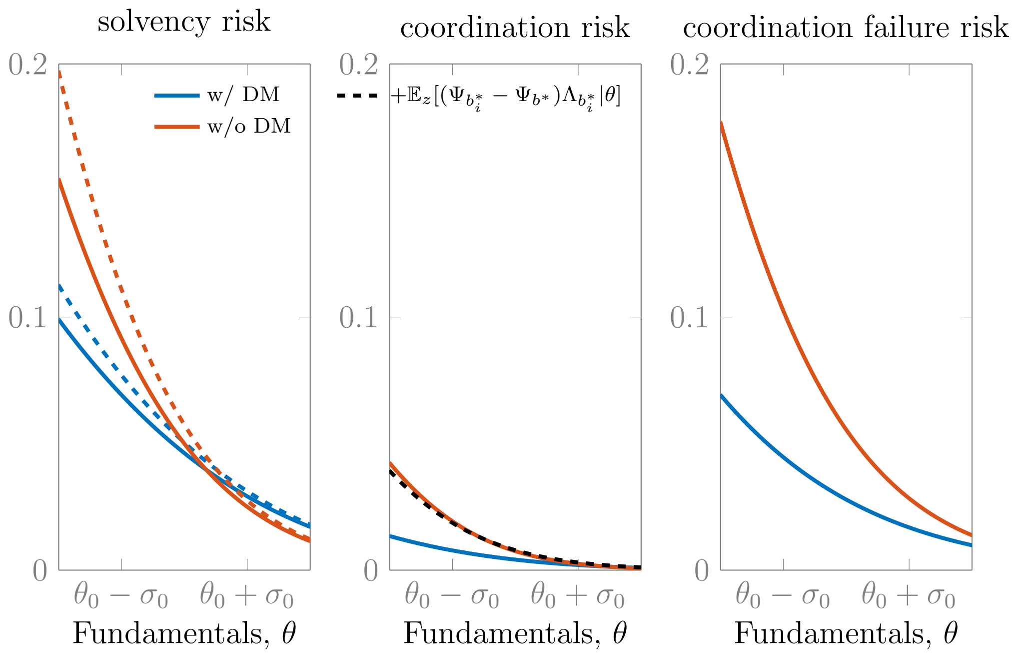
Blue curves correspond to the case in which the Sovereign is not constrained (debt management). Orange curves correspond to the case in which the Sovereign is constrained to choosing interest-inelastic funding rules (no debt management). The dashed curves on the left panel report the interest rate/default risk.

A Sovereign that takes coordination risk into account borrows (weakly) less than one who behaves as if all default risk were solvency risk (Figure 7, left panel). A rational sovereign internalises a higher default probability, and therefore higher funding costs. Since future consumption is decreasing in funding costs for all values of the fundamentals, expected future marginal utility, $\mathbb{E}_\theta\left[C^{-\rho}|z\right]$, rises, increasing the marginal cost of borrowing, (11), relative to the naive benchmark. Because of its higher funding costs, a rational sovereign may also anticipate a higher sensitivity of bond issuance to changes in b, $\partial Rb/\partial b$.[17] Therefore, the management of coordination risk makes the optimal funding rule steeper than its naive equivalent.

As a result, a Sovereign that manages coordination risk through the

[17] In the rational case, the derivative $\frac{\partial(Rb)}{\partial b}$ is equal to $\frac{\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa Rb-\mu)\right)\kappa Rb}{\frac{1}{R^2\beta^*\eta}-\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa Rb-\mu)\right)\kappa b}+R$, with $\mu \equiv (1-\psi)\,\theta_0+\psi z$. In the naive case, it is equal to $\frac{\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa Rb-\mu)\right)\kappa Rb}{\frac{1}{R^2\beta^*\eta}}+R$, where R is lower than in the rational case for all (b,z). Because the Gaussian pdf ϕ is non-monotonic, the two derivatives cannot be ranked unambiguously.

FIGURE 6: Debt management and the interest rate/risk decomposition given θ.



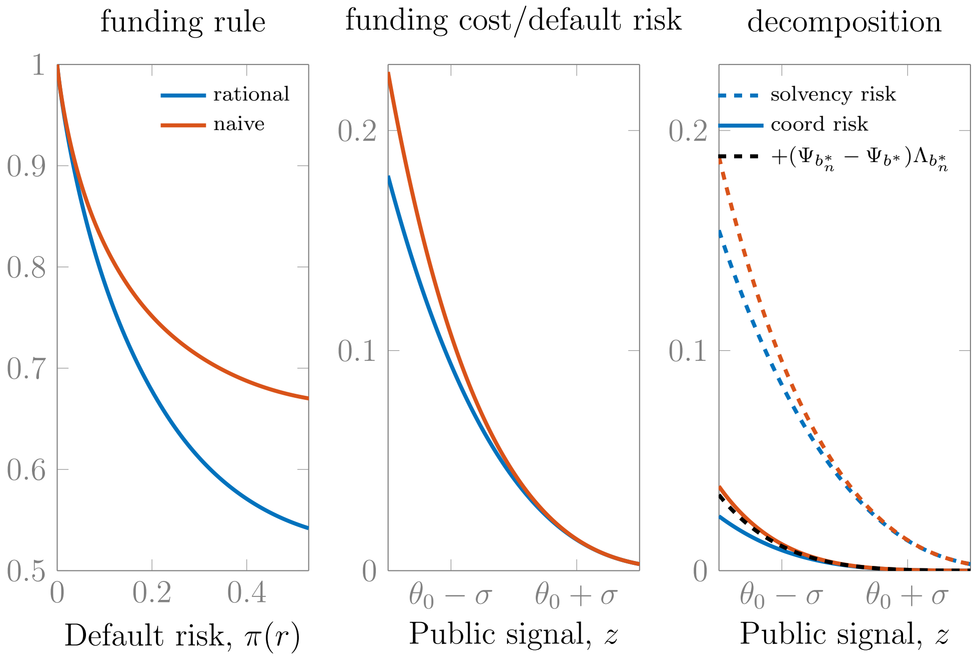
Blue curves correspond to the case in which the Sovereign is not constrained (debt management). Orange curves correspond to the case in which the Sovereign is constrained to choosing interest-inelastic funding rules (no debt management). The dashed curves on the left panel report the interest rate/default risk. Here, $\sigma_0 \equiv \sqrt{\alpha_0}$ denotes the standard deviation of the prior. Figure drawn for $y = 1$, $Y = 1.78$, $\rho = 1$, $\eta = 0.25$, $\beta = 1$, $\beta^* = 1$, $\omega = 0.4$, $\theta_0 = 0.08$, $1/\sqrt{\alpha_0} = 0.02$, $1/\sqrt{\alpha_z} = 0.03$.

optimal choice of funding rule faces (weakly) lower borrowing costs (centre panel), with the solvency component and the coordination component contributing similarly to the differential, and the coordination risk differential driven by the sunspot equivalent rather than by liquidity risk.

6 Discussion

Since the Latin American debt crisis of the 1980s and the subsequent Brady Plan to restructure outstanding bank loans into tradeable bonds that started in 1989, most debt issuance by sovereigns to private creditors has been through bonds sold at auction. At first glance, letting the Sovereign adopt a funding rule that can respond to debt funding costs may thus appear unrealistic from an operational perspective. After all, when an auction for government debt is announced, it comes with a particular bond

Figure 7: Coordination risk and debt management.



Blue curves correspond to the case in which the Sovereign is rational. Orange curves correspond to the case in which the Sovereign is naive. The dashed blue and orange curves on the right panel report solvency risk. The continuous curves on the right panel return coordination risk. The dashed black curve represents the contribution of the sunspot equivalent to the coordination risk differential.

issuance level, not with a schedule specifying how the government funding need may vary with interest rates.

In the model, however, the sovereign does indeed borrow by issuing non-contingent bonds. Once a funding rule b has been announced and financiers have submitted lending schedules l_b, for a given draw of the state variable z the market-clearing interest rate is pinned down as $R_b(z)$. The sovereign then issues bonds with face value $R_b(z)\,b(R_b(z))$. Thus, implementing a funding rule does not require designing any new financial instruments, only appropriately selecting the value of bond issuance.

The funding rules explored in this paper are a modelling device to represent debt management. As debt management requires only non-contingent bonds (as well as adequate asset buffers, as already discussed in the introduction) to reduce the likelihood of debt crises, it has an advantage over alternative ways to manage default risk that seek to make debt more state-contingent. It is still open to debate, for instance, whether introducing

GDP-linked bonds could become a viable option. First proposed by Shiller (1993) as a retirement savings vehicle, GDP-indexed bonds have been advocated for as a potentially useful tool to reduce the procyclicality of debt burdens (e.g. Borensztein and Mauro, 2004; Borensztein et al., 2005). A legacy of high government debt in the aftermath of the Great Financial Crisis has reignited the debate about these instruments (see, for instance, Barr et al., 2014; Benford et al., 2016; Blanchard et al., 2016; Bank of England, 2016). Despite consensus about the analytical case for GDP-linked securities, implementation has been held back by moral hazard concerns motivated by the link between payouts and an economic indicator produced and revised by the borrower (Borensztein, Obstfeld, and Ostry, 2018).

By modelling debt management as a funding rule, I have taken a reduced form approach. In particular, I have implicitly taken the size and composition of the asset buffer as given. Both would likely be affected by the usefulness of debt management. For example, by lowering the likelihood of debt crises, debt management should generate a precautionary motive for accumulating assets. And the need for liquidating assets in bad times calls for holding assets whose liquidation value is negatively correlated with fundamentals. These are interesting avenues for future research.

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Appendices

A Microfoundations

Consider a two-period economy, $t = 0, 1$. In each period, the economy produces a consumption good. There is a measure-one continuum of risk-averse identical households. The economy is inhabited by a benevolent Sovereign with access to international financial markets. Financial markets are populated by a measure-one continuum of risk-neutral financiers, indexed by $i \in [0, 1]$. The Sovereign can borrow from financiers by issuing a one-period discount bond. International debt contracts are not enforceable, so the Sovereign can default. Defaulting is costly.

A.1 The default rule

In the second period, the Sovereign chooses a haircut, h, subject to the resource constraint of the economy, $C + (1-h)rb \leq (1 - \theta\mathbb{1}\{h > 0\})Y$, where C denotes second-period private consumption; rb is the debt burden given the realised market-clearing rate r and Sovereign borrowing b (a scalar); and Y is future output. If the Sovereign defaults, there is an output loss of θY. This is a standard assumption in the sovereign default literature (e.g. Aguiar and Gopinath, 2006; Arellano, 2008; Yue, 2010), and can be rationalised as a reduced form approach to capture either the costs of being excluded from financial markets for some time after a default episode in a fully dynamic model or the costs associated with the banking crises that often accompany defaults. To ensure that the optimal haircut has a bang-bang nature, the output loss is independent of the haircut.

I assume that $h \in [0, \eta]$, with $\eta < 1$. This is because I want to generate an equilibrium with positive borrowing and default in the common knowledge benchmark of the model. This kind of equilibrium would not arise if I allowed $h = 1$, as financiers would not be willing to lend if they anticipated a full default.

The Sovereign is benevolent. It thus chooses h to solve the following problem:

$$\max_{h} \; C \text{ s.t. } C + (1-h)rb \leq (1 - \theta\mathbb{1}\{h > 0\})Y \text{ s.t. } h \in [0, \eta] \tag{A.1}$$

taking the default cost θ and the debt burden rb as given. Accordingly, the Sovereign defaults if and only if defaulting increases consumption. The optimal haircut h follows a threshold rule: $h = \eta$ if $\theta < \kappa rb$; $h \in \{0, \eta\}$ if $\theta = \kappa rb$ and $h = 0$ if $\theta > \kappa rb$, with $\kappa \equiv \eta/Y$. The default threshold κrb equates the default cost θY to the default benefit, ηrb. I assume that an indifferent Sovereign defaults.

A.2 Social welfare

Households are risk-averse, with preferences described by $\mathbb{E}\left[\frac{c^{1-\rho}}{1-\rho} + \beta\frac{C^{1-\rho}}{1-\rho}\right]$ where c is private consumption in the first period and $\rho > 0$. The expectation operator is conditional on all information available in the first period. Current consumption c must satisfy a market-clearing condition, that is, the first-period resource constraint $c \leq y + b$, which says that households consume current domestic output y plus the amount of resources the Sovereign borrows from financiers. Substituting for c and C in household preferences using the resource constraints and applying the optimal default rule derived in Section A.1 above returns:

$$\mathbb{E}\left[\frac{c^{1-\rho}}{1-\rho} + \beta \frac{C^{1-\rho}}{1-\rho}\right] = \mathbb{E}\left[\frac{(y+b)^{1-\rho}}{1-\rho} + \beta \frac{(Y - rb + Y(\kappa rb - \theta)\mathbb{1}\{\theta \leq \kappa rb\})^{1-\rho}}{1-\rho}\right].$$

consistent with (1).

A.3 The lower bound $\underline{b}$

Suppose that in the first period households have some endowment, $e_0^h > 0$, and they pay lump-sum taxes, T_0. The households' budget constraint is then given by $c + T_0 \leq e_0^h$. Let the sovereign have some endowment too, and let e_0^s denote the value of the endowment at the beginning of the first period, and e_1^s its value at the end of the first period. The Sovereign cannot accumulate assets, so $e_1^s \leq e_0^s$. It has to finance some spending, $G_0 > 0$, plus legacy liabilities, $r_{-1}b_{-1}$. The first-period Sovereign's budget constraint is $G_0 + r_{-1}b_{-1} \leq T_0 + b + (e_0^s - e_1^s)$. Denoting the gross financing need of the Sovereign as $GFN_0 \equiv (G_0 - T_0) + r_{-1}b_{-1}$ and rearranging the Sovereign's budget constraint returns $GFN_0 - (e_0^s - e_1^s) \leq b$. The left-hand side of this inequality is the net financing need of the Sovereign. Since $e_1^s \leq e_0^s$, assuming that $GFN_0 > e_0^s$ like in the high-debt euro area economies (see Figure 1), implies that the net financing need is positive. Hence, new borrowing b is bounded below, $b > GFN_0 - e_0^s \equiv \underline{b}$.

Combining the household and the sovereign budget constraints returns the resource constraint $c \leq \left(e_0^h - (r_{-1}b_{-1} - (e_0^s - e_1^s)) - G_0\right) + b$. Letting $y \equiv e_0^h - (r_{-1}b_{-1} - (e_0^s - e_1^s)) - G_0$ returns the resource constraint in the paper, $c \leq y + b$.

B Proofs

B.1 Proof of Proposition 1

The proof is already in the body of the paper.

B.2 The Sovereign does not believe that $\theta < 0$

Claim B.1. *Fix some arbitrarily small $\varepsilon_0 > 0$ and $\varepsilon_z > 0$. Let $P = (\theta_0, \alpha_0, \alpha_z)$, and*

$$\mathcal{A}(\varepsilon_0, \varepsilon_z) \equiv \left\{P \in (0,1) \times \mathbb{R}_+^2 : \Pr_\theta\{\theta \leq 0\} < \varepsilon_0 \text{ and } \varepsilon_0 > \varepsilon_0^*(P, \varepsilon_z)\right\},$$

where $\varepsilon_0^* = \Phi\left(-\theta_0 - \alpha_z \frac{\sqrt{\alpha_0 \alpha_z}}{\alpha_0 + \alpha_z} \Phi^{-1}(\varepsilon_z)\right)$ and $\Phi(x)$ and $\Phi^{-1}(y)$ denote the standard Gaussian cdf and its inverse, respectively. If $P \in \mathcal{A}(\varepsilon_0, \varepsilon_z)$, then $\Pr_z\{z \in \mathbb{R} : \Pr_\theta\{\theta \leq 0 | z\} \geq \varepsilon_0\} < \varepsilon_z$.

Proof. I can choose (θ_0, α_0) in such a way as to ensure that the prior probability of negative values be below the tolerance level ε_0. Accordingly, assume that $(\theta_0, \alpha_0) \in (0,1) \times \mathbb{R}_+$ are such that $\Pr_\theta\{\theta \leq 0\} < \varepsilon_0$.

Recall that $z = \theta + s/\sqrt{\alpha_z}$, where s is a standard normal random variable, independent of θ. By independence of θ and s, z has a Gaussian distribution with mean θ_0 and precision $1/\left(\frac{1}{\alpha_0}+\frac{1}{\alpha_z}\right) < \alpha_0$. As a result, the posterior distribution of θ conditional on z is also a Gaussian, with mean $\frac{\alpha_0}{\alpha_0+\alpha_z}\theta_0 + \frac{\alpha_z}{\alpha_0+\alpha_z}z$ and precision $\alpha_0+\alpha_z$. Therefore, $\Pr_\theta\{\theta<0|z\}$ is monotonically decreasing in z, $\Pr_\theta\{\theta<0|z\} \to 1$ as $z\to-\infty$ and $\Pr_\theta\{\theta<0|z\}\to 0$ as $z\to+\infty$.

Since $\Pr_\theta\{\theta\le 0|z\}$ is monotonically decreasing in z, there exists some z_0 such that $\Pr_\theta\{\theta\le 0|z\} < \varepsilon_0$ if and only if $z > z_0$. The threshold $z_0(\varepsilon_0)$ is defined as the unique real number that solves $\Pr_\theta\{\theta\le 0|z\} = \varepsilon_0$, $z_0 = -\frac{\alpha_0}{\alpha_z}\theta_0 - \frac{\Phi^{-1}(\varepsilon_0)}{\frac{\alpha_z}{\alpha_0+\alpha_z}\sqrt{\alpha_0+\alpha_z}}$ where Φ^{-1} denotes the inverse Gaussian cdf. The critical signal z_0 is decreasing in ε_0. Intuitively, for the posterior probability mass to the left of zero to rise, the posterior mean must fall.

By definition of z_0, $\{z\le z_0\} = \{z\in\mathbb{R} : \Pr_\theta\{\theta\le 0|z\}\ge\varepsilon_0\}$, so $\Pr_z\{z\in\mathbb{R} : \Pr_\theta\{\theta\le 0|z\}\ge\varepsilon_0\} = \Pr_z\{z\le z_0\}$. This probability is increasing in z_0 and thus it is decreasing in ε_0. Therefore, there exists some ε_0^* such that $\Pr_z\{z\le z_0\} < \varepsilon_z$ if and only if $\varepsilon_0 > \varepsilon_0^*$. The threshold $\varepsilon_0^*(P,\varepsilon_z)$ is defined as the unique real number that solves $\Pr_z\{z\le z_0(\varepsilon_0)\} = \varepsilon_z$, $\varepsilon_0^* = \Phi\left(-\theta_0 - \alpha_z\frac{\sqrt{\alpha_0\alpha_z}}{\alpha_0+\alpha_z}\Phi^{-1}(\varepsilon_z)\right)$. ■

B.3 Assumption 3

Lemma B.1. *Assumption 3 is a sufficient condition for the map $R(b,z)$, defined by (10), to be single-valued for all $(b,z)\in[\underline{b},\omega]\times\mathbb{R}$.*

Proof. Consider the indifference condition of financiers, (10), as a function of r for given (b,z), $G(r;b,z)\equiv\pi(r)-\Phi(\sqrt{\alpha}(\kappa rb-\mu(z)))$, with $\mu(z)\equiv(1-\psi)\theta_0+\psi z$. Since $\pi=\frac{1}{\eta}\left(1-\frac{1}{\beta^* r}\right)$ is increasing with $\pi\to 1$ as $r\to\bar{r}$ and $\pi\to 0$ as $r\to\underline{r}$ and Φ is increasing and $\Phi\in(0,1)$, it follows that $\lim_{r\to\underline{r}}G < \lim_{r\to\bar{r}}G$. As a result, $\frac{dG}{dr}>0$ for all $(r,b,z)\in(\underline{r},\bar{r})\times[\underline{b},\omega]\times\mathbb{R}$ is a sufficient condition for $G(r;b,z)=0$ to have a unique solution in $(\underline{r},\bar{r})$ for all pairs $(b,z)\in[\underline{b},\omega]\times\mathbb{R}$. Differentiating with respect to r returns $\frac{dG}{dr}=\frac{d\pi}{dr}-\kappa b\sqrt{\alpha}\phi(\sqrt{\alpha}(\kappa rb-\mu))$. Since the first term, $\frac{d\pi}{dr}=\frac{1}{r^2\beta^*\eta}$, is bounded below by $\frac{1}{\bar{r}^2\beta^*\eta}$ and the second term is bounded above by $\kappa\omega\sqrt{\alpha}\phi(0)$ because $b<\omega$, it follows that $\frac{1}{\bar{r}^2\beta^*\eta}>\kappa\omega\sqrt{\alpha}\phi(0)$ is a sufficient condition for $\frac{dG}{dr}>0$ for all $(r,b,z)\in(\underline{r},\bar{r})\times[\underline{b},\omega]\times\mathbb{R}$. ■

B.4 Assumption 4

Lemma B.2. *Let $V\equiv Rb$ denote the debt issuance function and $U\equiv\frac{1}{\bar{r}^2\beta^*\eta}-\kappa\omega\sqrt{\alpha}\phi(0)$. Assume that $\beta^*(1-\eta\Phi(1))>\kappa\omega\sqrt{\alpha}$. Then (i) $\frac{\partial V}{\partial b}\in\left(\underline{r},(\bar{r}\beta^*\eta U)^{-1}\right)$ and (ii) $\frac{\partial^2 V}{\partial b^2}>0$.*

Proof. Differentiating (10) returns $\frac{\partial R}{\partial b}=\frac{\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)\kappa R}{\frac{d\pi}{dr}-\kappa b\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)}$, with $\mu\equiv(1-\psi)\theta_0+\psi z$. By Assumption 3, $\frac{\partial R}{\partial b}>0$. Using $\frac{d\pi}{dr}=\frac{1}{r^2\beta^*\eta}$ and $\frac{\partial V}{\partial b}=\frac{\partial R}{\partial b}b+R$, $\frac{\partial V}{\partial b}=\frac{1}{\beta^*\eta R\left(\frac{1}{R^2\beta^*\eta}-\kappa b\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa Rb-\mu)\right)\right)}$. By Assumption 3, $\frac{1}{R^2\beta^*\eta}-\kappa b\sqrt{\alpha}\phi(\sqrt{\alpha}(\kappa Rb-\mu))>0$, and thus $\frac{\partial V}{\partial b}>0$. Since $\phi>0$ and $R>\underline{r}$, $\frac{\partial V}{\partial b}>\underline{r}$. In addition, using $R\in(\underline{r},\bar{r})$ and $V\in(\underline{rb},\bar{r}\omega)$, it follows that $\frac{\partial V}{\partial b}<(\bar{r}\beta^*\eta U)^{-1}$, with $(\bar{r}\beta^*\eta U)^{-1} > \bar{r}$. This proves part (i). As for part (ii), differentiating $\frac{\partial V}{\partial b}$ returns:

$$\frac{\partial^2 V}{\partial b^2} = \left(\frac{\partial V}{\partial b}\right)^2 \left(\frac{1}{R^2}\frac{\partial R}{\partial b} + \beta^*\eta\kappa\frac{\partial V}{\partial b}\sqrt{\alpha}\phi\left(\sqrt{\alpha}\left(\kappa V - \mu\right)\right)\left(-\sqrt{\alpha}\left(\kappa V - \mu\right)\sqrt{\alpha}\kappa V + 1\right)\right), \tag{B.2}$$

where $\frac{\partial R}{\partial b}$ and $\frac{\partial V}{\partial b} > 0$. The sign of $\frac{\partial^2 V}{\partial b^2}$ is ambiguous since $\sqrt{\alpha}\phi\left(\sqrt{\alpha}\left(\kappa V - \mu\right)\right)\left(-\sqrt{\alpha}\left(\kappa V - \mu\right)\right)$ could be negative. However, $\sqrt{\alpha}\phi\left(\sqrt{\alpha}\left(\kappa V - \mu\right)\right)\left(-\sqrt{\alpha}\left(\kappa V - \mu\right)\right) \in \left(-\sqrt{\alpha}\phi\left(1\right), \sqrt{\alpha}\phi\left(1\right)\right)$, because it is the derivative of a zero-mean Gaussian pdf with precision α evaluated at $\kappa V - \mu$. The lower bound $\sqrt{\alpha}\phi\left(1\right)$ is attained when $\sqrt{\alpha}\left(\kappa V - \mu\right) = 1$, or equivalently, using (10), for $R = 1/\beta^*\left(1 - \eta\Phi\left(1\right)\right)$. Hence, for (b, z) pairs such that $\sqrt{\alpha}\left(\kappa V - \mu\right) = 1$, the ambiguous term in (B.2) evaluates to:

$$-\sqrt{\alpha}(\kappa V-\mu)\sqrt{\alpha}\phi\big(\sqrt{\alpha}(\kappa V-\mu)\big)\sqrt{\alpha}\kappa V+\sqrt{\alpha}\phi\big(\sqrt{\alpha}(\kappa V-\mu)\big)=\left(1-\frac{\sqrt{\alpha}\kappa b}{\beta^*(1-\eta\Phi(1))}\right)\sqrt{\alpha}\phi(1).$$

Thus, $\beta^*\left(1 - \eta\Phi\left(1\right)\right) > \kappa\omega\sqrt{\alpha}$ is a sufficient condition for $\frac{\partial^2 V}{\partial b^2} > 0$, proving part (ii). ■

Lemma B.3. *Assumption 4 is a sufficient condition for the Sovereign's objective in problem (9) to be (strictly) concave in b on $[\underline{b}, \omega]$ for all $z \in \mathbb{R}$.*

Proof. Let $MB \equiv (y + b)^{-\rho}$ denote the marginal benefit of borrowing, and MC the marginal cost, given by (11). To establish concavity of the objective in problem (9), there suffices to show that $\frac{\partial MB}{\partial b} - \frac{\partial MC}{\partial b} < 0$ for all $(b, z) \in [\underline{b}, \omega] \times \mathbb{R}$. First, $-\frac{\partial MB}{\partial b} = \rho\left(y + b\right)^{-\rho-1} < 0$. Second, differentiating (11) returns:

$$\begin{aligned} -\frac{\partial MC}{\partial b} =& -\beta\frac{\partial^2 V}{\partial b^2}\mathbb{E}_\theta\left[\left(1 - h\right)C^{-\rho}|z\right] + \\ & -\beta\left(\frac{\partial V}{\partial b}\right)^2\left(\mathbb{E}_\theta\left[\left(1 - h\right)^2\rho C^{-\rho-1}|z\right] - \left(Y - V\right)^{-\rho}\kappa\eta\sqrt{\alpha}\phi\left(\sqrt{\alpha}\left(\kappa V - \mu\right)\right)\right), \end{aligned} \tag{B.3}$$

with C given by (2). Here, $-\frac{\partial^2 V}{\partial b^2}\mathbb{E}_\theta\left[\left(1 - h\right)C^{-\rho}|z\right] < 0$ and $-\left(\frac{\partial V}{\partial b}\right)^2 < 0$, by Lemma B.2. The term in parentheses on the second line of (B.3) is equal to:

$$\begin{aligned} &\int^{\kappa V}\left(1 - \eta\right)^2\rho\left(Y\left(1 - \theta\right) - \left(1 - \eta\right)V\right)^{-\rho-1}\sqrt{\alpha}\phi\left(\sqrt{\alpha}\left(\theta - \mu\right)\right)d\theta \\ &+ \left(Y - V\right)^{-\rho}\left(\int_{\kappa V}\frac{\rho}{Y - V}\sqrt{\alpha}\phi\left(\sqrt{\alpha}\left(\theta - \mu\right)\right)d\theta - \kappa\eta\sqrt{\alpha}\phi\left(\sqrt{\alpha}\left(\kappa V - \mu\right)\right)\right) > 0, \end{aligned}$$

where the inequality follows from the result that $\frac{\rho}{Y-V} > \kappa\eta = \frac{\eta^2}{Y}$ if $\rho > \eta^2$, since $1/(Y - V)$ is increasing in V and $V > 0$. It follows that $-\frac{\partial MC}{\partial b} < 0$, establishing strict concavity. ■

B.5 Proof of Lemma 1

Differentiating (10) with respect to b returns $\frac{\partial R}{\partial b} = \frac{\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)\kappa R}{\frac{d\pi}{dr}-\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)\kappa b} > 0$, and with respect to z returns $\frac{\partial R}{\partial z} = -\frac{\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)\psi}{\frac{d\pi}{dr}-\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)\kappa b} < 0$. The inequalities follow from $\frac{d\pi}{dr} > \sqrt{\alpha}\phi\left(\sqrt{\alpha}\left(\kappa V - \mu\right)\right)\kappa b$ for all $(b, z) \in [\underline{b}, \omega] \times R$ under Assumption 3. This proves the first part. The second part follows from $\frac{\partial V}{\partial b} = \frac{\partial R}{\partial b}b + R$, $\frac{\partial V}{\partial z} = \frac{\partial R}{\partial z}b$, and the first part. ■

B.6 The sign of $\frac{\partial^2 V}{\partial z \partial b}$

Lemma B.4. *The sign of $\frac{\partial^2 V}{\partial z \partial b}$ is ambiguous.*

Proof. Differentiating (10), $\frac{\partial R}{\partial z} = -\frac{\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)\psi}{\frac{d\pi}{dr}-\kappa b\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)}$ and $\frac{\partial V}{\partial z} = -\frac{\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)\psi b}{\frac{d\pi}{dr}-\kappa b\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)} < 0$, where the inequality follows from Assumption 3. Differentiating $\frac{\partial V}{\partial z}$ returns:

$$\begin{aligned}\frac{\partial^2 V}{\partial z \partial b} = \left(\frac{\partial V}{\partial b}\right)^2 & \left(\frac{1}{R^2}\frac{\partial R}{\partial z} + \beta^* \eta \kappa \frac{\partial V}{\partial z}\sqrt{\alpha}\phi\left(\sqrt{\alpha}\left(\kappa R b-\mu\right)\right)\left(-\sqrt{\alpha}\left(\kappa V-\mu\right)\sqrt{\alpha}\kappa V+1\right)\right. \\ & \left. -\beta^* \eta \psi \sqrt{\alpha}\phi\left(\sqrt{\alpha}\left(\kappa R b-\mu\right)\right)\left(-\sqrt{\alpha}\left(\kappa V-\mu\right)\right)\sqrt{\alpha}\kappa V\right), \end{aligned} \tag{B.4}$$

where $\frac{\partial R}{\partial z}$ and $\frac{\partial V}{\partial z} < 0$ by Lemma 1. By the same arguments used in the proof of Lemma B.3, $\sqrt{\alpha}\phi\left(\sqrt{\alpha}\left(\kappa R b-\mu\right)\right)\left(-\sqrt{\alpha}\left(\kappa V-\mu\right)\sqrt{\alpha}\kappa V+1\right) > 0$. The sign of $\frac{\partial^2 V}{\partial z \partial b}$ is ambiguous since the term on the second line of (B.4) could be negative, because it contains the derivative of a zero-mean Gaussian pdf with precision α evaluated at $\kappa V - \mu$. ■

B.7 Proof of Lemma 2

B.7.1 Useful results

The following two Lemmas are used in the proof of Lemma 2.

Lemma B.5. $\frac{\partial R}{\partial z} + \frac{\psi}{\kappa R}\frac{\partial R}{\partial b} = 0$; $\frac{\partial V}{\partial z} + \frac{\psi}{\kappa R}\frac{\partial V}{\partial b} = \frac{\psi}{\kappa}$ *and* $\frac{\partial^2 V}{\partial z \partial b} + \frac{\psi}{\kappa R}\frac{\partial^2 V}{\partial b^2} > 0$.

Proof. The first part follows from $\frac{\partial R}{\partial z} = -\frac{\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)\psi}{\frac{d\pi}{dr}-\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)\kappa b}$ and $\frac{\partial R}{\partial b} = \frac{\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)\kappa R}{\frac{d\pi}{dr}-\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)\kappa b}$. The second part follows from $\frac{\partial V}{\partial z} + \frac{\psi}{\kappa R}\frac{\partial V}{\partial b} = \frac{\partial R}{\partial z}b + \frac{\psi}{\kappa R}\left(\frac{\partial R}{\partial b}b + R\right)$ and using the first part. Using (B.2), (B.4) and parts (i) and (ii) returns

$$\frac{\partial^2 V}{\partial z \partial b} + \frac{\psi}{\kappa R}\frac{\partial^2 V}{\partial b^2} = \left(\frac{\partial V}{\partial b}\right)^2 \beta^* \eta \psi \sqrt{\alpha}\phi\left(\sqrt{\alpha}\left(\kappa V-\mu\right)\right) > 0,$$

which establishes the third part. ■

Lemma B.6. *The optimal ex-post financing need, $b_e^*(z) \in (\underline{b}, \omega)$, is a continuously differentiable function on $\mathbb{R}$, with $\frac{db_e^*}{dz} = -\frac{\partial^2 W}{\partial z \partial b} / \frac{\partial^2 W}{\partial b^2}$ and $W(b,z) \equiv \mathbb{E}_\theta\left[w\left(b, R(b,z), \theta\right)|z\right]$.*

Proof. By Lemma B.3, the objective function in problem (9), $W(b,z)$, is strictly concave in b on $[\underline{b}, \omega]$ for all $z \in \mathbb{R}$, and therefore the strong form of the maximum theorem applies. It follows that $b_e^*(z) \equiv \arg\max_{b \in [\underline{b}, \omega]} W(b,z)$, with R given by (10), is a continuous function on $\mathbb{R}$. Moreover, since the optimisation problem is convex, optimal funding is interior, $b_e^* \in (\underline{b}, \omega)$, and the first order condition uniquely pins down optimal borrowing as a function of z, $b_e^*(z) = \{b \in (\underline{b}, \omega) : \partial W / \partial b = 0\}$. By concavity of W in b on $[\underline{b}, \omega]$, the conditions required by the implicit function theorem are satisfied, and therefore the function b_e^* is continuously differentiable with $\frac{db_e^*}{dz} = -\frac{\partial^2 W}{\partial z \partial b} / \frac{\partial^2 W}{\partial b^2}$. ■

B.7.2 Proof

The function $R^* : \mathbb{R} \to (\underline{r}, \bar{r})$ is defined as $R^*(z) = R(b_e^*(z), z)$. Differentiating returns $\frac{dR^*}{dz} = \frac{\partial R}{\partial b}\frac{db_e^*}{dz} + \frac{\partial R}{\partial z}$. Using $\frac{db_e^*}{dz} = -\frac{\partial^2 W}{\partial z \partial b} / \frac{\partial^2 W}{\partial b^2}$ and the expressions for $\frac{\partial R}{\partial b}$ and $\frac{\partial R}{\partial z}$ in the proofs of Lemmas B.2 and B.4 returns: $\frac{dR^*}{dz} = \frac{\partial R}{\partial b}\left(-\frac{\partial^2 W}{\partial z \partial b} / \frac{\partial^2 W}{\partial b^2} - \frac{\psi}{\kappa R}\right)$. Since $\frac{\partial R}{\partial b} > 0$ and $\frac{\partial^2 W}{\partial b^2} < 0$ by Lemma B.3, R^* is strictly decreasing if and only if $-\frac{\partial^2 W}{\partial z \partial b} - \frac{\psi}{\kappa R}\frac{\partial^2 W}{\partial b^2} > 0$. I next verify that this is the case. To that end, observe that:

$$
\begin{aligned}
-\frac{\partial^2 W}{\partial z \partial b} - \frac{\psi}{\kappa R}\frac{\partial^2 W}{\partial b^2} =& \frac{\psi}{\kappa R}\rho (y+b)^{-\rho-1} + \left(\frac{\partial^2 V}{\partial z \partial b} + \frac{\psi}{\kappa R}\frac{\partial^2 V}{\partial b^2}\right)\mathbb{E}_\theta\left[(1-h)\,C^{-\rho}|z\right] \\
&+ \frac{\psi}{\kappa}\frac{\partial V}{\partial b}(1-\eta)\int^{\kappa V}\rho\left(Y(1-\theta) - (1-\eta)V\right)^{-\rho-1}\sqrt{\alpha}\phi\left(\sqrt{\alpha}(\theta-\mu)\right)d\theta \\
&+ \frac{\psi}{\kappa}\frac{\partial V}{\partial b}\rho (Y-V)^{-\rho-1}\left(1-\Phi\left(\sqrt{\alpha}(\kappa V-\mu)\right)\right).
\end{aligned}
$$

The terms on the first line are positive by concavity of the utility function and by Lemma B.5, respectively. The terms on the second and third line of this expression are positive by Lemma B.2, concavity of the utility function and the assumption that $Y > \omega/\beta^*(1-\eta)$, which ensures that future consumption be positive for all θ. It follows that $-\frac{\partial^2 W}{\partial z \partial b} - \frac{\psi}{\kappa R}\frac{\partial^2 W}{\partial b^2} > 0$. ■

B.8 Proof of Proposition 2

Given that the funding rule $b^* : (\underline{r}, \bar{r}) \to [\underline{b}, \omega]$ is constructed to be continuous on $(\underline{r}, \bar{r})$ and to support a unique market-clearing rate for all z, $R_{b^*} \equiv R_e^*$, b^* satisfies the conditions for admissibility in Definition 3, $b^* \in \mathcal{A}$. It thus suffices to show that b^* is a maximiser of the ex-ante problem, (8), that is,

$$
\mathbb{E}_\theta\left[\mathbb{E}_z\left[W\left(b^*\left(R_{b^*}(z)\right), z\right)|\theta\right]\right] \geq \mathbb{E}_\theta\left[\mathbb{E}_z\left[W\left(d_b(z), z\right)|\theta\right]\right] \text{ for all } b \in \mathcal{A}, \tag{B.5}
$$

where the map $d_b : \mathbb{R} \to [\underline{b}, \omega]$ is defined as $d_b(z) \equiv b(R_b(z))$.

By definition of b_e^*, $W(b_e^*(z), z) \geq W(d(z), z)$ for all $z \in \mathbb{R}$ and all maps $d : \mathbb{R} \to [\underline{b}, \omega]$. By construction of b^*, $b^*(R^*(z)) = b_e^*(z)$, and therefore $W(b^*(R_{b^*}(z)), z) \geq W(d(z), z)$ for all $z \in \mathbb{R}$ and all maps $d : \mathbb{R} \to [\underline{b}, \omega]$. Because this inequality holds for all $z \in \mathbb{R}$, it must also hold in expectation,

$$
\mathbb{E}_\theta\left[\mathbb{E}_z\left[W\left(b^*\left(R_{b^*}(z)\right), z\right)|\theta\right]\right] \geq \mathbb{E}_\theta\left[\mathbb{E}_z\left[W\left(d(z), z\right)|\theta\right]\right], \text{ for all maps } d : \mathbb{R} \to [\underline{b}, \omega]. \tag{B.6}
$$

Given that (B.6) holds for all $d : \mathbb{R} \to [\underline{b}, \omega]$, it also holds for any $d_b(z) \equiv b(R_b(z))$, with $b \in \mathcal{A}$. It follows that b^* satisfies (B.5). ■

C A variant with dispersed information

Consider the model presented in Section 2. Suppose that each financier receives a Gaussian private signal about θ, $x_{1,i} = \theta + \sigma_1 \varepsilon_{1,i}$, with $\sigma_1 > 0$. $\varepsilon_{1,i}$ is a standard normal shock, i.i.d. across lenders. Financiers also observe a second Gaussian signal, $x_{2,i} = \theta + \sigma_2 \varepsilon_{2,i} + \sigma_s s$, with

$\sigma_2 > 0$ and $\sigma_s > 0$. Here, $\varepsilon_{2,i}$ is a standard normal shock, i.i.d. across lenders; s is a standard normal shock. The three shocks, $\varepsilon_{1,i}$, $\varepsilon_{2,i}$ and s, are independent for all i. I let τ_1 denote the precision of the first signal, $\tau_1 \equiv 1/\sigma_1^2$, and τ_2 denote the precision of the second signal, $\tau_2 \equiv 1/\left(\sigma_2^2 + \sigma_s^2\right)$.

The shock s summarises correlated noise in private information, so it should be interpreted as capturing correlated movements in the beliefs of financiers. The presence of correlated noise guarantees that the interest rate does not restore common knowledge about the fundamentals, θ.[18] As an alternative, I could have introduced noise traders à la Grossman and Stiglitz (1976).

The two private signals, $x_{1,i}$ and $x_{2,i}$, represent a device to introduce heterogeneity of beliefs amongst financiers. A drawback of this assumption is that investors receive more information about the fundamentals, θ, than the Sovereign itself. If the Sovereign had better information than the private sector, for instance because it observed either θ or a noisy signal about θ prior to choosing borrowing, I would have had to contend with the additional complication of signaling, which is not the focus of this paper.[19] The two signals can be summarised by a sufficient statistic.

Lemma C.1. *The linear combination $x = \frac{\tau_1}{\tau_1+\tau_2}x_1 + \frac{\tau_2}{\tau_1+\tau_2}x_2$ is a sufficient statistic for the vector (x_1, x_2). Conditional on θ and s, x is Gaussian with mean $\theta + \lambda s$ and precision γ, with λ, $\gamma > 0$. Conditional on θ, x is a Gaussian with mean θ and precision $\tau_x < \gamma$.*

Financiers choose lending for any value of the equilibrium interest rate. Before making lending decisions, they know the funding rule, b, and they observe private information about the fundamentals, summarised by the sufficient statistic, x. They take the threshold default rule, (3), as given. The financier who has received signal x thus solves the following problem:

$$\max_{l_b} \mathbb{E}_\theta \left[-l_b + \left(1 - \eta \mathbb{1}\left\{\theta \le \kappa r b\left(r\right)\right\}\right) r l_b | x, r\right] \text{ s.t. } l_b \in [0, \omega], \text{ for all } r \in \mathcal{E}_b, \tag{C.7}$$

where $\mathcal{E}_b = \{r : r \in R_b(\theta, s) \text{ for some } (\theta, s)\}$ is the set of interest rates that can arise in equilibrium, and $R_b(\theta, s)$ is the set of market-clearing prices given a particular draw of the state vector, (θ, s).

Problem (C.7) is the dispersed information analogue of problem (6) in Section 4. Unlike in (6), however, the expectation operator in (C.7) is conditional on both x and r. This is because lenders understand and internalise that, by market clearing, the equilibrium interest rate conveys information about the fundamentals. Solving (C.7) returns optimal lending as a function of x and r, $l_b(x, r)$. Aggregating across financiers and using Lemma C.1 returns aggregate lending as $L_b(\theta, s, r) = \int l_b\left(x, r\right) d\Phi\left(\sqrt{\gamma}\left(x - \theta - \lambda s\right)\right)$.

I can now define an equilibrium. The equilibrium concept is mixture of perfect Bayesian and rational expectations equilibria, as in Tsyvinski, Mukherji, and Hellwig (2006).

[18] Having one signal with correlated noise would not have achieved the same result. Suppose foreign financiers observed only one signal, $x_i = \theta + \sigma_s s + \sigma\varepsilon_i$, with $\sigma > 0$. This is equivalent to observing a Gaussian signal about $\tilde{\theta} \equiv \theta + \sigma_s s$, with the prior on $\tilde{\theta}$ given by a Gaussian with mean $\bar{\theta}_0$ and precision $1/(\sigma_0^2 + \sigma_s^2)$, with $\sigma_0 \equiv 1/\sqrt{\alpha_0}$. The equilibrium interest rate function would then reveal $\tilde{\theta}$, thus restoring common knowledge about the fundamentals θ.

[19] See Angeletos, Hellwig, and Pavan (2006), Sandleris (2008) and more recently Phan (2017).

Definition 4. *An equilibrium is a posterior cumulative distribution function for financiers, $F_b(\theta, x, r)$, that is formulated according to Bayes' Law for all $r \in \mathcal{E}_b$; a lending map, $l_b(x, r)$, that solves the problem of financiers,* (C.7)*, for all $r \in \mathcal{E}_b$ when the expectations is computed according to F_b, and an interest rate map, $R_b(\theta, s)$, that returns the set of solutions to the market-clearing condition, $R_b(\theta, s) = \{r : L_b(\theta, s, r) = b\}$.*

Since the equilibrium definition encompasses the interest rate map, it allows to compute the joint probability of the equilibrium interest rate, R_b, and the state vector, (θ, s). Because θ is drawn from a Gaussian distribution with mean θ_0 and precision τ_0 while s is drawn from a standard Gaussian distribution, and the two distributions are independent, it follows that $\Pr_{(\theta,s)}\{R_b \le r\} = \int_{-\infty}^{+\infty} \left(\int_{\Theta_b(s,r)} d\Phi\left(\sqrt{\tau_0}(\theta - \theta_0)\right) \right) d\Phi(s)$, with $\Theta_b(r, s) \equiv \{\theta : R_b(\theta, s) \le r\}$.

Restrictions. I focus on equilibria such that interest rate map, R_b, depends on the fundamentals, θ, and the correlated shock, s, via a linear combination, $z = \theta + s/\sqrt{\tau_z}$, for some $\tau_z > 0$ exogenous to the choice of the funding rule, b. This restriction implies that the set of market-clearing rates is the same for all pairs (θ, s) such that $\theta + s/\sqrt{\tau_z} = z$. With a slight abuse of notation, I let $R_b(z)$ denote the interest rate map under this restriction. In addition, like in Section 4, I only study equilibria in which the market-clearing interest rate map, R_b, is an invertible function.

Together, these two restrictions imply that conditioning on r in problem (C.7) is equivalent to conditioning on z. The variable z thus works as a Gaussian endogenous public signal (EPS) about θ conveyed by the equilibrium interest rate, which allows me to characterise posterior beliefs and express them in closed form.

Moreover, the assumption that τ_z be exogenous to the choice of funding rule, b, means that the map between the fundamentals (θ, s) and the endogenous price signal z does not depend on b. Hence, there is no signal-jamming by the Sovereign: the realised equilibrium interest rate corresponding to a given pair (θ, s) is associated with the same signal z for any choice of funding rule, $R_b(\theta, s) = R_b\left(\theta + s/\sqrt{\tau_z}\right)$ for all b.

C.1 Equilibrium characterisation

In this Section I show that if the choice of borrowing is appropriately restricted to rules that do not feature too much complementarity (admissible rules), then the Sovereign can sustain an equilibrium such that (i) the market-clearing interest rate map, R_b, is an invertible function and (ii) it conveys a Gaussian signal about the fundamentals, $z = \theta + s/\sqrt{\tau_z}$, consistent with the restrictions set out above.

By Definition 4, I need to characterise four objects: the marginal signal map, x_b^*, the precision of the endogenous interest rate signal, τ_z, the posterior beliefs of financiers, and the interest rate function, R_b. I begin by guessing and verifying that that market-clearing interest rate, R_b, conveys a Gaussian endogenous public signal about the fundamentals, $z = \theta + \lambda s$. Recall that risk neutral financiers are only willing to lend provided the interest rate is at least as high as their reservation rate, $1/\beta^* (1 - \eta \Pr\{\theta \le \kappa r b | x, r\})$. Under this conjecture, conditioning on r is equivalent to conditioning on z, and the posterior beliefs of financiers are Gaussian with precision $\tau \equiv \tau_0 + \tau_z + \tau_x$ and mean $\delta_0\theta_0 + \delta_z z + \delta_x x$, where $\delta_0 \equiv \tau_0/\tau$ and so on. The posterior probability of default conditional on (x, z) is thus decreasing in x, and as a result, financiers lend if and only if $x \geq x_b^*(r)$, where x_b^* is the unique real number that makes a financier indifferent between lending and not lending, and therefore satisfies $\Pr_\theta\{\theta \leq \kappa r b(r) | x, z\} = \pi(r)$. Letting $\tau_z \equiv 1/\lambda^2$, it follows by Lemma C.1 that aggregate lending, l_b, is given by $\omega\Phi\left(\sqrt{\gamma}\left(z - x_b^*(r)\right)\right)$. The funding market clearing condition, $l_b = b$, then implies that $x_b^*(r) = z - \Phi^{-1}(b(r)/\omega)/\sqrt{\gamma}$. Substituting back into the indifference condition finally returns the equilibrium interest rate map:

$$R_b(z) = \left\{ r \ : \pi(r) = \Phi\left(\sqrt{\tau}\left(\kappa r b(r) - \delta_0\theta_0 - (\delta_x + \delta_z) z + \frac{\delta_x \Phi^{-1}\left(\frac{b(r)}{\omega}\right)}{\sqrt{\gamma}}\right)\right)\right\}, \tag{C.8}$$

which is the dispersed-information analogue of (7).

The two indifference conditions, (7) and (C.8), incorporate different posterior beliefs about the fundamentals, θ. Under common knowledge, the posterior is Gaussian with precision $\tau \equiv \tau_0 + \tau_z$ and mean $(\tau_0\theta_0 + \tau_z z)/\tau$, while with dispersed information the precision is $\tau \equiv \tau_0 + \tau_z + \tau_x$ and the mean is $(\tau_0\theta_0 + \tau_z z + \tau_x x)/\tau$. Moreover, posterior beliefs differ across financiers under dispersed information, so the posterior distribution that matters for the equilibrium interest rate is that of the marginal financier, x_b^*.

Because the identity of the marginal financier depends on the interest rate, changes in r affect the indifference condition of lenders through three channels as opposed to two. Suppose that a marginal increase in r results in a contraction in the demand for funds, so b falls. Market clearing requires that fewer investors be willing to lend, so $x_b^*(r) = z - \frac{\delta_x}{\sqrt{\gamma}}\Phi^{-1}\left(\frac{b(r)}{\omega}\right)$ has to increase for given z. The marginal financier becomes more optimistic about a default. This "market-clearing effect" reinforces the return effect. The reverse happens if b is increasing at some r, in which case the market-clearing effect dampens the return effect.

As in the variant with common knowledge presented in Section 4, a funding rule b is admissible if its corresponding market-clearing interest rate, R_b, is an invertible function. Definition 3 therefore still applies, although the appropriate market-clearing interest rate map is defined by (C.8) rather than by (7).

C.2 The best equilibrium

As far as the ex-ante problem, (8), and the ex-post problem, (9), are concerned, the model with dispersed information is identical to the model with common knowledge up to the market-clearing interest rate map. Accounting for the difference in the posterior beliefs of financiers, the results in Section 4 – Lemmas 1 and 2 and Proposition 2 – apply to the model with dispersed information.

To characterise the best equilibrium, one can thus use the same approach as under uncertainty and common knowledge, which requires solving the ex-post problem, (9), taking as given a market-clearing interest rate function defined by:

$$R(b,z) = \left\{ r : \pi(r) = \Phi\left(\sqrt{\tau}\left(\kappa r b - \delta_0\theta_0 - (\delta_z + \delta_x) z + \frac{\delta_x \Phi^{-1}\left(\frac{b}{\omega}\right)}{\sqrt{\gamma}}\right)\right)\right\}, \tag{C.9}$$

which is the dispersed-information analogue of (10). Because of the market-clearing effect, the ex-post equilibrium interest rate map, R, depends on b through two channels. First, higher b raises the default threshold, $\kappa r b$, for all z. Second, higher b lowers the posterior mean of the marginal financier, making her more pessimistic about the fundamentals, θ. While the first channel is active under both common knowledge and dispersed information, the second is unique to the dispersed-information variant. Both effects raise the probability of default for all z, thereby raising the reservation rate of the marginal financier.

C.3 Proofs

C.3.1 Proof of Lemma C.1

Financiers have a Gaussian prior with mean θ_0 and precision τ_0. The data they observe is given by $X = (x_1, x_2)'$. Conditional on θ, X is a bivariate Gaussian with mean $\mu_{X|\theta} = (\theta, \theta)'$ and variance-covariance matrix

$$\Sigma_{X|\theta} = \begin{bmatrix} \frac{1}{\tau_1} & 0 \\ 0 & \frac{1}{\tau_2} \end{bmatrix},$$

where $\tau_1 = \sigma_1^{-2}$ and $\tau_2 = 1/\left(\sigma_2^2 + \sigma_s^2\right)$. The unconditional distribution of X is also a bivariate Gaussian, with mean $\mu_X = (\theta_0, \theta_0)'$ and variance-covariance matrix given by:

$$\Sigma_X = \begin{bmatrix} \frac{1}{\tau_0} + \frac{1}{\tau_1} & \frac{1}{\tau_0} \\ \frac{1}{\tau_0} & \frac{1}{\tau_0} + \frac{1}{\tau_2} \end{bmatrix}.$$

By Bayes' rule, one obtains that the posterior distribution of θ given X is equal to:

$$\frac{\sqrt{\tau_0 + \tau_1 + \tau_2}}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left(\tau_0 + \tau_1 + \tau_2\right)\left(\theta - \frac{\tau_0\theta_0 + \tau_1 x_1 + \tau_2 x_2}{\tau_0 + \tau_1 + \tau_2}\right)^2 \right\},$$

which is the pdf of a Gaussian with mean $\frac{\tau_0\theta_0 + \tau_1 x_1 + \tau_2 x_2}{\tau_0 + \tau_1 + \tau_2}$ and precision $\tau_0 + \tau_1 + \tau_2$. This is the same posterior that would have arisen if instead of observing the two signals x_1 and x_2, financiers had observed one signal $x \equiv \frac{\tau_1}{\tau_1 + \tau_2} x_1 + \frac{\tau_2}{\tau_1 + \tau_2} x_2$, with precision $\tau_x \equiv \tau_1 + \tau_2$. Noticing that $\frac{\tau_1}{\tau_1+\tau_2} = \frac{\sigma_2^2 + \sigma_s^2}{\sigma_1^2 + \sigma_2^2 + \sigma_s^2}$ and that $\frac{\tau_2}{\tau_1+\tau_2} = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2 + \sigma_s^2}$ completes the proof of the first part of the Lemma.

Using the fact that $\frac{\tau_1}{\tau_1+\tau_2} + \frac{\tau_2}{\tau_1+\tau_2} = 1$ and the definitions of x_1 and x_2, it follows that:

$$\mathbb{E}[x|\theta, s] = \theta + \frac{\sigma_1^2 \sigma_s}{\sigma_1^2 + \sigma_2^2 + \sigma_s^2} s \text{ and } \mathbb{VAR}[x|\theta, s] = \frac{\sigma_1^2\left(\sigma_1^2\sigma_2^2 + \left(\sigma_2^2 + \sigma_s^2\right)^2\right)}{\left(\sigma_1^2 + \sigma_2^2 + \sigma_s^2\right)^2}.$$

Letting $\lambda \equiv \frac{\sigma_1^2 \sigma_s}{\sigma_1^2 + \sigma_2^2 + \sigma_s^2}$ and $\gamma \equiv \frac{\left(\sigma_1^2 + \sigma_2^2 + \sigma_s^2\right)^2}{\sigma_1^2\left(\sigma_1^2\sigma_2^2 + \left(\sigma_2^2 + \sigma_s^2\right)^2\right)}$ thus concludes the proof of the second part.

As for the third, it is straighforward to verify that $\mathbb{E}\left[x|\theta\right] = \theta$ and $\mathbb{VAR}\left[x|\theta\right] = 1/\tau_x$. This concludes the proof. ■

C.3.2 Proof of Lemmas 1 and 2 and Proposition 2 with dispersed information

Lemmas 1 and 2 and Proposition 2 continue to apply because Lemmas B.1–B.6 hold also under dispersed information. It suffices to replace α with τ and to redefine μ as $\delta_0\theta_0 + \delta_z z + \delta_x x$. Under dispersed information, the partial derivative of the ex-post market-clearing interest rate map is

$$\frac{\partial R}{\partial b} = \frac{\sqrt{\tau}\phi\left(\sqrt{\tau}\left(\kappa rb - \delta_0\theta_0 - (\delta_z + \delta_x)\, z + \frac{\delta_x \Phi^{-1}\left(\frac{b}{\omega}\right)}{\sqrt{\gamma}}\right)\right)\left(\kappa R + \frac{\delta_x}{\sqrt{\gamma}}\frac{1}{\omega\phi\left(\Phi^{-1}\left(\frac{b}{\omega}\right)\right)}\right)}{\frac{d\pi}{dr} - \sqrt{\tau}\phi\left(\sqrt{\tau}\left(\kappa rb - \delta_0\theta_0 - (\delta_z + \delta_x)\, z + \frac{\delta_x \Phi^{-1}\left(\frac{b}{\omega}\right)}{\sqrt{\gamma}}\right)\right)\kappa b},$$

reflecting the market-clearing effect of higher borrowing. ■