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ON THE DENSITY OF SUMSETS, II

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Abstract

Arithmetic quasidensities are a large family of real-valued set functions partially defined on the power set of $\mathbb{N}$, including the asymptotic density, the Banach density and the analytic density. Let $B \subseteq \mathbb{N}$ be a nonempty set covering $o(n!)$ residue classes modulo $n!$ as $n \to \infty$ (for example, the primes or the perfect powers). We show that, for each $\alpha \in [0, 1]$, there is a set $A \subseteq \mathbb{N}$ such that, for every arithmetic quasidensity μ, both A and the sumset $A + B$ are in the domain of μ and, in addition, $\mu(A + B) = \alpha$. The proof relies on the properties of a little known density first considered by Buck ['The measure theoretic approach to density', *Amer. J. Math.* **68** (1946), 560–580].

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1. Introduction

Let d be the *asymptotic* (or *natural*) *density* on the nonnegative integers $\mathbb{N}$ and $\mathrm{dom}(\mathsf{d})$ be the family of all sets $X \subseteq \mathbb{N}$ which possess asymptotic density, meaning that the limit of $|X \cap [1, n]|/n$ as $n \to \infty$ exists. We are going to show that, if $B \subseteq \mathbb{N}$ is nonempty and 'sufficiently small', then there is a family of sets of the form $A + B$ with A and $A + B$ both in $\mathrm{dom}(\mathsf{d})$ such that the corresponding asymptotic densities attain every value in the interval $[0, 1]$, where

$$A + B := \{x + y : x \in A,\ y \in B\}$$

is the *sumset* of A and B (see Section 2 for details and examples). Writing $\mathbb{P} := \{2, 3, 5, \ldots\}$ for the set of primes, we obtain as a special case the following result.

THEOREM 1.1. *For each* $\alpha \in [0, 1]$*, there exists* $A \in \mathrm{dom}(\mathsf{d})$ *such that* $A + \mathbb{P} \in \mathrm{dom}(\mathsf{d})$ *and* $\mathsf{d}(A + \mathbb{P}) = \alpha$.

In fact, our main result (Theorem 2.3) is much more general and stronger. It not only allows us to show that Theorem 1.1 holds with the primes replaced by a greater variety of sets and the asymptotic density d replaced by any in a large class of axiomatically defined 'densities' μ (including, among others, the Banach density, the analytic density and the logarithmic density), but it also does so *uniformly* in the choice of μ.

The result, whose proof relies on the properties of a little known density first considered by Buck [2], belongs to a vast literature on the interplay between sumsets and their 'largeness' (see, for example, [1, 5–7, 11, 13] and [12, Sections 4 and 5]). Most notably, an analogue of Theorem 1.1 with $\mathbb{P}$ replaced by a nonempty *finite* set $B \subseteq \mathbb{N}$ (and without the additional requirement that $A \in \mathrm{dom}(\mathsf{d})$) was proved by Faisant *et al.* in [5, Theorem 2.2]. A previous attempt to extend the latter to an *infinite* set B was made by Chu in [3, Theorem 1.5], but the proof turned out to be flawed [4]. So, Theorem 1.1 provides the first example in this direction (and is based on completely different ideas from [3, 5]).

1.1. Notation. We use $\mathbb{Z}$ for the integers and $\mathbb{N}^+$ for the positive integers. Given $X \subseteq \mathbb{N}$ and $q \in \mathbb{N}$, we define $q \cdot X := \{qx : x \in X\}$ and $X + q := X + \{q\}$. We let an *arithmetic progression* (AP) be a set of the form $k \cdot \mathbb{N} + h$ with $k \in \mathbb{N}^+$ and $h \in \mathbb{N}$, and we denote by $\mathscr{A}$ the family of all finite unions of APs. Finally, for each $a, b \in \mathbb{Z}$, we write $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$ for the *discrete interval* from a to b.

2. Preliminaries and main result

We say that a real-valued function $\mu^\star$ defined on the power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$ is an *arithmetic upper density* (on $\mathbb{N}$) if, for all $X, Y \subseteq \mathbb{N}$, the following conditions are satisfied:

(F1) $\mu^\star(X) \le \mu^\star(\mathbb{N}) = 1$;
(F2) $\mu^\star$ is *monotone*, that is, if $X \subseteq Y$, then $\mu^\star(X) \le \mu^\star(Y)$;
(F3) $\mu^\star$ is *subadditive*, that is, $\mu^\star(X \cup Y) \le \mu^\star(X) + \mu^\star(Y)$;
(F4) $\mu^\star(k \cdot X + h) = \mu^\star(X)/k$ for every $k \in \mathbb{N}^+$ and $h \in \mathbb{N}$.

Moreover, we call $\mu^\star$ an *arithmetic upper quasidensity* (on $\mathbb{N}$) if it satisfies (F1), (F3) and (F4).

REMARK 2.1. While there do exist nonmonotone arithmetic upper quasidensities [9, Theorem 1], such functions are not so interesting from the point of view of applications. Nevertheless, it seems meaningful to understand if monotonicity is critical to certain conclusions or can instead be dispensed with. This is our motivation for considering arithmetic upper quasidensities in spite of our main interest lying in the study of arithmetic upper densities (of course, the latter are a special case of the former).

We let the *conjugate* of an arithmetic upper quasidensity $\mu^\star$ be the function $\mu_\star : \mathcal{P}(\mathbb{N}) \to \mathbb{R} : X \mapsto 1 - \mu^\star(\mathbb{N} \setminus X)$, and we refer to the restriction μ of $\mu^\star$ to the set

$$\mathcal{D} := \{X \subseteq \mathbb{N} : \mu^\star(X) = \mu_\star(X)\}$$

as the *arithmetic quasidensity* induced by $\mu^\star$, or simply as an arithmetic quasidensity (on $\mathbb{N}$) if explicit reference to $\mu^\star$ is unnecessary. Accordingly, we call $\mathcal{D}$ the *domain* of μ and denote it by $\mathrm{dom}(\mu)$.

Arithmetic upper (quasi)densities and arithmetic (quasi)densities were introduced in [9] and further studied in [8, 10, 11], though we are adding here the adjective 'arithmetic' to emphasise that they assign precise values to APs (see Proposition 3.1(iv) below).

Notable examples of arithmetic upper densities include the upper asymptotic, upper Banach, upper analytic, upper logarithmic, upper Pólya and upper Buck densities (see [9, Section 6 and Examples 4, 5, 6 and 8]). In particular, we recall that the *upper Buck density* (on $\mathbb{N}$) is the function

$$\mathfrak{b}^\star : \mathcal{P}(\mathbb{N}) \to \mathbb{R} : \quad X \mapsto \inf_{A \in \mathscr{A},\, X \subseteq A} \mathsf{d}^\star(A),$$

where $\mathsf{d}^\star$ is the *upper asymptotic density* (on $\mathbb{N}$), that is, the function

$$\mathcal{P}(\mathbb{N}) \to \mathbb{R} : \quad X \mapsto \limsup_{n\to\infty} \frac{|X \cap [1, n]|}{n}.$$

We will write $\mathfrak{b}_\star$ and $\mathfrak{b}$, respectively, for the conjugate of and the density induced by $\mathfrak{b}^\star$.

REMARK 2.2. The asymptotic density d in Theorem 1.1 is just the density induced by $\mathsf{d}^\star$.

We are ready to state the main theorem of the paper, whose proof we postpone to Section 3.

THEOREM 2.3. *Let $B \subseteq \mathbb{N}$ be a nonempty set such that $\mathfrak{b}(B) = 0$. Then, for each $\alpha \in [0, 1]$, there exists $A \subseteq \mathbb{N}$ such that $A \in \mathrm{dom}(\mu)$ and $\mu(A + B) = \alpha$ for all arithmetic quasidensities μ.*

Note that the analogous statement to Theorem 2.3 may not hold for a set B with a nonzero upper Buck density and a real number $\alpha \in [\mathfrak{b}^\star(B), 1]$. For example, if $B \in \mathscr{A}$, then the upper Buck density of $A + B$ takes only a finite number of values as A varies over the subsets of $\mathbb{N}$.

The sets $B \subseteq \mathbb{N}$ such that $\mathfrak{b}(B) = 0$ have been studied in [10], where they are called *small sets*. Since $\mathfrak{b}$ is monotone and subadditive, it is clear that the family of small sets is closed under finite unions and subsets. Examples of small sets include the finite sets, the factorials, the perfect powers and the primes. One may be tempted to conjecture that a set $B \subseteq \mathbb{N}$ is small if (and only if) it is, in some sense, 'sufficiently sparse'. However, the property of being small depends on the distribution of B through the APs of $\mathbb{N}$ (see Proposition 3.1(v)), so much so that the set $\{n! + n : n \in \mathbb{N}\}$ is *not* small (its upper Buck density is 1), in spite of being sparse, by any standards.

To date, it is not known whether nonmonotone arithmetic quasidensities exist (see Remark 2.1). However, arithmetic quasidensities satisfy a weak form of monotonicity (implicit in the proof of Proposition 3.1) that will be enough for our goals.

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3. Proofs

To start with, we collect some basic properties of (upper and lower) quasidensities that will be used, possibly without further comment, in the proof of Theorem 2.3.

PROPOSITION 3.1. *Let $\mu_\star$ be the conjugate of an arithmetic upper quasidensity $\mu^\star$ on $\mathbb{N}$, and μ be the density induced by $\mu^\star$. Then the following hold:*

(i) $\mathfrak{d}_\star(X) \le \mu_\star(X) \le \mu^\star(X) \le \mathfrak{d}^\star(X)$ *for every* $X \subseteq \mathbb{N}$*;*
(ii) *if* $X \subseteq Y \subseteq \mathbb{N}$*, then* $\mathfrak{d}_\star(X) \le \mathfrak{d}_\star(Y)$*;*
(iii) $\mathscr{A} \subseteq \mathrm{dom}(\mathfrak{d}) \subseteq \mathrm{dom}(\mu)$ *and* $\mu(X) = \mathfrak{d}(X)$ *for every* $X \in \mathrm{dom}(\mathfrak{d})$*;*
(iv) *if* $k \in \mathbb{N}^+$ *and* $H \subseteq [\![0, k-1]\!]$*, then* $k \cdot \mathbb{N} + H \in \mathrm{dom}(\mathfrak{d})$ *and* $\mathfrak{d}(k \cdot \mathbb{N} + H) = |H|/k$*;*
(v) $\mathfrak{d}(X) = 0$ *if and only if* X *covers* $o(n!)$ *residue classes modulo* $n!$ *as* $n \to \infty$*;*
(vi) *if* $X \in \mathscr{A}$*, then* $X + Y \in \mathscr{A}$ *for every* $Y \subseteq \mathbb{N}$*.*

PROOF. See [11, Proposition 2.1] for items (i)–(iv) and [10, Proposition 2.6] for item (v). As for item (vi), let $X \in \mathscr{A}$ and $Y \subseteq \mathbb{N}$. There then exist $k \in \mathbb{N}^+$ and $H \subseteq [\![0, k-1]\!]$ such that $X = k \cdot \mathbb{N} + H$, and hence

$$X + Y = \bigcup_{y \in Y} (X + y) = \bigcup_{h \in H+Y} (k \cdot \mathbb{N} + h) = k \cdot \mathbb{N} + H',$$

where H' is the finite set $\bigcup_i \{\min((H+Y) \cap (k \cdot \mathbb{N} + i)\}$ and the union is extended over all $i \in [\![0, k-1]\!]$ such that $(H+Y) \cap (k \cdot \mathbb{N} + i) \neq \emptyset$ (with the understanding that an empty union is the empty set). □

We are ready for the proof of our main result. Note that the special case of a nonempty finite $B \subseteq \mathbb{N}$ was settled in [11, Theorem 1.2] by a different argument.

PROOF OF THEOREM 2.3. If $\alpha = 1$, then the conclusion is obvious (by taking $A = \mathbb{N}$). So, we assume from now on that $0 \le \alpha < 1$. We divide the remainder of the proof into a series of three claims.

Claim 1. There exist a sequence $(H_n)_{n\ge1}$ of (nonempty) subsets of $\mathbb{N}$ with $H_n \subseteq [\![0, n!-1]\!]$ and a sequence $(h_n)_{n\ge1}$ with $h_n \in H_n$ such that, for all $n \ge 1$, the following hold:

(i) $\mathfrak{d}^\star(n! \cdot \mathbb{N} + H_n' + B) \le \alpha < \mathfrak{d}^\star(n! \cdot \mathbb{N} + H_n + B)$, where $H_n' := H_n \setminus \{h_n\}$;
(ii) $n! \cdot \mathbb{N} + H_n' \subseteq H_{n+1} + (n+1)! \cdot \mathbb{N} \subseteq n! \cdot \mathbb{N} + H_n$.

To prove Claim 1(i), we proceed by induction. The base case is clear by taking $H_1 := \{0\}$ and $h_1 := 0$.

As for the inductive step, fix $m \ge 1$ and suppose we have already found a set $H_m \subseteq [\![0, m!-1]\!]$ and an integer $h_m \in H_m$ such that the claimed inequality holds for $n = m$. Since

$$m! \cdot \mathbb{N} + H_m = (m! \cdot \mathbb{N} + H_m') \cup ((m+1)! \cdot \mathbb{N} + h_m + m! \cdot [\![0, m]\!])$$

and $\mathfrak{d}^\star(m! \cdot \mathbb{N} + H'_m + B) \le \alpha < \mathfrak{d}^\star(m! \cdot \mathbb{N} + H_m + B)$, there is a minimal $k_{m+1} \in [\![0, m]\!]$ such that

$$\mathfrak{d}^\star((m! \cdot \mathbb{N} + H'_m) \cup ((m+1)! \cdot \mathbb{N} + h_m + m! \cdot [\![0, k_{m+1}]\!]) + B) > \alpha. \tag{3.1}$$

Consequently, we define

$$H_{m+1} := (H'_m + m! \cdot [\![0, m]\!]) \cup (h_m + m! \cdot [\![0, k_{m+1}]\!]) \quad \text{and} \quad h_{m+1} := h_m + k_{m+1} \cdot m!.$$

It is thus clear from (3.1) and the minimality of k_{m+1} that the claimed inequality is also true for $n = m + 1$. By induction, this is enough to complete the proof.

(ii) This is now a straightforward consequence of the recursive construction of the sequences $(H_n)_{n\ge1}$ and $(h_n)_{n\ge1}$, as given in the inductive step of the proof of item (i). This proves our claim.

Claim 2. Set $A := \bigcap_{n\ge1} A_n$, where $A_n := n! \cdot \mathbb{N} + H_n$. Then, $A \in \mathrm{dom}(\mathfrak{d})$.

To prove our claim, pick $n \in \mathbb{N}^+$. It follows from Claim 1(ii) that $A_n \setminus (n! \cdot \mathbb{N} + h_n) \subseteq A \subseteq A_n$. Considering that A_n and $A_n \setminus (n! \cdot \mathbb{N} + h_n)$ are both in $\mathscr{A}$, and $\mathscr{A}$ is contained in $\mathrm{dom}(\mathfrak{d})$, we obtain

$$\mathfrak{d}(A_n) - \frac{1}{n!} \le \mathfrak{d}_\star(A) \le \mathfrak{d}^\star(A) \le \mathfrak{d}(A_n). \tag{3.2}$$

However, Claim 1(ii) gives $A_{n+1} \subseteq A_n$. Since $\mathfrak{d}$ is monotone, it follows that $\mathfrak{d}(A_n)$ tends to a limit as $n \to \infty$. So, (3.2) implies that $\mathfrak{d}_\star(A) = \mathfrak{d}^\star(A) = \lim_n \mathfrak{d}(A_n)$ and hence, $A \in \mathrm{dom}(\mathfrak{d})$. This proves our claim.

Claim 3. $A + B \in \mathrm{dom}(\mathfrak{d})$ and $\mathfrak{d}(A + B) = \alpha$.

To prove our claim, fix $n \in \mathbb{N}^+$. We gather from Claim 1(ii) that

$$A_n \setminus (n! \cdot \mathbb{N} + h_n) + B \subseteq A + B \subseteq A_n + B.$$

Considering that, by Proposition 3.1(vi), $X \in \mathscr{A}$ yields $X + B \in \mathscr{A}$ and $\mathscr{A} \subseteq \mathrm{dom}(\mathfrak{d})$, it follows that

$$\mathfrak{d}(A_n \setminus (n! \cdot \mathbb{N} + h_n) + B) \le \mathfrak{d}_\star(A + B) \le \mathfrak{d}^\star(A + B) \le \mathfrak{d}(A_n + B). \tag{3.3}$$

Now, fix $\varepsilon > 0$. Since $\mathfrak{d}(B) = 0$, there exists $n_\varepsilon \in \mathbb{N}^+$ such that B covers at most $\varepsilon \cdot n!$ residue classes modulo $n!$ for all $n \ge n_\varepsilon$. Consequently, we obtain from the subadditivity of $\mathfrak{d}^\star$ and Claim 1(i) that

$$\mathfrak{d}(A_{n_\varepsilon} + B) \le \mathfrak{d}(A_{n_\varepsilon} \setminus (n_\varepsilon! \cdot \mathbb{N} + h_{n_\varepsilon}) + B) + \mathfrak{d}(n_\varepsilon! \cdot \mathbb{N} + h_{n_\varepsilon} + B) \le \alpha + \varepsilon.$$

However, the first inequality in the last display and Claim 1(i) imply

$$\mathfrak{d}(A_{n_\varepsilon} \setminus (n_\varepsilon! \cdot \mathbb{N} + h_{n_\varepsilon}) + B) \ge \mathfrak{d}(A_{n_\varepsilon} + B) - \mathfrak{d}(n_\varepsilon! \cdot \mathbb{N} + h_{n_\varepsilon} + B) > \alpha - \varepsilon. \tag{3.4}$$

Therefore, (2) and (3) imply that $\alpha - \varepsilon < \mathfrak{d}_\star(A + B) \le \mathfrak{d}^\star(A + B) \le \alpha + \varepsilon$ for every $\varepsilon > 0$, which suffices to conclude that $A + B \in \mathrm{dom}(\mathfrak{d})$ and $\mathfrak{d}(A + B) = \alpha$. This proves our claim.

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By Proposition 3.1(iii) and Claims 2 and 3, this is enough to finish the proof of the theorem.

PROOF OF THEOREM 1.1. This is now straightforward from Theorem 2.3 and Remark 2.2, when considering that the Buck density of the set of primes is zero (as already noted in Section 2). □

As a final remark, we point out that, *mutatis mutandis*, all the results of this paper carry over to arithmetic (upper) quasidensities on $\mathbb{Z}$, in the same spirit of [8–11].

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