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MAJORIZATION-SUBORDINATION THEOREMS FOR LOCALLY UNIVALENT FUNCTIONS, II

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Let $\mathfrak{S}$ denote the set of all normalized analytic univalent functions in the open unit disc D. Let $f(z)$, $F(z)$ and $\varphi(z)$ be analytic in $|z| < r$. We say that $f(z)$ is *majorized* by $F(z)$ in $|z| < r$ ($f(z) \ll F(z)$), if $|f(z)| \leqq |F(z)|$ in $|z| < r$; we say that $f(z)$ is *subordinate* to $F(z)$ in $|z| < r$ ($f(z) \prec F(z)$), if $f(z) = F(\varphi(z))$ where $|\varphi(z)| \leqq |z|$ in $|z| < r$.

Let $\mathfrak{U}_\alpha$ be the set of all locally univalent ($f'(z) \neq 0$) analytic functions in D with order $\leqq \alpha$ which are of the form $f(z) = z + \ldots$. The family $\mathfrak{U}_\alpha$ is known as the universal linear invariant family of order α [**6**]. A concise summary of and introduction to properties of linear invariant families which relate to the following material is contained in [**1**]. The present paper contains the proofs of some of the results announced in [**1**].

Majorization-subordination theory begins with Biernacki who showed in 1936 that if $f(z)$ is subordinate in D to $F(z)$ ($F(z) \in \mathfrak{S}$), then $f(z)$ is majorized by $F(z)$ in $|z| < 1/4$. Goluzin, Tao Shah, Lewandowski and MacGregor have examined various related problems since that time but always under the stipulation that the dominant function $F(z)$ is in $\mathfrak{S}$ (for greater detail see [**1**]).

In this paper we generalize the previously investigated problems by allowing $F(z)$ to be in $\mathfrak{U}_\alpha$. Our investigation shows that the important datum for majorization-subordination theory is *not* univalence, but the order of a linear invariant family. In particular, many classically derived estimates for univalent functions are true for functions of infinite valence.

1. Majorization of the derivatives. MacGregor [**4**] in 1967 investigated the effect that majorization by a univalent function has on the radius of majorization of the derivative. We prove corresponding results for majorization by a function in $\mathfrak{U}_\alpha$ and give a simplified proof that the result is sharp.

THEOREM 1. *Let $f(z)$ be majorized by $F(z)$ in D. If $F(z) \in \mathfrak{U}_\alpha$, $1 \leqq \alpha < \infty$, then $f'(z)$ is majorized by $F'(z)$ in*

$$|z| \leqq [(\alpha + 1)^{1/\alpha} - 1]/[(\alpha + 1)^{1/\alpha} + 1] = \tanh[(2\alpha)^{-1} \ln(\alpha + 1)].$$

The result is best possible for each α.

Proof. If $f(z)$ is majorized by $F(z)$ in D, then $f(z) = \varphi(z)F(z)$ where $|\varphi(z)| \leqq 1$ in D [**4**, Lemma 5]. Since [**5**, p. 168]

$$|\varphi'(z)| \leqq (1 - |\varphi(z)|^2)/(1 - |z|^2)$$

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and for functions $F(z)$ in $\mathfrak{U}_\alpha$ we have [**6**, p. 115, 1.10]

$$|F(z)| \leqq \frac{1}{2\alpha}(1-|z|^2)|F'(z)|\left[\left\{\frac{1+|z|}{1-|z|}\right\}^\alpha - 1\right],$$

it follows that

$$(1) \qquad |f'(z)| \leqq |F'(z)|\left\{|\varphi(z)| + \left[\frac{1-|\varphi(z)|^2}{2\alpha}\right]\left[\left(\frac{1+|z|}{1-|z|}\right)^\alpha - 1\right]\right\}.$$

However,

$$(2) \qquad |\varphi(z)| + \left[\frac{1-|\varphi(z)|^2}{2\alpha}\right]\left[\left(\frac{1+|z|}{1-|z|}\right)^\alpha - 1\right] \leqq 1$$

if

$$(3) \qquad \left(\frac{1+|z|}{1-|z|}\right)^\alpha - 1 \leqq \alpha.$$

Inequality (3) is equivalent to $|z| \leqq \tanh[(2\alpha)^{-1}\ln(\alpha+1)]$. Therefore, it follows from (1) and (2) that $f'(z)$ is majorized by $F'(z)$ in

$$|z| \leqq \tanh[(2\alpha)^{-1}\ln(\alpha+1)].$$

We now show that the result is best possible for each α. Consider the functions

$$F(z) = (2\alpha)^{-1}(1-[(1-z)/(1+z)]^\alpha) \quad \text{and} \quad \varphi(z) = (z+b)/(1+bz)$$

where $-1 \leqq b \leqq 1$. Let $f(z) \equiv \varphi(z)F(z)$. Clearly $F(z) \in \mathfrak{U}_\alpha$ majorizes $f(z)$ in D. Choose any r such that $\tanh[(2\alpha)^{-1}\ln(\alpha+1)] < r < 1$. We show for each such r that we can choose b so that $f'(r) > F'(r) > 0$. It therefore follows that $F'(z)$ cannot majorize $f'(z)$ outside of $|z| \leqq \tanh[(2\alpha)^{-1}\ln(\alpha+1)]$. We first note that

$$(4) \qquad \frac{F(r)}{F'(r)} = \frac{(1-r^2)}{2\alpha}\left[\left(\frac{1+r}{1-r}\right)^\alpha - 1\right] > \frac{1-r^2}{2}.$$

Since

$$f'(r) = F'(r)\left[\frac{r+b}{1+rb} + \frac{1-b^2}{(1+rb)^2}\cdot\frac{F(r)}{F'(r)}\right] \equiv F'(r)H(r,b)$$

and $H(r,1) \equiv 1$, we need only show that $\partial H(r,b)/\partial b|_{b=1} < 0$ in order to establish that $H(r, 1-\epsilon) > 1$ and hence that $f'(r) > F'(r) > 0$. But

$$\frac{\partial}{\partial b}H(r,b)|_{b=1} = \frac{2}{(1+r)^2}\left[\frac{1-r^2}{2} - \frac{F(r)}{F'(r)}\right],$$

which is negative by (4). Thus the result is best possible.

COROLLARY 1. *If* $f(z) \ll F(z)$ *in* D *and* $F(z) \in \mathfrak{U}_2$, *then* $f'(z) \ll F'(z)$ *in* $|z| \leqq 2 - \sqrt{3}$.

COROLLARY 2. *If* $f(z) \ll F(z)$ *in* D *and* $F(z) \in \mathfrak{U}_1$, *then* $f'(z) \ll F'(z)$ *in* $|z| \leqq 1/3$.

Pommerenke [**6**, p. 134] showed that $\mathfrak{U}_1$ is precisely the class of convex univalent functions. It is well-known that $\mathfrak{S}$ is a proper subset of $\mathfrak{U}_2$. Therefore, Corollary 1 is stronger than MacGregor's Theorem 1B, while Corollary 2 is his Theorem 1C.

2. The converse of the Biernacki problem. Lewandowski [**2**] in 1961 established a converse to the original Biernacki problem under the normalization $f(0) = 0$, $f'(0) \geqq 0$. He showed that majorization of $f(z)$ in D by $F(z)$ ($F(z) \in \mathfrak{S}$) implied that $f(z)$ is subordinate to $F(z)$ in $|z| < .21$. We remove the restriction of global univalence of $F(z)$ and substitute local univalence and finite order. Let $R(\alpha)$ be the 'radius of subordination' for functions majorized by a function in $\mathfrak{U}_\alpha$; that is, $R(\alpha)$ is the largest number such that if $f(z) \ll F(z)$ in D ($F(z) \in \mathfrak{U}_\alpha$) and $f'(0) \geqq 0$, then $f(z) \prec F(z)$ in $|z| < R(\alpha)$.

THEOREM 2. *Let $f'(0) \geqq 0$, $F(z) \in \mathfrak{U}_\alpha$, $1 \leqq \alpha < \infty$ and $f(z)$ be majorized by $F(z)$ in D. Let $R_2(\alpha)$ denote the root in* $[0, 1]$ *of the equation*

$$x(1 + x)^\alpha - (1 - x)^\alpha = 0.$$

Let α^* denote the root of

$$\frac{1}{x} - \left[\frac{x-1}{x+1}\right]^{x/2}\left[1 - \frac{1}{4}\left(\frac{x-1}{x+1}\right)^x\right]^{1/2} = 0. \tag{5}$$

If $1 \leqq \alpha \leqq \alpha^*$ let $R_2(\alpha)$ be the root of the equation

$$\frac{2x}{1+x^2} - \left[\frac{1-x}{1+x}\right]^\alpha\left[1 - \frac{1}{4}\left(\frac{1-x}{1+x}\right)^{2\alpha}\right]^{1/2} = 0$$

and let $R_2(\alpha)$ be $\alpha - (\alpha^2 - 1)^{1/2}$ if $\alpha \geqq \alpha^*$. Then the 'radius of subordination' for functions majorized by a function in $\mathfrak{U}_\alpha$ satisfies

$$R_1(\alpha) \leqq R(\alpha) \leqq R_2(\alpha).$$

Proof. A computation shows that $2.88 < \alpha^* < 2.89$.

We first show that $R(\alpha) \leqq R_2(\alpha)$ for all α, $1 \leqq \alpha < \infty$. Again we let $F(z) = (2\alpha)^{-1}[1 - ((1 - z)/(1 + z))^\alpha]$. If $f(z) = zF(z)$, then clearly $F(z)$ majorizes $f(z)$ in D and $f'(0) = 0$. It is easy to verify that $f(-\rho) > F(\rho) > 0$ for any ρ which satisfies $R_2(\alpha) < \rho < 1$.

Suppose that $f(z)$ were subordinate to $F(z)$ in $|z| < r$ where $R_2(\alpha) < r < 1$. Then $f(z) = F(\omega(z))$ where $\omega(z)$ is an analytic function satisfying $|\omega(z)| \leqq |z|$ in $|z| < r$. An analysis of $F(\omega(z)) = f(z)$ shows that $\omega(z)$ must be real if $z \in (-r, r)$.

If we restrict $F(z)$ to the real axis, it is an increasing real valued function. Thus for $R_2(\alpha) < \rho < r$, we have $\rho = |-\rho| \geqq |\omega(-\rho)| \geqq \omega(-\rho)$ and therefore $F(\rho) \geqq F(\omega(-\rho)) = f(-\rho)$. This is absurd since $f(-\rho) > F(\rho)$ for all $R_2(\alpha) < \rho < 1$. Therefore, for any α in $1 \leqq \alpha < \infty$, the radius of subordination $R(\alpha)$ cannot be greater than $R_2(\alpha)$ if $f(z)$ is majorized by $F(z)$ ($F(z) \in \mathfrak{U}_\alpha$).

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To establish a lower bound for $R(\alpha)$ we develop two preliminary bits of technical information. We first claim that $R_1(\alpha)$ is always less than or equal to the radius of convexity of the family $\mathfrak{U}_\alpha$ (which is $\alpha - (\alpha^2 - 1)^{1/2}$ [**6**, p. 133]). Since $R_1(\alpha)$ is precisely the radius of convexity for $\alpha \geqq \alpha^*$, we need only show that for $1 \leqq \alpha \leqq \alpha^*$, the root of

$$T(x) = \frac{2x}{1+x^2} - \left[\frac{1-x}{1+x}\right]^\alpha \left[1 - \frac{1}{4}\left(\frac{1-x}{1+x}\right)^{2\alpha}\right]^{1/2} = 0$$

is less than or equal to $\alpha - (\alpha^2 - 1)^{1/2}$. Since it is easy to show that $T(x)$ is a monotone increasing function on $[0, 1]$, it suffices to show that

$$T(\alpha - (\alpha^2 - 1)^{1/2}) \geqq 0$$

for $1 \leqq \alpha \leqq \alpha^*$. This follows upon noting that

$$T(\alpha - (\alpha^2 - 1)^{1/2}) = \frac{1}{\alpha} - \left[\frac{\alpha - 1}{\alpha + 1}\right]^{\alpha/2}\left[1 - \frac{1}{4}\left(\frac{\alpha-1}{\alpha+1}\right)^\alpha\right]^{1/2}$$

is a monotone decreasing function of α which is positive for $1 \leqq \alpha \leqq \alpha^*$ where α^* is the root of (5). A computation shows that $2.8 < \alpha^* < 2.9$.

We next claim that for any a such that $0 \leqq a \leqq R_1(\alpha)$,

$$\frac{a + R_1}{1 + aR_1} \leqq \left[\frac{1 - R_1}{1 + R_1}\right]^\alpha, \tag{6}$$

which is equivalent to showing that

$$\frac{2R_1}{1 + R_1^2} \leqq \left[\frac{1 - R_1}{1 + R_1}\right]^\alpha.$$

For $1 \leqq \alpha \leqq \alpha^*$ this is immediate since $R_1(\alpha)$ satisfies

$$\frac{2R_1}{1 + R_1^2} = \left[\frac{1 - R_1}{1 + R_1}\right]^\alpha \left[1 - \frac{1}{4}\left(\frac{1 - R_1}{1 + R_1}\right)^{2\alpha}\right]^{1/2} < \left[\frac{1 - R_1}{1 + R_1}\right]^\alpha.$$

For $\alpha \geqq \alpha^*$ we need only show (since $R_1(\alpha) = \alpha - (\alpha^2 - 1)^{1/2}$) that

$$\frac{1}{\alpha} \leqq \left[\frac{\alpha - 1}{\alpha + 1}\right]^{\alpha/2}$$

which, just as above, is immediate from (5) for all $\alpha \geqq \alpha^*$.

To show that $f(z)$ is subordinate to $F(z)$ in $|z| < r$ it suffices to show that $f(|z| < r) \subset F(|z| < r)$ [**3**, p. 163]. As previously remarked, if $f(z)$ is majorized by $F(z)$ in D, then $f(z) = \varphi(z)F(z)$ where $|\varphi(z)| \leqq 1$ in D and $\varphi(0) = a = f'(0) \geqq 0$. We examine two cases.

Case 1. $0 \leqq a \leqq R_1(\alpha)$ where $a = f'(0)/F'(0) = \varphi'(0)$. Since $F(z)$ is convex univalent in $|z| < R_1(\alpha)$, it is univalent there and hence with an easy modification of [**6**, (1.9)]

$$\frac{1}{2\alpha}\left[1 - \left(\frac{1-r}{1+r}\right)^\alpha\right] \leqq |F(z)| \leqq \frac{1}{2\alpha}\left[\left(\frac{1+r}{1-r}\right)^\alpha - 1\right] \tag{7}$$

for $|z| = r \leqq R_1(\alpha)$.

An easy application of the Schwarz lemma to $\varphi(z)$ yields that

$$|f(z)| \leqq |F(z)|(a + |z|)/(1 + a|z|).$$

Applying (6) and (7) yields

$$\begin{aligned}\max_{|z|=R_1} |f(z)| &\leqq \max_{|z|=R_1} |F(z)| \cdot \frac{a + R_1}{1 + aR_1} \\ &\leqq \left[\frac{R_1 + a}{1 + aR_1}\right]\left[\frac{1}{2\alpha}\right]\left[\left(\frac{1 + R_1}{1 - R_1}\right)^\alpha - 1\right] \\ &\leqq \left[\frac{1 - R_1}{1 + R_1}\right]^\alpha \frac{1}{2\alpha}\left[\left(\frac{1 + R_1}{1 - R_1}\right)^\alpha - 1\right] = \frac{1}{2\alpha}\left[1 - \left(\frac{1 - R_1}{1 + R_1}\right)^\alpha\right] \\ &\leqq \min_{|z|=R_1} |F(z)|.\end{aligned}$$

This implies that $f(|z| < R_1) \subset F(|z| < R_1)$ and hence that $f(z)$ is subordinate to $F(z)$ in $|z| < R_1(\alpha)$ which concludes the proof for Case 1.

Case 2. $R_1(\alpha) < a \leqq 1$: Fix z_0, $|z_0| = R_1(\alpha) = R_1$. Let $F(z_0) = w_0$ and let $\Lambda = F(|z| = R_1)$. Since $F(z)$ is convex and univalent in $|z| \leqq R_1$, Λ is a convex Jordan curve contained in the annulus

$$A = \left\{w : \frac{1}{2\alpha}\left[1 - \left(\frac{1 - R_1}{1 + R_1}\right)^\alpha\right] \leqq |w| \leqq \frac{1}{2\alpha}\left[\left(\frac{1 + R_1}{1 - R_1}\right)^\alpha - 1\right]\right\}.$$

A ray from the origin through w_0 intersects the inner and outer boundary of the annulus A at, say, c and b respectively. The circle with centre b which passes through the origin determines two new points d and e by its intersection with the inner boundary of A.

It follows easily that

$$\begin{aligned}\text{angle}(dw_0, 0) \geqq \text{angle}(db, 0) &= 2 \arcsin \frac{1}{2}\left[\frac{1 - R_1}{1 + R_1}\right]^\alpha \\ &= \arcsin\left[\frac{1 - R_1}{1 + R_1}\right]^\alpha\left[1 - \frac{1}{4}\left(\frac{1 - R_1}{1 + R_1}\right)^{2\alpha}\right]^{1/2} \\ &\geqq \arcsin \frac{2R_1}{1 + R_1^2}\end{aligned}$$

where we have used the fact that $T(R_1) \leqq 0$.

The function $h(z) = 1 - f(z)/F(z) = 1 - \varphi(z)$ has $\operatorname{Re} h(z) > 0$. Therefore, as is well-known, $|\arg h(z_0)| \leqq \arcsin 2R_1(1 + R_1^2)^{-1} \leqq \text{angle}(dw_0, 0)$. Furthermore, since $H(z) = (1 - a)(1 + z)(1 - az)^{-1}$ maps D onto a disc centred at 1 with radius 1, we have $h(D) \subset H(D)$. This implies that $h(z) = H(\omega(z))$ where $\omega(z)$ satisfies the Schwarz lemma. Hence

$$|h(z_0)| \leqq (1 - a)(1 + R_1)(1 - aR_1)^{-1}$$

which is less than 1 since $R_1 < a$. Therefore, $f(z_0) = w_0 - w_0h(z_0)$ is in the circular sector w_0d0ew_0 and hence in $F(|z| < R_1)$, since Λ is convex.

The point z_0 was arbitrary on $|z| = R_1$, consequently

$$f(|z| = R_1) \subset F(|z| \leqq R_1)$$

and by the maximum modulus principle $f(|z| < R_1) \subset F(|z| < R_1)$. Thus $f(z)$ is subordinate to $F(z)$ in $|z| < R_1$.

Thus in Cases 1 and 2, we have shown that $f(z)$ is subordinate to $F(z)$ in $|z| \leqq R_1(\alpha)$ which concludes the proof of the theorem.

COROLLARY 1. *If* $f(z) \ll F(z)$ *in* D, $F(z) \in \mathfrak{U}_1$, *and* $f'(0) \geqq 0$, *then* $f(z) \prec F(z)$ *in* $|z| < R$ *where* $.28 < R \leqq \sqrt{2} - 1$.

COROLLARY 2. *If* $f(z) \ll F(z)$ *in* D, $F(z) \in \mathfrak{U}_2$, *and* $f'(0) \geqq 0$, *then* $f(z) \prec F(z)$ *in* $|z| < R$ *where* $.21 < R < .3$.

Since $\mathfrak{S}$ is a proper subset of $\mathfrak{U}_2$, Corollary 2 is a strengthening of Lewandowski's original result [**2**]. Corollary 1 is a new result for the set of normalized convex univalent functions.

In part III of this paper we will present the long and tedious proof of

THEOREM 3. *Let* $f(z)$ *be subordinate to* $F(z)$ *in* D *and let* $f'(0) \geqq 0$. *If* $F(z) \in \mathfrak{U}_\alpha$, $1.65 \leqq \alpha < \infty$, *then* $f'(z)$ *is majorized by* $F'(z)$ *in* $|z| \leqq (\alpha + 1) - (\alpha^2 + 2\alpha)^{1/2}$. *The inequality is best possible.*

The problem investigated in Theorem 3 was first studied by Goluzin and given a complete solution in $\mathfrak{S}$ by Tao Shah (see [**1**] for further references).

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