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COLLOCATION METHODS FOR TWO DIMENSIONAL WEAKLY SINGULAR INTEGRAL EQUATIONS

IVAN G. GRAHAM

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Abstract

Given a Fredholm integral equation of the second kind, which is defined over a certain region $\bar{\Omega} \subseteq \mathbf{R}^2$, we define y_N^{I} and y_N^{II}, two different numerical approximations to its solution, using the collocation and iterated collocation methods respectively. We describe without proof some known results concerning the general convergence properties of y_N^{I} and y_N^{II} when the kernel and solution of the integral equation are smooth. Then, we prove rigorously order of convergence estimates for y_N^{I} and y_N^{II} which are applicable in the practically siginificant case when $\bar{\Omega}$ is a rectangle, and the kernel of the integral equation is weakly singular. These estimates are illustrated by the numerical solution of a two dimensional weakly singular equation which arises in electrical engineering.

1. Introduction

In this paper we shall discuss the numerical solution of two dimensional Fredholm integral equations of the second kind, which take the form

$$y(t) = f(t) + \int_{\bar{\Omega}} k(t, s)y(s)\, ds, \qquad t \in \bar{\Omega}. \tag{1.1}$$

Here $\Omega \subseteq \mathbf{R}^2$ is a domain (i.e. an open connected set) which is bounded, and $\bar{\Omega}$ denotes its closure. The functions k and f are given on $\bar{\Omega} \times \bar{\Omega}$ and $\bar{\Omega}$ respectively and our task is to determine the solution, y.

We shall abbreviate (1.1) using standard operator notation, as

$$y = f + Ky,$$

where K denotes the integral operator defined by

$$Ky(t) = \int_{\bar{\Omega}} k(t, s)y(s)\, ds, \qquad t \in \bar{\Omega},$$

and we shall assume that (1.1) has a unique continuous solution y on $\bar{\Omega}$. (Conditions sufficient to ensure this will be described in Section 2.)

We shall use the methods of collocation and iterated collocation to define two different approximations, y_N^{I} and y_N^{II}, to y. Specifically, we shall seek y_N^{I} in the form

$$y_N^{\mathrm{I}} = \sum_{i=1}^{N} a_i u_i, \tag{1.2}$$

where $\{u_1, \ldots, u_N\}$ is a certain set of piecewise constant basis functions defined on $\bar{\Omega}$, and the coefficients $\{a_1, \ldots, a_N\}$ are the solution set of the $N \times N$ linear system obtained by demanding that

$$y_N^{\mathrm{I}}(t_j) = f(t_j) + Ky_N^{\mathrm{I}}(t_j), \qquad j = 1, \ldots, N, \tag{1.3}$$

where $\{t_1, \ldots, t_N\} \subseteq \bar{\Omega}$ is some predetermined set of collocation points.

We then define y_N^{II} by the natural iteration,

$$y_N^{\mathrm{II}} = f + Ky_N^{\mathrm{I}}, \tag{1.4}$$

which, using (1.2), may also be written as

$$y_N^{\mathrm{II}} = f + \sum_{i=1}^{N} a_i K u_i.$$

In this paper, we shall examine the convergence properties of y_N^{I} and y_N^{II}. We shall state our main results below, but first an explanation concerning the construction of our basis set and collocation points is needed.

For each $N \in \mathbf{N}$ we introduce a mesh (partition) Π_N of $\bar{\Omega}$, consisting of N open, simply-connected, pairwise-disjoint subsets of $\bar{\Omega}$, $\{\Omega_i : i = 1, \ldots, N\}$, with the property that each Ω_i contains its centroid, and

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i.$$

For $i = 1, \ldots, N$, we then define u_i to be the function on $\bar{\Omega}$ which takes the value 1 on Ω_i, and 0 elsewhere. We assume that

$$\|\Pi_N\|_\infty \to 0, \quad \text{as } N \to \infty,$$

where

$$\|\Pi_N\|_\infty = \max_{i=1,\ldots,N} \; \sup_{t, t' \in \Omega_i} \|t - t'\|_\infty,$$

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and we also assume that

$$t_i \in \Omega_i,$$

for $i = 1, \ldots, N$.

It then follows, under fairly mild conditions on k and f, that, for sufficiently large N, y_N^{I} and y_N^{II} are well defined and converge to y (in the uniform norm).

In addition, it can be shown that, if y is suitably regular, then we have,

$$\|y - y_N^{\mathrm{I}}\|_\infty = O(\|\Pi_N\|_\infty), \tag{1.5}$$

while under additional regularity requirements on k and y, we also have,

$$\|y - y_N^{\mathrm{II}}\|_\infty = O\big(\|\Pi_N\|_\infty^2\big), \tag{1.6}$$

the final estimate also being dependent on each collocation point t_i being chosen as the centroid of the set Ω_i, for $i = 1, \ldots, N$.

Unfortunately, the regularity requirements on k and y which are needed for (1.5) and (1.6) to hold are rather strict (particularly in the case of (1.6)), and, in fact, may not be satisfied in practical situations, where singularities are often present. The main aim of this paper will be to derive order of convergence estimates for y_N^{I} and y_N^{II} (analogous to (1.5), and (1.6)) which are applicable in the practically important case when k has a weak singularity along the diagonal $t = s$, and when $\bar{\Omega}$ is a rectangle.

These results will be obtained in Theorem 10 of Section 4. As an illustration of the kind of information contained in Theorem 10, consider the prototype equations,

$$y(t) = f(t) + \int_0^d \int_0^1 |t - s|^{\alpha - 1} y(s)\, ds, \qquad t \in [0, 1] \times [0, d], \tag{1.7}$$

with $0 < \alpha < 1$, and

$$y(t) = f(t) + \int_0^d \int_0^1 \ln|t - s| y(s)\, ds, \qquad t \in [0, 1] \times [0, d], \tag{1.8}$$

where $|x|$ denotes the length of any vector $x \in \mathbf{R}^2$, and f is twice continuously differentiable on $[0, 1] \times [0, d]$. Theorem 10 then predicts that, for these prototype equations,

$$\|y - y_{N(\tau)}^{\mathrm{I}}\|_\infty = O(\|\Pi_{N(\tau)}\|_\infty),$$

and

$$\|y - y_{N(\tau)}^{\mathrm{II}}\|_\infty = O\big(\|\Pi_{N(\tau)}\|_\infty^{\beta + 1}\big),$$

where, in the case of equation (1.7), β is any number satisfying $0 < \beta < \alpha$, and, in the case of equation (1.8), β is any number satisfying $0 < \beta < 1$. Here $\Pi_{N(\tau)}$ is a family of rectangular meshes on $\bar{\Omega} = [0, 1] \times [0, d]$, which depend on a parameter τ in such a way that, as $\tau \to 0$, $N(\tau) \to \infty$, and, as $\tau \to 0$, the subsets of $\bar{\Omega}$ given by $\Pi_{N(\tau)}$ shrink in size in a suitably uniform manner.

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The theoretical section of this paper is organised as follows.

In Section 2, we discuss the basic properties of y_N^{I} and y_N^{II}, when $\bar{\Omega}$ is the closure of any general bounded domain, and we describe a set of circumstances under which (1.5) and (1.6) hold. In Section 3, we consider the case when $\bar{\Omega}$ is a rectangle, and when the kernel, k, of (1.1) has a weak singularity along the diagonal $t = s$, and we discuss the regularity of the solution to the resulting equation.

In Section 4, we prove the main result of the paper, Theorem 10. The ingredients of the proof are the regularity results of Section 3, two technical lemmas (Lemmas 6 and 8) concerning the approximation of functions over rectangular domains, and Lemma 9, which describes a special property possessed by the collocation points $\{t_1, \ldots, t_N\}$ when, for each $i = 1, \ldots, N$, t_i is chosen to be the centroid of Ω_i. These lemmas are also given in Section 4, prior to Theorem 10.

It may be observed from (1.3) and (1.4), that the calculation of y_N^{I} requires the evaluation of the two dimensional integrals,

$$\{Ku_i(t) \colon i = 1, \ldots, N\}, \tag{1.9}$$

at each of the collocation points, while the calculation of y_N^{II} at an arbitrary point $t \in \bar{\Omega}$, requires, in addition, the evaluation of each of the integrals (1.9) at that point. Due to the choice of the piecewise constant basis set, the integrals (1.9) have the particularly simple form

$$Ku_i(t) = \int_{\Omega_i} k(t, s)\, ds, \qquad i = 1, \ldots, N,$$

and may even be calculated analytically, if k is not too complicated. Thus the methods given here are demonstrably simple in construction and implementation, and yet possess quite respectable convergence rates, provided the basis set and the collocation points are carefully chosen. For these reasons we propose that these methods are attractive from a practical point of view, and to illustrate this point we use them to solve a two dimensional integral equation which arises in electrical engineering. This illustration is described in Section 5.

To date, the literature contains very little analysis on numerical methods for equations of the type (1.1). We remark, however, that the one dimensional analogues of the methods proposed here have been well studied. Results analogous to ours for one dimensional equations with smooth kernels and solutions are obtained by Sloan, Noussair and Burn [10], while Chandler [1] and Schneider [9] have studied product integration (of which iterated collocation is a special case) for one dimensional equations with weakly singular kernels.

2. Some general convergence results

In this section, it is our aim to state two theorems, Theorems 1 and 2, which describe the general convergence properties of y_N^{I} and y_N^{II}. The proofs of these theorems are given in [5].

In order to state these results, we must first introduce some function spaces. We denote by $L_\infty(\bar{\Omega})$ the space of essentially bounded functions on $\bar{\Omega}$. This space is a Banach space under the norm

$$\|\phi\|_\infty = \operatorname*{ess\,sup}_{t\in\Omega} |\phi(t)|.$$

Also, we let $C(\bar{\Omega})$ denote the set of functions which are bounded and uniformly continuous on Ω. Any function $\phi \in C(\bar{\Omega})$ has a unique continuous extension to the whole of $\bar{\Omega}$, and will henceforth be considered to be defined on $\bar{\Omega}$. $C(\bar{\Omega})$ is a Banach space under the norm

$$\|\phi\|_\infty = \sup_{t\in\Omega} |\phi(t)|.$$

For $m \in \mathbf{N}$, we let $C^m(\bar{\Omega})$ denote the space of all functions $\phi \in C(\bar{\Omega})$, which have the property that

$$\frac{\partial^{|\gamma|}\phi}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}} \in C(\bar{\Omega}),$$

for all multi-indices γ satisfying $|\gamma| \leqslant m$. (We use here the standard notation for multi-indices, see [7, p. 19]). Also, for $0 < \beta \leqslant 1$, we let $\mathrm{Lip}_\beta(\bar{\Omega})$ denote the space of all functions $\phi \in C(\bar{\Omega})$, which have the property that

$$\sup_{\substack{t\in\Omega\\ t+h\in\Omega}} |\phi(t+h) - \phi(t)| \leqslant C|h|^\beta,$$

for all non-zero $h \in \mathbf{R}^2$, with C independent of h. Both $C^m(\bar{\Omega})$ and $\mathrm{Lip}_\beta(\bar{\Omega})$ become Banach spaces when equipped with an appropriate norm [7, p. 25], but the definition of this norm will not be required in this paper. The above spaces are defined analogously when $\Omega \subseteq \mathbf{R}^n$, for any $n \geqslant 2$.

We then introduce the following assumptions on k and f, the given quantities in equation (1.1).

C1.
$$\sup_{t\in\bar{\Omega}} \int_{\bar{\Omega}} |k(t,s)|\,ds < \infty,$$

and

$$\lim_{t\to t'} \int_{\bar{\Omega}} |k(t,s) - k(t',s)|\,ds = 0, \qquad t' \in \bar{\Omega}.$$

C2. The homogeneous version of (1.1),

$$y(t) = \int_{\bar{\Omega}} k(t, s) y(s)\, ds$$

has no non-trivial solutions in $C(\bar{\Omega})$.

C3. $f \in C(\bar{\Omega})$.

REMARK. The assumption C1 is a convenient condition which ensures (see [3]) that K is compact as an operator from $L_\infty(\bar{\Omega})$ to $C(\bar{\Omega})$, and hence, also from $C(\bar{\Omega})$ to $C(\bar{\Omega})$. This fact, combined with C2, C3 and the Fredholm Alternative [6, p. 497] ensures the existence of a unique solution $y \in C(\bar{\Omega})$. Sufficient conditions for any given kernel to satisfy C1 have been investigated in [3].

We then have the following theorem.

THEOREM 1. *Let* C1, C2 *and* C3 *be satisfied. Then, for sufficiently large* N, y_N^{I} *exists in* $L_\infty(\bar{\Omega})$, y_N^{II} *exists in* $C(\bar{\Omega})$, *and*

$$\|y - y_N^{\mathrm{I}}\|_\infty \leqslant C_1 \|y - P_N y\|_\infty \to 0, \quad \text{as } N \to \infty,$$

and

$$\|y - y_N^{\mathrm{II}}\|_\infty \leqslant C_2 \|Ky - KP_N y\|_\infty \to 0, \quad \text{as } N \to \infty,$$

where C_1 *and* C_2 *are independent of* N, *and* P_N *is the interpolatory projection given, for* $\phi \in C(\bar{\Omega})$, *by*

$$P_N \phi = \sum_{i=1}^{N} \phi(t_i) u_i. \tag{2.1}$$

REMARK. From (1.3) and (1.4), it may be easily seen that y_N^{I} and y_N^{II} coincide at the collocation points $\{t_1, \ldots, t_N\}$, and we may think of y_N^{II} as a natural continuous interpolation to the piecewise constant approximation y_N^{I}.

Theorem 1 is a natural starting point from which we may derive order of convergence estimates for $\|y - y_N^{I}\|_\infty$ and $\|y - y_N^{\mathrm{II}}\|_\infty$, in terms of the mesh diameter $\|\Pi_N\|_\infty$. We shall see below that a large part in obtaining good convergence rates is played by the special set of points given by $t_i = (t_{i1}, t_{i2})$, say, where

$$\left.\begin{aligned} t_{i1} &= \frac{1}{A_i} \int_{\Omega_i} s_1\, ds_1\, ds_2, \\ t_{i2} &= \frac{1}{A_i} \int_{\Omega_i} s_2\, ds_1\, ds_2, \end{aligned}\right\} \qquad i = 1, \ldots, N, \tag{2.2}$$

where A_i is the area of Ω_i. That is, t_i is the centroid of Ω_i, for each $i = 1, \ldots, N$.

Then, the following result is a consequence of Theorem 1.

THEOREM 2. *Let* C1, C2 *and* C3 *be satisfied.*

(i) *If* $y \in C^1(\bar{\Omega})$, *then* (1.5) *holds.*

(ii) *If* $y \in C^1(\bar{\Omega})$, $\partial y/\partial t_1$, $\partial y/\partial t_2 \in \mathrm{Lip}_1(\bar{\Omega})$, $k \in \mathrm{Lip}_1(\overline{\Omega \times \Omega})$, *and the collocation points are chosen according to* (2.2), *then* (1.5) *and* (1.6) *hold.*

REMARK. The proof of Theorem 2, which is given in [5], uses Taylor's series methods, and hence requires that both k and y be fairly smooth. If k is weakly singular, however, then k is not even continuous, and it is obvious, before we even consider the regularity of y, that Theorem 2(ii) will be inapplicable to this case. The analogues of (1.5) and (1.6), for the weakly singular case, are proved in Section 4, using approximation theoretic arguments which are more sensitive to the regularity of both k and y than the Taylor's series methods. Before we can prove these analogues we need an accurate characterisation of the regularity of the solution y of (1.1), when k is weakly singular. This is the purpose of Section 3.

3. Regularity results for weakly singular equations

In this section, we describe the regularity of the solution y of (1.1), when k is weakly singular. Throughout this section and Section 4 we shall assume that

$$\bar{\Omega} = [0, 1] \times [0, d],$$

for some $d > 0$.

To define what is meant by a weakly singular kernel, we introduce the new assumption on k:

C1$'$. $k(t, s) = \psi_\alpha(|t - s|)$, for some $0 < \alpha \leqslant 1$, with $\psi_\alpha(x) = B(x)x^{\alpha-1}$, $0 < \alpha < 1$, and $\psi_1(x) = B(x)\ln x$, where $B \in C^1[0, R]$, with $R = \sup_{t,s \in \bar{\Omega}}|t - s|$.

Our regularity theory for y requires that f (the inhomogeneous term of (1.1)) be suitably smooth, and so we also introduce

C3$'$. $f \in C^2(\bar{\Omega})$.

REMARK. It is shown in [5] that C1$'$ implies C1. Since, C3$'$ trivially implies C3, it follows that any results which are true under C1, C2 and C3, will also be true under C1$'$, C2 and C3$'$.

Equations satisfying C1$'$, C2 and C3$'$ are the subject of a detailed singularity analysis in [5]. Among the results proved there, we find the following theorem.

THEOREM 3. *Let* C1′, C2 *and* C3′ *be satisfied. Then the solution* y *of* (1.1) *has the following properties.*

(i) $y \in C^1(\bar{\Omega})$.

(ii) $\partial y/\partial t_1, \partial y/\partial t_2 \in \mathrm{Lip}_\beta(\bar{\Omega})$,

where β *is any number satisfying* $0 < \beta < \alpha$.

It is clear then, that in the case that C1′, C2 and C3′ are satisfied, and $\bar{\Omega} = [0, 1] \times [0, d]$, the conditions of Theorem 2(i) are satisfied, but the conditions of Theorem 2(ii) are not satisfied, either by y or k.

4. Order of convergence estimates for weakly singular equations

In this section, we analyse the rates of convergence to zero of the quantities $\|y - y_N^{\mathrm{I}}\|_\infty$, and $\|y - y_N^{\mathrm{II}}\|_\infty$, in the case when C1′, C2, and C3′ are satisfied, and when $\bar{\Omega} = [0, 1] \times [0, d]$.

When proving boundedness results, we use C to denote a generic constant; the numerical value taken by C may vary from instance to instance.

Since the piecewise constant functions $\{u_1, \ldots, u_N\}$, which were used in the definition of y_N^{I} and y_N^{II}, are really just two dimensional splines (of order 1 or, equivalently, of degree 0) it is reasonable to expect that a tight numerical analysis of y_N^{I} and y_N^{II} will require some two dimensional spline approximation theory. Appealing to Munteanu and Schumaker [8] for such a theory, we must first define a certain family of rectangular meshes on $\bar{\Omega} = [0, 1] \times [0, d]$.

DEFINITION 4. *For each* $\tau \in (0, 1]$, *let there exist integers* $p(\tau)$, $q(\tau)$, *and meshes*

$$\Pi_{p(\tau)}\colon 0 = x_0(\tau) < x_1(\tau) < \cdots < x_{p(\tau)}(\tau) = 1,$$

and

$$\Pi_{q(\tau)}\colon 0 = y_0(\tau) < y_1(\tau) < \cdots < y_{q(\tau)}(\tau) = d,$$

with the property that, for some constants C_1, C_2,

$$C_1\tau \leqslant \underline{\Delta}_i(\tau) \leqslant \bar{\Delta}_i(\tau) \leqslant C_2\tau, \qquad i = 1, 2,\ \tau \in (0, 1],$$

where

$$\underline{\Delta}_1(\tau) = \min_{j=1,\ldots,p(\tau)} (x_j(\tau) - x_{j-1}(\tau)),$$

$$\bar{\Delta}_1(\tau) = \max_{j=1,\ldots,p(\tau)} (x_j(\tau) - x_{j-1}(\tau)),$$

$$\underline{\Delta}_2(\tau) = \min_{l=1,\ldots,q(\tau)} (y_l(\tau) - y_{l-1}(\tau)),$$

and

$$\bar{\Delta}_2(\tau) = \max_{l=1,\ldots,q(\tau)} (y_l(\tau) - y_{l-1}(\tau)).$$

In addition, suppose that, for $0 < \tau \leqslant \frac{1}{2}$,

$$\{x_j(2\tau): j = 0, \ldots, p(2\tau)\} \subseteq \{x_j(\tau): j = 0, \ldots, p(\tau)\},$$

and

$$\{y_l(2\tau): l = 0, \ldots, q(2\tau)\} \subseteq \{y_l(\tau): l = 0, \ldots, q(\tau)\}.$$

Then, with $N(\tau) = p(\tau)q(\tau)$, *we have, for each* $\tau \in (0, 1]$, *a mesh* $\Pi_{N(\tau)}$ *on* $\bar{\Omega}$, *given by*

$$\Pi_{N(\tau)} = \{(x_{j-1}(\tau), x_j(\tau)) \times (y_{l-1}(\tau), y_l(\tau)): j = 1, \ldots, p(\tau); l = 1, \ldots, q(\tau)\}.$$

We call such a family of meshes $\{\Pi_{N(\tau)}: \tau \in (0, 1]\}$ *an "M. S. family of meshes on* Ω".

We shall refer to the mesh $\Pi_{N(\tau)}$ as being made up of the subsets $\Omega_i(\tau)$ (or Ω_i when τ is understood), for $i = 1, \ldots, N(\tau)$, where, for definiteness, we adopt the indexation convention

$$\Omega_{(l-1)p(\tau)+j}(\tau) = (x_{j-1}(\tau), x_j(\tau)) \times (y_{l-1}(\tau), y_l(\tau)),$$

for $j = 1, \ldots, p(\tau)$, and $l = 1, \ldots, q(\tau)$.

REMARK 5. Let $\{\Pi_{N(\tau)}: \tau \in (0, 1]\}$ be an M.S. family of meshes on $\bar{\Omega}$.

(i) For each $i = 1, \ldots, N(\tau)$, the collocation point t_i, defined by (2.2) then turns out to be the centre of the rectangle $\bar{\Omega}_i(\tau)$.

(ii) It is clear from the definition of $\Pi_{N(\tau)}$, that

$$N(\tau) \to \infty \quad \text{as } \tau \to 0,$$

and

$$C_1\tau \leqslant \|\Pi_{N(\tau)}\|_\infty \leqslant C_2\tau,$$

so that

$$\|\Pi_{N(\tau)}\|_\infty \to 0, \quad \text{as } \tau \to 0.$$

(iii) Four examples of M. S. families of meshes are given in [8]. A particularly simple example is the case where, for $\tau \in (0, 1]$, $p(\tau)$ is chosen to be the integer which satisfies

$$\frac{1}{\tau} + \frac{1}{2} \geqslant p(\tau) > \frac{1}{\tau} - \frac{1}{2},$$

and $q(\tau)$ is set equal to $p(\tau)$. Then

$$\Pi_{p(\tau)}: 0 = x_0(\tau) < x_1(\tau) < \cdots < x_{p(\tau)}(\tau) = 1$$

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is given by

$$x_j(\tau) = j\tau, \qquad j = 0, \dots, p(\tau) - 1,$$
$$x_{p(\tau)}(\tau) = 1,$$

and

$$\Pi_{q(\tau)}: 0 = y_0(\tau) < y_1(\tau) < \cdots < y_{q(\tau)}(\tau) = d$$

is given by

$$y_l(\tau) = l\tau d, \qquad l = 0, \dots, q(\tau) - 1,$$
$$y_{q(\tau)}(\tau) = d.$$

(iv) A practically important subfamily of the family given in Remark 5(iii) is

$$\{\Pi_{N(\tau)}: \tau = n^{-1}, n = 1, 2, 3, \dots\}.$$

In this case, for each $\tau = n^{-1}$, we have $p(\tau) = q(\tau) = n$, and the mesh $\Pi_{N(\tau)}$ is just obtained simply by dividing $\bar{\Omega}$ into $N(\tau) = n^2$ subrectangles, each of dimensions $\frac{1}{n}$ by $\frac{d}{n}$.

From now on we shall let $\{\Pi_{N(\tau)}: \tau \in (0, 1]\}$ denote some fixed family of M.S. meshes on $\bar{\Omega}$. We shall let $P_{N(\tau)}$ denote the projection, analogous to (2.1), onto the space spanned by the set of piecewise constant functions defined on the mesh $\Pi_{N(\tau)}$, using the collocation points discussed in Remark 5(i). Then, for any $\tau \in (0, 1]$, $P_{N(\tau)}$ is a bounded operator from $C(\bar{\Omega})$ to $L_\infty(\bar{\Omega})$, with operator norm satisfying

$$\|P_{N(\tau)}\| \leqslant 1, \qquad \tau \in (0, 1]. \tag{4.1}$$

We shall define below a space of spline functions on a rectangular mesh over the two dimensional set $\bar{\Omega} = [0, 1] \times [0, d]$. These will be constructed as pointwise products of one dimensional splines. Thus, for $r \in \mathbf{N}$, and for any mesh

$$\Pi: a = x_0 < x_1 < \cdots < x_n = b$$

on any interval $[a, b]$, we define the one dimensional spline space $S_r(\Pi, [a, b])$ to be the space of all functions in $C^{r-2}[a, b]$, which reduce to polynomials of degree not greater than $r - 1$ on each $(x_{i-1}, x_i]$, for $i = 1, \dots, n$. If $r = 1$ this space includes functions discontinuous at one or more of the knots $\{x_i: i = 0, \dots, n\}$, and for definiteness, we assume that such functions are left continuous at each knot, and right continuous at a. We then define, for $r \in \mathbf{N}$, the two dimensional spline space $S_r(\Pi_{N(\tau)}, \bar{\Omega})$, by

$$S_r\big(\Pi_{N(\tau)}, \bar{\Omega}\big) = \big\{\xi: \xi(s_1, s_2) = \xi_1(s_1)\xi_2(s_2), \text{ for } (s_1, s_2) \in \bar{\Omega}, \text{ where}$$
$$\xi_1 \in S_r(\Pi_{p(\tau)}, [0, 1]), \text{ and } \xi_2 \in S_r(\Pi_{q(\tau)}, [0, d])\big\}.$$

We describe some important approximation theoretic properties of this two dimensional spline space in the next two lemmas.

LEMMA 6. *Let ψ_α be defined as in* C1′. *Then, for each $t \in \bar{\Omega}$, there exists a spline $u_{\alpha,t} \in S_1(\Pi_{N(\tau)}, \bar{\Omega})$, such that*

$$\int_{\bar{\Omega}} |\psi_\alpha(|t-s|) - u_{\alpha,t}(s)|\, ds \leqslant C\|\Pi_{N(\tau)}\|_\infty,$$

with C independent of t and τ.

PROOF. The proof follows from Munteanu and Schumaker [8, Lemma 5.5], where it is shown that there exists $u_{\alpha,t} \in S_1(\Pi_{N(\tau)}, \bar{\Omega})$ such that

$$\int_{\bar{\Omega}} |\psi_\alpha(|t-s|) - u_{\alpha,t}(s)|\, ds \leqslant C\left[\tau\int_{\bar{\Omega}} |\psi_\alpha(|t-s|)|\, ds + \sup_{0<\|h\|_\infty<\tau} \cdot \int_{\bar{\Omega}_h} |\psi_\alpha(|t-s-h|) - \psi_\alpha(|t-s|)|\, ds\right], \tag{4.2}$$

with C independent of t and τ, and where for $\varepsilon \in \mathbf{R}^2$,

$$\bar{\Omega}_\varepsilon = \{s \in \bar{\Omega} : s + \varepsilon \in \bar{\Omega}\}. \tag{4.3}$$

Now, for $t \in \bar{\Omega}$, we have, using C1′,

$$\int_{\bar{\Omega}} |\psi_\alpha(|t-s|)|\, ds \leqslant \begin{cases} \|B\|_\infty \int_{\bar{\Omega}} |t-s|^{\alpha-1}\, ds \leqslant C_1, & 0<\alpha<1, \\ \|B\|_\infty \int_{\bar{\Omega}} \ln|t-s|\, ds \leqslant C_2, & \alpha = 1, \end{cases} \tag{4.4}$$

with C_1, C_2 independent of t. Also, by slightly modifying the arguments of Kantorovich and Akilov [6, Theorem 4, p. 363], it may be shown (for full details see [5]), that

$$\int_{\bar{\Omega}_h} |\psi_\alpha(|t-s-h|) - \psi_\alpha(|t-s|)|\, ds \leqslant C|h|, \tag{4.5}$$

with C independent of t and h.

It follows then, on substitution of (4.4) and (4.5) into (4.2), and noting that $|h| \leqslant \sqrt{2}\,\|h\|_\infty$, that we have

$$\int_{\bar{\Omega}} |\psi_\alpha(|t-s|) - u_{\alpha,t}(s)|\, ds \leqslant C\tau,$$

with C independent of t and τ, and the required result follows from Remark 5(ii).

REMARK 7. Note that, by the triangle inequality, we have, from Lemma 6,

$$\int_{\bar{\Omega}} |u_{\alpha,t}(s)|\, ds \leqslant C\|\Pi_{N(\tau)}\|_\infty + \int_{\bar{\Omega}} |\psi_\alpha(|t-s|)|\, ds \leqslant C,$$

for some C which is independent of t and τ, where the final inequality follows from (4.4), and the observation that

$$\|\Pi_{N(\tau)}\|_\infty \leqslant (1 + d^2)^{1/2}, \qquad \tau \in (0, 1].$$

LEMMA 8. *Let C1′, C2 and C3′ be satisfied, and let y be the solution of* (1.1). *Then there exists a spline $\xi \in S_2(\Pi_{N(\tau)}, \bar{\Omega})$ such that*

$$\|y - \xi\|_\infty \leqslant C\|\Pi_{N(\tau)}\|_\infty^{\beta+1},$$

with C independent of τ.

PROOF. Note that, by Theorem 3, $y \in C^1(\bar{\Omega})$ and $\partial y/\partial t_1$, $\partial y/\partial t_2 \in \mathrm{Lip}_\beta(\bar{\Omega})$, for any β in the range $0 < \beta < \alpha$. It follows from Munteanu and Schumaker [8, Lemma 5.5], that there exists $\xi \in S_2(\Pi_{N(\tau)}, \bar{\Omega})$ such that

$$\|y - \xi\|_\infty \leqslant C\big[\tau^2\|y\|_\infty + \omega_2(y, \tau)\big], \tag{4.6}$$

with C independent of τ, where $\omega_2(y, \tau)$ is the two dimensional modulus of continuity given by

$$\omega_2(y, \tau) = \sup_{0 < \|h\|_\infty < \tau} \ \sup_{t \in \bar{\Omega}_{2h}} |y(t + 2h) - 2y(t + h) + y(t)|,$$

and $\bar{\Omega}_{2h}$ is defined by (4.3).

Now, it follows easily from the two-dimensional Taylor's theorem, and the known properties of y, that

$$\omega_2(y, \tau) \leqslant C\tau^{\beta+1} \tag{4.7}$$

with C independent of τ, and the required result follows on substitution of (4.7) into (4.6).

The next lemma highlights an important property of the choice of collocation points given in (2.2) and Remark 5(i).

LEMMA 9. *Let $\xi \in S_2(\Pi_{N(\tau)}, \bar{\Omega})$. Then*

$$\int_{\bar{\Omega}_i(\tau)} (\xi(s) - \xi(t_i))\, ds = 0, \qquad i = 1, \ldots, N(\tau).$$

REMARK. This result demonstrates the special role played by the points (2.2), and one consequence of it is the fact that the two dimensional approximate integration rule

$$\int_{\bar{\Omega}} \phi(s)\, ds \simeq \sum_{i=1}^{N(\tau)} \phi(t_i) \int_{\bar{\Omega}_i(\tau)} ds$$

turns out to be exact for functions in $S_2(\Pi_{N(\tau)}, \bar{\Omega})$, i.e. for functions which reduce to bilinear functions almost everywhere on each of the subsets $\bar{\Omega}_i$. In other words, this approximate integration rule is the two dimensional analogue of the product mid-point rule.

Proof of Lemma 9. Note that, for all $s \in \Omega_i$, and thus, for almost all $s \in \bar{\Omega}_i$, we have $\xi(s) = \xi_1(s_1)\xi_2(s_2)$, where ξ_1 and ξ_2 are linear. Note also that, by Remark 5(i), t_i is the mid point of Ω_i. Thus, to prove this lemma, it would be sufficient to show that

$$\int_0^d \int_0^1 ((a_1 s_1 + b_1)(a_2 s_2 + b_2) - (a_1 t_1 + b_1)(a_2 t_2 + b_2))\, ds_1 ds_2 = 0,$$

where a_1, b_1, a_2, b_2 are constants, and $t_1 = \frac{1}{2}$, $t_2 = \frac{d}{2}$.

Now,

$$\begin{aligned} \int_0^d \int_0^1 &((a_1 s_1 + b_1)(a_2 s_2 + b_2) - (a_1 t_1 + b_1)(a_2 t_2 + b_2))\, ds_1 ds_2 \\ &= a_1 a_2 \int_0^d \int_0^1 (s_1 s_2 - t_1 t_2)\, ds_1 ds_2 + a_1 b_2 \int_0^d \int_0^1 (s_1 - t_1)\, ds_1 ds_2 \\ &\quad + a_2 b_1 \int_0^d \int_0^1 (s_2 - t_2)\, ds_1 ds_2, \end{aligned} \tag{4.8}$$

and

$$\int_0^d \int_0^1 (s_1 - t_1)\, ds_1 ds_2 = d \int_0^1 (s_1 - t_1)\, ds_1 = 0, \tag{4.9}$$

since $t_1 = \frac{1}{2}$, and, similarly,

$$\int_0^d \int_0^1 (s_2 - t_2)\, ds_1 ds_2 = 0. \tag{4.10}$$

Also,

$$\begin{aligned} \int_0^d \int_0^1 (s_1 s_2 - t_1 t_2)\, ds_1 ds_2 &= \int_0^d \left(\frac{1}{2} s_2 - t_1 t_2 \right) ds_2 \\ &= \frac{1}{4} d^2 - d t_1 t_2 = 0, \end{aligned} \tag{4.11}$$

and the required result follows on combination of (4.9), (4.10) and (4.11) with (4.8).

Let $y^{\mathrm{I}}_{N(\tau)}$, $y^{\mathrm{II}}_{N(\tau)}$ denote the approximations to y defined by the collocation and iterated collocation methods respectively, using the M.S. family of meshes $\{\Pi_{N(\tau)}: \tau \in (0, 1]\}$. Note that, in view of Remark 5(ii), Theorem 1 holds with N replaced by $N(\tau)$, provided that the phrase "N sufficiently large" is replaced by the phrase "τ sufficiently small". This fact will be used in the proof of the following theorem, which is the main result of the paper.

THEOREM 10. *Let* C1′, C2 *and* C3′ *be satisfied. Then*

(i) $$\|y - y^{\mathrm{I}}_{N(\tau)}\|_\infty = O(\|\Pi_{N(\tau)}\|_\infty),$$

and

(ii) $$\|y - y^{\mathrm{II}}_{N(\tau)}\|_\infty = O\big(\|\Pi_{N(\tau)}\|_\infty^{\beta+1}\big),$$

for any β *in the range* $0 < \beta < \alpha$.

PROOF. By definition of the projection $P_{N(\tau)}$, it follows that

$$\|y - P_{N(\tau)}y\|_\infty \leqslant \sup_{\substack{t,\, t' \in \bar{\Omega} \\ \|t - t'\|_\infty < \|\Pi_{N(\tau)}\|_\infty}} |y(t) - y(t')|,$$

and since it was proved in Theorem 3 that $y \in C^1(\bar{\Omega})$, an easy application of Taylor's theorem yields

$$\|y - P_{N(\tau)}y\|_\infty \leqslant C\|\Pi_{N(\tau)}\|_\infty, \tag{4.12}$$

with C independent of τ, and the result (i) follows from Theorem 1.

To obtain (ii), recall Lemma 6, and write, for $t \in \bar{\Omega}$,

$$\begin{aligned} Ky(t) - KP_{N(\tau)}y(t) &= \int_{\bar{\Omega}} \psi_\alpha(|t - s|)(y(s) - P_{N(\tau)}y(s))\, ds \\ &= \int_{\bar{\Omega}} (\psi_\alpha(|t - s|) - u_{\alpha,t}(s))(y(s) - P_{N(\tau)}y(s))\, ds \\ &\quad + \int_{\bar{\Omega}} (u_{\alpha,t}(s))(y(s) - P_{N(\tau)}y(s))\, ds \\ &= I_1(t) + I_2(t), \quad \text{say.} \end{aligned} \tag{4.13}$$

Now, using Hölder's inequality, we have, for $t \in \bar{\Omega}$,

$$|I_1(t)| \leqslant \int_{\bar{\Omega}} |\psi_\alpha(|t - s|) - u_{\alpha,t}(s)|\, ds \|y - P_{N(\tau)}y\|_\infty,$$

and it follows from Lemma 6, and (4.12), that

$$|I_1(t)| \leqslant C\|\Pi_{N(\tau)}\|_\infty^2, \tag{4.14}$$

with C independent of t and τ.

Also, since $u_{\alpha,t} \in S_1(\Pi_{N(\tau)}, \bar{\Omega})$, we have, for some $c_1, \ldots, c_n$ which are constant with respect to s,

$$\begin{aligned} I_2(t) &= \sum_{i=1}^{N(\tau)} c_i \int_{\bar{\Omega}_i} (y(s) - P_{N(\tau)}y(s))\, ds \\ &= \sum_{i=1}^{N(\tau)} c_i \int_{\bar{\Omega}_i} (I - P_{N(\tau)})(y - \xi)(s)\, ds, \end{aligned}$$

where ξ is any function in $S_2(\Pi_{N(\tau)}, \bar{\Omega})$, using Lemma 9. Thus, making use again of Hölder's inequality, we have, for $t \in \bar{\Omega}$,

$$\begin{aligned} |I_2(t)| &= \left| \int_{\bar{\Omega}} u_{\alpha,t}(s)(I - P_{N(\tau)})(y - \xi)(s)\, ds \right| \\ &\leqslant \int_{\bar{\Omega}} |u_{\alpha,t}(s)|\, ds \|(I - P_{N(\tau)})(y - \xi)\|_\infty \\ &\leqslant C\|y - \xi\|_\infty, \end{aligned}$$

with C independent of t and τ, where we have used (4.1) and Remark 7. Thus, by Lemma 8,

$$|I_2(t)| \leqslant C\|\Pi_{N(\tau)}\|_\infty^{\beta+1}, \tag{4.15}$$

with C independent of t and τ, and β any number such that $0 < \beta < \alpha$.

Combining (4.14) and (4.15) with (4.13) we obtain

$$\|Ky - KP_{N(\tau)}y\|_\infty \leqslant C\|\Pi_{N(\tau)}\|_\infty^{\beta+1},$$

with C independent of τ, and (ii) then follows from Theorem 1.

5. A numerical example

In this section we solve numerically the integral equation,

$$y(t_1, t_2) = C_0 + \lambda \int_0^d \int_0^1 \ln\sqrt{(t_1 - s_1)^2 + (t_2 - s_2)^2}\, y(s_1, s_2)\, ds_1 ds_2,$$
$$\text{for } (t_1, t_2) \in [0, 1] \times [0, d], \tag{5.1}$$

where C_0 and λ are scalars and $d > 0$. This equation arises in the problem of determining the cross-sectional distribution of current in an infinitely long rectangular conducting bar which carries an alternating current [4]. Let the cross section of the conductor have length a, and breadth b. Let g denote the conductivity of the material of the bar, let μ denote the permeability of free space, and let ω denote the angular frequency of the alternating current. Then it is shown in [4] that the parameters λ and d depend on these quantities, and are given by

$$\lambda = (a^2\mu g\omega/2\pi)i,$$
$$d = b/a.$$

It is also shown in [4] that C_0 has no physical significance, and may be considered to be a scaling factor in the problem. To understand this point more clearly, note that if we modify (5.1) by replacing C_0 by $2C_0$, then the solution to the equation thus obtained is merely twice the solution of (5.1).

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We have solved equation (5.1) by the collocation and iterated collocation methods, using the family of meshes $\{\Pi_{N(\tau)}: \tau = n^{-1}, n = 2, 3, 4, 5, 6, 7, 8\}$ over $[0, 1] \times [0, d]$, which were described in Remark 5(iv). That is, for each $\tau = 2^{-1}$, $3^{-1}, \ldots, 8^{-1}$, $y^{\mathrm{I}}_{N(\tau)}$ and $y^{\mathrm{II}}_{N(\tau)}$ where defined by (1.3) and (1.4) using the mesh obtained by dividing $[0, 1] \times [0, d]$ into n^2 subrectangles, each of dimension $\frac{1}{n}$ by $\frac{d}{n}$. The collocation points $\{t_i: i = 1, \ldots, N(\tau)\}$ were chosen to be the mid-points of each subrectangle as described in Remark 5(i). For this problem, the numerical calculation of $y^{\mathrm{I}}_{N(\tau)}$ and $y^{\mathrm{II}}_{N(\tau)}$ was facilitated by the fact that, due to the relative simplicity of the logarithmic kernel, the necessary two dimensional integrals $\{Ku_i(t): i = 1, \ldots, N(\tau)\}$, for $t \in [0, 1] \times [0, d]$ can be given analytically.

We shall set $d = 0.5$, and $\lambda = 0.266425 \times 10^2$. These values correspond to the particular physical situation of a copper bar, of cross-section 0.1 by 0.05, carrying an alternating current of frequency 60 (all units being in the RMKS system). We fix the scaling factor C_0 so that the average of the values of the piecewise constant approximation, $y^{\mathrm{I}}_{N(5^{-1})}$ over the domain $[0, 1] \times [0, d]$ is unity.

We shall display here only our results for $y^{\mathrm{II}}_{N(\tau)}$, $\tau = 2^{-1}, 4^{-1}, 6^{-1}$ and 8^{-1}, and we shall use them to obtain experimental rates of convergence, for comparison with the theoretical estimates of Theorem 10(ii). We do not display results for $y^{\mathrm{I}}_{N(\tau)}$, but note that $y^{\mathrm{I}}_{N(\tau)}$ coincides with $y^{\mathrm{II}}_{N(\tau)}$ at the collocation points, and hence converges at the same rate as $y^{\mathrm{II}}_{N(\tau)}$ at those points. Further numerical results are given in [5]. The same problem has been solved by a different method in [2]. In Table 1 we give the values of $y^{\mathrm{II}}_{N(\tau)}$ at the four points $(0, 0)$, $(\frac{1}{2}, 0)$, $(0, \frac{d}{2})$, $(\frac{1}{2}, \frac{d}{2})$ of the domain $[0, 1] \times [0, d]$.

TABLE 1

Values of $y^{\mathrm{II}}_{N(\tau)}$

τ	$y^{\mathrm{II}}_{N(\tau)}$ at $(0, 0)$	$y^{\mathrm{II}}_{N(\tau)}$ at $(\frac{1}{2}, 0)$	$y^{\mathrm{II}}_{N(\tau)}$ at $(0, \frac{d}{2})$	$y^{\mathrm{II}}_{N(\tau)}$ at $(\frac{1}{2}, \frac{d}{2})$
2^{-1}	$1.270 + 5.857i$	$0.9466 + 1.062i$	$1.140 + 3.931i$	$0.7277 - 2.186i$
4^{-1}	$1.900 + 4.421i$	$1.026 + 1.244i$	$1.399 + 2.347i$	$-0.2349 - 0.4783i$
6^{-1}	$2.188 + 4.055i$	$1.166 + 1.171i$	$1.680 + 1.840i$	$-0.1187 - 0.3580i$
8^{-1}	$2.327 + 3.929i$	$1.211 + 1.134i$	$1.841 + 1.647i$	$-0.09286 - 0.3467i$

According to Theorem 10, if $\tau = n^{-1}$, we have the theoretical prediction

$$\|y - y^{\mathrm{II}}_{N(\tau)}\|_\infty \leqslant C\|\Pi_{N(\tau)}\|_\infty^{\beta+1} = C\frac{1}{n^{\beta+1}}, \tag{5.2}$$

for any β in the range $0 < \beta < 1$. To estimate the experimental rate of convergence, we conjecture that

$$\left(y - y^{\mathrm{II}}_{N(\tau)}\right)(t) = c(t)\frac{1}{n^{\lambda}}, \tag{5.3}$$

where the $c(t)$ are complex constants which depend on $t \in [0, 1] \times [0, d]$, and $\lambda > 0$ is to be determined. Using the computed values of $y^{\mathrm{II}}_{N(\tau)}$ for $\tau = 2^{-1}, 4^{-1}, 6^{-1}$ and (5.3), we obtain three equations in the unknowns $y(t)$, $c(t)$ and λ. Eliminating $y(t)$ and $c(t)$, we obtain a non-linear equation in λ, which we solve using the secant method. A second approximation to λ is obtained in the same way using the numerical values of $y^{\mathrm{II}}_{N(\tau)}$, for $\tau = 4^{-1}, 6^{-1}, 8^{-1}$. The approximate values of λ thus obtained are given in Table 2. The values of λ obtained by the first approximation are rather erratic in comparison to the prediction (5.2). This is possibly because the asymptotic convergence rate proposed in (5.3) will only hold for sufficiently large values of n. The values of λ obtained by the second approximation conform more satisfactorily to the prediction (5.2), at least in the cases of points $(0, 0)$, $(\frac{1}{2}, 0)$ and $(0, \frac{d}{2})$. The exceptionally high value of λ at $(\frac{1}{2}, \frac{d}{2})$ may be seen as evidence that the global prediction, (5.2), although probably sharp on the edges of the domain (where the solution is singular), is likely to be pessimistic at points in the interior of the domain (where the solution is smooth).

TABLE 2
Approximations to λ

	At point $(0, 0)$	At point $(\frac{1}{2}, 0)$	At point $(0, \frac{d}{2})$	At point $(\frac{1}{2}, \frac{d}{2})$
1st approx. to λ using $y^{\mathrm{II}}_{N(2^{-1})}, y^{\mathrm{II}}_{N(4^{-1})}, y^{\mathrm{II}}_{N(6^{-1})}$	1.3	$(\simeq 0.0)$	0.9	3.3
2nd approx. to λ using $y^{\mathrm{II}}_{N(4^{-1})}, y^{\mathrm{II}}_{N(6^{-1})}, y^{\mathrm{II}}_{N(8^{-1})}$	1.6	1.9	1.4	4.0

For each of the four points t, the second approximation to λ (given in Table 2) was used along with the values of $y^{\mathrm{II}}_{N(6^{-1})}(t)$ and $y^{\mathrm{II}}_{N(8^{-1})}(t)$ to calculate the constants $c(t)$ in (5.3). The value of $c(t)$ was then used to estimate the maximum absolute error in $y^{\mathrm{II}}_{N(8^{-1})}$. The results are given in Table 3.

TABLE 3

Estimated errors in $y^{II}_{N(\delta^{-1})}$

	At point $(0, 0)$	At point $(\frac{1}{2}, 0)$	At point $(0, \frac{d}{2})$	At point $(\frac{1}{2}, \frac{d}{2})$
Estimated error in $y^{II}_{N(\delta^{-1})}$	0.32	0.080	0.12	0.013

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References

[1] G. A. Chandler, "Product integration methods for weakly singular second kind integral equations", submitted for publication.

[2] F. A. Dewar, *Solution of Fredholm integral equations by the Galerkin and iterated Galerkin methods* (M. Sc. Thesis, University of New South Wales, 1980).

[3] I. G. Graham and I. H. Sloan, "On the compactness of certain integral operators", *J. Math. Anal. Appl.* 68 (1979), 580–594.

[4] I. G. Graham, "Some application areas for Fredholm integral equations of the second kind", in *The application and numerical solution of integral equations* (eds. R. S. Anderssen, F. R. de Hoog, and M. A. Lukas) (Alphen aan den Rijn: Sijthoff and Noordhoff, 1980).

[5] I. G. Graham, *The numerical solution of Fredholm integral equations of the second kind* (Ph. D. Thesis, University of New South Wales, 1980).

[6] L. V. Kantorovich and G. P. Akilov, *Functional analysis in normed spaces* (London/New York: Pergamon, 1964).

[7] A. Kufner, O. John, and S. Fucik, *Function spaces* (Leyden: Noordhoff International, 1977).

[8] M. J. Munteanu and L. L. Schumaker, "Direct and inverse theorems for multidimensional spline approximation", *Indiana Univ. Math. J.* 23 (1973), 461–470.

[9] C. Schneider, "Product integration for weakly singular integral equations", *Math. of Comp.*, to appear.

[10] I. H. Sloan, E. Noussair and B. J. Burn, "Projection methods for equations of the second kind", *J. Math. Anal. Appl.* 69 (1979), 84–103.

School of Mathematics
University of New South Wales
Kensington
NSW 2033