

Scale-dependent anisotropy, energy transfer and intermittency in bubble-laden turbulent flows

Tian Ma^{1,2,†}, Bernhard Ott³, Jochen Fröhlich³ and Andrew D. Bragg^{1,†}

¹Department of Civil and Environmental Engineering, Duke University, Durham, NC 27708, USA

²Helmholtz-Zentrum Dresden – Rossendorf, Institute of Fluid Dynamics, 01328 Dresden, Germany

³Technische Universität Dresden, Institute of Fluid Mechanics, 01062 Dresden, Germany

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Data from direct numerical simulations of disperse bubbly flows in a vertical channel are used to study the effect of the bubbles on the carrier-phase turbulence. We developed a new method, based on an extension of the barycentric map approach, that allows us to quantify and visualize the anisotropy and componentiality of the flow at any scale. Using this we found that the bubbles significantly enhance anisotropy in the flow at all scales compared with the unladen case, and that for some bubble cases, very strong anisotropy persists down to the smallest scales of the flow. The strongest anisotropy observed was for the cases involving small bubbles. Concerning the energy transfer among the scales of the flow, our results indicate that for the bubble-laden cases, the energy transfer is from large to small scales, just as for the unladen case. However, there is evidence of an upscale transfer when considering the transfer of energy associated with particular components of the velocity field. Although the direction of the energy transfer is the same with and without the bubbles, the behaviour of the energy transfer is significantly modified by the bubbles, suggesting that the bubbles play a strong role in altering the activity of the nonlinear term in the flow. The skewness of the velocity increments also reveals a strong effect of the bubbles on the flow, changing both its sign and magnitude compared with the single-phase case. We also consider the normalized forms of the fourth-order structure functions, and the results reveal that the introduction of bubbles into the flow strongly enhances intermittency in the dissipation range, but suppresses it at larger scales. This strong enhancement of the dissipation-scale intermittency has significant implications for understanding how the bubbles might modify the mixing properties of turbulent flows.

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† Email addresses for correspondence: tian.ma@hzdr.de, andrew.bragg@duke.edu

1. Introduction

Turbulence and multiphase flows are two of the most challenging topics in fluid mechanics and when combined they pose a formidable challenge, even in the dilute dispersed regime (Balachandar & Eaton 2010). The focus here is on liquid flows laden with dispersed bubbles, which can be particularly challenging since the bubbles can strongly alter the liquid phase turbulence (Mudde 2005; Lohse 2018; Elghobashi 2019). In particular, the bubbles can modify the turbulence due to production effects arising from the bubble wakes (Riboux, Risso & Legendre 2010; Lai & Socolofsky 2019), enhanced local turbulent kinetic energy dissipation rates in the vicinity of the bubble surfaces (Santarelli, Roussel & Fröhlich 2016; Masuk, Salibindla & Ni 2021) and modulation of the liquid mean velocity profile due to interphase momentum transfer, resulting in an alteration of shear-induced turbulence (Lu & Tryggvason 2013; du Cluzeau, Bois & Toutant 2019; Cifani, Kuerten & Geurts 2020; Bragg *et al.* 2021). Mathai, Lohse & Sun (2020) highlighted particular ways in which the classical scenario for single-phase turbulence, based on single-point statistical analysis, is modified due to the bubbles moving relative to the fluid. Turbulence arising from this relative motion is often referred to as bubble-induced turbulence (BIT) and its effects can be captured in the Reynolds-averaged Navier–Stokes modelling framework through the inclusion of additional source terms in the relevant transport equations (Fox 2014; Joshi & Nandakumar 2015; Ma 2017; Liao *et al.* 2019; Ma, Lucas & Bragg 2020a).

While significant progress has been made in understanding and characterizing how the bubbles influence the single-point turbulence statistics of the liquid phase, less attention has been paid to the influence of the bubbles on the multiscale/multipoint flow statistics. Those that have considered this aspect have only focused on the kinetic energy spectrum of the liquid velocity fluctuations, with a key observation being that in BIT dominated flows, a power-law behaviour for the energy spectrum arises with exponent -3 in both the wavenumber and frequency domain (Lance & Bataille 1991; Roghair *et al.* 2011; Mendez-Diaz *et al.* 2013; Ma *et al.* 2017). This behaviour was also reported in Pandey, Ramadugu & Perlekar (2020), who investigated the energy budget equations in wavenumber space and explained the -3 slope as arising due to a balance between kinetic energy production due to the bubbles and viscous dissipation. In their analysis they evaluated the nonlinear scale-to-scale energy flux term for bubbly flows, showing a forward (downscale) energy cascade, just as also occurs for single-phase turbulence in three dimensions. An issue with their analysis, however, is that they included the values of the flow at grid points occupied by the bubbles when evaluating the statistics of the carrier phase, thereby contaminating the fluid statistics. A similar finding concerning an average forward energy cascade was reported by Lai *et al.* (2018), who performed direct numerical simulation (DNS) of bubbles rising vertically in a channel flow filled with initially quiescent water. They also showed that the energy cascade is highly anisotropic, and that there is a strong inverse energy cascade for eddies orientated vertically in the flow.

There have been very few studies exploring the multiscale properties of bubble-laden turbulent flows in physical space, e.g. using structure function analysis. Rensen, Luther & Lohse (2005) performed hot-film anemometry measurements in the Twente water tunnel and computed the longitudinal second- and fourth-order structure functions with the aid of Taylor's hypothesis. They found an increase of the second-order structure function for the two-phase case compared with the single-phase case under the same bulk Reynolds number, and that this increase was more pronounced at the small scales than the large scales. Their fourth-order structure function results revealed an increase of the intermittency at the small scales of the flow when the flow contained bubbles, even for a relatively low gas void fraction (0.5%). Similar behaviour was also observed in Biferale

et al. (2012) when comparing the small-scale properties of boiling and non-boiling convective turbulent flows.

An important aspect yet to be quantified is how the bubbles affect the anisotropy of the flow at different scales. For single-phase turbulence, the energy containing scales in many flows such as those with shear, rotation and buoyancy are anisotropic (Biferale & Procaccia 2005). Phenomenological theories of turbulence predict a return to isotropy at small enough scales (Kolmogorov 1941*b*; Frisch 1995; Sreenivasan & Antonia 1997). However, measurements have revealed persistent small-scale anisotropy (Pumir & Shraiman 1995; Shen & Warhaft 2000; Ouellette *et al.* 2006; Carter & Coletti 2017). In contrast to single-phase flow, where energy is often injected into the flow at large scales, bubbles can inject energy into the flow at the scale of their size, which usually corresponds to the small scales of the turbulence. Since the bubbles have a preferential direction of motion due to buoyancy, this could lead to the injection of strong anisotropy into the flow at the small scales, leading to strong departures from the behaviour of the single-phase case. The study of Pandey *et al.* (2020) was based on Fourier space analysis with averaging over spherical shells in wavevector space, and so did not permit them to explore the anisotropy of the flow at different scales.

Another important point is that, in Pandey *et al.* (2020), the flow had no background turbulence (i.e. all the turbulence was generated by the bubbles), and hence it was not possible to consider how the bubbles modify the turbulence compared with the single-phase case. In order to more fully understand how the bubbles modify the properties of the turbulence, it is desirable to consider a configuration in which the unladen flow is already turbulent, and then one can explore how the bubbles modify the properties of the turbulence when they are introduced.

In the present work we seek to advance the understanding of the properties of bubble-laden turbulent flows across its range of scales. To do this, data from DNS of finite-size bubbles in a turbulent channel flow with a prescribed bulk Reynolds number are utilized, for different bubble sizes and for mono- and bidisperse cases. A new method is developed based on the barycentric map (Banerjee *et al.* 2007), and applied to the DNS data to quantify the anisotropy of the bubble-laden turbulence flow across the range of its scales. By computing the structure functions of various orders, the direction-dependent liquid velocity fluctuations are also explored at different scales, as is the scale-to-scale energy transfer and intermittency. These results provide new insights into the properties of bubble-laden turbulent flows, and how they differ from the single-phase counterpart at different scales in the flow.

2. Direct numerical simulations

2.1. Database

The DNS data we use are from the studies of Santarelli & Fröhlich (2015, 2016) who simulated the motion of many thousands of bubbles at low Eötvös number in a vertical turbulent channel flow. The bubbles are handled using an immersed boundary method, and are modelled as rigid spherical objects with a no-slip condition enforced at their surface, representing the behaviour of air bubbles rising in contaminated water. Compared with other simulations of this type (see the related references in Mathai *et al.* 2020), these simulations are substantially closer to applications in that they involve a turbulent background flow, contaminated fluid, realistic density ratio ρ^G/ρ^L (where ρ^G and ρ^L are the gas and liquid densities, respectively), higher bubble Reynolds number, a much larger domain and a much larger number of bubbles.

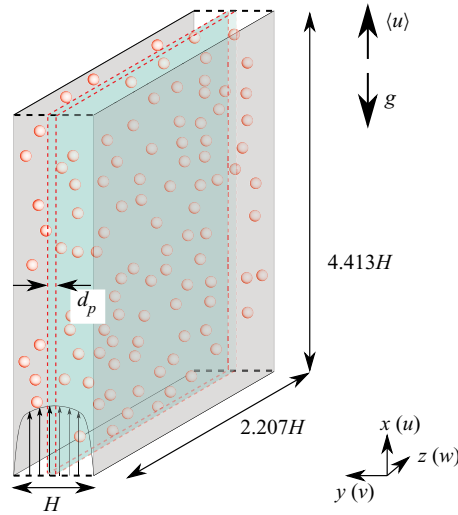


Figure 1. Schematic representation of the DNS configuration (not to scale). The marked region shows the location where the structure functions are calculated. (Main part of the picture reprinted from Ma *et al.* 2017.)

As shown in figure 1, the vertical flow takes place between two flat walls separated by the distance H , and the size of the computational domain is $L_x \times L_y \times L_z = 4.41H \times H \times 2.21H$. Here, x denotes the streamwise coordinate, y the wall-normal coordinate and z the spanwise coordinate, and the corresponding unit basis vectors are $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$, respectively. The numerical grid employed has the same spacing $\Delta = H/232$ in all directions, resulting in $1024 \times 232 \times 512$ grid points in the x, y and z directions, respectively. A no-slip condition was applied at the walls, and periodic boundary conditions were applied in the x and z directions. The gravitational acceleration acts in the direction $-\mathbf{e}_x$ with magnitude g , and the bulk velocity U_b was kept constant by instantaneously adjusting a volume force, equivalent to a pressure gradient, thus imposing a desired bulk Reynolds number $Re_b = U_b H / \nu$, where ν is the kinematic viscosity of the fluid. The DNS were all conducted with $Re_b = 5263$.

The data used in this work were obtained for three monodisperse cases (SmMany, SmFew, LaMany) and one bi-disperse case labelled BiDisp, of the same void fraction as SmMany and LaMany with half the void fraction consisting of smaller bubbles and the other half of larger bubbles. Additionally, a single-phase simulation labelled Unladen was performed under the same conditions for comparison. Table 1 provides an overview of all cases with the corresponding labels.

The ratios of Δ/η for all the cases are given table 1, where η is the Kolmogorov scale based on the averaged dissipation in the channel centre obtained in Santarelli & Fröhlich (2016). For single-phase turbulence, $\Delta/\eta \lesssim 2.1$ is considered the requirement for a DNS to be well resolved in terms of accurately capturing the dissipation rate in the flow (Pope 2000). Based on this criterion, our DNS are well resolved. For accurately capturing high-order moments of small-scale, single-phase turbulence, it is known that finer grids are required, and this likely also applies to bubble-laden turbulent flows. Given these resolution constraints, we restrict our focus in this paper to moments of order four or less. We expect that the grid resolutions of the DNS are sufficient to fully resolve the second- and third-order flow statistics which are the main focus of our paper. For the fourth-order moments, it is possible that the DNS is slightly under-resolved and these

Parameter	Unladen	SmMany	SmFew	LaMany	BiDisp(Sm)	BiDisp(La)
N_p	—	2880	384	913	1440	546
α_b	—	2.14 %	0.29 %	2.14 %	1.07 %	1.07 %
d_p/H	—	0.052	0.052	0.076	0.052	0.076
Ar	—	38171	38171	114528	38171	114528
Re_p	—	235.5	268.3	475.2	233.6	463.6
C_D	—	0.89	0.705	0.666	0.93	0.703
Δ/η	0.41	1.5	0.99	1.8		1.7

Table 1. Parameters of the cases used for the present study according to Santarelli & Fröhlich (2016). The labels *BiDisp(Sm)* and *BiDisp(La)* denote the results for the bi-disperse case, *BiDisp*, where averaging has been restricted to small and large, respectively. Here, N_p is the number of bubbles, α_b is the bulk void fraction, d_p the bubble diameter, $Ar \equiv |\rho^G - \rho^L|gd_p^3/(\rho^L v^2)$ the Archimedes number. The values of Re_p , the bubble Reynolds number based on d_p and the bubble to fluid relative velocity, as well as C_D the drag coefficient, are obtained from the DNS; Δ/η is the ratio of grid resolution and Kolmogorov length $\eta \equiv (v^3/\epsilon)^{1/4}$.

results must therefore be treated with some caution. However, as will be shown later, the SmFew case is the one which displays greatest flow intermittency at the small scales, but this case has $\Delta/\eta = 0.99$, which is more than twice as fine as required to resolve the dissipation, and may be sufficient for resolving the fourth-order moment. A study similar to Yeung, Sreenivasan & Pope (2018) will need to be conducted in future work in order to precisely consider the effect of the grid resolution on higher-order statistics in bubble-laden turbulent flows. Another related point is that in Dodd & Jofre (2019) it was shown that the viscous length scale of the flow at a droplet interface in a turbulent flow, $\delta_v = v\sqrt{\rho^G/\tau_\Sigma}$ (τ_Σ is the mean interfacial shear stress), can be less than η , and that the grid resolution must be sufficiently fine to resolve this length scale. However, since δ_v is small, $\alpha_b \leq 2.14\%$, and we only consider flow data at points where there are no bubbles (see below), then the contribution to the flow data we are analysing arising from flow in the viscous layer around the bubbles in our flow will be very small. Hence, in the regions the flow that are the focus of our analysis, the grid resolution criterion is determined by η , not δ_v . If, by contrast, we were to consider the flow properties close to the surface of the bubbles, such as was considered in Dodd & Jofre (2019), then it would indeed be necessary to ensure that the grid is sufficiently fine to resolve δ_v .

2.2. Data processing

A standard way to analyse the multiscale properties of turbulence is to use the fluid velocity increments $\Delta \mathbf{u}'(\mathbf{x}, \mathbf{r}, t) \equiv \mathbf{u}'(\mathbf{x} + \mathbf{r}, t) - \mathbf{u}'(\mathbf{x}, t)$, where $\mathbf{u}' \equiv \mathbf{u} - \langle \mathbf{u} \rangle$ is the fluctuating fluid velocity, \mathbf{r} is the separation vector and $\langle \cdot \rangle$ denotes an ensemble average (estimated using appropriate space and time averages). The calculation of velocity increments in a bubble-laden flow is, however, delicate, since the phase boundaries can interrupt the fluid flow signal. To overcome this non-continuous velocity signal challenge, different methods have been used in the literature, such as smoothing the discontinuities by a Gauss function (Lance & Bataille 1991); considering only intervals between bubbles where the velocity signal is continuous (Martínez *et al.* 2010; Roghair *et al.* 2011; Mendez-Díaz *et al.* 2013); and measuring the wake behind a rising swarm of bubbles, where there are no bubbles (Riboux *et al.* 2010). Here, we use a method ideally suited for interface-resolved DNS of disperse flows proposed in our previous study (Ma *et al.* 2017). In this method, the fluid velocity is recorded along grid lines in the spanwise direction whose wall-normal location

lies within the centre region $0.474H < y < 0.526H$ highlighted in [figure 1](#). This width corresponds to the smaller bubble diameter ([table 1](#)), which is a sufficiently thin region for these lines to be considered statistically equivalent. For each line, data were recorded whenever the entire line was free from bubbles, and we recorded 1 000 000 instances of this for each case. With this method, one cannot compute $\Delta \mathbf{u}'(\mathbf{x}, \mathbf{r}, t)$ for arbitrary \mathbf{r} , but can compute $\Delta \mathbf{u}'(\mathbf{x}, r_3 \mathbf{e}_3, t)$, allowing us to perform an extensive investigation into the scale-dependent properties of bubble-laden turbulent flows. Note that in the data analysis that follows, $\langle \mathbf{u} \rangle$ is computed based only on these lines of data, and therefore does not correspond to the average over the entire flow field. The mean velocity makes a negligible direct contribution to the statistics of $\Delta \mathbf{u}'(\mathbf{x}, r_3 \mathbf{e}_3, t)$ since the mean velocity is almost constant over the region $0.474H < y < 0.526H$ in which the data are collected.

To test for statistical convergence, we computed higher-order moments of $\Delta \mathbf{u}'(\mathbf{x}, r_3 \mathbf{e}_3, t)$ over the first and second halves of the dataset consisting of 1 000 000 lines of data. The results showed negligible differences, indicating that the 1 000 000 realizations are sufficient for statistical convergence of the results.

2.3. Reynolds number in bubble-laden turbulent flows

Before analysing the properties of the bubble-laden turbulent channel flows at different scales we first consider the Reynolds numbers of the different cases, since this gives insight into the range of excited scales of motion in the turbulent flow. As mentioned in § 2, the bulk Reynolds number $Re_b = U_b H / \nu$ was kept fixed at $Re_b = 5263$. However, since we are interested in the properties of the fluctuating component of the velocity field, it is more informative for our purposes to consider a Reynolds number based on the fluctuating velocity field. To that end we consider the Reynolds number $Re_H \equiv u^* H / \nu$, where $u^* \equiv \sqrt{(2/3)k_c}$, and k_c is the turbulent kinetic energy (TKE) $k \equiv (1/2)\langle u'_i u'_i \rangle$ evaluated at the channel centre.

In [figure 2](#) we plot Re_H vs A_2 , where $A_2 = a_{ji} a_{ij}$ is the second invariant of the Reynolds-stress anisotropy tensor (evaluated at the channel centre), $a_{ij} = \langle u'_i u'_j \rangle / k - (2/3)\delta_{ij}$, that quantifies the magnitude of the large-scale anisotropy in the flow. Plotting the results in this way gives further insight into how the range of excited scales in the flow is also related to the flow anisotropy. The results show that Re_H varies significantly across the cases. In the SmFew case, it is only slightly larger than the unladen case as the bubble size d_p is small and the bulk void fraction α_b is quite low. However, as d_p and α_b are increased, Re_H increases significantly, implying that as d_p and α_b are increased (at least over the range we consider), the range of excited scales in the flow also increases, and hence the flow becomes increasingly multiscale. The increase cannot continue indefinitely, however, since when d_p and α_b become sufficiently large, the problem becomes analogous to flow through a porous medium, for which the flow Reynolds number cannot be very large. This is also related to the fact that in such a regime, H is no longer the relevant length scale in the flow Reynolds number, but rather the inter-bubble distance becomes the appropriate length scale. In the range of parameters considered for the bubble-laden cases, [figure 2](#) suggests that Re_H monotonically decreases with increasing A_2 . While one may speculate that this is due to the differing properties of the bubble wakes for different sized bubbles, the explanation for the observed trend is not entirely clear since the behaviour may also depend on wake interactions which are more difficult to understand.

3. Multiscale anisotropy and second-order structure function

A systematic approach for analysing the multiscale anisotropy of a turbulent flow is to use the irreducible representations of the SO(3) group, which consists of projecting the

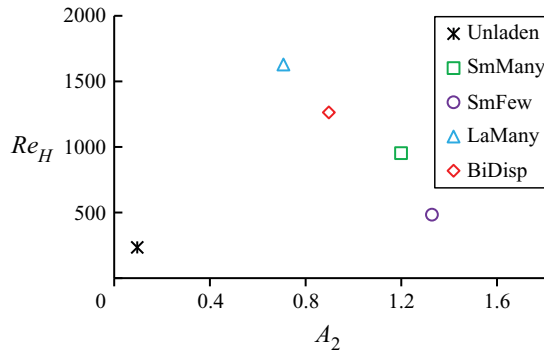


Figure 2. Reynolds number, Re_H plotted vs the second anisotropy invariant.

multipoint turbulent correlation functions onto the space of spherical harmonics (Arad, L’vov & Procaccia 1999; Biferale & Toschi 2001; Biferale & Procaccia 2005). However, such an analysis requires information on the full three-dimensional flow field, something that is usually not obtainable from experiments. Furthermore, as discussed in § 2, phase boundaries interrupt the flow field in multiphase flows, introducing further challenges in applying this method.

Due to these challenges, many investigations on anisotropy in turbulent flows focus directly on the structure function tensor (which are essentially moments of the velocity increments), comparing the longitudinal and transverse components in order to discern the level of anisotropy in the flow. In single-phase flows, the main focus was on scrutinizing the postulate of local isotropy and its implications (Kolmogorov 1941b, K41 for brevity). In general, experiments and numerical simulations do not strictly confirm the convergence toward isotropy predicted within K41 theory as a function of the scale (Dhruva, Tsuji & Sreenivasan 1997; Kurien & Sreenivasan 2000; Shen & Warhaft 2002). In the context of the single-point Reynolds stresses, the Lumley triangle (Lumley & Newman 1977) provided a powerful way to quantify and visualize anisotropy in the flow. It would be desirable to have something analogous to this for the structure functions, which would then provide a way to quantify and visualize anisotropy in the flow at different scales.

3.1. Second-order structure function and its anisotropy

Consider the second-order structure function

$$D_{ij}(\mathbf{r}, t) \equiv \langle \Delta u'_i(\mathbf{r}, t) \Delta u'_j(\mathbf{r}, t) \rangle. \quad (3.1)$$

Hereafter, we will suppress the time argument since we are focusing on statistically stationary flows. The Cartesian coordinate system chosen is depicted in figure 1, and as discussed in § 2, our DNS data only allow us to compute the velocity increments for separations in the spanwise direction, i.e. $\mathbf{r} = r_3 \mathbf{e}_3$ (and $r \equiv \|\mathbf{r}\| = r_3$). In this case the longitudinal structure function is $D_{LL}(r_3) = D_{33}(r_3)$, and for an incompressible, isotropic flow we would have

$$D_{11}^{iso} = D_{22}^{iso} = D_{33} + \frac{r}{2} \frac{\partial}{\partial r} D_{33}; \quad D_{ij}^{iso} = 0 \text{ (for } i \neq j\text{)}. \quad (3.2a,b)$$

Figure 3 displays the results for D_{11}, D_{22}, D_{33} as a function of r/H for all of the DNS cases. We also plot the isotropic form of the results based on (3.2a,b) for the cases Unladen

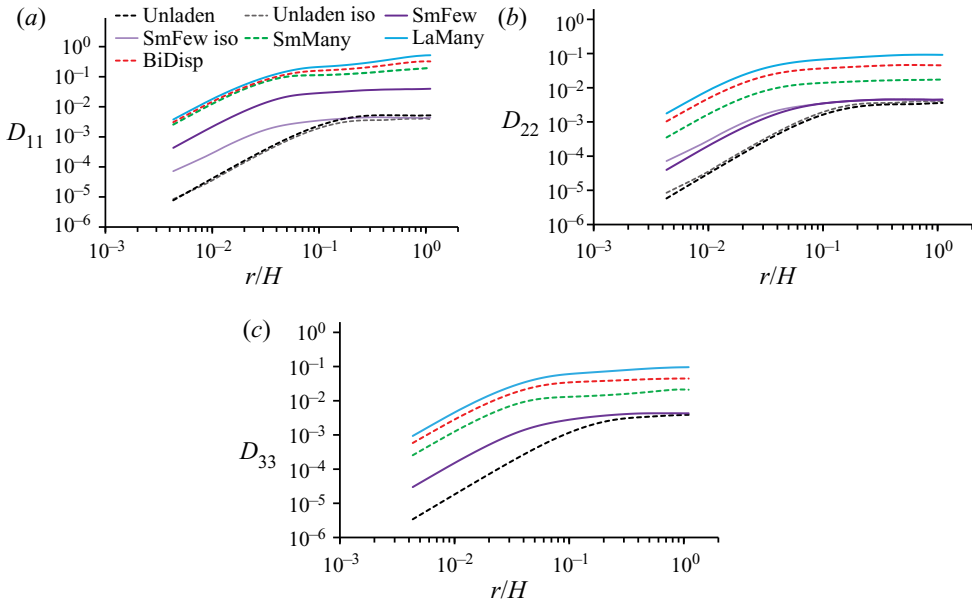


Figure 3. DNS results for the second-order transverse (a,b) and longitudinal (c) structure functions.

(labelled Unladen iso) and SmFew (labelled SmFew iso). The results show that, in general, the introduction of bubbles into the flow leads to strong enhancements of the fluctuations in all three directions of the flow. For $r/H \geq O(1)$, $D_{\gamma\gamma} \approx 2\langle u'_\gamma u'_\gamma \rangle$ as expected (no index summation is implied), and are consistent with the values obtained by Santarelli & Fröhlich (2016) for the corresponding Reynolds normal stresses in the channel centre. Compared with the unladen case, the enhancement of the structure function level is in the sequence SmFew, SmMany, BiDisp to LaMany, which corresponds to increasing averaged bubble Reynolds number and/or gas void fraction. This holds for all three directions across all scales from around one channel width to the smallest dissipative scales, and shows that the bubbles modify fluctuations in the flow at scales both larger and smaller than the bubble length scale d_p . Moreover, the increase of the structure functions in the bubble-laden cases is more pronounced at the smaller scales than at the larger scales. This behaviour is in close agreement with the experimental results of Rensen *et al.* (2005) who used Taylor’s hypothesis to construct the results. We also note that, although D_{22} is very similar for the Unladen and the SmFew cases at the large scales, it is significantly different for these cases at the smaller scales (and similarly for D_{33}), with the bubbles significantly enhancing the smaller-scale fluctuations in the flow.

For the unladen case, departures from $D_{11} = D_{11}^{iso}$ and $D_{22} = D_{22}^{iso}$ are not too strong, with stronger departures at the larger scales, consistent with the Reynolds-stress behaviour in the channel centre $\langle u'_1 u'_1 \rangle / \langle u'_2 u'_2 \rangle \approx 1.47$. As r is decreased the flow becomes more isotropic, although, as we shall show later, the small scales do not actually reach an isotropic state. In contrast, for the bubble-laden cases, there are strong departures from isotropy at all scales. For example, for the SmFew case, D_{11} shows significant deviations from D_{11}^{iso} at all scales, and for D_{22} the deviations from the isotropic form D_{22}^{iso} actually become stronger as one goes to scales smaller than the bubble length scale ($d_p \approx 0.05H$ for this case), while the larger scales behave more isotropically. The return to isotropy can therefore be strongly violated for bubbly turbulent flows. This is not surprising, however,

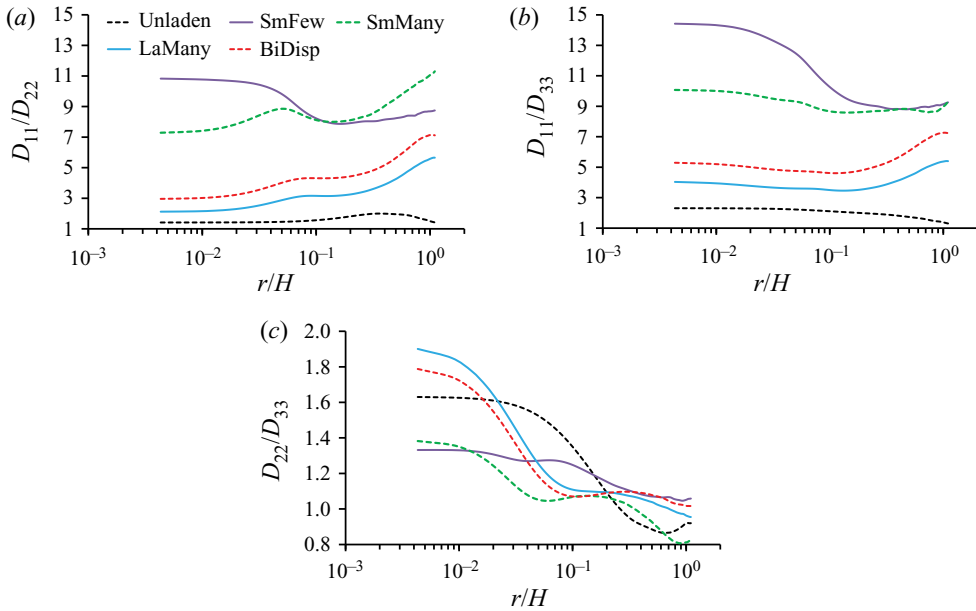


Figure 4. Ratio of D_{TT}/D_{TT} (a) and D_{TT}/D_{LL} (b,c) vs separation distance for all cases considered.

since the mean velocity of the bubbles is unidirectional, and the fact that $d_p \ll H$ means that the bubbles can directly inject fluctuations into the flow at the small scales, in contrast to the unladen case where the fluctuations are injected into the flow at large scales due to the mean shearing of the flow.

Figure 4 shows the ratios of the different structure function components in order to see more clearly the anisotropy of the flow. For an isotropic system we would have $D_{11}/D_{22} = 1$ at all scales, and for the unladen case D_{11}/D_{22} is quite close to 1, especially at smaller scales, while for the bubble-laden cases D_{11}/D_{22} deviates strongly from 1, reaching values $O(10)$. Except for the SmFew case, the curves for D_{11}/D_{22} in the bubble-laden cases tend to decrease with decreasing r , but with a bump at scale $r = O(d_p)$, indicating the injection of anisotropy at the bubble scale. In isotropic turbulence, $D_{11}/D_{33} \rightarrow 1$ and $D_{22}/D_{33} \rightarrow 1$ for $r \geq O(H)$, while $D_{11}/D_{33} \rightarrow 2$ and $D_{22}/D_{33} \rightarrow 2$ for $r/H \rightarrow 0$ (Pope 2000). While deviations from these are not too strong for the unladen case, strong departures are observed for the bubble-laden cases at all scales. The departures are strongest for the ratios involving D_{11} , which is to be expected since this is the direction of the mean trajectory of the bubbles. Moreover, for each of the ratios plotted in figure 4, the bubble-laden cases reveal a bump at $r = O(d_p)$, indicating the injection of anisotropy into the flow due to the bubbles and their anisotropic motion in the flow due to the buoyancy force acting on them.

While comparisons of the ratios of D_{11}, D_{22}, D_{33} are a standard way to analyse the multiscale anisotropy in the flow (van de Water & Herweijer 1999; Brugger *et al.* 2018), this method provides limited quantitative insight into the degree of anisotropy. For example, while the behaviour of these ratios is well known for an isotropic flow in the limits $r/H \rightarrow 0, r/H \geq O(1)$, their behaviour for intermediate r/H is not known and cannot be determined simply by the condition of isotropy. It is therefore desirable to provide a simple measure of anisotropy based on D_{ij} that applies at all scales, and also helps to visualize the anisotropic behaviour. In the context of the Reynolds-stress tensor, this can be achieved using either the Lumley triangle (Lumley & Newman 1977) or, more

recently, the barycentric map (BAM) proposed by Banerjee *et al.* (2007). However, as we describe in the next subsection, these methods cannot be directly applied to quantify multiscale anisotropy associated with D_{ij} , and modification is required.

3.2. Quantifying and visualizing scale-dependent anisotropy

Characterizing the scale-dependent anisotropy associated with D_{ij} is considerably more involved than that based on the Reynolds stress $\langle u_i u_j \rangle$. This is because the relationship between componentiality and isotropy breaks down at sub-integral scales. For example, in the case of the Reynolds-stress tensor, its components in the isotropic state are $\langle u'_i u'_j \rangle^{iso} \equiv \langle u'_m u'_m \rangle \delta_{ij} / 3$, according to which the components in all three Cartesian directions are the same (a ‘three-component flow’). In the case of D_{ij} , its isotropic state is (assuming the flow is incompressible)

$$D_{ij}^{iso} = D_{LL} \delta_{ij} + \frac{r}{2} \left(\delta_{ij} - \frac{r_i r_j}{r^2} \right) \frac{\partial}{\partial r} D_{LL}. \tag{3.3}$$

At the large scales $(\partial/\partial r)D_{LL} = 0$, and we have $D_{ij}^{iso} = D_{LL} \delta_{ij} = \langle u'_m u'_m \rangle \delta_{ij} / 3$, and hence the isotropic state corresponds to a three-component flow. On the other hand, in the limit $r \rightarrow 0$ one has $D_{LL} = \langle \epsilon \rangle r^2 / 15\nu$ (Pope 2000) and

$$D_{ij}^{iso} = 2D_{LL} \delta_{ij} - \frac{r_i r_j}{r^2} D_{LL}. \tag{3.4}$$

In this case the isotropic state does not correspond to a three-component flow, but rather the components transverse to \mathbf{r} are twice as large as those parallel to \mathbf{r} . Hence, in general there is no correspondence between the componentiality of the flow and isotropy. For this reason, measures such as $D_{ij} - (D_{mm} \delta_{ij} / 3)$, which have previously been used to define scale-dependent anisotropy (e.g. Brugger *et al.* 2018), do not in fact quantify isotropy, but rather only quantify deviations from the three-component state.

In view of these considerations, anisotropy must be quantified by the deviation of D_{ij} from D_{ij}^{iso} , rather than from $D_{mm} \delta_{ij} / 3$. To this end, we define two normalized tensors A^{iso} and A with components

$$A_{ij}^{iso}(\mathbf{r}) \equiv \frac{D_{ij}^{iso}}{D_{kk}^{iso}}, \tag{3.5}$$

and

$$A_{ij}(\mathbf{r}) \equiv \frac{D_{ij}}{D_{kk}}, \tag{3.6}$$

respectively. We may represent these symmetric tensors in their respective eigenframes, and then re-arrange to write them in the form

$$A^{iso} = (\lambda_1 - \lambda_2) \mathbf{P}_{1c} + (\lambda_2 - \lambda_3) \mathbf{P}_{2c} + \lambda_3 \mathbf{P}_{3c}, \tag{3.7}$$

$$A = (\mu_1 - \mu_2) \mathbf{Q}_{1c} + (\mu_2 - \mu_3) \mathbf{Q}_{2c} + \mu_3 \mathbf{Q}_{3c}, \tag{3.8}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3$ are the ordered eigenvalues of A^{iso} (due to isotropy we have $\lambda_1 = \lambda_2$) with corresponding normalized eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and $\mu_1 \geq \mu_2 \geq \mu_3$ are the ordered eigenvalues of A with corresponding normalized eigenvectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$. The basis tensors

in (3.7) and (3.8) are defined as

$$P_{1c} \equiv p_1 p_1; \quad P_{2c} \equiv p_1 p_1 + p_2 p_2; \quad P_{3c} \equiv p_1 p_1 + p_2 p_2 + p_3 p_3, \quad (3.9)$$

$$Q_{1c} \equiv q_1 q_1; \quad Q_{2c} \equiv q_1 q_1 + q_2 q_2; \quad Q_{3c} \equiv q_1 q_1 + q_2 q_2 + q_3 q_3, \quad (3.10)$$

and represent the one-component, two-component and three-component limiting states of the tensor. For example, if the tensor A has $\mu_1 \neq 0, \mu_2 = \mu_3 = 0$ then $A = \mu_1 Q_{1c}$ so that the tensor is one-component, while if the tensor has $\mu_1 = \mu_2, \mu_3 = 0$ then $A = \mu_2 Q_{2c}$ so that the tensor is two-component etc. The two sets of basis tensors P_{1c}, P_{2c}, P_{3c} and Q_{1c}, Q_{2c}, Q_{3c} are related by rotation matrices and contain information on the orientation of the tensors. Our aim is to use a BAM to visualize the componentiality of the tensors, and the information on the componentiality is contained within the coefficients multiplying the basis tensors in (3.7) and (3.8). The BAM (like the Lumley triangle) therefore does not convey information about the orientation of the tensor in space, which is contained in the basis tensors.

The coordinates of a BAM are used to represent the componentiality of the flow, and to construct such a map, new variables are introduced corresponding to the coefficients in (3.7) and (3.8) normalized by the largest associated eigenvalue

$$\mathcal{I}_{1c} = (\lambda_1 - \lambda_2)/\lambda_1; \quad \mathcal{I}_{2c} = (\lambda_2 - \lambda_3)/\lambda_1; \quad \mathcal{I}_{3c} = \lambda_3/\lambda_1, \quad (3.11)$$

$$\mathcal{J}_{1c} = (\mu_1 - \mu_2)/\mu_1; \quad \mathcal{J}_{2c} = (\mu_2 - \mu_3)/\mu_1; \quad \mathcal{J}_{3c} = \mu_3/\mu_1. \quad (3.12)$$

Using these variables, the coordinates for a BAM can be defined as (Banerjee *et al.* 2007)

$$(x_{BAM}^{iso}, y_{BAM}^{iso}) = \mathcal{I}_{1c}(x_{1c}, y_{1c}) + \mathcal{I}_{2c}(x_{2c}, y_{2c}) + \mathcal{I}_{3c}(x_{3c}, y_{3c}), \quad (3.13)$$

$$(x_{BAM}, y_{BAM}) = \mathcal{J}_{1c}(x_{1c}, y_{1c}) + \mathcal{J}_{2c}(x_{2c}, y_{2c}) + \mathcal{J}_{3c}(x_{3c}, y_{3c}). \quad (3.14)$$

Here, $(x_{1c}, y_{1c}) = (1, 0)$, $(x_{2c}, y_{2c}) = (0, 0)$ and $(x_{3c}, y_{3c}) = (1/2, \sqrt{3}/2)$ are the three corner points corresponding to the limiting states of componentiality, which are chosen to be corner points of an equilateral triangle as this aids the interpretation of the BAM results. By design, according to their definitions we have $\mathcal{I}_{1c} + \mathcal{I}_{2c} + \mathcal{I}_{3c} = 1$, $\mathcal{J}_{1c} + \mathcal{J}_{2c} + \mathcal{J}_{3c} = 1$ and at each corner of the triangle, only one of the coefficients is finite. As such, the location of a given coordinate in the triangle provides direct visual information regarding the componentiality of the flow at the scale considered, and the relative contribution of each of the three limiting states.

With this approach, $A^{iso}(\mathbf{r})$ and $A(\mathbf{r})$ can be mapped to a location in the BAM, and the linear distance between the coordinates corresponding to these two tensors then gives a measure of the anisotropy at that scale. In particular, the anisotropy at a given scale can be quantified by

$$C_{ani}(\mathbf{r}) \equiv \sqrt{(x_{BAM} - x_{BAM}^{iso})^2 + (y_{BAM} - y_{BAM}^{iso})^2}. \quad (3.15)$$

For an isotropic flow, $A = A^{iso}$, and consistent with this our definition yields $C_{ani} = 0 \forall r$. The point in the BAM representing the state $A = A^{iso}$ will in general depend upon scale. At the large scales of an isotropic flow, $\mathcal{I}_{1c} = \mathcal{I}_{2c} = 0$ and $\mathcal{I}_{3c} = 1$, so that the coordinates for A^{iso} are $(x_{BAM}, y_{BAM}) = (x_{3c}, y_{3c})$. Therefore, at the large scales $C_{ani}(\mathbf{r})$ recovers the property that anisotropy is related to distance from the top of the triangle which corresponds to the three-component state, just as for the single-point Reynolds-stress tensor (Banerjee *et al.* 2007). At smaller scales, A^{iso} is not located at the point $(x_{BAM}, y_{BAM}) = (x_{3c}, y_{3c})$, and its location in the triangle will depend upon r .

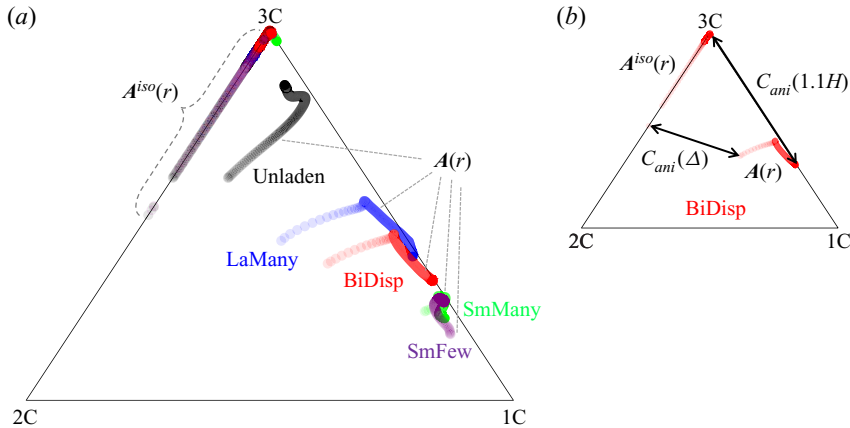


Figure 5. Representation of A (3.6) and its isotropic form A^{iso} (3.5) in the BAM (a) for all cases considered. The transparency of a given colour reflects the scale size, with lighter/darker denoting smaller/larger scale. (b) Plot to illustrate the meaning of $C_{ani}(r)$, defined in (3.15), using data for the BiDisp case. In this plot, $C_{ani}(r)$ is depicted by the double-arrow lines, with different lines corresponding to C_{ani} at different scales r . The start and end points for a given line correspond to points in the BAM representing $A(r)$ and $A^{iso}(r)$, and the length of the line denotes the value of C_{ani} . The lines shown correspond to the values of C_{ani} for the smallest scale $r = \Delta$ and largest scale $r = 1.1H$ accessible with our dataset, and illustrate that the anisotropy is stronger at $r = 1.1H$ than it is at $r = \Delta$.

Provided that $(\partial/\partial r)D_{LL} \geq 0$ (true for a homogeneous flow), then all projections of D_{ij}^{iso} in the plane orthogonal to r are equivalent and are greater than or equal to D_{LL} . As a result, the point representing A^{iso} is confined to the left side of the triangle, corresponding to a state of axisymmetric contraction. Moreover, as r is decreased, this point moves monotonically away from the three-component state towards the two-component state. The linear distance from this point to the point in the triangle representing A represents the amount of anisotropy. This is illustrated in figure 5.

We note that in the traditional Lumley triangle, it is the fact the trace of the Reynolds-stress anisotropy tensor, a_{ii} , is zero that allows its properties to be represented in a two-dimensional triangle using two independent eigenvalues of a_{ij} . This restriction does not apply to the BAM approach, which is why we are able to represent the properties of A^{iso} and A in a two-dimensional map, even though the trace of these tensors is not zero.

3.3. Application of the new method

We now turn to apply the new method described in § 3.2 to our DNS results. The trajectories for A^{iso} in figure 5(a) all lie on the left side of the triangle, corresponding to the state of axisymmetric contraction, associated with the fact that $D_{11}^{iso} = D_{22}^{iso} \geq D_{33}$. The trajectories for A^{iso} at large r are near to the three-component upper corner of the triangle, however, for the largest r/H for which we have data, $(\partial/\partial r)D_{LL} \neq 0$ and so the exact three-component state $A^{iso} = I/3$ is not observed. As r is decreased, the trajectory for A^{iso} moves down the left side of the triangle, towards the two-component corner. As discussed in § 3.2, the reason why A^{iso} is located along this edge of the triangle is that $D_{11}^{iso} = D_{22}^{iso} > D_{33}$ when $(\partial/\partial r)D_{33} > 0$, corresponding to a state of axisymmetric contraction.

The results in figure 5(a) for A show that for all cases, at the large scales the flow is closer to a state of axisymmetric expansion than contraction (unlike A^{iso}), due to the

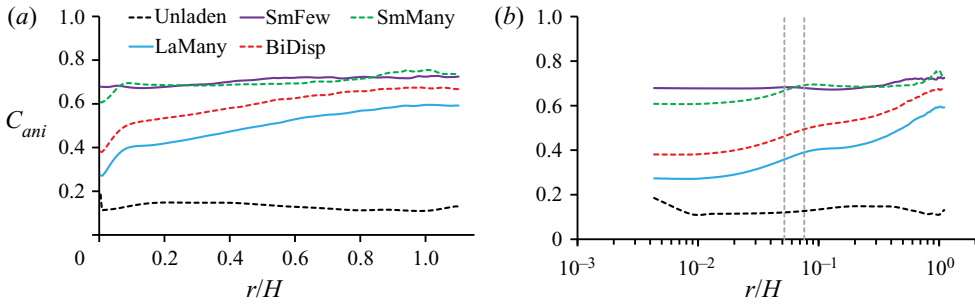


Figure 6. Anisotropy measures C_{ani} (a) linear plot, (b) the same data with semi-log plot. The two vertical dashed lines in (b) show $r = d_p$ for smaller and larger bubbles, respectively.

streamwise fluctuations being larger than those for the other directions. However, the bubble cases are much closer to the one-component corner of the triangle at the large scales than the unladen case. This is due to the fact that in addition to the streamwise pressure gradient acting on the flow, the bubbles also experience a strong buoyancy force in this direction which causes their streamwise fluctuations to be much larger than the fluctuations in the other directions. The trajectories of \mathcal{A} in the triangle are highly nonlinear, and show a tendency to migrate towards the axisymmetric contraction side of the triangle as r is decreased, consistent with an approach to isotropy as r is decreased, although never obtaining an isotropic state. The exception to this is the SmFew case for which the trajectory of \mathcal{A} actually approaches the one-component corner of the triangle as r is decreased. This surprising behaviour is consistent with that observed in figure 4. For each of the bubble cases, and most noticeably for the LaMany and BiDisp cases, there is a cusp point at which the trajectory of C_{ani} as a function of r in the triangle changes direction. This turning point occurs at $r = O(d_p)$.

A quantitative measure of the scale-wise anisotropy is provided by C_{ani} , introduced in § 3.2. This quantity provides a linear measure of anisotropy, and is zero for an isotropic flow. For illustration, in figure 5(b) we show the trajectories of \mathcal{A} and \mathcal{A}^{iso} for the BiDisp case and join the starting and ending points of these trajectories with double-arrow lines. The length of the line connecting the starting points represents $C_{ani}(r = 1.1H)$, while the length of the line connecting the ending points represents $C_{ani}(r = \Delta)$, where Δ is the grid spacing in the spanwise direction. The plot shows that $C_{ani}(1.1H) > C_{ani}(\Delta)$, such that the flow is more isotropic at the smaller scales in BiDisp case.

In figure 6 we show the results for C_{ani} as a function of r/H for all cases. In general, C_{ani} monotonically decreases with decreasing r , showing that the flow becomes more isotropic at decreasing scales. However, for these cases isotropy is never fully recovered, with anisotropy persisting into the dissipative range scales. Such behaviour for unladen turbulent flows has also been observed experimentally, including at much higher Reynolds numbers, as seen in studies by Kurien & Sreenivasan (2000), Antonia, Zhou & Romano (2002) and Carter & Coletti (2017). Furthermore, we note that for $r \leq 0.02H$ (most clearly seen in the semi-log plot of figure 6b) the return to isotropy is interrupted for the unladen case, and C_{ani} actually becomes larger as r is further decreased, implying increasing anisotropy at these scales. Similar behaviour has also been observed in both experimental (Carter & Coletti 2017) and numerical studies (Meneveau 1991; Bos, Liechtenstein & Schneider 2007), where it was suggested to be due to anisotropic intermittency at the dissipative scales.

The results in [figure 6](#) show that the departure from isotropy is in general much more pronounced for the bubble-laden cases than for the unladen case. This is again due to the fact that the bubbles introduce a significant source of momentum into the flow in the direction parallel to gravity due to the buoyancy force they experience. For the unladen case, the anisotropy is quite weak in the channel centre because the mean shear that generates the anisotropy is weak in that region of the flow. We also notice from [figure 6\(a\)](#) that the level of anisotropy decreases with increasing bubble Reynolds number, Re_p . In particular, the anisotropy across the scales generally decreases in the sequence SmMany (SmFew), BiDisp to LaMany, which corresponds to increasing Re_p . This behaviour is likely due to both the Re_p -dependent structure of the wakes produced by the bubbles (see the supplementary material to Santarelli & Fröhlich 2016) and also the path of the rising bubbles which becomes more chaotic and less uni-directional as Re_p increases (Horowitz & Williamson 2010; Ern *et al.* 2012). It is also interesting to note that the cases SmFew and SmMany which have similar Re_p , but different α_b , have a similar level of anisotropy. This suggests that Re_p plays more of a role than α_b in determining the contribution of the bubbles to the flow anisotropy, at least over some regimes of α_b (of course it cannot be true in general since, for example, when $\alpha_b = 0$ the bubbles have no effect on the flow).

Another observation prompted by [figure 6](#) is that the shape of the C_{ani} curve is remarkably similar for the three cases involving the highest α_b (SmMany, LaMany and BiDisp). This shape may be approximately divided into three regimes: for $r > O(d_p)$, a range in which the anisotropy gradually reduces as r is decreased; a bump (seen more clearly in the semi-log plot) is observed at the scale of the bubble $r = O(d_p)$; and a third regime at $r < O(d_p)$ where the anisotropy again reduces, but at a much faster rate with decreasing r than it does in the first regime. This rapid reduction in anisotropy in the last regime is never fully successful, however, with significant anisotropy persisting at the smallest scales. For the SmFew case, the first two regimes can also be identified, however, we observe scale-independent anisotropy for $r < O(d_p)$. A possible reason for this difference is that due to the low gas void fraction in the SmFew case, scales at $r < O(d_p)$ are influenced more strongly than the other cases by the single-phase behaviour, which as discussed, leads C_{ani} to actually increase at the smallest scales. Moreover, we notice that the rate at which the anisotropy decays with decreasing r is stronger with increasing Re_p , even though the actual value of the anisotropy becomes weaker with increasing Re_p . This is in contrast with the phenomenological notion from single-phase homogeneous anisotropic turbulence that the rate of return is typically faster for more strongly anisotropic flows, and is related to the nature of the slow part of the pressure–strain interaction term (Chung & Kim 1995). The present observation implies, however, that in bubbly flows the additional rapid pressure–strain term arising from the bubble-induced force production (Ma *et al.* 2020b) may play an important role in determining the scale-dependent anisotropy in bubbly turbulent flows. More extensive DNS data would be required in order to investigate this in detail.

The results in [figure 6\(b\)](#) also show that there is no clear difference in the scale at which the bump occurs in the C_{ani} curve when comparing cases with small and large bubbles. One reason for this is simply that if the injection of fluctuations into the flow by the bubbles occurs primarily at $r = O(d_p)$, then, unless d_p differs significantly, one might not be able to discern a clear difference in the scale of the injection. This is the case for the present data, since the small and large bubbles have diameters in the ratio 0.052 : 0.076, i.e. their ratio is order 1.

4. High-order structure functions

4.1. Energy transfer and third-order structure functions

The third-order structure function is of particular significance since it is related to the mean nonlinear energy transfer among the scales of a turbulent flow (Alexakis & Biferale 2018). In particular, in the Kármán–Howarth-type equation governing $D_{ii}(r_3, t)$, the inter-scale energy transfer term is (Hill 2001)

$$\mathcal{F}(r_3, t) = \sum_{\gamma=1}^3 \mathcal{F}_{\gamma}(r_3, t) \equiv \sum_{\gamma=1}^3 \frac{\partial}{\partial r_{\gamma}} D_{\gamma ii} \Big|_{r=r_3 e_3}, \quad (4.1)$$

where $D_{\gamma ii}(\mathbf{r}, t) \equiv \langle \Delta u'_{\gamma}(\mathbf{r}, t) \Delta u'_i(\mathbf{r}, t) \Delta u'_i(\mathbf{r}, t) \rangle$. Note that this definition does not assume isotropy of the flow, but applies to anisotropic flows as considered here.

When $\mathcal{F} < 0$, this corresponds to the nonlinear term supplying energy to the scale, while $\mathcal{F} > 0$ corresponds to the nonlinear term removing energy from the scale. In a single-phase, three-dimensional turbulent flow, there is on average a downscale flux of energy with $\mathcal{F} < 0$ in the inertial and dissipation ranges, representing the transfer of energy injected into the flow at the large scales down to the smallest scales where it is dissipated due to viscous stresses. However, the presence of bubbles in the flow could change this picture, since bubbles produce kinetic energy at scales $O(d_p)$ and some of this may then be transferred up towards the larger scales, as well as some being transferred down towards smaller scales. It is of great interest to understand this basic question of how the bubbles modify the energy transfer mechanisms and behaviour in the turbulent flow compared with the single-phase behaviour and mechanisms of strain self-amplification and vortex stretching (Carbone & Bragg 2020; Johnson 2020). However, since our DNS data were obtained for a fixed streamwise and wall-normal location, we are only able to compute the \mathcal{F}_3 contribution in (4.1), and thereby cannot fully determine \mathcal{F} . Nevertheless, we can still partially explore the question of the effect of the bubbles on the energy transfer by considering \mathcal{F}_3 .

In figure 7(a) we plot \mathcal{F}_3 for the unladen case. The results show that at larger scales $\mathcal{F}_3 > 0$, while at smaller scales $\mathcal{F}_3 < 0$, corresponding to energy being taken from the large scales and passed down to the small scales via the nonlinear transfer term. At this low Reynolds number, there is no inertial range over which \mathcal{F}_3 is constant, indicating that there is no cascade in the strict sense. In figure 7(b) we plot \mathcal{F}_3 for all the cases. The first thing that is apparent from these results is that the nonlinear energy transfer is in general much stronger for the bubble cases than the unladen case. For example, for the LaMany case, the peak magnitude of \mathcal{F}_3 is three orders of magnitude larger than for the unladen case. However, this is mainly due simply to the fact that the bubble-laden cases have more turbulent kinetic energy in the flow (see figure 3). The second thing is that the direction of the energy transfer for the bubble cases is the same as the unladen case, indicating that energy is passed downscale from the large scales to the small scales in the flow. The study of Pandey *et al.* (2020) explored a homogeneous bubbly turbulent flow using a Fourier space analysis and also observed that there is a downscale energy transfer, although their flow was considerably different to ours, and as noted in the introduction, there are some issues with their analysis.

Our results indicate then that the introduction of bubbles into the turbulent flow does not lead to an upscale energy transfer, at least for the cases we have considered and for the \mathcal{F}_3 contribution. One possible reason for this is that the energy being sent downscale from the large scales (where it is produced due to the mean-shear production), simply overwhelms any energy being transferred upscale from the bubbles, so that overall the

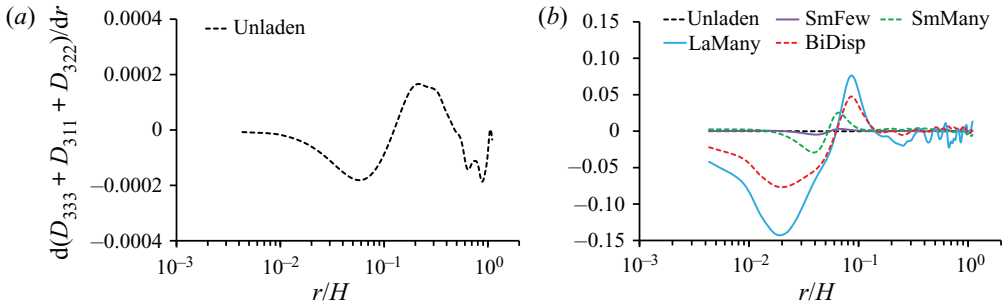


Figure 7. Contribution \mathcal{F}_3 to the nonlinear energy transfer term: (a) unladen case, (b) all the cases.

energy transfer is downscale. However, since in most of these bubble cases the dominant source of TKE comes from bubble induced production (Santarelli & Fröhlich 2016), rather than mean-shear production, this does not seem likely. A second possible reason is that since the scales at which the bubbles are producing kinetic energy, $O(d_p)$, are also scales at which the viscous forces in the flow are strong, then most of the kinetic energy produced by the bubbles may be simply dissipated before it is able to be transferred upscale by nonlinear forces in the flow. A third possible reason relates to the fact that although the bubble length scale is d_p , they may inject energy at scales significantly larger than this. Indeed, the wakes produced by bubbles can have a length that is $O(10d_p)$. For the Reynolds number of our DNS the scale separation between the bubble diameter and the large scales of the flow is not very large, and therefore the bubbles may actually directly inject TKE into scales on the order of the large scales of the flow, and this energy is then able to be sent downscale via the nonlinear energy transfer term. If this third explanation is true, then an interesting implication is that if BIT flows can develop with $d_p \lll \ell_I$ (where ℓ_I is the integral length scale of the flow), then the maximum scale at which the bubbles will inject energy into the flow will be much smaller than ℓ_I . In this case, an upscale energy transfer must occur at scales $\gg d_p$ since otherwise the large scales would have no source of energy.

In figure 8 we plot one of the contributions to \mathcal{F}_3 , namely dD_{333}/dr . In contrast to figure 7, this result shows that for the SmFew and SmMany cases there is an upscale transfer of energy, with energy being extracted from the small scales and passed to the large scales. This reveals a subtle effect of the bubbles; while overall the bubble cases still exhibit a downscale energy transfer, the energy transfer associated with particular components of the velocity field do exhibit an upscale energy transfer. By contrast, the LaMany case shows $dD_{333}/dr < 0$ at the small scales, corresponding to a downscale energy transfer. This differing behaviour is likely because for this case Re_p is much larger than for the small bubbles, leading to stronger turbulent mixing in the flow, and a behaviour closer to single-phase flow.

Figure 9 shows $D_{333}/D_{33}^{3/2}$ (the skewness of the longitudinal velocity increment), as well as the unnormalized form D_{333} . For the present unladen case, in the channel centre we find $D_{333}/D_{33}^{3/2} \approx -0.4$ for $r \rightarrow 0$, which is in close agreement with the value found in isotropic turbulence (Ishihara *et al.* 2007; Davidson 2015). The Reynolds number of the channel flow is too low for there to be a clear inertial range, and therefore the unladen data for D_{333} do not show evidence of the ‘four-fifths law’ $D_{333} = -(4/5)\langle \epsilon \rangle r$ predicted for the inertial range of isotropic turbulence by Kolmogorov (1941a). For the cases involving bubbles, a first observation is that $D_{333}/D_{33}^{3/2}$ is positive for all cases at $r = O(d_p)$, and for the SmFew and SmMany cases it remains positive in the limit $r \rightarrow 0$, while for others it

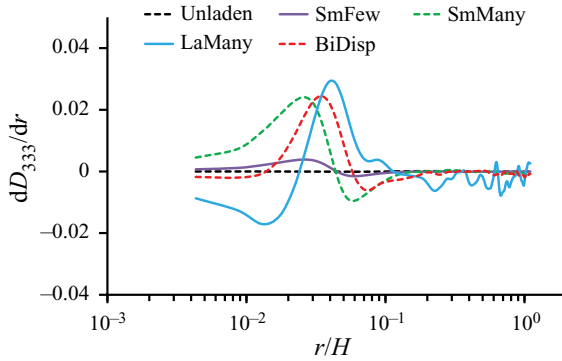


Figure 8. Derivative of D_{333} , the third-order longitudinal structure function.

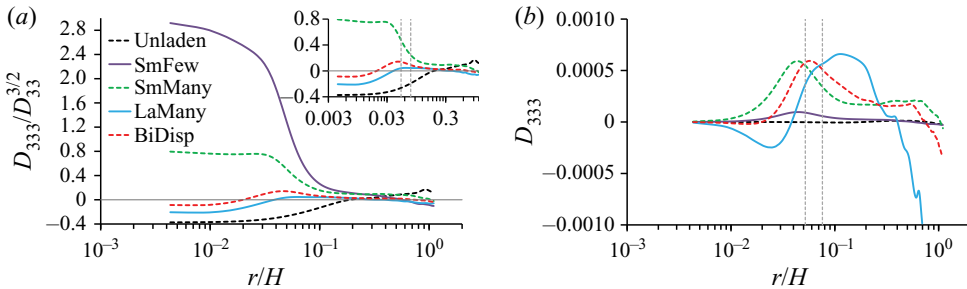


Figure 9. Third-order longitudinal structure functions: (a) skewness; (b) unnormalized. The inset in (a) shows zoomed for the cases excluding SmFew. The two vertical dashed lines in the inset of (a) and in (b) show $r = d_p$ for smaller and larger bubbles, respectively. The horizontal line in (a) is the value of 0.

becomes negative in this limit. The data for D_{333} indicate that the fluctuations are much larger for the bubble cases than for the unladen case, especially those with higher volume fractions, while the results for the skewness show that the bubble cases have a skewness that is either larger or smaller in magnitude than the unladen case, depending upon the case. Note that, the skewness normalizes out the amount of energy in the longitudinal component, and so gives a clearer indicator as to how the bubbles modify the behaviour of the nonlinear transfer in the flow. The SmFew case is particularly interesting, since it exhibits a very large value of the skewness for $r < O(d_p)$, even though this case has the smallest volume fraction α_b . However, as expected, the results for D_{333} for this case reveal that the magnitude of D_{333} is much smaller for the SmFew case than for the other three bubble cases that have higher values of α_b . This shows how sensitive certain properties of the turbulent flow are to the presence of the bubbles, even for relatively low α_b . It also implies that the bubbles could have a significant effect on tracer particle dispersion in turbulent flows, whose irreversibility is intimately related to the asymmetry in the velocity increment distributions (Jucha *et al.* 2014; Bragg, Ireland & Collins 2016; Bragg 2017).

The transverse structure functions D_{111} and D_{222} were also considered for each case. The associated skewness values were found to be very close to zero for all cases (not shown here).

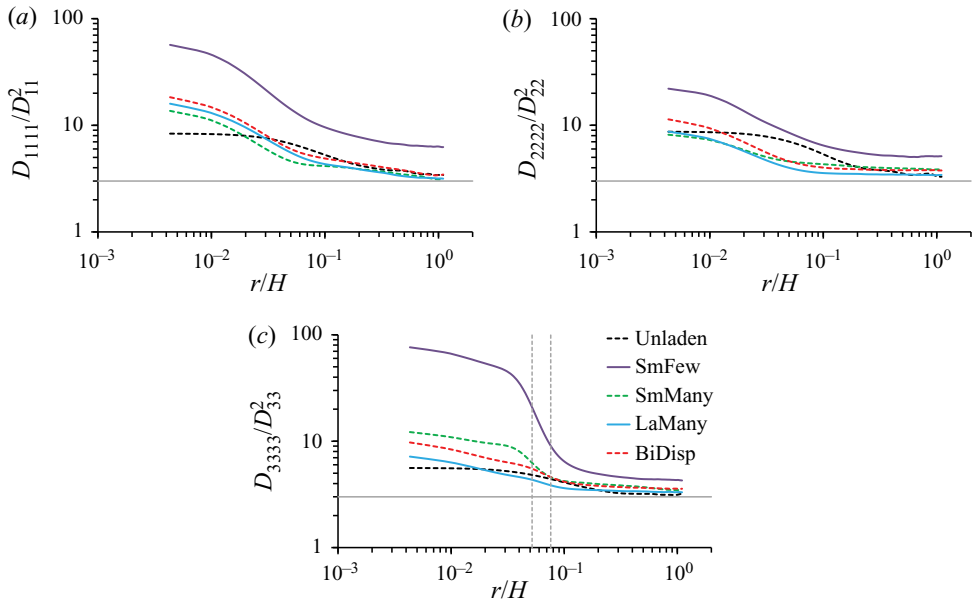


Figure 10. Normalized fourth-order transverse (a,b) and longitudinal (c) structure functions, corresponding to the flatness of the velocity increments. The two vertical dashed lines in (c) show $r = d_p$ for smaller and larger bubbles, respectively. The horizontal lines in (a,b,c) correspond to the Gaussian value of 3 for the flatness.

4.2. Fourth-order structure functions

We now consider the fourth-order structure functions to explore how the bubbles influence the intermittency of the background turbulence. In figure 10(a–c) we consider the normalized fourth-order moments (i.e. the flatness of the velocity increments) in each of the directions, denoted by D_{1111}/D_{11}^2 , D_{2222}/D_{22}^2 and D_{3333}/D_{33}^2 . If the velocity increments had Gaussian probability distributions, then these quantities would be equal to three at all scales. Departures from three indicate non-Gaussianity in the velocity increment distributions, while spatial dependence of these quantities indicates that the velocity increments are not only non-Gaussian, but also intermittent (Frisch 1995).

For the unladen case, the results in figure 10(a–c) show that, at the large scales, the quantities are close to three, but increase as r decreases. Furthermore, the results show that the departures from Gaussianity are stronger for the transverse components than they are for the longitudinal component, also observed in isotropic turbulence (Ishihara, Gotoh & Kaneda 2009). The effect of the bubbles on these quantities is non-trivial. At the very smallest scales, the bubble cases show much stronger non-Gaussianity and intermittency in the turbulent flow than for the unladen case. A similar finding was also observed in the experiment of Rensen *et al.* (2005) for their fully developed turbulent bubbly flow and in Biferale *et al.* (2012) when comparing the small-scale properties of boiling and non-boiling convective turbulent flows. However, with the exception of the SmFew case, the bubbles tend to suppress intermittency in the flow at scales comparable to their diameter, for the streamwise and wall-normal velocity directions. Roughly speaking, it is only at scales smaller than $O(d_p)$ that the flatness starts to increase significantly.

The flatness results for the SmFew case reveal much stronger non-Gaussianity and intermittency in the flow compared with the other three bubble cases that have much higher gas void fraction (≈ 7 times than SmFew). Some insight into this behaviour can

be found by considering the ways in which the bubbles modify the turbulent flow. The most significant qualitative modification the bubbles make to the turbulent flow is the wakes they generate, and the flow properties in these wakes will typically be significantly different from the properties of the background turbulence. When the number of bubbles is not too large (but still large enough to make a statistically significant contribution to the flow properties), there are relatively few regions in the flow where the flow differs due to the wakes from everywhere else (i.e. these regions are ‘rare’), breaking self-similarity in the flow and enhancing intermittency compared with the case without the bubbles. As the number of bubbles increases, the regions of the flow occupied by the wakes becomes less rare, and hence their contribution to the intermittency reduces. Moreover, as the number of bubbles/size of the bubbles increases, the possibility for wake interaction increases, and this can enhance mixing in the flow, which would reduce the impact of the wakes on the flow intermittency.

A striking result implied by the results in [figure 10\(a–c\)](#) concerns the relationship between Re_H and the level of intermittency. For example, as shown in [figure 2](#), the LaMany case has a much larger Re_H than the SmFew case, yet the latter exhibits much stronger intermittency in the flow than the former. This is in contrast to single-phase turbulence where small-scale intermittency increases with increasing Reynolds number (Frisch 1995). This again indicates that the mechanisms generating intermittency in bubble-laden turbulent flows are significantly different from those in single-phase turbulence, and are connected to the bubble wakes, as explained previously.

Finally, we remind the reader of the point made in [§ 2](#) that the fourth-order moments might be slightly under-resolved by the DNS. Consequently, these fourth-order results should be viewed with some caution, and future work is needed to examine the resolution criteria more thoroughly.

5. Conclusions

In this paper we have analysed various properties of bubble-laden turbulent flows across the range of scales of the flow, including the anisotropy, energy transfer and intermittency in the flow.

To explore the anisotropy of the flow at different scales, we developed an extension of the BAM approach that was previously developed for analysing the one-point Reynolds-stress tensor (Banerjee *et al.* 2007). In our approach, the anisotropy at any scale may be quantified and visualized, as well as providing information on the componentiality of the flow at the scale. Using this we were able to explore how the bubbles modify anisotropy in the flow. We found that the bubbles significantly enhance anisotropy at all scales compared with the unladen case, and that for some bubble cases, very strong anisotropy persists down to the smallest scales of the flow. The strongest anisotropy was observed for the cases involving small bubbles. Deviations from the behaviour that would be expected for an isotropic flow occur not only in the streamwise/gravity direction, but also in the other directions.

Concerning the energy transfer among the scales of the flow, the DNS data did not allow us to thoroughly explore this, but we were able to consider a number of its important aspects. Our results indicate that for the bubble-laden cases, the energy transfer is from large to small scales, just as for the unladen case. However, there is evidence of an upscale transfer when considering the transfer of energy of particular components of the velocity field, rather than the full kinetic energy involving all three components of the flow. Similar findings were also reported in Lai *et al.* (2018) where the energy cascade in bubbly flows was shown to be highly anisotropic. We also conjectured, however, that in a bubble-laden

channel flow at sufficiently high Reynolds number, and with channel height sufficiently large compared with the bubble size, there may exist an upscale transfer of energy at scales much larger than the bubble diameter. Our DNS does not lie in the parameter range required to observe such behaviour.

We also considered the normalized form of the fourth-order structure functions, corresponding to the flatness of the fluid velocity increments. These results revealed that the introduction of bubbles into the flow strongly enhances the intermittency of the turbulence in the dissipation range, but suppresses it at scales comparable to the bubble diameter. The SmFew case, however, shows enhancements of intermittency at all scales. We interpreted the effect of the bubbles on the flow intermittency in terms of the contributions of the bubble wakes to the overall properties of the turbulence. This strong enhancement of the dissipation-scale intermittency has significant implications for understanding how the bubbles might modify the mixing properties of turbulent flows, and the associated large velocity gradients. These will be the subject of future investigations.

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Author ORCIDs.

-  Tian Ma <https://orcid.org/0000-0002-8406-4698>;
-  Andrew D. Bragg <https://orcid.org/0000-0001-7068-8048>.

REFERENCES

- ALEXAKIS, A. & BIFERALE, L. 2018 Cascades and transitions in turbulent flows. *Phys. Rep.* **767**, 1–101.
- ANTONIA, R.A., ZHOU, T. & ROMANO, G.P. 2002 Small-scale turbulence characteristics of two-dimensional bluff body wakes. *J. Fluid Mech.* **459**, 67.
- ARAD, I., L'VOV, V.S. & PROCACCIA, I. 1999 Correlation functions in isotropic and anisotropic turbulence: the role of the symmetry group. *Phys. Rev. E* **59** (6), 6753.
- BALACHANDAR, S. & EATON, J.K. 2010 Turbulent dispersed multiphase flow. *Annu. Rev. Fluid Mech.* **42**, 111–33.
- BANERJEE, S., KRAHL, R., DURST, F. & ZENGER, CH. 2007 Presentation of anisotropy properties of turbulence, invariants versus eigenvalue approaches. *J. Turbul.* **8**, N32.
- BIFERALE, L., PERLEKAR, P., SBRAGAGLIA, M. & TOSCHI, F. 2012 Convection in multiphase fluid flows using lattice Boltzmann methods. *Phys. Rev. Lett.* **108** (10), 104502.
- BIFERALE, L. & PROCACCIA, I. 2005 Anisotropy in turbulent flows and in turbulent transport. *Phys. Rep.* **414** (2–3), 43–164.
- BIFERALE, L. & TOSCHI, F. 2001 Anisotropic homogeneous turbulence: hierarchy and intermittency of scaling exponents in the anisotropic sectors. *Phys. Rev. Lett.* **86** (21), 4831.
- BOS, W.J.T., LIECHTENSTEIN, L. & SCHNEIDER, K. 2007 Small-scale intermittency in anisotropic turbulence. *Phys. Rev. E* **76** (4), 046310.
- BRAGG, A.D. 2017 Analysis of the forward and backward in time pair-separation probability density functions for inertial particles in isotropic turbulence. *J. Fluid Mech.* **830**, 63–92.
- BRAGG, A.D., IRELAND, P.J. & COLLINS, L.R. 2016 Forward and backward in time dispersion of fluid and inertial particles in isotropic turbulence. *Phys. Fluids* **28** (1), 013305.
- BRAGG, A.D., LIAO, Y., FRÖHLICH, J. & MA, T. 2021 Assessment of the validity of a log-law for wall-bounded turbulent bubbly flows. *Int'l J. Heat Fluid Flow* (in press).
- BRUGGER, P., KATUL, G.G., DE ROO, F., KRÖNIGER, K., ROTENBERG, E., ROHATYN, S. & MAUDER, M. 2018 Scalewise invariant analysis of the anisotropic Reynolds stress tensor for atmospheric surface layer and canopy sublayer turbulent flows. *Phys. Rev. Fluids* **3** (5), 054608.

- CARBONE, M. & BRAGG, A.D. 2020 Is vortex stretching the main cause of the turbulent energy cascade? *J. Fluid Mech.* **883**, R2.
- CARTER, D.W. & COLETTI, F. 2017 Scale-to-scale anisotropy in homogeneous turbulence. *J. Fluid Mech.* **827**, 250–284.
- CHUNG, M.K. & KIM, S.K. 1995 A nonlinear return-to-isotropy model with Reynolds number and anisotropy dependency. *Phys. Fluids* **7** (6), 1425–1437.
- CIFANI, P., KUERTEN, J.G.M. & GEURTS, B.J. 2020 Flow and bubble statistics of turbulent bubble-laden downflow channel. *Intl J. Multiphase Flow* **126**, 103244.
- DU CLUZEAU, A., BOIS, G. & TOUTANT, A. 2019 Analysis and modelling of Reynolds stresses in turbulent bubbly up-flows from direct numerical simulations. *J. Fluid Mech.* **866**, 132–168.
- DAVIDSON, P.A. 2015 *Turbulence: An Introduction for Scientists and Engineers*. Oxford University Press.
- DHRUVA, B., TSUJI, Y. & SREENIVASAN, K.R. 1997 Transverse structure functions in high-Reynolds-number turbulence. *Phys. Rev. E* **56** (5), R4928.
- DODD, M.S. & JOFRE, L. 2019 Small-scale flow topologies in decaying isotropic turbulence laden with finite-size droplets. *Phys. Rev. Fluids* **4**, 064303.
- ELGHOBASHI, S. 2019 Direct numerical simulation of turbulent flows laden with droplets or bubbles. *Annu. Rev. Fluid Mech.* **51** (1), 217–244.
- ERN, P., RISSO, F., FABRE, D. & MAGNAUDET, J. 2012 Wake-induced oscillatory paths of bodies freely rising or falling in fluids. *Annu. Rev. Fluid Mech.* **44**, 97–121.
- FOX, R.O. 2014 On multiphase turbulence models for collisional fluid-particle flows. *J. Fluid Mech.* **742**, 368–424.
- FRISCH, U. 1995 *Turbulence: The Legacy of A.N. Kolmogorov*. Cambridge University Press.
- HILL, R.J. 2001 Equations relating structure functions of all orders. *J. Fluid Mech.* **434**, 379–388.
- HOROWITZ, M. & WILLIAMSON, C.H.K. 2010 The effect of Reynolds number on the dynamics and wakes of freely rising and falling spheres. *J. Fluid Mech.* **651**, 251–294.
- ISHIHARA, T., GOTOH, T. & KANEDA, Y. 2009 Study of high-Reynolds number isotropic turbulence by direct numerical simulation. *Annu. Rev. Fluid Mech.* **41**, 165–180.
- ISHIHARA, T., KANEDA, Y., YOKOKAWA, M., ITAKURA, K. & UNO, A. 2007 Small-scale statistics in high-resolution direct numerical simulation of turbulence: Reynolds number dependence of one-point velocity gradient statistics. *J. Fluid Mech.* **592**, 335.
- JOHNSON, P.L. 2020 Energy transfer from large to small scales in turbulence by multiscale nonlinear strain and vorticity interactions. *Phys. Rev. Lett.* **124**, 104501.
- JOSHI, J.B. & NANDAKUMAR, K. 2015 Computational modeling of multiphase reactors. *Annu. Rev. Chem. Biomol.* **6** (1), 347–378.
- JUCHA, J., XU, H., PUMIR, A. & BODENSCHATZ, E. 2014 Time-reversal-symmetry breaking in turbulence. *Phys. Rev. Lett.* **113**, 054501.
- KOLMOGOROV, A.N. 1941a Dissipation of energy in locally isotropic turbulence. *Dokl. Akad. Nauk SSSR* **32**, 16–18.
- KOLMOGOROV, A.N. 1941b The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *Dokl. Akad. Nauk SSSR* **30**, 299–303.
- KURIEN, S. & SREENIVASAN, K.R. 2000 Anisotropic scaling contributions to high-order structure functions in high-Reynolds-number turbulence. *Phys. Rev. E* **62** (2), 2206.
- LAI, C.C.K., FRAGA, B., CHAN, R. & DODD, M. 2018 Energy cascade in a homogeneous swarm of bubbles rising in a vertical channel. In *CTR Annual Research Briefs*, pp. 55–64.
- LAI, C.C.K. & SOCOLOFSKY, S.A. 2019 The turbulent kinetic energy budget in a bubble plume. *J. Fluid Mech.* **865**, 993–1041.
- LANCE, M. & BATAILLE, J. 1991 Turbulence in the liquid phase of a uniform bubbly air-water flow. *J. Fluid Mech.* **222**, 95–118.
- LIAO, Y., MA, T., KREPPER, E., LUCAS, D. & FRÖHLICH, J. 2019 Application of a novel model for bubble-induced turbulence to bubbly flows in containers and vertical pipes. *Chem. Engng Sci.* **202**, 55–69.
- LOHSE, D. 2018 Bubble puzzles: from fundamentals to applications. *Phys. Rev. Fluids* **3** (11), 110504.
- LU, J. & TRYGGVASON, G. 2013 Dynamics of nearly spherical bubbles in a turbulent channel upflow. *J. Fluid Mech.* **732**, 166–189.
- LUMLEY, J.L. & NEWMAN, G.R. 1977 The return to isotropy of homogeneous turbulence. *J. Fluid Mech.* **82** (1), 161–178.
- MA, T. 2017 A contribution to turbulence modelling in bubbly flows. PhD thesis, Institute of Fluid Mechanics, Technische Universität Dresden, TUDpress Verlag der Wissenschaften GmbH, Dresden.
- MA, T., LUCAS, D. & BRAGG, A.D. 2020a Explicit algebraic relation for calculating Reynolds normal stresses in flows dominated by bubble-induced turbulence. *Phys. Rev. Fluids* **5**, 084305.

- MA, T., LUCAS, D., JAKIRLIĆ, S. & FRÖHLICH, J. 2020b Progress in the second-moment closure for bubbly flow based on direct numerical simulation data. *J. Fluid Mech.* **883**, A9.
- MA, T., SANTARELLI, C., ZIEGENHEIN, T., LUCAS, D. & FRÖHLICH, J. 2017 Direct numerical simulation-based Reynolds-averaged closure for bubble-induced turbulence. *Phys. Rev. Fluids* **2**, 034301.
- MARTÍNEZ, M.J., CHEHATA, G.D., VAN GILS, D., SUN, C. & LOHSE, D. 2010 On bubble clustering and energy spectra in pseudo-turbulence. *J. Fluid Mech.* **650**, 287–306.
- MASUK, A.U.M., SALIBINDLA, A.K.R. & NI, R. 2021 Simultaneous measurements of deforming hinze-scale bubbles with surrounding turbulence. *J. Fluid Mech.* **910**, A21.
- MATHAI, V., LOHSE, D. & SUN, C. 2020 Bubbly and buoyant particle-laden turbulent flows. *Annu. Rev. Condens. Matter Phys.* **11**, 529–559.
- MENDEZ-DIAZ, S., SERRANO-GARCÍA, J.C., ZENIT, R. & HERNÁNDEZ-CORDERO, J.A. 2013 Power spectral distributions of pseudo-turbulent bubbly flows. *Phys. Fluids* **25** (4), 043303.
- MENEVEAU, C. 1991 Dual spectra and mixed energy cascade of turbulence in the wavelet representation. *Phys. Rev. Lett.* **66** (11), 1450.
- MUDEDE, R.F. 2005 Gravity-driven bubbly flows. *Annu. Rev. Fluid Mech.* **37**, 393–423.
- OUELLETTE, N.T., XU, H., BOURGOIN, M. & BODENSCHATZ, E. 2006 Small-scale anisotropy in lagrangian turbulence. *New J. Phys.* **8** (6), 102.
- PANDEY, V., RAMADUGU, R. & PERLEKAR, P. 2020 Liquid velocity fluctuations and energy spectra in three-dimensional buoyancy-driven bubbly flows. *J. Fluid Mech.* **884**, R6.
- POPE, S.B. 2000 *Turbulent Flows*, 1st edn. Cambridge University Press.
- PUMIR, A. & SHRAIMAN, B.I. 1995 Persistent small scale anisotropy in homogeneous shear flows. *Phys. Rev. Lett.* **75** (17), 3114.
- RENSEN, J., LUTHER, S. & LOHSE, D. 2005 The effect of bubbles on developed turbulence. *J. Fluid Mech.* **538**, 153–187.
- RIBOUX, G., RISSO, F. & LEGENDRE, D. 2010 Experimental characterization of the agitation generated by bubbles rising at high Reynolds number. *J. Fluid Mech.* **643**, 509–539.
- ROGHAIR, I., MERCADO, J.M., VAN SINT ANNALAND, M., KUIPERS, H., SUN, C. & LOHSE, D. 2011 Energy spectra and bubble velocity distributions in pseudo-turbulence: numerical simulations vs experiments. *Intl J. Multiphase Flow* **37**, 1093–1098.
- SANTARELLI, C. & FRÖHLICH, J. 2015 Direct numerical simulations of spherical bubbles in vertical turbulent channel flow. *Intl J. Multiphase Flow* **75**, 174–193.
- SANTARELLI, C. & FRÖHLICH, J. 2016 Direct numerical simulations of spherical bubbles in vertical turbulent channel flow. Influence of bubble size and bidispersity. *Intl J. Multiphase Flow* **81**, 27–45.
- SANTARELLI, C., ROUSSEL, J. & FRÖHLICH, J. 2016 Budget analysis of the turbulent kinetic energy for bubbly flow in a vertical channel. *Chem. Engng Sci.* **141**, 46–62.
- SHEN, X. & WARHAFT, Z. 2000 The anisotropy of the small scale structure in high Reynolds number ($R_\lambda \sim 1000$) turbulent shear flow. *Phys. Fluids* **12** (11), 2976–2989.
- SHEN, X. & WARHAFT, Z. 2002 Longitudinal and transverse structure functions in sheared and unsheared wind-tunnel turbulence. *Phys. Fluids* **14** (1), 370–381.
- SREENIVASAN, K.R. & ANTONIA, R.A. 1997 The phenomenology of small-scale turbulence. *Annu. Rev. Fluid Mech.* **29** (1), 435–472.
- VAN DE WATER, W. & HERWEIJER, J.A. 1999 High-order structure functions of turbulence. *J. Fluid Mech.* **387**, 3–37.
- YEUNG, P.K., SREENIVASAN, K.R. & POPE, S.B. 2018 Effects of finite spatial and temporal resolution in direct numerical simulations of incompressible isotropic turbulence. *Phys. Rev. Fluids* **3**, 064603.