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ON REAL NONISOMORPHIC BANACH SPACES WITH HOMEOMORPHIC GEOMETRIC STRUCTURE SPACES

RYOTARO TANAKA

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Abstract

We give an example of a pair of real Banach spaces such that they are neither linearly isomorphic nor isomorphic with respect to the structure of Birkhoff–James orthogonality, but have mutually homeomorphic geometric structure spaces.

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1. Introduction and preliminaries

This paper is concerned with a basic problem in the theory of geometric nonlinear classification of Banach spaces. It was recently pointed out in [16, Example 4.14] that there exists a pair of complex Banach spaces (in fact, uniform algebras) (X, Y) with the following properties: X is neither linearly isomorphic to Y nor $\overline{Y}$ and X is not isomorphic to Y (nor $\overline{Y}$) with respect to the structure of Birkhoff–James orthogonality, but X and Y have mutually homeomorphic geometric structure spaces, where $\overline{Y}$ is the complex conjugate of Y. In particular, this example solved [15, Problem 6.16] in the positive. The main objective of the present paper is to obtain a similar example in the real case. Namely, we give a pair of real Banach spaces such that they are neither linearly isomorphic nor isomorphic with respect to the structure of Birkhoff–James orthogonality, but have mutually homeomorphic geometric structure spaces. This answers [15, Problem 6.15] affirmatively.

In the rest of this paper, all Banach spaces are assumed to be real. As usual, Banach spaces X and Y are *linearly isomorphic* if there exists a linear homeomorphism between X and Y, in which case, we write $X \cong Y$.

We recall the notion of Birkhoff–James orthogonality which was first introduced by Birkhoff [3] and was significantly developed by James [6, 7]. Let X be a Banach

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space and let $x, y \in X$. Then, x is said to be *Birkhoff–James orthogonal* to y, denoted by $x \perp_{BJ} y$, if $\|x + \lambda y\| \geq \|x\|$ for each $\lambda \in \mathbb{R}$. Obviously, it is equivalent to the usual orthogonality in Hilbert spaces. Moreover, as the existing results indicate, the behaviour of Birkhoff–James orthogonality is closely related to the geometric structure of Banach spaces; see [1] for a comprehensive survey on generalised orthogonality types.

Very recently, the nonlinear equivalence based on the structure of Birkhoff–James orthogonality has been studied in [2, 5, 13, 14, 16]. Let X, Y be Banach spaces. Then, X and Y are *isomorphic with respect to the structure of Birkhoff–James orthogonality*, denoted by $X \sim_{BJ} Y$, if there exists a (possibly nonlinear) bijection $T : X \to Y$ such that $x \perp_{BJ} y$ if and only if $Tx \perp_{BJ} Ty$. In this direction, some classification results as well as descriptions of Birkhoff–James orthogonality preservers are known.

The geometric structure space $\mathfrak{S}(X)$ of a Banach space X was first introduced in [14, Definition 3.4 and Proposition 3.5] for classifying the class of spaces of continuous functions $C_0(K)$, and the family of classical sequence spaces c_0 and ℓ_p by their structure of Birkhoff–James orthogonality. It is defined as the set

$$\mathfrak{S}(X) = \Big\{ \bigcup_{f \in \Phi^*(F)} \ker f : F \text{ is a maximal face of } B_X \Big\},$$

where $\Phi^*(F) = \{f \in B_{X^*} : f(x) = 1 \text{ for each } x \in F\}$, equipped with the closure operator

$$S^= = \Big\{ I \in \mathfrak{S}(X) : \bigcap_{J \in S} J \subset I \Big\}.$$

Moreover, it was shown in [14, Proposition 3.5] that:

(i) $\emptyset^= = \emptyset$;
(ii) $S \subset S^=$;
(iii) $(S^=)^= = S^=$; and
(iv) $S_1 \subset S_2$ implies $S_1^= \subset S_2^=$.

However, $(S_1 \cup S_2)^= \subset S_1^= \cup S_2^=$ does not hold in general, that is, the closure operator $S \mapsto S^=$ does not necessarily induce a topology on $\mathfrak{S}(X)$. The closure space $\mathfrak{S}(X)$ is said to be *topologisable* if the closure operator $S \mapsto S^=$ satisfies the Kuratowski closure axioms, or equivalently, if the set $\{S \subset \mathfrak{S}(X) : S^= = S\}$ fulfils the axioms of closed sets. It is known that $\mathfrak{S}(X)$ is not topologisable whenever X is a reflexive smooth Banach space with $\dim X \geq 2$ [14, Theorem 3.8], and that $\mathfrak{S}(C_0(K))$ is topologisable and homeomorphic to K [14, Theorem 5.2].

The theory of geometric structure spaces was further developed in [15], where the nonlinear equivalence of Banach spaces based on geometric structure spaces was introduced and studied. Let X, Y be Banach spaces. Then, X and Y are *isomorphic with respect to geometric structure spaces*, denoted by $X \sim_{\mathfrak{S}} Y$, if they have homeomorphic geometric structure spaces, that is, if there exists a bijection $\Phi : \mathfrak{S}(X) \to \mathfrak{S}(Y)$ satisfying $\Phi(S^=) = \Phi(S)^=$ for each $S \subset \mathfrak{S}(X)$. It was shown in [15] that $C_0(K)$-spaces are isometrically classified under '$\sim_{\mathfrak{S}}$' [15, Theorem 3.16], and that c_0 and ℓ_p have mutually different geometric structure spaces [15, Theorem 6.14]. These apply to classification results under '$\sim_{BJ}$' via the following fact. If X, Y are Banach spaces and if $X \sim_{BJ} Y$, then $X \sim_{\mathfrak{S}} Y$ [14, Theorem 3.10]. This allows us to find the difference between the structure of Birkhoff–James orthogonality of two given Banach spaces by using their geometric structure spaces.

With this notation, the main result of this paper is stated as follows.

THEOREM 1.1. *There exists a pair of real Banach spaces* (X, Y) *such that* $X \not\cong Y$ *and* $X \nsim_{BJ} Y$*, but* $X \sim_{\mathfrak{S}} Y$.

Here, it should be noted that if either of X, Y is finite dimensional, or both of X, Y are reflexive and smooth, then $X \sim_{\mathfrak{S}} Y$ implies that $X \cong Y$ [15, Theorems 4.3 and 6.5]. Hence, the desired pair must be constructed outside of these classes.

2. Results

Let K be a compact Hausdorff space. Then, the symbol $C(K)$ denotes the Banach space of all continuous functions on K, where the norm of $f \in C(K)$ is defined by

$$\|f\|_\infty = \sup\{|f(t)| : t \in K\}.$$

For each nonzero regular Borel measure μ on K, let $Z(K, \mu)$ be the hyperplane of $C(K)$ given by

$$Z(K, \mu) = \left\{f \in C(K) : \int f \, d\mu = 0\right\}.$$

The desired example will be constructed by using $C(K)$ and $Z(K, \mu)$ for a special pair (K, μ). To be precise, we adopt the connected compact Hausdorff space K, constructed by Koszmider [9, Section 5], such that $C(K)$ is infinite-dimensional and not isomorphic to any of its hyperplanes. In this setting, we automatically have $C(K) \not\cong Z(K, \mu)$ for any nonzero regular Borel measure μ on K.

Next, we find a regular Borel (probability) measure μ on K such that $C(K) \nsim_{BJ} Z(K, \mu)$, but $C(K) \sim_{\mathfrak{S}} Z(K, \mu)$. It will turn out that the required property for such a μ is nonatomicity. Let μ be a nonnegative Borel measure on K. Then, a Borel subset E of K is called an *atom* of μ if $\mu(E) > 0$, and whenever F is a Borel subset of E, either $\mu(F) = 0$ or $\mu(E \setminus F) = 0$. A Borel measure μ is said to be *atomic* if it has an atom. As was noted by Knowles [8, page 63], if E is an atom of μ, then $\mu(\{t\}) = \mu(E) > 0$ for some $t \in E$. Hence, μ is atomic if and only if $\mu(\{t\}) > 0$ for some $t \in K$.

According to [8, Theorem 1], if K is *perfect*, that is, if K has no isolated point, then there exists a nonatomic regular Borel probability measure on K. We remark that Koszmider's space K is perfect since it is connected and Hausdorff. Hence, we can find a nonatomic regular Borel probability measure on Koszmider's space K.

The following lemma will be needed for proving both $C(K) \nsim_{BJ} Z(\mu, K)$ and $C(K) \sim_{\mathfrak{S}} Z(\mu, K)$ provided that μ is a nonatomic regular Borel probability measure on K.

LEMMA 2.1. *Let K be a compact Hausdorff space and let μ be a nonatomic regular Borel probability measure on K. If $t \in K$ and if U is an open neighbourhood of t, then there exists an $f \in Z(K, \mu)$ such that $f(t) = \|f\|_\infty = 1$ and $f(K \setminus U) = \{0\}$.*

PROOF. First, we recall that the support for μ is given by

$$\operatorname{supp}(\mu) = \{t \in K : \mu(U) > 0 \text{ for each open neighbourhood } U \text{ of } t\}.$$

If $t \notin \operatorname{supp}(\mu)$, then $\mu(U) = 0$ for some open neighbourhood U of t. In particular, each $s \in U$ does not belong to $\operatorname{supp}(\mu)$. Hence, $K \setminus \operatorname{supp}(\mu)$ is an open subset of K and $\operatorname{supp}(\mu)$ is a closed subset of K. We also note that $\mu(\operatorname{supp}(\mu)) = 1$. Indeed, since μ is regular, for each $\varepsilon > 0$, there exists an open subset U of K such that $\operatorname{supp}(\mu) \subset U$ and $\mu(U) < \mu(\operatorname{supp}(\mu)) + \varepsilon$. Set $F = K \setminus U$. Since $F \subset K \setminus \operatorname{supp}(\mu)$, for each $x \in F$, there exists an open neighbourhood U_x of x such that $\mu(U_x) = 0$. Then, $(U_x)_{x \in F}$ is an open covering for a compact set F and has a finite subcovering $U_{x_1}, \ldots, U_{x_n}$. It follows from the monotonicity and subadditivity of μ that

$$\mu(F) \subset \mu\Big(\bigcup_{j=1}^{n} U_{x_j}\Big) \leq \sum_{j=1}^{n} \mu(U_{x_j}) = 0.$$

Therefore, $\mu(F) = 0$, which implies that $\mu(U) = 1$. This proves that $1 - \varepsilon < \mu(\operatorname{supp}(\mu))$ for arbitrary $\varepsilon > 0$. Hence, $\mu(\operatorname{supp}(\mu)) = 1$ holds.

Now, let $t \in K$ and let U be an open neighbourhood of t. Suppose first that $t \notin \operatorname{supp}(\mu)$. Set $F = \operatorname{supp}(\mu) \cup (K \setminus U)$. Then, by Urysohn's lemma, there exists an $f \in C(K)$ such that $0 \leq f \leq \mathbf{1}$, $f(t) = 1$ and $f(F) = \{0\}$. In particular, it follows from $f|\operatorname{supp}(\mu) = 0$ that $f \in Z(K, \mu)$.

Next, suppose that $t \in \operatorname{supp}(\mu)$. Then, $\mu(U) > 0$. Since

$$\mu(\{t\}) = \inf\{\mu(V) : V \text{ is an open neighbourhood of } t\} = 0$$

by the nonatomicity and regularity of μ, there exists an open neighbourhood V of t such that $V \subset U$ and $0 < \mu(V) < 2^{-1}\mu(U)$. By Urysohn's lemma, we have a $g \in C(K)$ such that $0 \leq g \leq \mathbf{1}$, $g(t) = 1$ and $g(K \setminus V) = \{0\}$. Set $W = \{s \in K : g(s) > 2^{-1}\}$. Since $\overline{W} \subset \{s \in K : g(s) \geq 1/2\}$, it follows that $\overline{W} \subset V$. Moreover, the estimation $0 < \mu(W) \leq \mu(\overline{W}) \leq \mu(V) < 2^{-1}\mu(U)$ ensures that $\mu(U \setminus \overline{W}) > 2^{-1}\mu(U)$. Now, by the inner regularity of μ, we can find a compact subset F of K such that $F \subset U \setminus \overline{W}$ and $\mu(F) > 2^{-1}\mu(U)$. By Urysohn's lemma, there exist $h, k \in C(K)$ such that $0 \leq h \leq \mathbf{1}, 0 \leq k \leq \mathbf{1}, h(t) = 1, k(F) = \{1\}$ and $h(K \setminus W) = k((K \setminus U) \cup \overline{W}) = \{0\}$. Since $W_1 = \{s \in K : h(s) > 2^{-1}\}$ is an open neighbourhood of t, it follows that

$$0 < 2^{-1}\mu(W_1) \leq \int_{W_1} h \, d\mu \leq \int h \, d\mu = \int_W h \, d\mu \leq \mu(W).$$

Furthermore, we obtain

$$\int k \, d\mu \geq \int_F k \, d\mu = \mu(F) > \mu(W).$$

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Set $u = h - \alpha k$, where

$$\alpha = \frac{\int h \, d\mu}{\int k \, d\mu} \in (0, 1).$$

Then, $u \in Z(K, \mu)$ and $u(K \setminus U) = \{0\}$. Finally, $hk = 0$ guarantees that $u(t) = \|u\|_\infty = 1$. □

REMARK 2.2. It also follows from the preceding lemma that $Z(K, \mu)$ separates the points of K.

Using Lemma 2.1, we can prove that $C(K) \not\sim_{BJ} Z(K, \mu)$ for nonatomic regular Borel probability measures μ on K. Recall that an element x of a Banach space X is called a *right symmetric point* for the Birkhoff–James orthogonality if $y \in X$ and $y \perp_{BJ} x$ imply that $x \perp_{BJ} y$. The early study on local symmetry of Birkhoff–James orthogonality can be found in [17], while the term 'symmetric point' first appeared in [11, 12]. It is obvious that the right symmetry of a point is stable under Birkhoff–James orthogonality preservers. Moreover, it can be shown that $\mathbf{1} \in C(K)$ is a right symmetric point for the Birkhoff–James orthogonality. Indeed, if $f \in C(K)$ and if $f \perp_{BJ} \mathbf{1}$, then $f(K)$ contains both nonpositive and nonnegative numbers; otherwise,

$$\min\{\|f + 2^{-1}\|f\|_\infty \mathbf{1}\|, \|f - 2^{-1}\|f\|_\infty \mathbf{1}\|\} \leq 2^{-1}\|f\|_\infty,$$

which contradicts $f \perp_{BJ} \mathbf{1}$. Hence, $\|\mathbf{1} + \lambda f\|_\infty \geq 1$ for each $\lambda \in \mathbb{R}$, that is, $\mathbf{1} \perp_{BJ} f$. This shows that $\mathbf{1}$ is right symmetric.

To summarise, it will turn out that $C(K) \not\sim_{BJ} Z(K, \mu)$ once it has been proved that $Z(\mu, K)$ contains no nonzero right symmetric points for Birkhoff–James orthogonality. Now, we are ready to prove the following theorem.

THEOREM 2.3. *Let K be a connected compact Hausdorff space and let μ be a nonatomic regular Borel probability measure on K. Then, $Z(K, \mu)$ contains no nonzero right symmetric points for Birkhoff–James orthogonality. Consequently, $C(K) \not\sim_{BJ} Z(K, \mu)$.*

PROOF. Let f be a nonzero element of $Z(K, \mu)$. Since Birkhoff–James orthogonality is homogeneous, we may assume that $\|f\|_\infty = 1$. Moreover, from $\int f \, d\mu = 0$ and the intermediate value theorem, $f(t) = 0$ for some $t \in K$. Let $U^- = \{s \in K : f(s) < 0\}$ and $U^+ = \{s \in K : f(s) > 0\}$, and let

$$\begin{aligned} U_n^- &= \{s \in K : f(s) < -1/n\}, \quad U_n^+ = \{s \in K : f(s) > 1/n\}, \\ F_n^- &= \{s \in K : f(s) \leq -1/n\}, \quad F_n^+ = \{s \in K : f(s) \geq 1/n\}, \\ V_n &= K \setminus (F_n^- \cup F_n^+) = \{s \in K : -1/n < f(s) < 1/n\}, \end{aligned}$$

for each $n \in \mathbb{N}$. We divide the argument into the two cases.

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Case (I): $\mu(U^-) = 0$. In this case, $\mu(U^+) = 0$; otherwise,

$$\int f\,d\mu = \int_{K\setminus U^-} f\,d\mu \geq \int_{U^+} f\,d\mu > 0$$

by the (inner) regularity of μ, which contradicts $f \in Z(K,\mu)$. Since $t \in V_4$, by Lemma 2.1, there exists a $g_1 \in Z(K,\mu)$ such that $g_1(t) = \|g_1\|_\infty = 1$ and $g_1(K \setminus V_4) = \{0\}$. Moreover, by Urysohn's lemma, we have $h_1, k_1 \in C(K)$ satisfying $0 \leq h_1 \leq \mathbf{1}$, $h_1(F_2^-) = \{1\}$, $h_1(K \setminus U_4^-) = \{0\}$, $0 \leq k_1 \leq \mathbf{1}$, $k_1(F_2^+) = \{1\}$, and $k_1(K \setminus U_4^+) = \{0\}$. We note that $k_1, h_1 \in Z(K,\mu)$ by $\mu(U_4^-) = \mu(U_4^+) = 0$. Now, set $f_1 = g_1 - h_1 + k_1 \in Z(K,\mu)$. Since $g_1h_1 = g_1k_1 = h_1k_1 = 0$, it turns out that $\|f_1\|_\infty = 1$. It follows from $f(t) = 0$ that

$$\|f_1 + \lambda f\|_\infty \geq |(f_1 + \lambda f)(t)| = |g_1(t)| = 1$$

for each $\lambda \in \mathbb{R}$; that is, $f_1 \perp_{BJ} f$. Moreover,

$$|(f - 4^{-1}f_1)(s)| \leq |f(s)| + 4^{-1}|f_1(s)| < 2^{-1} + 4^{-1} = 3/4$$

for each $s \in K \setminus (F_2^- \cup F_2^+) = V_2$,

$$|(f - 4^{-1}f_1)(s)| = |(f + 4^{-1}h_1)(s)| = |f(s) + 4^{-1}| = -f(s) - 4^{-1} \leq 1 - 4^{-1} = 3/4$$

for each $s \in F_2^-$ and

$$|(f - 4^{-1}f_1)(s)| = |(f - 4^{-1}k_1)(s)| = |f(s) - 4^{-1}| = f(s) - 4^{-1} \leq 1 - 4^{-1} = 3/4$$

for each $s \in F_2^+$. This shows that $\|f - 4^{-1}f_1\|_\infty \leq 3/4$; that is, $f \not\perp_{BJ} f_1$.

Case (II): $\mu(U^-) > 0$. In this case, we have $\mu(U^+) > 0$ by an argument similar to that at the beginning of Case (I). Since $U^- = \bigcup_n F_n^-$ and $U^+ = \bigcup_n F_n^+$, we obtain $\mu(F_n^-) > 0$ and $\mu(F_n^+) > 0$ for sufficiently large n with $n \geq 2$. By Lemma 2.1, there exists a $g_2 \in Z(K,\mu)$ such that $g_2(t) = \|g_2\|_\infty = 1$ and $g_2(K \setminus V_{2n}) = \{0\}$. Moreover, Urysohn's lemma generates $h_2, k_2 \in C(K)$ such that $0 \leq h_2 \leq \mathbf{1}$, $h_2(F_n^-) = \{1\}$, $h_2(K \setminus U_{2n}^-) = \{0\}$, $0 \leq k_2 \leq \mathbf{1}$, $k_2(F_n^+) = \{1\}$ and $k_2(K \setminus U_{2n}^+) = \{0\}$. We note that $g_2h_2 = g_2k_2 = h_2k_2 = 0$. Set

$$\alpha = \frac{\int h_2\,d\mu}{\int k_2\,d\mu}, \quad f_2 = g_2 + \frac{1}{\max\{1,\alpha\}}(-h_2 + \alpha k_2).$$

It follows that $\alpha > 0$, $f_2 \in Z(K,\mu)$ and $\|f\|_\infty = f_2(t) = g_2(t) = 1$. Hence, $f_2 \perp_{BJ} f$ holds. Further, we derive $f \not\perp_{BJ} f_2$ since

$$|(f - (2n)^{-1}f_2)(s)| \leq |f(s)| + (2n)^{-1}|f_2(s)| < n^{-1} + (2n)^{-1} = 3/(2n) < 1$$

for each $s \in K \setminus (F_n^- \cup F_n^+) = V_n$,

$$\begin{aligned}|(f - (2n)^{-1}f_2)(s)| &= \left|\left(f + \frac{1}{2n\max\{1,\alpha\}}h_2\right)(s)\right| = \left|f(s) + \frac{1}{2n\max\{1,\alpha\}}\right| \\ &= -f(s) - \frac{1}{2n\max\{1,\alpha\}} \leq 1 - \frac{1}{2n\max\{1,\alpha\}} < 1\end{aligned}$$

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for each $s \in F_n^-$ and

$$|(f-(2n)^{-1}f_2)(s)| = \left|\left(f - \frac{\alpha}{2n\max\{1,\alpha\}}k_2\right)(s)\right| = \left|f(s) - \frac{\alpha}{2n\max\{1,\alpha\}}\right|$$
$$= f(s) - \frac{\alpha}{2n\max\{1,\alpha\}} \leq 1 - \frac{\alpha}{2n\max\{1,\alpha\}} < 1$$

for each $s \in F_n^+$. This completes the proof. □

Finally, we show that $C(K) \sim_{\mathfrak{S}} Z(K,\mu)$ whenever μ is a nonatomic regular Borel probability measure on K. In this direction, our first aim is to identify $\mathfrak{S}(Z(K,\mu))$. To this end, we begin with the following well-known fact which can be proved by a combination of the Mazur separation theorem and Milman's partial converse to the Krein–Milman theorem.

Lemma 2.4. *Let K be a compact Hausdorff space and let M be a closed subspace of $C(K)$. Then, $\mathrm{ext}(B_{M^*}) \subset \{\pm\delta_t|M : t \in K\}$, where B_{M^*} is the unit ball of the dual space M^* of M and δ_t is the evaluation functional at $t \in K$, that is, $\delta_t(f) = f(t)$ for each $f \in C(K)$.*

We make use of the preceding lemma for identifying the support functionals for maximal faces of $B_{Z(K,\mu)}$. For this purpose, we need the notion of weak peak points for subspaces of $C(K)$. Recall that a point $t \in K$ is called a *weak peak point* for a subspace M of $C(K)$ if, for each open neighbourhood U of t, there exists an $f \in M$ such that $f(t) = \|f\|_\infty = 1$ and $|f(s)| < 1$ whenever $s \in K \setminus U$. The Bishop–de Leeuw theorem states that the set of weak peak points for a uniform algebra coincides with its Choquet boundary (see, for example, [10, Section 8]).

In analogy with [16, Lemma 3.2], we have the following lemma (see also [4, Lemma 3.2]).

Lemma 2.5. *Let K be a compact Hausdorff space, let M be a closed subspace of $C(K)$ and let $t \in K$ be a weak peak point for M. Then, $F_t = \delta_t^{-1}(1) \cap B_M$ is a maximal face of B_M and $\Phi^*(F_t) = \{\delta_t|M\}$.*

Proof. Let F be a proper face of B_M containing F_t and let $f \in F$. Further let $A_f = \{s \in K : |f(s)| = 1\}$. Then, $t \in A_f$. Indeed, if $t \notin A_f$, then we have a $g \in M$ such that $g(t) = \|g\|_\infty = 1$ and $|g(s)| < 1$ whenever $s \in A_f$, since t is a weak peak point for M and $K \setminus A_f$ is an open neighbourhood of t. Moreover, $2^{-1}(f+g) \in F$ by $g \in F_t \subset F$ and the convexity of F. It follows that $|2^{-1}(f(s)+g(s))| = 1$ for some $s \in K$, which implies that $|f(s)| = |g(s)| = 1$. However, this is impossible by the choice of g. Hence, $t \in A_f$. Now, we note that $f(t)^{-1}f \in F_t$ and $2^{-1}(f + f(t)^{-1}f) \in F$. It turns out that $f(t) = 1$ by

$$\left|\frac{1+f(t)^{-1}}{2}\right| = \left\|\frac{1}{2}(f + f(t)^{-1}f)\right\|_\infty = 1.$$

Therefore, $f \in F_t$, that is, $F = F_t$. This proves the maximality of F_t.

Now, we note that $\Phi^*(F_t)$ is a weakly* closed proper face of B_{M^*}, which together with the Krein–Milman theorem implies that $\Phi^*(F_t) = \overline{\mathrm{co}}^{w^*}(\mathrm{ext}(\Phi^*(F_t)))$.

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Let $\rho \in \mathrm{ext}(\Phi^*(F_t))$. By the maximality of F_t, we obtain $F_t = \rho^{-1}(1) \cap B_X$. Moreover, by $\mathrm{ext}(\Phi^*(F_t)) \subset \mathrm{ext}(B_{M^*})$ and Lemma 2.4, there exist an $s \in K$ such that $\rho = \delta_s|M$ or $\rho = -\delta_s|M$. If $s \neq t$, then we have an $f \in M$ such that $f(t) = \|f\|_\infty = 1$ and $|f(s)| < 1$. However, this leads to $f \in F_t$ and $1 = |\rho(f)| = |f(s)| < 1$, which is a contradiction. Thus, $s = t$. Finally, let g be an arbitrary element of F_t. If $\rho = -\delta_t|M$, then $(-g)(t) = \rho(g) = 1$ by $g \in F_t = \rho^{-1}(1) \cap B_M$, that is, $-g \in F_t$. However, this means that $0 = 2^{-1}(g + (-g)) \in F_t$, which contradicts $F_t \subset S_X$. Hence, $\rho = \delta_t|M$. This shows that $\mathrm{ext}(\Phi^*(F_t) = \{\delta_t|M\}$ and $\Phi^*(F_t) = \overline{\mathrm{co}}^{w^*}(\mathrm{ext}(\Phi^*(F_t))) = \{\delta_t|M\}$. The proof is complete. □

We need another auxiliary lemma.

Lemma 2.6. *Let X be a Banach space and let B be a weakly* closed subset of B_{X^*} such that $\|x\| = \max\{|\rho(x)| : \rho \in B\}$ for each $x \in X$. Then, for each convex subset C of S_X, there exists a $\rho \in B$ such that $C \subset \rho^{-1}(1) \cap B_X$ or $C \subset \rho^{-1}(-1) \cap B_X$. In particular, each maximal face of B_X has the form $\rho^{-1}(1) \cap B_X$ or $(-\rho)^{-1}(1) \cap B_X$ for some $\rho \in B$.*

Proof. We first note that $B_0 = \{\pm\rho : \rho \in B\}$ is also weakly* closed. Take arbitrary finitely many elements $x_1, \ldots, x_n$ of C. Since $x_0 = n^{-1}\sum_{j=1}^n x_j \in C$, we have a $\rho \in B$ such that $|\rho(x_0)| = 1$. It follows that $\rho(x_0) = \rho(x_1) = \cdots = \rho(x_n)$. Hence, $\rho(x_0)^{-1}\rho \in B_0$ and $(\rho(x_0)^{-1}\rho)(x_1) = \cdots = (\rho(x_0)^{-1}\rho)(x_n) = 1$. Now, set $B_x = \{\rho \in B_0 : \rho(x) = 1\}$ for each $x \in C$. Then, B_x is a nonempty weakly* closed subset of B_0. Moreover, the family $(B_x)_{x\in C}$ has the finite intersection property. Therefore, $\bigcap_{x\in C} B_x \neq \emptyset$ by the weak* compactness of B_0. Now, we obtain $C \subset \rho^{-1}(1) \cap B_X$ for an arbitrary $\rho \in \bigcap_{x\in C} B_x \subset B_0$, as desired. □

We conclude this paper with the following theorem which completes the construction of an example of a pair of Banach spaces (X, Y) such that $X \not\cong Y$ and $X \nsim_{BJ} Y$, but $X \sim_{\mathfrak{S}} Y$.

Theorem 2.7. *Let K be a compact Hausdorff space and let μ be a nonatomic regular Borel probability measure on K. Then, $\mathfrak{S}(Z(K,\mu)) = \{I_t : t \in K\}$, where $I_t = \ker(\delta_t|Z(K,\mu))$. Moreover, $C(K) \sim_{\mathfrak{S}} Z(K,\mu)$.*

Proof. By Lemma 2.1, each $t \in K$ is a weak peak point for $Z(K,\mu)$. From this and Theorem 2.5, it turns out that $F_t = \delta_t^{-1}(1) \cap B_{Z(K,\mu)}$ is a maximal face of $B_{Z(K,\mu)}$ with $\Phi^*(F_t) = \{\delta_|Z(K,\mu)\}$ for each t. Hence,

$$I_t = \ker(\delta_t|Z(K,\mu)) = \bigcup_{\rho\in\Phi^*(F_t)} \ker\rho \in \mathfrak{S}(Z(K,\mu))$$

for each $t \in K$, that is, $\mathfrak{S}(Z(K,\mu)) \supset \{I_t : t \in K\}$.

For the converse, we note that $B = \{\delta_t : t \in K\}$ is a weakly* closed subset of $B^*_{Z(K,\mu)}$ such that $\|f\|_\infty = \max\{|\delta_t(f)| : t \in K\}$ for each $f \in Z(K,\mu)$. Therefore, by Lemma 2.6, each maximal face of $B_{Z(K,\mu)}$ has the form F_t or $-F_t$. Combining this with the preceding paragraph, it follows that $\mathfrak{S}(Z(K,\mu)) \subset \{I_t : t \in K\}$. This proves that $\mathfrak{S}(Z(K,\mu)) = \{I_t : t \in K\}$.

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Next, let $\iota(t) = I_t$ for each $t \in K$. Since $Z(K,\mu)$ separates the points of K, the mapping ι is a bijection from K onto $\mathfrak{S}(Z(K,\mu))$. Suppose that $A \subset K$. If $t_0 \in \overline{A}$, then there exists a net $(t_a)_a \subset A$ that converges to t_0. It follows that $f(t_0) = 0$ whenever $f \in \bigcap_{t\in A} I_t$, which implies that $I_{t_0} \in \{I_t : t \in A\}^=$. Hence, $\{I_t : t \in \overline{A}\} \subset \{I_t : t \in A\}^=$. Conversely, if $t_0 \notin \overline{A}$, then setting $U = K \setminus \overline{A}$ yields an open neighbourhood of t. By Lemma 2.1, there exists an $f \in Z(K,\mu)$ such that $f(t_0) = \|f\|_\infty = 1$ and $f(K \setminus U) = \{0\}$. Since $K \setminus U = \overline{A}$, we see that $f \in \bigcap_{t\in A} I_t \setminus I_{t_0}$. Therefore, $I_{t_0} \notin \{I_t : t \in A\}^=$. This shows that $\{I_t : t \in \overline{A}\} \supset \{I_t : t \in A\}^=$, that is, $\iota(\overline{A}) = \iota(A)^=$. From this, ι is a (closure space) homeomorphism from K onto $\mathfrak{S}(Z(K,\mu))$. Moreover, $\mathfrak{S}(C(K))$ and K are homeomorphic (as closure spaces) by [14, Theorem 5.2]. Thus, $\mathfrak{S}(C(K))$ and $\mathfrak{S}(Z(K,\mu))$ are also homeomorphic. This completes the proof. □

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RYOTARO TANAKA, Katsushika Division, Institute of Arts and Sciences,
Tokyo University of Science, Tokyo 125-8585, Japan
e-mail: r-tanaka@rs.tus.ac.jp