





Design Tasks and Their Complexity for the European Train Control System with Hybrid Train Detection

Stefan Engels , Tom Peham , Judith Przigoda, Nils Przigoda , and Robert Wille ,

Abstract

Railway networks have become increasingly important in recent times, especially to move freight and public transportation from road traffic and planes to more environmentally friendly trains. Since expanding the global railway network is time and resource-consuming, maximizing the rail capacity on the existing infrastructure is desirable. However, simply running more trains is infeasible as certain constraints enforced by the train control system must be satisfied. The capacity of a network depends (amongst others) on the distance between trains allowed by this safety system. While most signaling systems rely on fixed blocks defined by costly hardware, new specifications provided by Level 2 with Hybrid Train Detection of the European Train Control System (ETCS L2 HTD), formerly known as ETCS Hybrid Level 3, allow the usage of virtual subsections. This additional degree of freedom allows for shorter train following times and, thus, more trains on existing railway tracks. On the other hand, new design tasks arise on which automated methods might be helpful for designers of modern railway networks. However, although first approaches exist that solve design problems arising within ETCS L2 HTD, neither formal descriptions nor results on the computational complexity of the corresponding design tasks exist. In this paper, we fill this gap by providing a formal description of design tasks for ETCS L2 HTD and proof that these tasks are NP-complete or NP-hard, respectively. By that, we are providing a solid basis for the future development of methods to solve those tasks, which will be integrated into the *Munich Train Control Toolkit* available open-source on GitHub at <https://github.com/cda-tum/mtct>.

Index Terms

Train control systems, ETCS Level 2 HTD, design automation, design tasks, formalization, complexity

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I. INTRODUCTION

Railway networks are an important part of both public transportation and logistics, e.g., to reduce carbon emissions in comparison to cars and planes. Unlike road traffic, trains have relatively long braking distances. Hence, trains cannot operate on sight due to safety constraints. Moreover, switches have to be safely set for trains to turn. Because of that, a control system is needed for efficient and safe rail traffic. To allow or deny a train to use a specific part of the track, signaling systems are essential. For this, many national train control systems have been developed. The European Union Agency for Railways has listed about 40 systems alone in Europe [1]. In Germany, for example, *PZB 90* (intermittent train protection) and *LZB* (continuous train control) are widely used [2]. For reasons of interoperability, these national control systems are unified in the *European Train Control System* (ETCS) [3], [4]. It is specified in the *Control Command and Signalling Technical Support Instrument* (CCS TSI) [5], in particular the set of specifications listed in its appendix A [6]. Even for metro lines, a radio-based system analog to ETCS (Level 2) is being implemented, namely the *Communication Based Train Control* (CBTC) [7], [8]. Similarly, major standardized systems have been implemented in China (*Chinese Train Control System*, CTCS) and North America (*Positive Train Control*, PTC) [9].

Currently, most railway systems in the world rely on block signaling, where the whole railway network is divided into blocks equipped with means of train detection to determine whether a given block is currently occupied by a train. Within a block, at most, there can only be one train at a time [11]. This paper focuses on ETCS, even though the model and key components likely carry over to other modern control systems. ETCS can be implemented in various levels [3], [4]. Previously, the specifications distinguished three levels, however with the CCS TSI 2023 [5] Level 2 and Level 3 have been combined.

Level 1 (L1) relies on signals and hardware on the tracks (e.g., Eurobalises and Euroloops) which transmit position and signaling information at discrete points (Eurobalise) or predefined intervals (Euroloop). The trackside fully handles the detection of block occupations (e.g., using axle counters), see Figure 1a, in which the following train is slowing down because it receives the moving authority into TTD2 only at the next Eurobalise.

In *Level 2* (L2), there is a permanent GSM-R connection between the train and the control system. By doing so, the train can continuously receive the maximal permitted speed. Signals are now optional but are often included as a fallback system. The train receives its moving authority through the GSM-R connection and is no longer bound to the discrete positions of Eurobalises.

At the same time, train location and integrity can be managed at different parts of the control system. Classically, reporting block sections as free works as in L1. However, due to the aforementioned difference in transmitting the moving authority the trains in Figure 1b can follow each other more closely.

The block section borders are equipped with *Trackside Train Detection* (TTD) systems, e.g., axle counters, called TTD sections. Thus far, those blocks often have been defined by geography, economic considerations, or as a trade-off between efforts of installing axle counters and possible benefits. Because of this, the length of the corresponding TTD sections ranges from some meters (e.g., around turnouts) or some hundred meters (e.g., at train stations) to several kilometers (e.g., in remote areas). Of course, this significantly affects the efficiency of the railway networks.

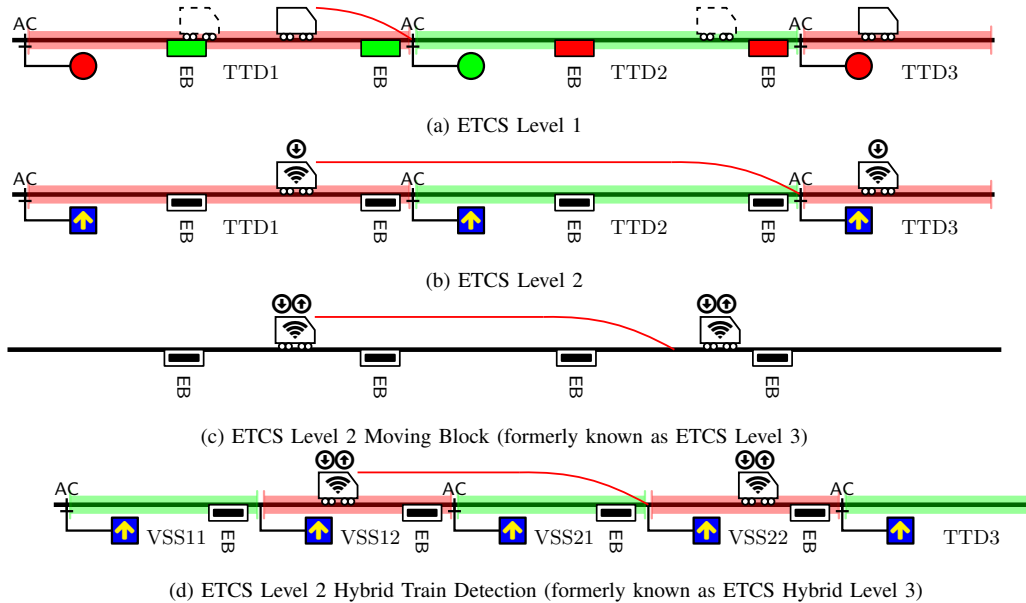


Fig. 1: Schematic drawings of various ETCS levels by Engels et al. [10] published under a CC BY 4.0 license

With the introduction of *Moving Block*, formerly known as ETCS Level 3, those principles have changed for the first time since the 19th century. The train reports the exact position itself, and no trackside detection is necessary. For this, trains must be equipped with an integrity system that detects if a train is still complete. Only in this case, it can safely release parts of the track. Rather than relying on fixed blocks, trains can follow at their braking distances since no hardware predefines any blocks, see Figure 1c. However, in practice, such systems impose new challenges [12].

Hence, Level 2 can also be used with a compromise of both, classical and moving blocks. In *Level 2 Hybrid Train Detection* (L2 HTD), formerly known as Hybrid Level 3, TTD sections that already exist are divided into smaller *Virtual Subsections* (VSS), which do not require physical axle counters anymore and, hence, allow for a much higher degree of freedom in the utilization of existing railway networks, see Figure 1c. Independently of trackside hardware, the occupation of these virtual subsections is observed by the position and integrity data obtained from the trains [13].

This additional degree of freedom allows for shorter train following times and, thus, more trains on existing railway tracks. At the same time, new design tasks arise on which automated methods might be helpful for designers of modern railway networks. However, thus far, design automation methods for ETCS, on the one hand, mainly focus on blocks characterized by hardware (e.g., axle counters). Because of this, they use general measures of capacity, energy efficiency, or economic efficiency independently from a specific schedule to propose a signaling layout [14], [15], [16], [17], [18], [19], [20], [21], [22]. Thus, the respective solution is expected to work well on various schedules. On the other hand, train routing was considered separately, assuming that the block layout was already decided on. Approaches are mainly separated in macroscopic routing [23] and track allocation on a microscopic

level, i.e., taking into account the constraints imposed by the control system and sections. Track allocation was considered in general [24], [25], [26] and specifically focusing on railway stations [27], [28], [29]. Besides that, recent research by Schlechte et al. optimizes time tables on dedicated moving block systems [30], which was later extended to use lazy constraints to speed up computation [31]. However, implementing a moving block system in practice is significantly harder than using virtual subsections, in which case only the design but not the control logic changes [12].

While all these methods can also be applied for ETCS L2 HTD, because virtual subsections and TTD sections have essentially the same consequences in final operation (assuming no system fails), they are not utilizing the additional degree of freedom introduced by virtual subsections. Using ETCS L2 HTD, changing a previous block layout defined by TTD sections even *after* building the network is now possible. In particular, it might be reasonable to consider a desired schedule directly because changing a block layout with a new schedule is easier than before. However, corresponding considerations are just at the beginning.

For ETCS L2 HTD, specifically, the contributions to the “Hybrid ERTMS/ETCS Level 3 Case Study” provided an important step in this direction [32]. Here, the main goal was to detect potential ambiguities in the specification to fix them in later versions. Submitted papers do this using iUML-B, Event-B, Formal B, Electrum, SysML/KAOS, and SPIN [32]. However, while these formalizations provided initial (non-ambiguous) interpretations of the ETCS L2 HTD, no precise design tasks, e.g., how to generate new layouts, can be derived from that. To the best of our knowledge, only research proposed in [33], [34], [10] explicitly considered the automatic design of hybrid train detection systems. But even those papers specified the design tasks at a rather high level, and neither formal descriptions nor results on the computational complexity of those tasks have been provided.

Overall, all related work on the development of design methods for ETCS L2 HTD has been conducted without a clear definition nor a formal understanding of the complexity of the underlying task. As a consequence, we currently have heuristics that usually generate non-optimal results, on the one hand, [34] and exact solutions which significantly suffer from huge run-times and severely limited scalability on the other [33], [10], but do not know whether this is caused by insufficient methodology or complexity reasons. Similar scheduling problems (e.g., [24]) have been proven to be NP-hard by using the alternative graph introduced in [35]. At the same time, we are not aware of formal complexity results on the design tasks considered in [33], [34], [10].

In this paper, we aim to shed light on this. To this end, we first introduce a graph model and review the relevant concepts of ETCS L2 HTD on this in Section II. Afterward, we discuss three essential design tasks resulting from the new degrees of freedom and provide clear, i.e., formal, problem definitions and examples in Section III. This includes proofs showing that these new design tasks remain complex, i.e., are NP-complete or NP-hard, respectively. By this, we provide a proper and valuable theoretical foundation (both on the problem description and its complexity) for upcoming solution-oriented research in this area. Arising methods for the automated design of such systems will be made available within the *Munich Train Control Toolkit* (MTCT) hosted on GitHub (<https://github.com/cda-tum/mtct>).

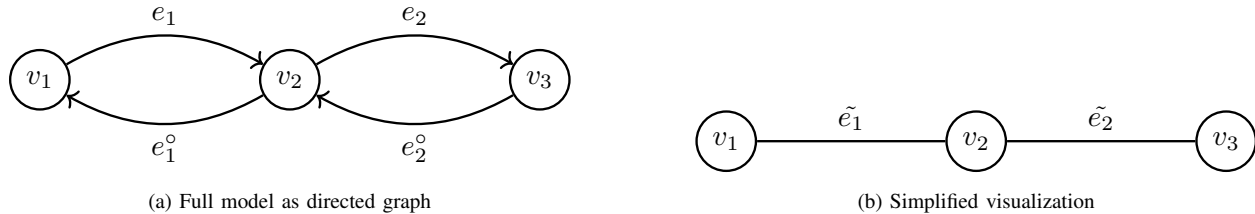


Fig. 2: Graphs representing a single track

II. PRELIMINARIES

In this section, we provide a basic definition of railway networks using graph terminology, which, step by step, is extended to cover the relevant properties. Within this, we define trains, their routes, and components related to the train control system. Based on that, in Section III we then formalize the design tasks, which might arise in the context of ETCS L2 HTD.

A. Basic Definition of Railway Networks

In the following, we derive a definition for a railway network and valid movements on it that provide the basis for formulating corresponding design and verification tasks. For this, we introduce a directed graph $G = (V, E)$ together with further properties and constraints. In a straightforward and abstract view, edges E represent tracks that connect to the nodes V representing certain *points of interests* of the network (e.g., turnouts, crossings, signals, and stations). While railway tracks do not have a direction, a notion of direction will be important when introducing trains to prevent corner cases that are cumbersome to exclude in an undirected setting. Unless special operational constraints limit the direction on a track, we add edges for both directions; see Figure 2a representing a single track on which trains can move from left to right and from right to left. However, as said, edges (v_1, v_2) and (v_2, v_1) , for example, represent the same track segment. For edge $e = (u, v) \in E$, we denote its reverse edge $e^\circ = (v, u)$ if existent. In particular, e and e° represent the same track segment and $(e^\circ)^\circ = e$.

Remark II.1. *To simplify notation and visualization, we let \tilde{e} denote both e and e° and visualize this by an undirected edge. Hence, the graph in Figure 2a simplifies to Figure 2b. Whenever we use an undirected edge \tilde{e} , it should be clear from the context to which of the two directed edges we refer. While this notation is rather imprecise, we believe it will improve the readability of some concepts as long as we bear in mind that they represent two edges simultaneously.*

Of course, an edge $e \in E$ has a natural length, e.g., in meters, which we denote by a mapping $\text{len}: E \rightarrow \mathbb{R}_{>0}$. Note that $\text{len}(e) = \text{len}(e^\circ)$ since they represent the same track segment.

Example II.2. *Fig. 3 shows a simple railway layout with a siding. Because of this, it is possible that two trains driving in opposite directions can pass each other. For example, the train driving from v_1 to v_5 can use the turnout at v_2 via v_6 and v_7 and come back to the main track using turnout v_4 and, then, drive towards v_5 . The second*

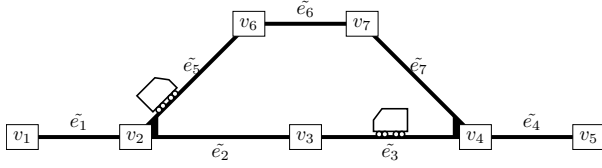


Fig. 3: Siding as a graph

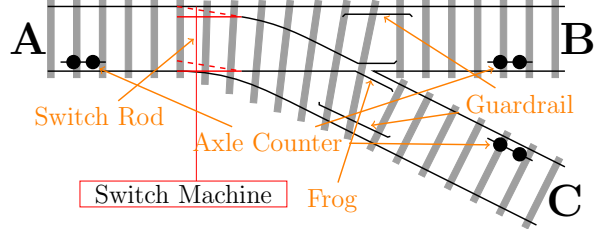


Fig. 4: Simple turnout

train, driving from v_5 to v_1 , can drive via v_4 , v_3 , and v_2 to its final destination v_1 . In this example, nodes v_2 and v_4 represent simple turnouts.

Turnouts usually do not allow trains to arbitrarily switch between the conjoining tracks. Instead, the possible ways over the turnout depend (besides others), e.g., on the direction the train is coming from. More precisely, Figure 4 shows a schematic drawing of a simple turnout as a standard right-hand switch with all its details. A train coming from the left-hand side (marked with an **A**) can take two ways: either to **B** or **C**. In contrast, a train coming from the right-hand side (either from **C** or **B**) can take only one way, namely to **A**. These restrictions on movements are not yet incorporated within the described graph framework. Hence, we introduce a successor function $s_v: \delta^{in}(v) \rightarrow \mathcal{P}(\delta^{out}(v))$ for every vertex $v \in V$, where $\delta^{in}(v)$ and $\delta^{out}(v)$ are the sets of edges entering and leaving v respectively. Then, $s_v(e_{in})$ represents the possible successor edges when entering v from a certain edge $e_{in} \in \delta^{in}(v)$.

Example II.3. In the case of Example II.2, the successor functions $\{s_v\}_{v \in V}$ are defined by

$$s_{v_3}(e) = \begin{cases} \{\tilde{e}_3\}, & \text{if } e = \tilde{e}_2 \\ \{\tilde{e}_2\}, & \text{if } e = \tilde{e}_3 \end{cases} \quad s_{v_7}(e) = \begin{cases} \{\tilde{e}_7\}, & \text{if } e = \tilde{e}_6 \\ \{\tilde{e}_6\}, & \text{if } e = \tilde{e}_7 \end{cases} \quad (1)$$

$$s_{v_6}(e) = \begin{cases} \{\tilde{e}_6\}, & \text{if } e = \tilde{e}_5 \\ \{\tilde{e}_5\}, & \text{if } e = \tilde{e}_6 \end{cases} \quad s_{v_1}(\tilde{e}_1) = s_{v_5}(\tilde{e}_4) = \emptyset \quad (2)$$

for the nodes corresponding to straight tracks without any turnouts and

$$s_{v_2}(e) = \begin{cases} \{\tilde{e}_2, \tilde{e}_5\}, & \text{if } e = \tilde{e}_1 \\ \{\tilde{e}_1\}, & \text{if } e \in \{\tilde{e}_2, \tilde{e}_5\} \end{cases} \quad s_{v_4}(e) = \begin{cases} \{\tilde{e}_3, \tilde{e}_7\}, & \text{if } e = \tilde{e}_4 \\ \{\tilde{e}_4\}, & \text{if } e \in \{\tilde{e}_3, \tilde{e}_7\} \end{cases} \quad (3)$$

for the functions corresponding to turnouts.

Using the above, we can now define a simple railway network as follows:

Definition II.4 (Railway Network). A railway network $N = (G, \text{len}, \{s_v\}_{v \in V})$ is defined by

- a directed graph $G = (V, E)$ with vertices V being the set of points of interest and edges E being the railway tracks between the aforementioned points of interest,

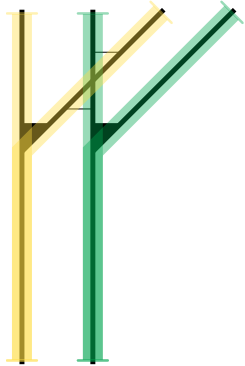


Fig. 5: Double junction, or level junction



Fig. 6: Double slip switch at Munich Central Station by Láczy [38] published under a CC BY-SA 2.0 license

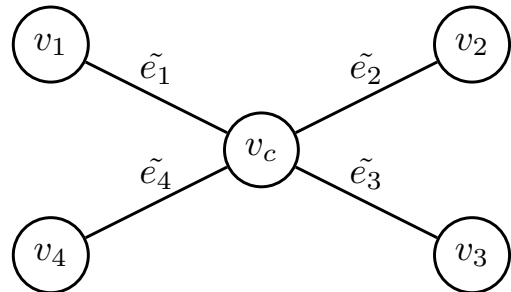


Fig. 7: Graph of crossing

- a mapping $\text{len}: E \rightarrow \mathbb{R}_{>0}$ denoting the length of each edge such that $\text{len}(e) = \text{len}(e^\circ)$ for every pair of edges $e, e^\circ \in E$, and
- a family of mappings $\{s_v\}_{v \in V}$, where $s_v: \delta^{\text{in}}(v) \rightarrow \mathcal{P}(\delta^{\text{out}}(v))$ represents the valid movements over v .

However, railway networks do not only consist of turnouts where just three tracks are connected. For example, they might also cross each other. Reasons for this might be double junctions, where simple *crossings* occur, as depicted in Figure 5 or *slip switches* as depicted in Figure 6. In these cases, the underlying graph structure is the same; however, the respective successor functions differ. In general, tracks that cross each other can be modeled as shown in Figure 7, where node v_c corresponds to the crossing point.

In the case of a simple *crossing*, e.g., where the green and yellow lines cross each other in Figure 5, a train can move from v_1 (v_2) via v_c to v_3 (v_4) or vice versa, but it is not possible that a train moves from v_1 via v_c to v_2 or v_4 . To model this movement restriction, the corresponding successor function s_{v_c} is given by

$$s_{v_c}(\tilde{e}_1) = \{\tilde{e}_3\}, \quad s_{v_c}(\tilde{e}_2) = \{\tilde{e}_4\}, \quad (4)$$

$$s_{v_c}(\tilde{e}_3) = \{\tilde{e}_1\} \quad \text{and} \quad s_{v_c}(\tilde{e}_4) = \{\tilde{e}_2\}. \quad (5)$$

On the other hand, *double slip switches* as, e.g., shown in Fig. 6 are quite common in the entry to bigger stations as their usage saves place, which is normally very limited in such. Their switch machines make it possible to change the track when going from one side to the other. In this case, the successor function at v_c changes to

$$s_{v_c}(e) = \begin{cases} \{\tilde{e}_2, \tilde{e}_3\}, & \text{if } e \in \{\tilde{e}_1, \tilde{e}_4\} \\ \{\tilde{e}_1, \tilde{e}_4\}, & \text{if } e \in \{\tilde{e}_2, \tilde{e}_3\} \end{cases}. \quad (6)$$

Likewise, a *single slip switch* looks similar to double slip switches but allows fewer movements, i.e., for 2 out of 4 entry points, there is no possibility to change the track. More precisely, we can model this using

$$s_{v_c}(\tilde{e}_1) = \{\tilde{e}_2, \tilde{e}_3\}, \quad s_{v_c}(\tilde{e}_3) = \{\tilde{e}_1\}, \quad (7)$$

$$s_{v_c}(\tilde{e}_2) = \{\tilde{e}_1, \tilde{e}_4\} \quad \text{and} \quad s_{v_c}(\tilde{e}_4) = \{\tilde{e}_2\}. \quad (8)$$

These examples demonstrate that any real-world railway network can be modeled using Definition II.4. Even if further not-mentioned points of interest occur, they can be represented analog to the above examples.

Note that the above definition is somewhat similar to the double-vertex graph introduced by Montigel [36] even though the successor functions are included in the double-vertices. At the same time, the choice to use these successor functions is similar with how valid train movements are modeled in [37]. Because of how their specific MIP model is constructed, they do not require directed edges, which we believe is vital in our solution-independent description. Hence, the necessity to define train schedules, valid movements, and the possibility to add new sections differs from their work.

In principle, it is not important which of the above infrastructure models is used. We choose successor functions, because we find this graph model to be more intuitive for the purpose of this work. The upcoming design tasks could easily also be formulated on, e.g., Montigel’s double-vertex graph model.

Using the above definition, we can also formally define valid track sequences, which could be traversed by a train and, hence, establish a connection between the edges of the graph and its successor functions.

Definition II.5 (Track Sequence). *For some $k \in \mathbb{N}$, a track sequence is a k -tuple $(e_1, \dots, e_k) \in E^k$ ($e_i = (v_i, w_i)$) such that $v_{i+1} = w_i$ for all $i = 1, \dots, k - 1$. The track sequence is valid if, and only if, $e_{i+1} \in s_{w_i}(e_i)$ for all $i = 1, \dots, k - 1$.*

Example II.6. *Consider again the network depicted in Figure 3. Assume that all edges e_i are facing right and e_i° are facing left. Then, (e_2°, e_5, e_6) is a track sequence since they form a path in the graph. However, the movement from e_2° to e_5 is forbidden by the successor function describing the logic of turnout v_2 . Hence, the track sequence is not valid. On the other hand, (e_1, e_5, e_6) is a valid track sequence.*

Finally, observe that the above description does not model reality up to every detail. For example, the defined model cannot currently model dead-end stations. For this, trains must be allowed to turn around on certain edges after a predefined waiting time. Moreover, different train control systems are implemented throughout a train network. In particular, if one considers a maximal section of the network fully equipped with an ETCS L2 HTD system, trains might enter and leave the network from and to sections with lower-level control systems. Modeling these “borders” of the railway might also be beneficial, i.e., adding places where trains can enter and leave the graph at a predefined rate. Eventually, these can be added as additional constraints. However, we believe to have covered the most important properties in the description above. Hence, it is reasonable to develop further work on the presented formulation first and, if they turn out promising, add further constraints afterward.

B. Trains and Their Schedules

Until now, we have focused on modeling the railway network without considering the trains and their movements. In this section, we define trains as well as their movements and schedules.

Definition II.7 (Train). A train $\text{tr} := (l^{(\text{tr})}, v_{\max}^{(\text{tr})}, a^{(\text{tr})}, d^{(\text{tr})})$ is a tuple, where

- $l^{(\text{tr})} \in \mathbb{R}_{>0}$ defines its length and,
- $v_{\max}^{(\text{tr})}$ defines its maximal possible speed, and
- $a^{(\text{tr})}$ and $d^{(\text{tr})}$ denote the maximal acceleration and deceleration respectively.

Train lengths do not necessarily coincide with the length of edges within a railway network $N = (G, \text{len}, \{s_v\}_{v \in V})$. In particular, trains might only be located on parts of an edge. For this, we denote a *track interval* $\text{iv} := (e, \lambda, \mu) \in E \times [0, 1] \times [0, 1]$ such that $\lambda \leq \mu$, where λ denotes the start and μ the end of the section in relative terms. A *track range* $\text{rg} := (\text{iv}_1, \dots, \text{iv}_k)$ is then given by a series of track intervals $\text{iv}_i = (e_i, \lambda_i, \mu_i)$ such that (e_1, \dots, e_k) is a valid track sequence according to Definition II.5, $\lambda_i = 0$ for every $1 < i \leq k$, and $\mu_i = 1$ for every $1 \leq i < k$. Naturally, the lengths of the edges carry over to track intervals and ranges by setting $\text{len}(\text{iv}) := (\mu - \lambda) \cdot \text{len}(e)$ and

$$\text{len}(\text{rg}) := \sum_{i=1}^k \text{len}(\text{iv}_i) = \sum_{i=1}^k (\mu_i - \lambda_i) \cdot \text{len}(e_i). \quad (9)$$

If $k \geq 2$, then we can also write

$$\text{len}(\text{rg}) = (1 - \lambda_1) \cdot \text{len}(e_1) + \sum_{i=2}^{k-1} \text{len}(e_i) + \mu_k \cdot \text{len}(e_k). \quad (10)$$

Definition II.8 (Train Route). For a given train tr , a train route $R^{(\text{tr})} := (\text{rg}_t^{(\text{tr})})_{t \geq 0}$ is a series of track ranges for every time $t \geq 0$, such that $\text{len}(\text{rg}_t^{(\text{tr})}) = l^{(\text{tr})}$.

Given some track range $\text{rg} := ((e_1, \lambda_1, \mu_1), \dots, (e_k, \lambda_k, \mu_k))$, we denote

$$\text{occ}(\text{rg}) := \{e_1, \dots, e_k\} \quad (11)$$

to be the occupied track segments. Naturally

$$\text{occ}_t^{(\text{tr})} := \text{occ}(\text{rg}_t^{(\text{tr})}) \quad (12)$$

corresponds to the segments occupied by train tr at time $t \geq 0$.

As of now, trains can theoretically move and jump around a railway network arbitrarily because the different time steps are not linked with each other. Consider two track ranges $\text{rg}^{(1)} = ((e_1^{(1)}, \lambda_1^{(1)}, \mu_1^{(1)}), \dots, (e_l^{(1)}, \lambda_l^{(1)}, \mu_l^{(1)}))$ and $\text{rg}^{(2)} = ((e_1^{(2)}, \lambda_1^{(2)}, \mu_1^{(2)}), \dots, (e_k^{(2)}, \lambda_k^{(2)}, \mu_k^{(2)}))$. They are

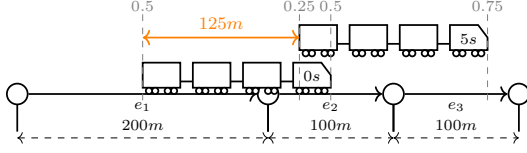


Fig. 8: A train tr at two overlapping time steps t and $t + \varepsilon$

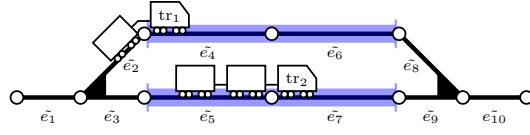


Fig. 9: Example train station

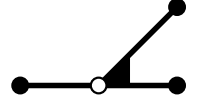


Fig. 10: Turnout

said to *intersect (in order)* if there exists an $s \in \{1, 2, \dots, \min\{l, k\}\}$ such that $e_{l-s+i}^{(1)} = e_i^{(2)}$ for all $i \in \{1, \dots, s\}$, i.e., the end of $rg^{(1)}$ and start of $rg^{(2)}$ overlap. We then define

$$\text{dist} \left(rg^{(1)}, rg^{(2)} \right) := \underbrace{\sum_{i=1}^{l-s} \left(\mu_i^{(1)} - \lambda_i^{(1)} \right) \text{len} \left(e_i^{(1)} \right)}_{\text{length of non-overlapping edges of } rg^{(1)}} + \underbrace{\left(\lambda_1^{(2)} - \lambda_{l-s+1}^{(1)} \right) \text{len} \left(e_{l-s+1}^{(1)} \right)}_{\text{length on overlapping edge}} \quad (13)$$

to be their *distance*. Using this, we can characterize the speed of a train by the following definition.

Definition II.9 (Train Speed). A train $tr := \left(l^{(tr)}, v_{max}^{(tr)}, \left(rg_t^{(tr)} \right)_{t \geq 0} \right)$ is said to respect its maximal speed if for every $t \geq 0$ and $0 < \varepsilon < \frac{l^{(tr)}}{v_{max}^{(tr)}}$ it holds that $rg_t^{(tr)}$ and $rg_{t+\varepsilon}^{(tr)}$ intersect in order with

$$\text{dist} \left(rg_t^{(tr)}, rg_{t+\varepsilon}^{(tr)} \right) \leq \varepsilon \cdot v_{max}^{(tr)}. \quad (14)$$

Its speed at time t is then given by

$$v_t^{(tr)} := \frac{d}{d\varepsilon} \left[\text{dist} \left(rg_t^{(tr)}, rg_{t+\varepsilon}^{(tr)} \right) \right]_{\varepsilon=0} = \lim_{\varepsilon \rightarrow 0} \frac{\text{dist} \left(rg_t^{(tr)}, rg_{t+\varepsilon}^{(tr)} \right)}{\varepsilon}. \quad (15)$$

Moreover, the change in speed is bound by

$$-a^{(tr)} \leq \frac{d}{dt} \left(v_t^{(tr)} \right) \leq a^{(tr)}. \quad (16)$$

Of course, further details of train movements can be modeled within this framework as additional constraints. For example, the train's acceleration and deceleration properties might depend on its current speed and/or gradient of the underlying track. Hence, braking curves defined in the official specifications can also be considered [39].

Example II.10. Consider Figure 8 with an 150m-long train positioned at $((e_1, 0.5, 1), (e_2, 0, 0.5))$, which moved to $((e_2, 0.25, 1), (e_3, 0, 0.75))$ within 5 seconds. Then, the distance $\text{dist} \left(rg_{0s}^{(tr)}, rg_{5s}^{(tr)} \right)$ is given by $\underbrace{(1 - 0.5) \cdot 200m}_{\mu_1^{(0s)} - \lambda_1^{(0s)} \text{len}(e_1)} + \underbrace{(0.25 - 0) \cdot 100m}_{\lambda_1^{(5s)} - \lambda_2^{(0s)} \text{len}(e_2)} = 125m$. Hence, we can approximate its speed by $\frac{125m}{5s} = 25 \frac{m}{s} = 90 \frac{km}{h}$.

Of course, a train does not drive anywhere on a network but has a fixed set of stations to visit or possibly even a fixed schedule. We denote a *train station* to be a subset of edges $S \subseteq E$. A train tr is said to be *in station* S at time t if $\text{occ}_t^{(tr)} \subseteq S$. It *stops* in S if additionally $v_t^{(tr)} = 0$. For example, a station can include all tracks of a certain real-world train station. If necessary, we can also restrict it to only cover a subset, e.g., if some platforms should be used solely by regional and others only by long-distance trains.

Example II.11. Consider Figure 9, a station $S = \{\tilde{e}_4, \tilde{e}_5, \tilde{e}_6, \tilde{e}_7\}$, and the two trains depicted. Here, tr_1 occupies \tilde{e}_2 and \tilde{e}_4 . Since $\tilde{e}_2 \notin S$, tr_1 is not in the train station. On the other hand, tr_2 occupies $\tilde{e}_5 \in S$ and $\tilde{e}_7 \in S$ and, hence, is in the train station.

Combining the above and adding time constraints, we are now ready to define train schedules.

Definition II.12 (Train Schedule). Let tr_1, \dots, tr_m be m trains on a railway network N . For every train tr_i , a train schedule consists of a list of m_i stations $S_1^{(tr_i)}, \dots, S_{m_i}^{(tr_i)} \subseteq E$ together with time intervals $[\underline{t}_1^{(tr_i)}, \bar{t}_1^{(tr_i)}], \dots, [\underline{t}_{m_i}^{(tr_i)}, \bar{t}_{m_i}^{(tr_i)}]$. A train tr_i respects its schedule if, for every $j \in \{1, \dots, m_i\}$ and $t \in [\underline{t}_j^{(tr_i)}, \bar{t}_j^{(tr_i)}]$, tr_i stops in $S_j^{(tr_i)}$ as defined above. In general, we assume that $\underline{t}_1^{(tr_i)} = 0$ for every train, so that $S_1^{(tr_i)}$ denotes its starting position.

C. Sections and Virtual Subsections in Railway Networks

As mentioned, trains cannot drive independently from each other, and running into each other is an unwanted situation. For this, a control system ensures that a block is occupied by at most one train, which still needs to be defined formally. To this end, *Trackside Train Detection* (TTD) systems, e.g., axle counters, are used that collect corresponding information on the occupation of blocks. Accordingly, corresponding blocks are frequently also called TTD sections.

Unfortunately, an edge $e \in E$ of the graph definition from above does not match a TTD section. A TTD section starts and ends with, e.g., axle counters. Three axle counters are usually used for a turnout: one for each of the three tracks that meet at the turnout (see also Figure 4). In this case, the switch's three-track segments (edges) belong to the same block. To model such TTD sections, we introduce another helping function on the vertices, also called border function, $C_V: V \rightarrow \{0 = \text{no border}, 2 = \text{TTD-Border}\}$, which indicates if a certain vertex is equipped with trackside train detection and, thus, corresponds to a border between two TTD sections. A simple turnout would typically correspond to one central node of value 0 and three outer nodes of value 2, in which case the respective edges will have a rather small length corresponding to the distance between the turnout's center and the axle counters. This is depicted in Figure 10: A black filled node \bullet visualizes a vertex which represents a TTD-Border, i. e., C_V evaluates to 2 = TTD-Border. A white filled circle \circ visualizes a vertex which represents no TTD-Border, i. e., C_V evaluates to 0 = no border. Abstractly, we refer to a vertex $v \in V$ with $C_V(v) = k$ to be a border of level k . Further, we define by $L'_k(G) := (V'_k, E'_k)$ an *undirected* graph interchanging the role of vertices and edges and additionally ignoring borders of level $\geq k$.

Formally, we define $V'_k := E$ and

$$E'_k := \{\{e_1, e_2\} \subseteq V'_k : e_2 = e_1^\circ \text{ or } \tilde{e}_1 \cap \tilde{e}_2 \cap C_K \neq \emptyset\}, \quad (17)$$

where $C_K := \{v \in V : C_V(v) < k\}$ are the nodes with border value less than k . Now, if two nodes within $L'_2(G)$ are connected, then their corresponding track segments are not separated by any TTD; hence, they fall within one TTD section. In particular, if we set

$$TTDs := \{T_1, \dots, T_m\} = CC(L'_2(G)), \quad (18)$$

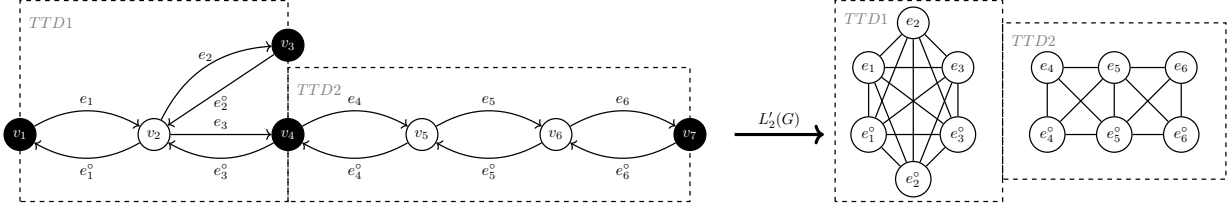


Fig. 11: TTD sections

where $CC(\cdot)$ denotes the connected components, then T_1, \dots, T_m correspond exactly to the TTD sections. Edges within a section $T_i \in TTDs$ are not separated by any TTD, whereas a TTD section $T_i \in T$ is separated from any other TTD section $T_j \in TTDs - \{T_i\}$ by corresponding borders. Naturally, this provides a partition of the edges of the original graph G .

Example II.13. Consider a graph with 7 vertices as depicted in Figure 11. Vertices v_1 to v_4 represent a simple turnout as in Figure 4 connected to a straight track up to v_7 . As before, \bullet corresponds to a TTD border and \circ to no border. Figure 11 shows how the network transforms into $L_2^*(G)$, which consists of two connected components. The vertices of each connected component (i.e., edges in the original graph) compose one TTD section. The corresponding sections are marked in both graphs in Figure 11.

With the introduction of hybrid train detection, sections do not have to be separated by TTDs. In particular, this allows splitting existing TTD sections into virtual subsections (VSS). To also model these, we add a VSS-Border element to the border function as follows: $C_V: V \rightarrow \{0 = \text{no border}, 1 = \text{VSS-Border}, 2 = \text{TTD-Border}\}$. Similarly to Equation (18) the VSS sections V_1, \dots, V_n are given by the connected components of $L_1'(G)$, i.e.,

$$\{V_1, \dots, V_n\} := CC(L_1'(G)). \quad (19)$$

A key concept in train control is the division of the tracks into multiple blocks. A block is only allowed to be occupied by at most one train. In the context of ETCS L2 HTD, these blocks are given by the VSS sections that function essentially analog to their TTD counterparts. Hence, the respective control condition can be formalized as follows and depicted in Figure 12.

Definition II.14 (VSS Condition). Let V_1, \dots, V_n be the VSS-sections as defined in Equation (19) and tr_1, \dots, tr_m be the trains operating on the respective railway network. The train control system ensures the following condition for every $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$, $k \in \{1, \dots, m\} - \{j\}$, and $t \geq 0$:

$$V_i \cap \text{occ}_t^{(tr_j)} \neq \emptyset \Rightarrow V_i \cap \text{occ}_t^{(tr_k)} = \emptyset. \quad (20)$$

Overall, given a railway network N , a certain number of trains, and a train schedule, the following properties are desired whenever considering a design task:

- Every train respects its maximal speed (Definition II.9) and schedule (Definition II.12).
- The VSS condition is satisfied (Definition II.14).

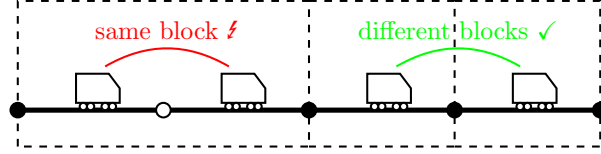


Fig. 12: VSS condition

Again, note that one can, in theory, model the behavior of the control system as detailed as desired. For example, one should consider that the entire braking distance in front of a train should not be occupied in the VSS condition (Definition II.14). This could, e.g., be achieved by including this in the train's occupation as in [10]. Moreover, interlockings could be considered using additional constraints. Nevertheless, we believe the level of detail provided in this paper suffices for defining relevant design tasks.

III. DESIGN TASKS AND THEIR COMPLEXITY

In this section, we define different design tasks that might occur in ETCS L2 HTD systems and could be valuable to automate. They are formulated independent from whichever solution method might be chosen to solve them.

A. Verification of Train Schedules on ETCS L2 HTD Layouts

Note that a desired train schedule might be impossible on a particular railway network since only one train can be located on each TTD/VSS section, a restricted track speed, or a limited train speed. Even disregarding the timing aspect, a desired routing might not be satisfiable. Hence, verifying whether a given setup is realizable on a given track layout is attractive.

VERIFICATION OF TRAIN ROUTES (TR-VER)

Input: A railway network $N = (G, \text{len}, \{s_v\}_{v \in V})$; border function $C_V: V \rightarrow \{0, 1, 2\}$; trains $\text{tr}_1, \dots, \text{tr}_m$; a list of stations $S_0^{(\text{tr}_i)}, \dots, S_{m_i}^{(\text{tr}_i)}$ for every train together with minimal stopping times $\Delta t_0^{(\text{tr}_1)}, \dots, \Delta t_{m_i}^{(\text{tr}_i)} \in \mathbb{R}^{\geq 0}$.

Task: Do there exist train routes $R^{(\text{tr}_1)}, \dots, R^{(\text{tr}_m)}$ such that the VSS condition is fulfilled, all trains respect their speed, and train tr_i starts in $S_0^{(\text{tr}_i)}$ and stops for at least time $\Delta t_j^{(\text{tr}_i)}$ in station $S_j^{(\text{tr}_i)}$? If yes, provide a certificate.

We can also add a predefined schedule to the problem and obtain the following variant.

VERIFICATION OF TRAIN SCHEDULES (TS-VER)

Input: A railway network $N = (G, \text{len}, \{s_v\}_{v \in V})$; border function $C_V: V \rightarrow \{0, 1, 2\}$; trains $\text{tr}_1, \dots, \text{tr}_m$; a train schedule for tr_1 to tr_m .

Task: Do there exist train routes $R^{(\text{tr}_1)}, \dots, R^{(\text{tr}_m)}$ such that the VSS condition is fulfilled and all trains respect their speed and schedule? If yes, provide a certificate.

Example III.1. Consider the train network in Figure 13 and three trains tr_1 to tr_3 . As before, \bullet corresponds to a TTD border and \circ to no border. Also, we are given one real-world train station $S_0 := \{\tilde{e}_5, \tilde{e}_{10}\}$, which is marked

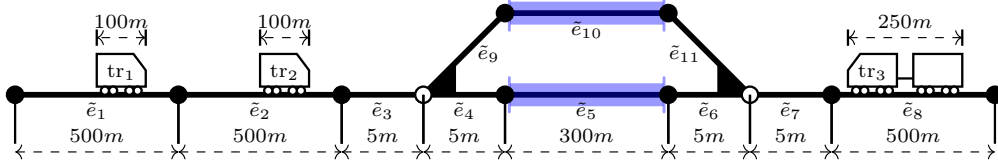


Fig. 13: Infeasible verification task ($t = 0$)

in blue. Furthermore the starting positions are given as “stations” $S_1 := \{\tilde{e}_1\}$, $S_2 := \{\tilde{e}_2\}$, and $S_3 := \{\tilde{e}_8\}$. Then, consider the following schedule

$$\text{tr}_1: S_1 \text{ for } t \in [0, 1] \text{ and } S_0 \text{ for } t \in [240, 300]$$

$$\text{tr}_2: S_2 \text{ for } t \in [0, 1] \text{ and } S_0 \text{ for } t \in [120, 300]$$

$$\text{tr}_3: S_3 \text{ for } t \in [0, 1] \text{ and } S_0 \text{ for } t \in [180, 300],$$

where time is measured in seconds. Observe that the station S_0 only consists of two blocks; hence a maximum of two trains can stop at a time. However, the schedule requires all three trains to be stopped at the station at time $t = 200$, for example. Hence, there are no train routes such that the VSS condition is fulfilled, the trains respect their schedule, and the verification task returns infeasibility.

Observe that the routing on a fixed VSS layout might differ slightly from systems using trackside oriented train detection. This is because trains equipped with *train integrity monitoring* (TIM) can report VSS sections free independently of their corresponding TTD sections. Any train can follow closely. At the same time, trains without TIM need TTD hardware (e.g., axle counters) to report a section as free. Hence, a train can only follow if a non-TIM train has cleared an entire TTD section.

If all trains are equipped with TIM, the situation simplifies. In that case, there is no logical difference between TTD and VSS sections and the block layout is fixed for the verification tasks defined above. Hence, TS-VER and TR-VER are essentially equivalent to the respective tasks on classical layouts in this (simplified) case. Scheduling under such classical signaling systems is known to be NP-hard [35], [24]. In particular, it is reasonable to assume that TS-VER and TR-VER have the same complexity. At the same time, there are minor differences to the scheduling problems considered, so that the proof does not carry over formally. While the following complexity result is not unexpected, we formally include them for reasons of completeness. Moreover, it serves as a basis for the following design tasks in Sections III-B and III-C, which are tailored to ETCS L2 HTD.

Theorem III.2. *TS-VER and TR-VER are NP-complete.*

Proof. Given a certificate, we can check feasibility using the definitions in polynomial time, hence, TR-VER and TS-VER are in NP.

To show NP-hardness, we first polynomially reduce 3SAT, which is known to be NP-complete [40], to TR-VER. The proof is inspired by Yu’s paper on optimal multi-robot path planning [41]. However, the proof simplifies in

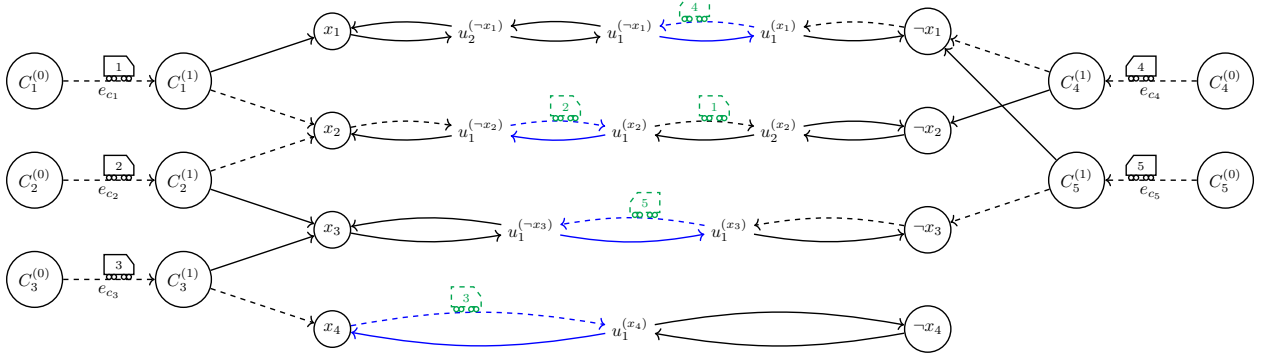


Fig. 14: Reduction to TR-VER

our setting because stations can consist of multiple edges compared to only one node in his paper. For reduction, we are given an instance of 3SAT on variables x_1, \dots, x_n defined by clauses $\mathcal{C} = \{C_1, \dots, C_m\}$, each consisting of at most three literals. From this instance, we construct a railway network. We add two nodes, $c_j^{(0)}$ and $c_j^{(1)}$, for every clause and connect them by a single directed edge $e_{c_j} = (c_j^{(0)}, c_j^{(1)})$. Furthermore, we have $2n$ nodes corresponding to all literals, i.e., x_i and $\neg x_i$. Denote $\mathcal{L} := \{x_1, \dots, x_n, \neg x_1, \dots, \neg x_n\}$ to be the set of all literals. An edge $(c_j^{(1)}, l)$ is added for $l \in \mathcal{L}$ if, and only if, $l \in C_j$.

Next, we add tracks connecting any literal with its negation in both directions, including various nodes in between. For a literal $l \in \mathcal{L}$ let $\#l := |\{c \in \mathcal{C} : l \in C\}|$ be the number of clauses l appears in. We add nodes $u_1^{(l)}, \dots, u_{\#l}^{(l)}$ for every $l \in \mathcal{L}$ and additionally characterize the node corresponding to $\neg l$ as $u_{\#l+1}^{(l)}$. For every $k \in \{1, \dots, \#l\}$, we add $(u_k^{(l)}, u_{k+1}^{(l)})$ and $(u_{k+1}^{(l)}, u_k^{(l)})$. Finally, we complete the connection by adding edges $(u_1^{(x_i)}, u_1^{(\neg x_i)})$ for every $i \in \{1, \dots, n\}$.

The resulting network consists of

$$2m + \underbrace{2n}_{\leq 6m} + \underbrace{\sum_{l \in \mathcal{L}} \#l}_{=3m} \leq 11m = \mathcal{O}(m) \quad (21)$$

nodes and

$$m + 3m + \underbrace{2n}_{\leq 6m} + 2 \underbrace{\sum_{i=1}^n (\#x_i + \#\neg x_i)}_{=3m} \leq 16m = \mathcal{O}(m) \quad (22)$$

edges. In particular, there is no exponential blow-up in the problem size, and the reduction is polynomial.

As an example consider the 3SAT instance $\bigwedge_{i=1}^5 C_i$ with

$$\begin{aligned} C_1 &= x_1 \vee x_2, & C_2 &= x_2 \vee x_3, & C_3 &= x_3 \vee x_4, \\ C_4 &= \neg x_1 \vee \neg x_2 \text{ and} & C_5 &= \neg x_1 \vee \neg x_3. \end{aligned} \quad (23)$$

We choose an instance with strictly less than three literals per clause to get a cleaner figure focusing on the relevant. The reduction is analog even for general SAT instances. Figure 14 shows the graph resulting from the 3SAT instance in Equation 23 together with respective trains, which we add in the next proof step.

First, for valid train movements, we do not allow trains to turn around on the connecting two-way track; hence we characterize the respective successor functions accordingly. More precisely, for every $l \in \mathcal{L}$, $k \in \{1, \dots, \#l + 1\}$, and $e \in \delta^{in} \left(u_k^{(l)} \right)$ it holds

$$s_{u_k^{(l)}}(e) = \left\{ e' \in \delta^{out} \left(u_k^{(l)} \right) : e' \neq e^\circ \right\}. \quad (24)$$

For all other (not yet specified) pairs of vertices v and edges e , there are no restrictions, i.e., $s_v(e) = \delta^{out}(v)$.

Finally, we add one train tr_{c_j} per clause $c_j \in \mathcal{C}$. To simplify, we assume that $\text{len}(e) = 1$ for every edge e , every vertex is a TTD border, i.e., $C_V(v) = 2$ for every vertex v , and every train tr_{c_j} has length $l^{(\text{tr}_{c_j})} = 1$ and maximal speed $v_{max}^{(\text{tr}_{c_j})} = 1$ unit length per time unit. Each train has two stations associated with it, namely, it starts at $S_0^{(\text{tr}_{c_j})} = \{e_{c_j}\}$ and wants to visit

$$S_1^{(\text{tr}_{c_j})} = \left\{ e_l := \left(u_1^{(-l)}, u_1^{(l)} \right) : l \in c_j \right\} \quad (25)$$

at any time. Also, set $\Delta t_i^{(\text{tr}_{c_j})} = 0$ for every $i \in \{0, 1\}$ and $j \in \{1, \dots, m\}$. In Figure 14, black trains represent the starting positions corresponding to the respective clauses. The stations S_1 are marked in blue.

Now assume we are given an oracle for TR-VER and a respective solution to the instance defined above. Note that whenever a train enters the track between some literal l and its negation, it can only advance towards $-l$ and never leave. In particular, a train tr can pass at most one literal $l \in \mathcal{L}$ and visit at most one of the edges in $S_1^{(\text{tr})}$. Once a train visits edge e_l , no train can ever visit e_{-l} , because they would need to meet somewhere on the track going in opposite directions. Since for edge e , its reverse e° is always in the same TTD section, this is not possible in a feasible routing. In particular, it is well-defined to set x_i to be true if some train occupies e_{x_i} at any time, and x_i to be false if some train occupies e_{-x_i} at any time. Variables that are not set by a train can be chosen arbitrarily, e.g., we can choose them to be true. By doing so, we have found a feasible solution to the original 3SAT instance.

On the other hand, assume we are given a feasible solution to the 3SAT instance. Choose $l_j \in c_j$ such that l_j evaluates to True for every $j \in \{1, \dots, m\}$. We route train tr_{c_j} to e_{l_j} on the railway network. Note that at most $\#l$ trains can enter through a node corresponding to $l \in \mathcal{L}$. Since there are exactly $\#l$ edges “behind” the station, in theory, all these trains can visit the corresponding station in a feasible routing at some point in time and then move to one of the later edges to stop without interfering without any other train. Hence, such a routing is possible. Equivalently, if the oracle for TR-VER concludes infeasibility, the respective 3SAT instance is itself infeasible.

Combining the above, we have argued that a 3SAT instance is feasible if, and only if, the corresponding instance of TR-VER is. Hence, we have polynomially reduced 3SAT to TR-VER, proving the latter’s NP-hardness.

Exemplary, in Figure 14, a feasible solution to TR-VER is depicted with dotted lines. It translates to a feasible solution of Equation 23, namely $x_2 = x_4 = \text{True}$ and $x_1 = x_3 = \text{False}$.

When considering the additional timing aspect of TS-VER, we only need to slightly change the reasoning from above. We extend the stations to also include the buffer edges behind them, i.e.,

$$S_1^{(\text{tr}_{c_j})} = \bigcup_{l \in c_j} \left(\{e_l\} \cup \bigcup_{i=1}^{\#l} \left\{ \left(u_i^{(l)}, u_{i+1}^{(l)} \right) \right\} \right). \quad (26)$$

In particular, once a train enters its respective station, it will never leave it again because (in fact) it cannot do that. Note that for train tr_{c_j} to reach the furthest point on the edges corresponding to $l \in c_j$, there is only one

path, namely $(c_j^{(0)}, c_j^{(0)}, l = u_{\#-l+1}^{(-l)}, u_{\#-l}^{(-l)}, \dots, u_1^{(-l)}, u_1^{(l)}, \dots, u_{\#l+1}^{(l)})$, consisting of $\#l + \#-l + 3$ edges. If we were to route the trains one after another (which is possible because they all start on different edges and their paths cannot cross any starting position of another train), they would take time

$$T_{max} := \sum_{j=1}^m \max_{l \in c_j} (\underbrace{\#l}_{\leq m} + \underbrace{\#-l}_{\leq m} + 3) \leq m \cdot (2m + 3) = \mathcal{O}(m^2) \quad (27)$$

Hence, if we choose that every train has to stop in $S_1^{(\text{tr}_{c_j})}$ in the interval $[T_{max}, \infty]$ (∞ can be replaced by any number $\geq T_{max}$), then the schedule is satisfiable if, and only if, a feasible routing exists when neglecting the timing aspect. Hence, an oracle that solves TS-VER would in turn also solve 3SAT. This constitutes another polynomial-time reduction from 3SAT to TS-VER, which proves NP-hardness of the latter.

From the above reasoning, it can be inferred that TR-VER and TS-VER are NP-complete, which concludes the proof. \square

The attentive reader might object that many parts of real-world railway networks result in planar graphs within our framework and, thus, considering arbitrary graphs in this proof might be too general. Note, however, that we can restrict ourselves to the Planar Monotone 3SAT, which was also proven to be NP-complete [42]. Analog to the reasoning by Yu, we obtain planar instances of the respective verification tasks when applying the same reduction [41]. Hence, the reasoning also holds even when restricting TS-VER and TR-VER to planar graphs.

Finally, observe that in a verification setting no VSS is created. Instead, the layout is fixed. Hence, the above results essentially carry over to fixed-block systems such as ETCS L2.

B. Generation of VSS Layouts

As mentioned before, the potential of hybrid train detection is that virtual subsections can be added without requiring expensive track-side detection hardware. In particular, what adding VSS sections means within our framework needs to be formalized. Namely, they correspond to inserting a VSS border vertex to separate a given edge into two, or rather split both directional edges at the same position as follows.

Definition III.3 (VSS Addition Operator). *For an edge $e_\star = (v_1, v_2) \in E$ and a scalar $\rho \in (0, 1)$, the VSS addition operator $\psi_{e_\star}^\rho$ takes as an input a railway network $N = (G, \text{len}, \{s_v\}_{v \in V})$, a border function C_V , and a set of stations S . It then adds a VSS border $\rho \cdot \text{len}(e_\star)$ away from v_1 . More formally, it outputs a new network $N' = (G', \text{len}', \{s'_v\}_{v \in V'})$, a border function $C_{V'}$, and a set of stations S' as follows:*

The vertices are expanded by a new vertex denoting a new VSS border, i.e.,

$$V' = V \cup \{v_{new}\} \quad C_{V'}(v) = \begin{cases} C_V(v), & \text{if } v \in V \\ 1, & \text{if } v = v_{new} \end{cases}. \quad (28)$$

Edges e_\star and possibly e_\star° are split at the new vertex into $e_1 := (v_1, v_{new})$ and $e_2 := (v_{new}, v_2)$ plus possibly their reverse edges, i.e., using

$$E^{\Delta^-} := \{e_\star, e_\star^\circ\} \cap E \quad E^{\Delta^+} := \begin{cases} \{e_1, e_2\}, & \text{if } e_\star^\circ \notin E \\ \{e_1, e_2, e_1^\circ, e_2^\circ\}, & \text{if } e_\star^\circ \in E \end{cases} \quad (29)$$

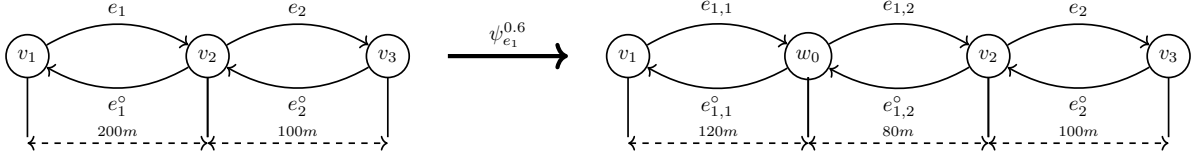


Fig. 15: VSS addition operator

we have

$$E' = (E - E^{\Delta^-}) \cup E^{\Delta^+}. \quad (30)$$

The lengths are updated according to parameter ρ , i.e.,

$$\text{len}'(e) = \begin{cases} \text{len}(e), & \text{if } e \in E \\ \rho \text{len}(e_\star), & \text{if } e \in \{e_1, e_1^\circ\} \\ (1 - \rho) \text{len}(e_\star), & \text{if } e \in \{e_2, e_2^\circ\} \end{cases}. \quad (31)$$

The successor functions $\{s'_v\}_{v \in V'}$ are updated accordingly to represent the same movements in practice, i.e., using $E_1 := \{e \in \delta_E^{\text{in}}(v_1) : e_\star \in s_{v_1}(e)\}$ and $E_2 := \{e \in \delta_E^{\text{in}}(v_2) : e_\star^\circ \in s_{v_2}(e)\}$ we have

$$s'_v = s_v \text{ for all } v \in V' - \{v_1, v_2, v_{\text{new}}\} \quad s'_{v_{\text{new}}}(e) = \begin{cases} \{e_2\}, & \text{if } e = e_1 \\ \{e_1^\circ\}, & \text{if } e = e_2^\circ \end{cases} \quad (32)$$

$$s'_{v_1}(e) = \begin{cases} (s_{v_1}(e) - \{e_\star\}) \cup \{e_1\}, & \text{if } e \in E_1 \\ s_{v_1}(e_1^\circ), & \text{if } e = e_1^\circ \\ s_{v_1}(e), & \text{otherwise} \end{cases} \quad s'_{v_2}(e) = \begin{cases} (s_{v_2}(e) - \{e_\star\}) \cup \{e_2^\circ\}, & \text{if } e \in E_2 \\ s_{v_2}(e_\star), & \text{if } e = e_2 \\ s_{v_2}(e), & \text{otherwise} \end{cases} \quad (33)$$

If a station includes e_\star or its reverse, it is updated accordingly, i.e., using $f(e_\star) := \{e_1, e_2\}$, $f(e_\star^\circ) := \{e_1^\circ, e_2^\circ\}$, and

$$\Delta^-(S) := \{e_\star, e_\star^\circ\} \cap S \quad \Delta^+(S) := \bigcup_{e \in \Delta^-(S)} f(e) \quad (34)$$

we have

$$S' = \{(S - \Delta^-(S)) \cup \Delta^+(S) : S \in \mathcal{S}\}. \quad (35)$$

We denote by Ψ the set of all VSS addition operators.

Example III.4. Consider the graph depicted in Figure 15. Assume, we want to separate the track between v_1 and v_2 into two VSS sections, which corresponds to adding a VSS border, say w_0 . The first section has a length of $0.6 \cdot 200m = 120m$, the latter of $80m$. The transformation is shown in Figure 15 and corresponds to the operator $\psi_{e_1}^{0.6}$, which is equivalent to $\psi_{e_1^\circ}^{0.4}$.

By leaving the TTDs fixed and only adding VSS sections, we can change the block structure at a low cost. Because of this, it might be possible to change the virtual subsections when a new train schedule is introduced.

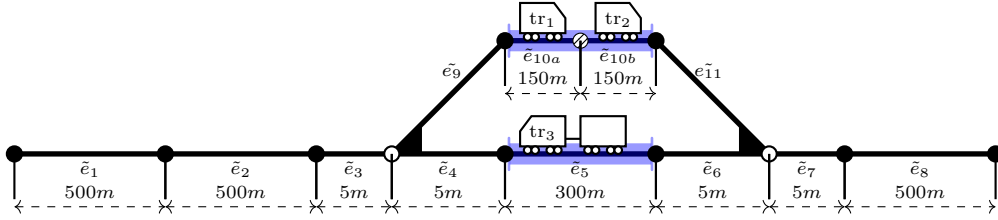


Fig. 16: Generation of VSS layout ($t = 300$)

Hence, we can somewhat “overfit” to the current schedule and do not have to solely rely on schedule-independent KPIs (e.g., headway). This leads to the following design task of generating VSS layouts.

GENERATION OF VSS LAYOUT (VSS-GEN)

Input: A railway network $N = (G, \text{len}, \{s_v\}_{v \in V})$; border function $C_V: V \rightarrow \{0, 1, 2\}$; trains $\text{tr}_1, \dots, \text{tr}_m$; a train schedule for tr_1 to tr_m .

Task: Determine a minimal number of VSS addition operators $\psi^{(1)}, \dots, \psi^{(k)} \in \Psi$ (and train routes $R^{(\text{tr}_1)}, \dots, R^{(\text{tr}_m)}$) such that on $\psi^{(k)} \circ \dots \circ \psi^{(1)}(\cdot)$ the VSS condition is fulfilled and all trains respect their speed and schedule or output that no such operators exist.

If possible, we try to keep the number of additional virtual subsections small to keep it tractable for the control system during operation. In particular, we do not require a hard optimum, because one section more or less will likely not pose any additional costs. On the other hand, it might be interesting to still have a minimal number of subsections to ensure a given schedule. In a later stage, one could then add more subsections to improve robustness; if the generated VSS layout from this design task is minimal, there might be more space to add subsections to improve robustness.

Example III.5. Again, consider the setting from Example III.1 and Figure 13. If we separate \tilde{e}_{10} into two virtual subsections by including a VSS border (\otimes), the train station S_0 has three blocks. In fact, we can now route the trains as shown in Figure 16, i.e., in the end, trains tr_1 and tr_2 can occupy the newly created subsections and train tr_3 the other track. Hence, by adding one VSS border, we have made the previously infeasible schedule feasible. Note that we could also have added additional subsections, e.g., split track \tilde{e}_5 and have tr_3 occupy two blocks in the end. However, as said, our objective is to add rather few VSS borders, and one additional VSS border suffices to make the schedule feasible.

Since VSS-GEN is not a decision problem, NP-completeness is not defined in this context. However, NP-hardness can be shown by relating it to one of the previously defined verification tasks, where we use the more general definition that a problem is NP-hard if it can be reduced from some NP-complete problem, hence, not requiring the NP-hard problem to be of decision type [43].

Theorem III.6. *VSS-GEN is NP-hard.*

Proof. Consider an instance of TS-VER. Assume we are given an oracle to solve any instance of VSS-GEN. The given TS-VER instance constitutes valid input to VSS-GEN. Let k^* be the number of VSS addition operators used in an optimal solution of VSS-GEN on the respective input or let $k^* = \infty$ if no feasible solution exists. Then, the solution to the given TS-VER instance is “yes” if, and only if, $k^* = 0$. In particular, we have polynomially reduced TS-VER to VSS-GEN. The claim follows directly from Theorem III.2. \square

Remark III.7. *Instead of adding additional vertices in a continuous setting using VSS addition operators, we might also consider a discretized version, which might be easier to handle computationally. In this setting, we would require the railway network N to consist of all nodes that could possibly become a VSS border. Adding a VSS border is then equivalent to changing $C_V(v)$ from 0 to 1 for some $v \in V$ with $|\Gamma(v)| \leq 2$, where Γ denotes the neighborhood (ignoring the direction of edges).*

C. Schedule Optimization Using the Potential of VSS

In a final design task, we want to analyze the potential of VSS sections for optimizing schedules. In a simplified version (ignoring, e.g., guaranteed changes at a station), several trains should visit a particular set of train stations as fast as possible. On the other hand, we might have a maximal number of VSS sections the control system can safely handle. Hence, we are interested in the optimal position of a predefined number of VSS borders.

SCHEDULE OPTIMIZATION (VSS-OPT)

Input: A railway network $N = (G, \text{len}, \{s_v\}_{v \in V})$; border function $C_V: V \rightarrow \{0, 1, 2\}$; trains $\text{tr}_1, \dots, \text{tr}_m$; a list of stations $S_0^{(\text{tr}_i)}, \dots, S_{m_i}^{(\text{tr}_i)}$ for every train together with minimal stopping times $\Delta t_0^{(\text{tr}_1)}, \dots, \Delta t_{m_i}^{(\text{tr}_i)} \in \mathbb{R}^{\geq 0}$; an upper bound K_{max} .

Task: Generate a feasible VSS layout with at most K_{max} VSS borders and a schedule, such that train tr_i starts in $S_0^{(\text{tr}_i)}$, stops for at least time $\Delta t_j^{(\text{tr}_i)}$ in station $S_j^{(\text{tr}_i)}$, and the trains travel times are minimized. As an objective one might, e.g., minimize the (weighted) sum of travel times or the longest travel time.

Theorem III.8. *VSS-OPT is NP-hard.*

Proof. Let an instance of TR-VER be given and assume an oracle for VSS-OPT. By additionally choosing $K_{max} = 0$, this constitutes valid input to VSS-OPT. If the oracle to VSS-OPT returns a feasible solution, then the answer to the respective TR-VER-instance is “yes”, otherwise “no”. In particular, we have polynomially reduced TR-VER to VSS-OPT. The claim follows directly from Theorem III.2. \square

IV. CONCLUSIONS

In this paper, we formalized important properties of ETCS L2 HTD and corresponding design tasks. Moreover, we argued that the framework could eventually also model more realistic train movements as additional constraints by stating some examples of possible extensions in Section II. While the verification tasks in Section III-A are

analog to older level systems (since the block layout is fixed), the two latter tasks (Sections III-B and III-C) make use of the new freedom provided by virtual subsections. Furthermore, we proved that these tasks are NP-hard (or NP-complete, respectively). By this, we provided a theoretical foundation for further developing corresponding design methods for ETCS L2 HTD. These methods will be integrated into the Munich Train Control Toolkit (MTCT) hosted on GitHub (<https://github.com/cda-tum/mtct>).

The results confirm that previously proposed heuristics and exact methods (presented, e.g., in [33], [34]), which only generate non-optimal results or have a severely limited scalability, respectively, do not suffer from insufficient methodology but complexity reasons. Overall, for the first time, a solid basis for developing (ideally scalable) design automation solutions incorporating the freedom allowed by virtual subsections within ETCS L2 HTD is provided. These methods might be a valuable addition to the already existing work on fixed-block control systems summarized in Section I. It will also be interesting to see how these methods tailored to hybrid train detection systems perform compared to previously designed methods that do not consider the specific schedule for generating signaling layouts.

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