

An Algebraic Representation of the Fixed-Point Closure of $*$ -Continuous Kleene Algebras

A Categorical Chomsky-Schützenberger Theorem

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Abstract

The family $\mathcal{R}X^*$ of regular subsets of the free monoid X^* generated by a finite set X is the standard example of a $*$ -continuous Kleene algebra. Likewise, the family $\mathcal{C}X^*$ of context-free subsets of X^* is the standard example of a μ -continuous Chomsky algebra, i.e. an idempotent semiring that is closed under a well-behaved least fixed-point operator μ . For arbitrary monoids M , $\mathcal{C}M$ is the closure of $\mathcal{R}M$ as a μ -continuous Chomsky algebra, more briefly, the fixed-point closure of $\mathcal{R}M$. We provide an algebraic representation of $\mathcal{C}M$ in a suitable product of $\mathcal{R}M$ with C'_2 , a quotient of the regular sets over an alphabet Δ_2 of two pairs of bracket symbols. Namely, $\mathcal{C}M$ is isomorphic to the centralizer of C'_2 in the product of $\mathcal{R}M$ with C'_2 , i.e. the set of those elements that commute with all elements of C'_2 . This generalizes a well-known result of Chomsky and Schützenberger (1963) and admits us to denote all context-free languages over finite sets $X \subseteq M$ by regular expressions over $X \cup \Delta_2$ interpreted in the product of $\mathcal{R}M$ and C'_2 . More generally, for any $*$ -continuous Kleene algebra K the fixed-point closure of K can be represented algebraically as the centralizer of C'_2 in the product of K with C'_2 .

Keywords: idempotent semiring, fixed-point-closure, regular language, context-free language, $*$ -continuous Kleene algebra, μ -continuous Chomsky algebra, Chomsky-Schützenberger theorem

1. Introduction

The classical Chomsky-Schützenberger theorem of Formal Language Theory relates the set $\mathcal{C}X^*$ of context-free languages of the free monoid X^* over the finite set X to the set $\mathcal{R}(X \cup \Delta_2)^*$ of regular languages over an extension of X by a set $\Delta_2 = \{b, d, p, q\}$ of two bracket pairs b, d and p, q . It says that every $L \in \mathcal{C}X^*$ is the image of the intersection $R \cap D_2(X)$ of a regular language $R \in \mathcal{R}(X \cup \Delta_2)^*$ with the Dyck language $D_2(X) \in \mathcal{C}(X \cup \Delta_2)^*$ of balanced bracketed strings under the homomorphism $h_{X^*} : (X \cup \Delta_2)^* \rightarrow X^*$ that keeps elements of X fixed and maps those of Δ_2 to the unit of X^* . This result of Chomsky and Schützenberger (1963) can be stated as

$$(CST) \quad \mathcal{C}X^* \subseteq \{h_{X^*}(R \cap D_2(X)) \mid R \in \mathcal{R}(X \cup \Delta_2)^*\}.$$

Intuitively, for $L = h_{X^*}(R \cap D_2(X))$, the set $R \cap D_2(X)$ consists of the sentences of L enriched by begin- and end-markers of their phrases according to some context-free grammar for L , i.e. by

their linearized parse trees; the two bracket pairs of Δ_2 are used to encode a begin- and end-marker for each phrase category of the grammar. The reverse inclusion

$$(\text{reverse CST}) \quad \mathcal{C}X^* \supseteq \{ h_{X^*}(R \cap D_2(X)) \mid R \in \mathcal{R}(X \cup \Delta_2)^* \}$$

is trivial, since the intersection of a context-free set with a regular one and the homomorphic image of a context-free set are context-free. (It will no longer be trivial in the generalizations considered below.) Thus, balanced brackets are at the heart of the extension from regular to context-free sets.

To extend this relation $\mathcal{C}X^* = \{ h_{X^*}(R \cap D_2(X)) \mid R \in \mathcal{R}(X \cup \Delta_2)^* \}$ from finitely generated free monoids X^* to arbitrary monoids M , we define $\mathcal{R}M$ and $\mathcal{C}M$ as suitable closures of the semiring $\mathcal{F}M$ of finite subsets of M (with union as addition and elementwise product as multiplication) within the semiring $\mathcal{P}M$ of all subsets of M . Namely, the family $\mathcal{C}M$ of context-free subsets of M is the closure of $\mathcal{F}M$ under components of least solutions of finite systems of inequations

$$y_1 \geq p_1(y_1, \dots, y_n), \dots, y_n \geq p_n(y_1, \dots, y_n) \quad (1)$$

between variables y_i and polynomials $p_i \in \mathcal{C}M[y_1, \dots, y_n]$. Notice that each such inequation system does have a least solution in $\mathcal{P}M$, so we can add the components of this solution to $\mathcal{C}M$. By adding further unknowns and inequations for the coefficients from $\mathcal{C}M$, we can ultimately do with polynomials $p_i \in \mathcal{F}M[y_1, \dots, y_n]$ in (1). The family $\mathcal{R}M$ of regular sets of M is the closure of $\mathcal{F}M$ under components of least solutions of finite systems of inequations $y_i \geq p_i(y_1, \dots, y_n)$ with right-linear polynomials $p_i \in \mathcal{R}M[y_1, \dots, y_n]$, i.e. where p_i is a sum of monomials A or Ay with $A \in \mathcal{R}M$ and $y \in \{y_1, \dots, y_n\}$. These definitions make $\mathcal{C}M$ resp. $\mathcal{R}M$ be closed under least fixed-points of arbitrary resp. right-linear polynomials and don't assume that M be finitely generated.

In classical Formal Language Theory, regular and context-free languages are defined by regular and context-free grammars. A *context-free grammar* $G = (X, Y, P, S)$ over the set of *terminal symbols* X is a finite set $P \subseteq Y \times (X \cup Y)^*$ of *grammar rules* or productions (y, α) , where Y is a set of variables or *nonterminal symbols*, disjoint from X , and a specific *start symbol* $S \in Y$. The productions give rise to a binary rewrite relation \Rightarrow_G on $(X \cup Y)^*$ by $w \Rightarrow_G w'$ iff for some $u, v \in (X \cup Y)^*$ and $(y, \alpha) \in P$, $w = uyv$ and $w' = u\alpha v$. Each $y \in Y$ leads to a language $L(G, y) = \{ w \in X^* \mid y \Rightarrow_G^+ w \}$, where \Rightarrow_G^+ is the transitive closure of \Rightarrow_G , and $L(G) = L(G, S)$ is the language defined by G . A *right-linear grammar* is a context-free grammar where $P \subseteq Y \times (X^* \cup X^*Y)$.

Notice that P gives rise to a system (1) with $Y = \{y_1, \dots, y_n\}$ and $S = y_1$ where p_i is the sum over all monomials $\alpha' \in \mathcal{F}X^*[y_1, \dots, y_n]$ that arise from the right-hand side of a grammar rule (y_i, α) by replacing each $x \in X$ occurring in α by the singleton $\{x\} \in \mathcal{F}X^*$. The languages $(L(G, y_1), \dots, L(G, y_n))$ are the least solution of the system (1) derived from G in this way. Conversely, a polynomial system (1) with parameters from $\mathcal{F}X^*$ gives rise to a context-free grammar $G = (X, Y, P, y_1)$ with $Y = \{y_1, \dots, y_n\}$ and P consisting of those (y_i, α) where α arises from a monomial α' of p_i by replacing each parameter in α' by one of its members $w \in X^*$. The least solution (L_1, \dots, L_n) of the given polynomial system agrees with the languages $(L(G, y_1), \dots, L(G, y_n))$ of the grammar so derived from it.

The grammatical way to define $\mathcal{C}X^*$ can be extended from free monoids X^* to arbitrary monoids M , following Hopkins (2008a). The *free extension* $M[Y]$ of a monoid M by a set Y disjoint from M consist of the set of all finite sequences $m_0 y_1 m_1 \dots y_k m_k$, where $m_0, \dots, m_k \in M$ and $y_1, \dots, y_k \in Y$, with the sequence m_0 of length 1 with $m_0 = 1$ as unit, and with the operation defined by

$$(m_0 y_1 m_1 \dots y_k m_k)(m'_0 y'_1 m'_1 \dots y'_{k'} m'_{k'}) = m_0 y_1 m_1 \dots y_k (m_k \cdot^M m'_0) y'_1 m'_1 \dots y'_{k'} m'_{k'}$$

for $k, k' \in \mathbb{N}$, $m_0, \dots, m'_k \in M$, $y_1, \dots, y'_{k'} \in Y$ as product; for disjoint sets X and Y , there is an isomorphism $X^*[Y] \simeq (X \cup Y)^*$, obtained by identifying $y_1 y' \in X^*[Y]$ with $yy' \in Y^*$. A context-free grammar $G = (Y, P, S)$ over the monoid M then consists of a set Y of variables, disjoint from M , a finite set $P \subseteq Y \times M[Y]$ of productions, and a main variable $S \in Y$. As for free monoids,

G gives rise to a binary relation \Rightarrow_G on $M[Y]$ such that $w \Rightarrow_G w'$ iff for some $\alpha, \beta \in M[Y]$ and $(y, \gamma) \in P$, $w = \alpha y \beta$ and $w' = \alpha \gamma \beta$. For each variable $y \in Y$, G defines a subset $L(G, y) = \{m \in M \mid y \Rightarrow_G^+ m\}$ of M , where \Rightarrow_G^+ is the transitive closure of \Rightarrow_G , and $L(G) := L(G, S)$. This gives us a grammatical definition of a set

$$\mathcal{C}M = \{L(G) \mid G \text{ a context-free grammar over } M\}.$$

The equivalence between the two definitions of $\mathcal{C}M$ can be shown by relating context-free grammars over M with finite systems of polynomial inequations with parameters from $\mathcal{F}M$, as in the case of free monoids. (For an equivalence proof, see also Remark 2 in Leiß and Hopkins (2018).) The regular subsets $\mathcal{R}M$ can be defined analogously by context-free grammars with right-linear productions. Independently of the aims of the present article, $\mathcal{R}M$ and $\mathcal{C}M$ with non-free monoid M are useful to study rational or context-free transductions T between a free input monoid X^* and a free output monoid Z^* as subsets of $M = X^* \times Z^*$.

The components of least solutions in $\mathcal{P}X^*$ of polynomial systems (1) with right-linear polynomials $p_i \in \mathcal{R}X^*[y_1, \dots, y_n]$ can be denoted by regular expressions in the elements of X . The regular expressions over X , defined by

$$r, s ::= 0 \mid 1 \mid x \mid (r + s) \mid (r \cdot s) \mid r^* \quad \text{with } x \text{ ranging over } X,$$

are interpreted in the Kleene algebra $\mathcal{R}X^*$ by taking \emptyset for 0, the singleton containing the unit of X^* for 1, $\{x\}$ for $x \in X$, binary union for $+$, elementwise concatenation for \cdot , and iteration (aka monoid closure) for $*$. If the right-linear polynomial p_n of (1) is $q(y_1, \dots, y_{n-1}) + Ay_n$ and r denotes the parameter $A \in \mathcal{R}X^*$, then the least solution of $y_n \geq p_n(y_1, \dots, y_n)$ is denoted by $(r^* \cdot q)(y_1, \dots, y_{n-1})$. Substituting y_n by $(r^* \cdot q)$ in the remaining inequations, and iterating this process, the first component of the least solution of (1) is named by a regular expression over X , free of unknowns y_1, \dots, y_n .

In a similar way, the components of least solutions of arbitrary polynomial inequations (1) with parameters from $\mathcal{C}X^*$ can be named by regular μ -terms over X ,

$$r, s ::= 0 \mid 1 \mid x \mid y \mid (r + s) \mid (r \cdot s) \mid r^* \mid \mu y. r, \quad \text{with } x \text{ ranging over } X,$$

the extension of regular expressions over X by an infinite set of variables y and a unary least-fixed point operator μ binding a variable; then $(\mu y_n. p_n)(y_1, \dots, y_{n-1})$ is a term denoting the least solution of $y_n \geq p_n(y_1, \dots, y_n)$ of (1), relative to given values for y_1, \dots, y_{n-1} in $\mathcal{C}X^*$. One of the results of this article is an algebraic notation for context-free languages that is close to the regular expressions and does not use a binding operator.

Now, the Chomsky-Schützenberger theorem and its reverse extend readily to the families $\mathcal{R}M$ of regular and $\mathcal{C}M$ of context-free subsets of an arbitrary finitely generated monoid M as

$$\mathcal{C}M = \{h_M(R \cap D_2(M)) \mid R \in \mathcal{R}(M[\Delta_2])\}. \quad (2)$$

We here assume M to be finitely generated to have $D_2(M) \in \mathcal{C}(M[\Delta_2])$, but this assumption can be avoided by a different formulation using the pure Dyck language $D_2 \in \mathcal{C}\Delta_2^*$ of brackets only,

$$\mathcal{C}M = \{\pi_1(R \cap (M \times D_2)) \mid R \in \mathcal{R}(M \times \Delta_2^*)\},$$

where $\pi_1 : M \times \Delta_2^* \rightarrow M$ is the first projection.

In a number of semi-published contributions, Hopkins (1993) has tried to turn this dependence of $\mathcal{C}M$ on $\mathcal{R}(M[\Delta_2])$ into a dependence on $\mathcal{R}M$, express it in more algebraic terms, and generalize it to an algebraic construction of the fixed-point-closure of a $*$ -continuous Kleene algebra K . The idea is to let elements of M and Δ_2 commute with each other, i.e. replace $M[\Delta_2]$ by $M \times \Delta_2^*$ or $\mathcal{R}(M[\Delta_2])$ by a suitable product $\mathcal{R}M \otimes_{\mathcal{R}} C_2$ of $\mathcal{R}M$ with a “bra-ket algebra” C_2 , and perform the balance-check $\cap D_2(M)$ and the bracket-erasure h_M of (2) by algebraic calculations in C_2 . The “bra-ket algebra”

$$C_2 = \mathcal{R}\Delta_2^* / \langle bd = 1 = pq, bq = 0 = pd, db + qp = 1 \rangle$$

is the quotient of the regular sets $\mathcal{R}\Delta_2^*$ by a congruence that is generated by three conditions^a: by “match” conditions $bd = 1 = pq$ that allow us to shorten strings by erasing all pairs bd or pq of matching adjacent brackets, by “mismatch” conditions $bq = 0 = pd$ that allow us to throw away all strings containing a bracket mismatch bq or pd , and by a “completeness” condition $db + qp = 1$ which ensures that corresponding bra-ket algebras C_n based on $n > 2$ bracket pairs can be embedded in C_2 .^b The product $\mathcal{R}M \otimes_{\mathcal{R}} C_2$ is, intuitively, the smallest *-continuous Kleene algebra extension of $\mathcal{R}M$ and C_2 in which elements of $\mathcal{R}M$ commute with those of C_2 . The algebraic version of the Chomsky-Schützenberger theorem and its reverse then is that $\mathcal{C}M$ is isomorphic to the centralizer of C_2 in $\mathcal{R}M \otimes_{\mathcal{R}} C_2$,

$$\mathcal{C}M \simeq Z_{C_2}(\mathcal{R}M \otimes_{\mathcal{R}} C_2), \quad (3)$$

i.e. to the set of elements that commute with every member of C_2 . All elements of $\mathcal{R}M \otimes_{\mathcal{R}} C_2$ can be denoted by regular expressions in the generators of M and C_2 , and it is easy to define a subset of expressions that denote the members of the centralizer of C_2 . This is the above-mentioned algebraic notation system for context-free sets that is much closer to the familiar one for regular sets than context-free grammars or regular μ -terms are.

The goal of the present work is to replace C_2 in (3) by a simpler algebra, the “polycyclic *-continuous Kleene algebra on two generators”,

$$C'_2 = \mathcal{R}\Delta_2^* / \langle bd = 1 = pq, bq = 0 = pd \rangle,$$

and instead of (3) prove

$$\mathcal{C}M \simeq Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2), \quad (4)$$

our first main result. The congruence used in C'_2 is generated by just the bracket match and mismatch conditions. These, but not the completeness condition, can be seen as monoid equations, provided we extend Δ_2^* by an annihilating monoid element 0. Doing so gives us the “polycyclic monoid on two generators”,

$$P'_2 = (\Delta_2 \cup \{0\})^* / \{x0 = 0 = 0x \mid x \in \Delta_2 \cup \{0\}\} \cup \{bd = 1 = pq, bq = 0 = pd\}.$$

This allows us to treat C'_2 as $\mathcal{R}P'_2$ modulo $\{0\} = \emptyset$ and $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$ as $\mathcal{R}(M \times P'_2)$ modulo $\{(1, 0)\} = \emptyset$. Moreover, it lets us understand the centralizer of C'_2 in $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$: it contains (the equivalence classes of) those $R \in \mathcal{R}(M \times P'_2)$ where $R \subseteq M \times \{0, 1\}$.

Intuitively, therefore, (4) says that the context-free sets $L \in \mathcal{C}M$ correspond to the regular sets $R \in \mathcal{R}(M \times P'_2)$ whose elements (m, t') have $t' \in \{0, 1\}$: elements $(m, 1) \in R$ come from an element $m \in M \cap L$ with a parse tree $t \in D_2 \subseteq \Delta_2^*$ with respect to a context-free grammar for L , reduced to $t' = 1$ in P'_2 ; elements $(m, 0) \in R$ come from unparseable $m \in M \setminus L$ and are thrown away in the quotient of $\mathcal{R}(M \times P'_2)$ by $\{(1, 0)\} = \emptyset$.

However, the algebraic form (4) of the Chomsky-Schützenberger theorem and its reverse for monoids does not yet give an algebraic representation of the fixed-point closure of an arbitrary *-continuous Kleene algebra K , which need not be of the form $\mathcal{R}M$ for some monoid M . Equipped with suitable notions of \mathcal{R} - and \mathcal{C} -morphisms, the categories of *-continuous Kleene algebras (Kozen (1981)) and μ -continuous Chomsky algebras (Grathwohl et al. (2013)), respectively, are subcategories $\mathbb{D}\mathcal{R}$ and $\mathbb{D}\mathcal{C}$ of the category \mathbb{D} of dioids (i.e. idempotent semirings). In fact, $\mathbb{D}\mathcal{R}$ and $\mathbb{D}\mathcal{C}$ are the Eilenberg-Moore categories of certain monads on the category of monoids. The Kleene algebras $\mathcal{R}M$ and the Chomsky algebras $\mathcal{C}M$, with monoid M , form the so-called Kleisli subcategories of $\mathbb{D}\mathcal{R}$ and $\mathbb{D}\mathcal{C}$. We want to extend (4) from their Kleisli subcategories to the categories $\mathbb{D}\mathcal{R}$ and $\mathbb{D}\mathcal{C}$ themselves. In an equivalent categorical formulation of Hopkins (2008a) and Leiß and Hopkins (2018), the objects of $\mathbb{D}\mathcal{R}$ and $\mathbb{D}\mathcal{C}$, called \mathcal{R} -dioids and \mathcal{C} -dioids there,

^aThe equations are semiring equations, in which elements $w \in \Delta_2^*$ stand for the singleton sets $\{w\}$ and 0 for the empty set.

^bThe term “completeness” refers to the fact that the embedding is based on a complete suffix code of n opening brackets by two opening brackets and a complete prefix code of n closing brackets by two closing brackets, see Remark 7 below.

are dioids in which the regular resp. context-free subsets U of their multiplicative monoids have least upper bounds $\sum U$ with respect to the partial order given by addition. As part of a broader programme of algebraization of formal language theory, Hopkins (2008b) shows that there is an adjunction

$$Q_{\mathcal{R}}^{\mathcal{C}} : \mathbb{D}\mathcal{R} \rightarrow \mathbb{D}\mathcal{C} \quad \text{and} \quad Q_{\mathcal{C}}^{\mathcal{R}} : \mathbb{D}\mathcal{C} \rightarrow \mathbb{D}\mathcal{R}$$

between these categories, where $Q_{\mathcal{C}}^{\mathcal{R}}$ is the forgetful functor (restricting \sum to regular subsets) and $Q_{\mathcal{R}}^{\mathcal{C}}$ maps an \mathcal{R} -dioid K to its fixed-point- or \mathcal{C} -closure in $\mathbb{D}\mathcal{C}$, which extends K by a set of order-theoretic ideals of K . In yet unpublished work, Hopkins tries to show that the \mathcal{C} -closure of an arbitrary \mathcal{R} -dioid K can also be algebraically described as

$$Q_{\mathcal{R}}^{\mathcal{C}}(K) \simeq Z_{C_2}(K \otimes_{\mathcal{R}} C_2). \quad (5)$$

Based on (4), our second main result is that, again, C_2 can be replaced by the simpler algebra C'_2 , yielding a ‘‘categorical’’ Chomsky-Schützenberger theorem

$$Q_{\mathcal{R}}^{\mathcal{C}}(K) \simeq Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2). \quad (6)$$

In contrast to the case of the Kleisli subcategories, we here really need the categorical notion of product of \mathcal{R} -dioids. A detailed presentation of the algebraic and categorical background used here is given in Hopkins and Leib (2018).

The rest of this paper is organized as follows. Section 2 gives the background of categories $\mathbb{D}\mathcal{A}$ of \mathcal{A} -dioids for certain subfunctors \mathcal{A} of the powerset functor $\mathcal{P} : \mathbb{M} \rightarrow \mathbb{M}_{\leq}$, where \mathbb{M} (resp. \mathbb{M}_{\leq}) is the category of (partially ordered) monoids and (monotone) homomorphisms. Section 2 also collects definitions and known results on products and quotients of \mathcal{A} -dioids and introduces polycyclic \mathcal{A} -dioids $C'_{n,\mathcal{A}}$ with n generators. Section 3 proves the algebraic version $\mathcal{C}M \subseteq Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$ of the Chomsky-Schützenberger theorem, Section 4 the algebraic version of the reverse Chomsky-Schützenberger theorem, $\mathcal{C}M \supseteq Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$. These are combined in Section 5 to a proof that $Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$ is the \mathcal{C} -closure of $\mathcal{R}M$; for $M = X^*$, this leads to a set of regular expressions over $X \cup \Delta_2$ that evaluate to the context-free sets of $\mathcal{C}M$. Section 6 finally proves the general representation result: for an arbitrary \mathcal{R} -dioid K , the \mathcal{C} -closure of K is isomorphic to $Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$. In the Conclusion, we also sketch where Hopkins’ proof of (5) uses the completeness assumption and how it is avoided in our proof of (6).

2. The Category of \mathcal{A} -Dioids and \mathcal{A} -Morphisms

Let \mathbb{M} be the category of monoids $(M, \cdot, 1)$ and homomorphisms between monoids, and \mathbb{M}_{\leq} the category of partially ordered monoids $(M, \cdot, 1, \leq)$ and monotone homomorphisms between them. We consider subfunctors of the powerset-functor $\mathcal{P} : \mathbb{M} \rightarrow \mathbb{M}_{\leq}$ and partially order them by $\mathcal{A} \leq \mathcal{A}'$ iff for all monoids M , $\mathcal{A}M \subseteq \mathcal{A}'M$. A *monadic operator*^c is a functor $\mathcal{A} : \mathbb{M} \rightarrow \mathbb{M}_{\leq}$ such that for all monoids M, N ,

(A₀) $\mathcal{A}M$ is a set of subsets of M ,

(A₁) $\mathcal{A}M$ contains all singleton subsets of M ,

(A₂) $\mathcal{A}M$ is closed under products, hence $(\mathcal{A}M, \cdot, \{1\}, \subseteq)$ is a partially ordered monoid, where

$$A \cdot B := \{ a \cdot b \mid a \in A, b \in B \} \quad \text{for } A, B \in \mathcal{A}M,$$

(A₃) $\mathcal{A}M$ is closed under unions of sets from $\mathcal{A}((\mathcal{A}M, \cdot, \{1\}))$, and

^cThis generalizes the definition in Hopkins (2008a), Hopkins and Leib (2018), which demand (A₁) $\mathcal{A}M$ contains all finite subsets of M . The generalization serves to cover the example under (1) in Example 1 below.

(A₄) if $f : M \rightarrow N$ is a homomorphism, so is $\mathcal{A}f : \mathcal{A}M \rightarrow \mathcal{A}N$, where for $U \subseteq M$,

$$(\mathcal{A}f)(U) := \{ f(u) \mid u \in U \}.$$

We write $\mathcal{A}M$ for both the selected set of subsets of M and the partially ordered monoid $(\mathcal{A}M, \cdot, \{1\}, \subseteq)$ or just the monoid $(\mathcal{A}M, \cdot, \{1\})$. For the lifting $\mathcal{A}f$ of f we often write \tilde{f} .

An \mathcal{A} -dioid $(D, \cdot, 1, \leq)$ is a partially ordered monoid which is \mathcal{A} -complete, i.e. each $U \in \mathcal{A}D$ has a least upper bound, $\sum U \in D$, and \mathcal{A} -distributive, i.e.

$$(\sum U)(\sum V) = \sum(UV) \quad \text{for all } U, V \in \mathcal{A}D.$$

An \mathcal{A} -morphism $f : D \rightarrow D'$ between \mathcal{A} -dioids D and D' is a monotone homomorphism such that $f(\sum U) = \sum(\mathcal{A}f)(U)$ for all $U \in \mathcal{A}D$. Let $\mathbb{D}\mathcal{A}$ be the category of \mathcal{A} -dioids and \mathcal{A} -morphisms between \mathcal{A} -dioids.

Example 1 (Hopkins (2008a)). The following functors are monadic operators:

- (1) \mathcal{S} , where $\mathcal{S}M$ is the set of all singleton subsets of M ; $\mathbb{D}\mathcal{S}$ is the category \mathbb{M} of monoids (with equality as partial order) and (monoid-) homomorphisms.
- (2) \mathcal{F} , where $\mathcal{F}M$ is the set of all finite subsets of M ; $\mathbb{D}\mathcal{F}$ is the category \mathbb{D} of dioids and dioid-homomorphisms.
- (3) \mathcal{R} , where $\mathcal{R}M$ is the set of all regular subsets of M ; $\mathbb{D}\mathcal{R}$ is the category of *-continuous Kleene algebras and morphisms between them, see Kozen (1981), Hopkins (2008a).
- (4) \mathcal{C} , where $\mathcal{C}M$ is the set of all context-free subsets of M ; $\mathbb{D}\mathcal{C}$ is the category of μ -continuous Chomsky algebras and morphisms between them, see Grathwohl et al. (2013), Leib and Hopkins (2018).
- (5) \mathcal{T} , where $\mathcal{T}M$ is the set of all Turing-subsets (r.e.) of M . Notice that (A₃) generalizes the well-known fact that the union of an r.e. family of r.e. subsets of \mathbb{N} is an r.e. subset of \mathbb{N} .
- (6) \mathcal{P}_{\aleph_0} , where $\mathcal{P}_{\aleph_0}M$ is the set of countable subsets of M ; $\mathbb{D}\mathcal{P}_{\aleph_0}$ is the category of closed semirings, see Kozen (1990).
- (7) \mathcal{P} , the power set operator; $\mathbb{D}\mathcal{P}$ is the category of quantales with unit (Rosenthal (1990)).

The sets $\mathcal{R}M$, $\mathcal{C}M$, $\mathcal{T}M$ can be defined by generalizing the grammatical approach of doing so for free monoids $M = X^*$ (Hopkins (2008a)). We use the more algebraic, equivalent definitions for \mathcal{R} and \mathcal{C} given in Hopkins and Leib (2018). $\mathcal{R}M$ is the closure of $\mathcal{F}M$ under (binary) union, elementwise product, and iteration $*$, i.e. if $A \in \mathcal{R}M$, so is $A^* := \bigcup \{ A^n \mid n \in \mathbb{N} \}$, where $A^0 = \{1\}$, $A^{n+1} = A^n \cdot A$. $\mathcal{C}M$ is the closure of $\mathcal{F}M$ under components of least solutions of polynomial systems over $\mathcal{C}M$, i.e. the components A_1, \dots, A_n of the least solution in $\mathcal{P}M$ of a system of inequations

$$x_1 \geq p_1(x_1, \dots, x_n), \dots, x_n \geq p_n(x_1, \dots, x_n)$$

with polynomials p_i in x_1, \dots, x_n with parameters from $\mathcal{C}M$ all belong to $\mathcal{C}M$. \diamond

A Kleene Algebra $(K, +, \cdot, *, 0, 1)$ is an idempotent semiring with a unary operation $*$ such that a^*b is the least solution x of $ax + b \leq x$ and ba^* the least solution of $xa + b \leq x$, where $y \leq z$ is defined by $y + z = z$ (Kozen (1994)). The Kleene algebra K is *-continuous if

$$ac^*b = \sum \{ ac^m b \mid m \in \mathbb{N} \} \quad \text{for all } a, b, c \in K.$$

A Chomsky algebra (Grathwohl et al. (2013)) is an idempotent semiring K that is algebraically closed, i.e. where every finite system of inequations $x_1 \geq p_1(x_1, \dots, x_n), \dots, x_n \geq p_n(x_1, \dots, x_n)$ with polynomials $p_i \in K[x_1, \dots, x_n]$ has a least solution $a_1, \dots, a_n \in K$. If semiring terms, i.e. terms in $0, 1, +, \cdot$, are extended to μ -terms by adding a unary fixed-point operator μ , the least solution of $x \geq p(x, x_1, \dots, x_k)$ can be named by the term $(\mu x.p)(x_1, \dots, x_k)$. A Chomsky algebra

is μ -continuous, if for all μ -terms $\mu x.p$ with parameters from K ,

$$a(\mu x.p)b = \sum \{ a \cdot p^m(0) \cdot b \mid m \in \mathbb{N} \} \quad \text{for all } a, b \in K.$$

In the $*$ - and μ -continuity conditions, the existence of the least upper bounds on the right is asserted, not assumed. Hence, these conditions are related to the \mathcal{R} - and \mathcal{C} -dioid axioms by the existence assertions $U = \{ c^m \mid m \in \mathbb{N} \} \in \mathcal{R}K$ resp. $U = \{ p^m(0) \mid m \in \mathbb{N} \} \in \mathcal{C}K$ and the \mathcal{A} -distributivity assertion, for $\mathcal{A} = \mathcal{R}$ resp. $\mathcal{A} = \mathcal{C}$,

$$a(\sum U)b = \sum aUb, \quad \text{for all } a, b \in K,$$

which (assuming \mathcal{A} -completeness) is equivalent to the \mathcal{A} -distributivity condition above. Of course, by $\mathcal{A}K$ we mean \mathcal{A} applied to the multiplicative monoid of K . We refer to Hopkins (2008a) and Leiß and Hopkins (2018) for the details of the equivalence between $*$ -continuous Kleene algebras and \mathcal{R} -dioids and between μ -continuous Chomsky algebras and \mathcal{C} -dioids, respectively, and use the latter terminology in the following.

A dioid $(D, +, \cdot, 0, 1)$ is an idempotent semiring. Idempotency of $+$ provides a partial order \leq on D , via $d \leq d'$ iff $d + d' = d'$, with 0 as least element. Distributivity makes \cdot monotone with respect to \leq , and $+$ guarantees a least upper bound $\sum U = d_1 + \dots + d_n$ for each finite subset $U = \{d_1, \dots, d_n\}$ of D . Let \mathbb{D} be the category of dioids with dioid homomorphisms.

For $\mathcal{F} \leq \mathcal{A}$, (A_3) implies that $\mathcal{A}M$ is an idempotent semiring with

$$0 := \bigcup \emptyset, \quad A + B := \bigcup \{A, B\}, \quad \text{for } A, B \in \mathcal{A}M.$$

In this case, every \mathcal{A} -dioid $(D, \cdot, 1, \leq)$ becomes a dioid $(D, +, \cdot, 0, 1)$, using $a + b := \sum \{a, b\}$ and $0 := \sum \emptyset$, and every \mathcal{A} -morphism is a dioid-homomorphism. Hence we then view $\mathbb{D}\mathcal{A}$ as a subcategory of \mathbb{D} .

Let us collect a number of results of that will be frequently used below:

Theorem 1 (Hopkins (2008a)). *Let \mathcal{A} be a monadic operator with $\mathcal{F} \leq \mathcal{A}$. Let M, N be monoids and D be an \mathcal{A} -dioid.*

- (1) $\mathcal{A}M$ is an \mathcal{A} -dioid.
- (2) $\mathcal{A}M$ is the free \mathcal{A} -dioid extension of M . (That is, any homomorphism $f : M \rightarrow D$ to an \mathcal{A} -dioid D extends uniquely to an \mathcal{A} -morphism $f^* : \mathcal{A}M \rightarrow D$ such that $f(m) = f^*(\{m\})$.)
- (3) The least-upper-bound-operator $\sum : \mathcal{A}D \rightarrow D$ is an \mathcal{A} -morphism.
- (4) If $f : M \rightarrow N$ is a homomorphism, its lifting $\mathcal{A}f : \mathcal{A}M \rightarrow \mathcal{A}N$ is an \mathcal{A} -morphism.
- (5) If $f : M \rightarrow N$ is a surjective homomorphism, so is its lifting $\mathcal{A}f : \mathcal{A}M \rightarrow \mathcal{A}N$.
- (6) If M is a submonoid of N , then $\mathcal{A}M \subseteq \mathcal{A}N$.

In fact, by Theorem 16 of Hopkins (2008b), $\mathcal{A} : \mathbb{M} \rightarrow \mathbb{D}\mathcal{A}$ and the forgetful functor $\widehat{\mathcal{A}} : \mathbb{D}\mathcal{A} \rightarrow \mathbb{M}$ form an adjunction and combine to a monad $T_{\mathcal{A}} = (\widehat{\mathcal{A}} \circ \mathcal{A}, \eta, \mu) : \mathbb{M} \rightarrow \mathbb{M}$ with $m \in M \mapsto \{m\} \in \mathcal{A}M$ as unit η and $U \in \mathcal{A}\mathcal{A}M \mapsto \bigcup U \in \mathcal{A}M$ as product μ . Therefore, \mathcal{A} -dioids are best viewed as Eilenberg-Moore T -algebras for this monad $T_{\mathcal{A}}$, i.e. two-sorted structures

$$D = (D, \cdot, 1, \sum : \mathcal{A}D \rightarrow D)$$

with \sum as “structure map” (see Mac Lane (1971)). The free \mathcal{A} -dioids $\mathcal{A}M$ with monoid M are the objects of the Kleisli-category of this monad $T_{\mathcal{A}}$.

Besides the free \mathcal{A} -dioid extensions $\mathcal{A}M$ of monoids M , further examples of \mathcal{A} -dioids can be obtained from given ones by suitable quotient- and tensor product constructions as defined later. Some categories $\mathbb{D}\mathcal{A}$ are closed under matrix semiring formation, in particular:

Example 2. Let J be a non-empty set and $D = (D, +^D, \cdot^D, 0^D, 1^D)$ a \mathcal{P} -dioid. Then

$$\text{Mat}_{J,J}(D) = \langle D^{J \times J}, +, \cdot, 0, 1 \rangle$$

is a \mathcal{P} -dioid, where $D^{J \times J}$ is the set of functions from $J \times J$ to D , 0 is the constant function with value 0^D , 1 the function with $1(i, i) = 1^D$ and $1(i, j) = 0^D$ for $i \neq j$, and $+$ and \cdot are given by $(f + g)(i, j) = f(i, j) +^D g(i, j)$ and $(f \cdot g)(i, j) = \sum \{ f(i, k) \cdot^D g(k, j) \mid k \in J \}$, for all $i, j \in J$.

$\text{Mat}_{J,J}(D)$ is \mathcal{P} -complete, because any $U \subseteq \text{Mat}_{J,J}(D)$ has a least upper bound $\sum U$ defined componentwise by $(\sum U)(i, j) = \sum \{ f(i, j) \mid f \in U \}$ using $\sum : \mathcal{P}D \rightarrow D$; the \mathcal{P} -distributivity of $\text{Mat}_{J,J}(D)$ follows from the definition of matrix product and the \mathcal{P} -distributivity of D :

$$\begin{aligned} (g \cdot (\sum U) \cdot h)(i, j) &= \sum \{ g(i, k) \cdot^D (\sum \{ f(k, l) \mid f \in U \}) \cdot^D h(l, j) \mid k, l \in J \} \\ &= \sum \{ g(i, k) \cdot^D f(k, l) \cdot^D h(l, j) \mid f \in U, k, l \in J \} \\ &= \sum \{ (g \cdot f \cdot h)(i, j) \mid f \in U \}. \end{aligned}$$

Elements of $\text{Mat}_{J,J}(D)$ will be called $J \times J$ -matrices over D and often are denoted by A, B ; as usual, we write $A_{i,j}$ instead of $A(i, j)$ etc. If M is a monoid, $\text{Mat}_{J,J}(\mathcal{P}M)$ is a \mathcal{P} -dioid. In particular, since $\mathbb{B} \simeq \mathcal{P}M$ for the trivial monoid M , the boolean matrices $\text{Mat}_{J,J}(\mathbb{B})$ form a \mathcal{P} -dioid. We identify $\text{Mat}_{J,J}(\mathbb{B})$ with the \mathcal{P} -dioid of binary relations on J , and often write $(i, j) \in A \subseteq J \times J$ instead of $A_{i,j} = 1$. \diamond

Remark 2. Similarly, $\mathbb{D}\mathcal{F}$ is closed under $\text{Mat}_{n,n}(\cdot)$ for finite n and $\mathbb{D}\mathcal{P}_{\aleph_0}$ under $\text{Mat}_{J,J}(\cdot)$ for countable J . It is an open question whether $\text{Mat}_{n,n}(D)$ is an \mathcal{A} -dioid for an arbitrary monadic operator \mathcal{A} and \mathcal{A} -dioid D . The least upper bound $\sum U$ of a set $U \subseteq \text{Mat}_{n,n}(D)$ has to satisfy $(\sum U)_{i,j} = \sum \{ A_{i,j} \mid A \in U \}$, but $U \in \mathcal{A}(\text{Mat}_{n,n}(D))$ does not seem to imply $\{ A_{i,j} \mid A \in U \} \in \mathcal{A}D$, so it is already unclear if the componentwise suprema exist and $\text{Mat}_{n,n}(D)$ is \mathcal{A} -complete. Defining the iteration A^* of a matrix A of dimension $n \times n$ by a formula of Conway (1971), Kozen (1994) shows that for a Kleene algebra K , $\text{Mat}_{n,n}(K)$ is a Kleene algebra. Ésik et al. (2015) claim that the same is true for $*$ -continuous Kleene algebras, i.e. $\mathbb{D}\mathcal{R}$, referring to Kozen (1990), who however treats the case $\mathbb{D}\mathcal{P}_{\aleph_0}$ of closed semirings. We are aware of no published proof that the category of \mathcal{R} -dioids is closed under $\text{Mat}_{n,n}(\cdot)$, but the proof for the category of \mathcal{C} -dioids in Leiß (2016) can be modified to obtain one. (A simpler one may be submitted elsewhere.)

Lemma 3. Let $f : M \rightarrow D$ be a homomorphism from a monoid M to a dioid D . Suppose that for each $L \in \mathcal{A}M$, its image $(\mathcal{A}f)(L) \in \mathcal{A}D$ has a least upper bound in D , $\widehat{L} := \sum \{ f(m) \mid m \in L \}$, and that $\widehat{\cdot} : \mathcal{A}M \rightarrow D$ is a surjective homomorphism. Then D is an \mathcal{A} -dioid.

Proof. By Theorem 1, the map $\mathcal{A}(\widehat{\cdot}) : \mathcal{A}\mathcal{A}M \rightarrow \mathcal{A}D$ is a surjective homomorphism. Therefore, we can define a least-upper-bound operator \sum that makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{A}\mathcal{A}M & \xrightarrow{\mathcal{A}(\widehat{\cdot})} & \mathcal{A}D \\ \downarrow \cup & & \downarrow \sum \\ \mathcal{A}M & \xrightarrow{\widehat{\cdot}} & D. \end{array}$$

For each $V \in \mathcal{A}D$ there is $U \in \mathcal{A}\mathcal{A}M$ such that $V = \widehat{L} \mid L \in U$. Since $\cup U \in \mathcal{A}M$, its image under $(\mathcal{A}f) = \widehat{f}$ has by assumption a least upper bound, $\widehat{\cup U} = \sum \{ f(m) \mid m \in \cup U \} \in D$. Then

$$\sum V := \widehat{\cup U}$$

is an upper bound of V in D , since for each $L \in U$, $L \subseteq \bigcup U$, hence $\tilde{f}(L) \subseteq \tilde{f}(\bigcup U)$ and $\widehat{L} \leq \widehat{\bigcup U}$. If $e \in D$ is any upper bound of V , then for each $m \in \bigcup U$ there is $L \in U$ with $m \in L$, so

$$f(m) \leq \widehat{L} \leq e.$$

Hence e is an upper bound of $\{f(m) \mid m \in \bigcup U\}$ and so $\widehat{\bigcup U} \leq e$. Therefore, ΣV is the least upper bound of V in D , and D is \mathcal{A} -complete.

To show that D is \mathcal{A} -distributive, suppose $V_1, V_2 \in \mathcal{A}D$. Since $\mathcal{A}(\widehat{\cdot}) : \mathcal{A}\mathcal{A}M \rightarrow \mathcal{A}D$ is surjective, there are $U_1, U_2 \in \mathcal{A}\mathcal{A}M$ such that $V_i = \{\widehat{L} \mid L \in U_i\}$ and $\Sigma V_i = \widehat{\bigcup U_i}$ for $i = 1, 2$. Then $V_1 V_2 \in \mathcal{A}D$ has a least upper bound $\Sigma(V_1 V_2)$, and since $(\Sigma V_1)(\Sigma V_2)$ is an upper bound of $V_1 V_2$, we have $\Sigma(V_1 V_2) \leq (\Sigma V_1)(\Sigma V_2)$. For the reverse inequation, use that $\widehat{\cdot}$ and \bigcup are homomorphisms:

$$\begin{aligned} \Sigma(V_1 V_2) &= \Sigma\{\widehat{L_1 L_2} \mid L_1 \in U_1, L_2 \in U_2\} \\ &= \Sigma\{\widehat{L_1 L_2} \mid L_1 \in U_1, L_2 \in U_2\} \\ &= \Sigma\{\widehat{L} \mid L \in U_1 U_2\} \\ &= \Sigma\{\Sigma\{f(m) \mid m \in L\} \mid L \in U_1 U_2\} \\ &\geq \Sigma\{f(m) \mid m \in \bigcup(U_1 U_2)\} \\ &= \widehat{\bigcup(U_1 U_2)} \\ &= (\bigcup U_1 \cdot \bigcup U_2)^\widehat{\cdot} \\ &= \widehat{\bigcup U_1} \cdot \widehat{\bigcup U_2} \\ &= (\Sigma V_1)(\Sigma V_2). \end{aligned} \quad \square$$

For monadic operators $\mathcal{A} \leq \mathcal{B}$, the \mathcal{B} -completion of an \mathcal{A} -dioid D is a \mathcal{B} -dioid \overline{D} together with an embedding \mathcal{A} -morphism $\eta_D : D \rightarrow \overline{D}$ such that the following ‘universal property’ holds: any \mathcal{A} -morphism f from D to a \mathcal{B} -dioid D' extends uniquely to a \mathcal{B} -morphism $\tilde{f} : \overline{D} \rightarrow D'$, in the sense that $f = \tilde{f} \circ \eta_D$:

$$\begin{array}{ccc} & & D' \\ & \nearrow f & \vdots \tilde{f} \\ D & \xrightarrow{\eta_D} & \overline{D} \end{array}$$

By the universal property, the \mathcal{B} -completion of D is unique up to a \mathcal{B} -isomorphism.

The \mathcal{C} -completion of a Kleene algebra or an \mathcal{B} -dioid is analogous to the algebraic closure of a field, but instead of adjoining roots to polynomials, we adjoin least fixed points. The \mathcal{C} -completion shall therefore also be called the \mathcal{C} -closure or *fixed-point closure* of the Kleene algebra.

Proposition 4. For monoids M , the \mathcal{C} -completion of $\mathcal{B}M$ is $\mathcal{C}M$ with the inclusion as $\eta_{\mathcal{B}M}$.

Proof. $\mathcal{C}M$ is a \mathcal{C} -dioid by Theorem 1 of Hopkins (2008a). If $f : \mathcal{B}M \rightarrow C$ is an \mathcal{B} -morphism to a \mathcal{C} -dioid C , we define $\tilde{f} : \mathcal{C}M \rightarrow C$ by

$$\tilde{f}(L) = \Sigma\{f(\{m\}) \mid m \in L\}, \quad \text{for } L \in \mathcal{C}M.$$

It is routine to check that \tilde{f} is the only \mathcal{C} -morphism $h : \mathcal{C}M \rightarrow C$ with $f = h \circ \eta_{\mathcal{B}M}$. \square

In the rest of this paper, we give an algebraic construction of the \mathcal{C} -closure $\mathcal{C}M$ of $\mathcal{R}M$ for monoids M and generalize this to a construction of the \mathcal{C} -closure of an arbitrary \mathcal{R} -dioid K .

2.1 Quotients in $\mathbb{D}\mathcal{A}$

From now on, we assume $\mathcal{F} \leq \mathcal{A}$, so that all \mathcal{A} -dioids are dioids. For a partial order (D, \leq) , the *down-closure* of $U \subseteq D$ is $U^\downarrow := \{d \in D \mid d \leq u \text{ for some } u \in U\}$.

If $D = (D, +^D, \cdot^D, 0^D, 1^D)$ is a dioid and ρ a dioid-congruence on D , then D/ρ , the set of congruence classes of ρ , is a dioid under the operations

$$d/\rho + d'/\rho := (d +^D d')/\rho, \quad d/\rho \cdot d'/\rho := (d \cdot^D d')/\rho, \quad 0 := 0^D/\rho, \quad 1 := 1^D/\rho.$$

The partial order \leq on D/ρ derived from $+$ is

$$d/\rho \leq d'/\rho : \iff (d +^D d')/\rho = d'/\rho.$$

An \mathcal{A} -congruence on an \mathcal{A} -dioid D is a dioid-congruence ρ on D such that for all $U, U' \in \mathcal{A}D$, if $(U/\rho)^\downarrow = (U'/\rho)^\downarrow$, then $(\Sigma U)/\rho = (\Sigma U')/\rho$.

Proposition 5 (Hopkins and Leiß (2018)). *If D is an \mathcal{A} -dioid and ρ an \mathcal{A} -congruence on D , then D/ρ is an \mathcal{A} -dioid and the canonical map $d \mapsto d/\rho$ is an \mathcal{A} -morphism.*

Proof. For each $V \in \mathcal{A}(D/\rho)$, there is $U \in \mathcal{A}D$ such that $V = U/\rho = \{d/\rho \mid d \in U\}$. Since ρ is an \mathcal{A} -congruence, $\Sigma(U/\rho) := (\Sigma U)/\rho$ is well-defined and a least upper bound of V . (This needs $\mathcal{F} \leq \mathcal{A}$.) \square

If D is an \mathcal{A} -dioid and $E \subseteq D \times D$, there is a least \mathcal{A} -dioid-congruence ρ on D with $E \subseteq \rho$, the intersection of all \mathcal{A} -dioid-congruences on D above E .

The \mathcal{A} -dioid $\mathcal{A}(X^*/\rho)$ of the quotient monoid X^*/ρ is isomorphic to the quotient $\mathcal{A}X^*/\tilde{\rho}$ of the free \mathcal{A} -dioid $\mathcal{A}X^*$ by a suitable \mathcal{A} -congruence $\tilde{\rho}$. If ρ is determined by a set E of monoid equations, $\tilde{\rho}$ is determined by the corresponding set of dioid equations between singleton sets:

Proposition 6 (Hopkins and Leiß (2018)). *Let ρ be a congruence on the monoid M and $\mathcal{A}\rho$ the least \mathcal{A} -congruence on $\mathcal{A}M$ above $\{(\{m\}, \{m'\}) \mid (m, m') \in \rho\}$. Then $\mathcal{A}(M/\rho) \simeq \mathcal{A}M/\mathcal{A}\rho$.*

It is shown in Hopkins and Leiß (2018) that $\mathbb{D}\mathcal{A}$ has coequalizers and the quotient D/ρ of an \mathcal{A} -dioid D by an \mathcal{A} -congruence ρ on D is the coequalizer of two suitable \mathcal{A} -morphisms $f, g : N \rightrightarrows D$. The kernel $\ker(f)$ of an \mathcal{A} -morphism $f : D \rightarrow D'$ between \mathcal{A} -dioids is an \mathcal{A} -congruence on D .

2.2 The Polycyclic \mathcal{A} -Dioids $C'_{n,\mathcal{A}}$

Let $\Delta_n = \{p_0, \dots, p_{n-1}, q_0, \dots, q_{n-1}\}$ be a set of n pairs of ‘‘brackets’’, $p_0, q_0, \dots, p_{n-1}, q_{n-1}$. We decompose $\Delta_n = P_n \cup Q_n$ into the set $P_n = \{p_i \mid i < n\}$ of ‘‘opening’’ brackets and the set $Q_n = \{q_i \mid i < n\}$ of ‘‘closing’’ brackets. We also use $\langle i \mid$ for p_i and $\mid i \rangle$ for q_i .

Let \mathcal{A} be a monadic operator such that $\emptyset \in \mathcal{A}M$ for each monoid M . The *polycyclic \mathcal{A} -dioid* $C'_{n,\mathcal{A}}$ is the quotient $C'_{n,\mathcal{A}} = \mathcal{A}\Delta_n^*/\rho_n$ of $\mathcal{A}\Delta_n^*$ by the \mathcal{A} -congruence ρ_n generated by the relations

$$p_i q_j = \delta_{i,j} \quad \text{for } i, j < n, \tag{7}$$

where $\delta_{i,j}$ is the Kronecker δ . These equations allow us to algebraically distinguish matching brackets, where $p_i q_j = 1$, from non-matching ones, where $p_i q_j = 0$. Recall that in $\mathcal{A}\Delta_n^*$, elements of Δ_n^* are interpreted by their singleton sets, 0 by the empty set, so that ρ_n is the least

\mathcal{A} -congruence containing $(\{p_i q_i\}, \{1\})$ and $(\{p_i q_j\}, \emptyset)$ for $i, j < n, i \neq j$. The *bra-ket algebra* $C_{n, \mathcal{A}}$ of Hopkins (2007) further assumes a ‘‘completeness’’ condition, $1 = \sum \{q_i p_i \mid i < n\}$, i.e.

$$C_{n, \mathcal{A}} = \mathcal{A} \Delta_n^* / \langle \{p_i q_j = \delta_{i,j} \mid i, j < n\} \cup \{1 = \sum \{q_i p_i \mid i < n\}\} \rangle.$$

We will only use the case $\mathcal{A} = \mathcal{R}$ and abbreviate $C_{n, \mathcal{R}}$ and $C'_{n, \mathcal{R}}$ by C_n and C'_n , respectively.

Remark 7. The reason to call $1 = q_0 p_0 + \dots + q_{n-1} p_{n-1}$ the *completeness condition* is as follows. A map $c : \{0, \dots, n-1\} \rightarrow \{0, 1\}^*$ is a *prefix code* if none of the code words c_0, \dots, c_{n-1} is a prefix of another one. A prefix code c is *complete* if for each $w \in \{0, 1\}^*$, either none or both successors $w0, w1$ of w are prefixes of code words. By induction on the word length, define $p_w, q_w \in \Delta_2^*$ by $p_\varepsilon = \varepsilon = q_\varepsilon$, $p_{iw} = p_w p_i$ and $q_{iw} = q_i q_w$ for $i < 2$. Then the map c is a prefix code iff in C_2 , $p_{c_i} q_{c_j} = \delta_{i,j}$ for all $i, j < n$, and a complete prefix code iff additionally $\sum_{i < n} q_{c_i} p_{c_i} = 1$. In the latter case, $p_i \mapsto p_{c_i}, q_i \mapsto q_{c_i}$ ($i < n$) is an embedding of C_n into C_2 , for $n \geq 2$. For example, $c_0 = 0, c_1 = 10, c_2 = 11$ is a complete prefix code $c : \{0, 1, 2\} \rightarrow \{0, 1\}^*$, leading to an embedding of C_3 into C_2 by mapping p_0, p_1, p_2 to $p_0, p_0 p_1, p_1 p_1$ and q_0, q_1, q_2 to $q_0, q_1 q_0, q_1 q_1$. The code words of different closing brackets aren’t prefixes of each other, those of different opening brackets aren’t suffixes of another.

As we omit the completeness condition, we can view the semiring equations (7) as *monoid* equations, interpreted in monoids with an annihilating element 0, which leads to a different representation of $C'_{n, \mathcal{A}}$ that will be useful later.

The *polycyclic monoid generated by P_n* is the quotient $P'_n := (\Delta_n \cup \{0\})^* / \rho_n$, where ρ_n now is the monoid congruence generated by

$$\{p_i q_j = \delta_{i,j} \mid i, j < n\} \cup \{x0 = 0 = 0x \mid x \in \Delta_n \cup \{0\}\}.$$

Every string $w \in (\Delta_n \cup \{0\})^*$ can be reduced to a ‘‘normal form’’ $nf_n(w)$, using the equations generating ρ_n as rules to shorten a string. The reduction either runs into a bracket mismatch and returns 0, or finds matching brackets only and returns 1, or ends in a string with all closing brackets in front of all opening brackets. Hence we can identify P'_n with $(Q_n^* P_n^* \cup \{0\}, \cdot, 1)$, where

$$u \cdot v := nf_n(uv), \quad u, v \in Q_n^* P_n^* \cup \{0\}.$$

The *Cayley graph* of the polycyclic monoid P'_n is the graph whose nodes are the elements $x \in P'_n$ with edges $x \xrightarrow{\delta} x \cdot \delta$, for $\delta \in \Delta_n$.

Example 3. A drawing of a part of the Cayley graph of P'_2 is given in Figure 1, not showing the sink node 0 and edges connected to it. For example, there is no edge $q_0 \xrightarrow{p_0} 1$ since $q_0 p_0 \neq 1$. As the picture indicates, the subgraph with nodes in P'_2 corresponds to a stack $P_2^* \simeq \{0, 1\}^*$, with p_i for *push*(i) and q_i for *pop*(i).^d For $u \in \Delta_2^*$, an equation $u = 1$ holds in P'_2 iff u is the label of a path in the Cayley graph starting and ending in the node 1. The set of all those u is of course the *pure Dyck language* $D_2 \in \mathcal{C} \Delta_2^*$ of balanced strings over Δ_2 , i.e. the least solution in $\mathcal{P} \Delta_2^*$ of

$$y \geq 1 + p_0 y q_0 + p_1 y q_1 + y y.$$

It is easy to define a pushdown automaton accepting $\{u \in \Delta_2^* \mid u = 1 \text{ in } P'_2\}$, keeping on its stack the ρ -normal form of the input sequence read. \diamond

Nivat and Perrot (1970) initiated the study of the connection between polycyclic monoids and formal languages.

^dActually, the graph contains infinitely many stacks: for each $\vec{q} \in Q_2^*$ the subgraph with nodes in $\{\vec{q}\} P_2^*$ forms a stack.

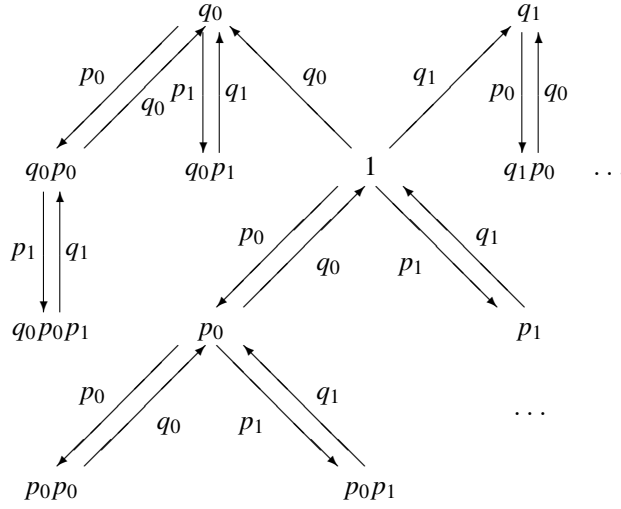


Figure 1. Part of the Cayley graph of P'_2 .

Remark 8. The polycyclic dioid C'_n has an interpretation in the algebra of binary relations on the Cayley graph of P'_n , arising from the interpretation of p_i and q_i as the transition relations $\xrightarrow{p_i}$ and $\xrightarrow{q_i}$ shown in the graph. Under this interpretation, the completeness condition $1 = \sum_{i < n} q_i p_i$ is not valid, as node 1 is not related to itself by any of the relations $\xrightarrow{q_i p_i}$. We will later use a restriction of this interpretation to the “stack part” P_n^* of P'_n . It may seem that a constant π representing an “empty stack” test is needed to model a stack properly by

$$1 = \pi + \sum \{ q_i p_i \mid i < n \}, \quad \pi \pi = \pi, \quad \bigwedge_{i < n} (\pi q_i = 0 = p_i \pi).$$

But in our case, this turns out to be unnecessary, since computations popping from the empty stack can be interpreted by the empty transition relation (cf. the proof of Theorem 17).

The map from P'_n to $\mathcal{A} \Delta_n^*$ given by $0 \mapsto \emptyset$, $1 \mapsto \{1\}$, and $p_i \mapsto \{p_i\}$, $q_i \mapsto \{q_i\}$ for $i < n$, extends to a homomorphism from P'_n to the multiplicative monoid of $C'_{n, \mathcal{A}}$. Since P'_n has no non-trivial congruences (by Nivat and Perrot (1970)), this is an embedding of P'_n into $C'_{n, \mathcal{A}}$, for $\mathcal{F} \leq \mathcal{A}$.

In the original definition $C'_n = \mathcal{R} \Delta_n^* / \rho_n$, we take the quotient of the regular sets $\mathcal{R} \Delta_n^*$ under the semiring congruence ρ_n , where 1 stands for $\{1\}$ and 0 for \emptyset . We now take the regular sets $\mathcal{R} P'_n$ of P'_n and remove the annihilating element:

Proposition 9. *If ν is the least \mathcal{R} -congruence on $\mathcal{R} P'_n$ containing $(\{0\}, \emptyset)$, then $C'_n \simeq \mathcal{R} P'_n / \nu$. Moreover, for $B, B' \in \mathcal{R} P'_n$ we have $B / \nu = B' / \nu$ iff $B \setminus \{0\} = B' \setminus \{0\}$.*

Proof. We first prove the second statement. \Leftarrow : Since $\{0\} / \nu = \emptyset / \nu$ is the least element of $\mathcal{R} P'_n / \nu$, for $B \in \mathcal{R} P'_n$ we have

$$B / \nu = \sum \{ \{w\} / \nu \mid w \in B \} = \sum \{ \{w\} / \nu \mid w \in B \setminus \{0\} \}.$$

Hence if $B, B' \in \mathcal{R} P'_n$ with $B \setminus \{0\} = B' \setminus \{0\}$, then $B / \nu = B' / \nu$.

\Rightarrow : One shows by induction on the regular operations that

$$v_0 := \{ (B, B') \in \mathcal{R}P'_n \times \mathcal{R}P'_n \mid B \setminus \{0\} = B' \setminus \{0\} \}$$

is an \mathcal{R} -congruence containing $(\{0\}, \emptyset)$, and obviously $v \subseteq v_0$.

For the first statement, notice that a congruence class A/ρ_n of $C'_n = \mathcal{R}\Delta_n^*/\rho_n$ can be represented by a set of reduced strings, $\{nf_n(w) \mid w \in A\} \setminus \{0\}$, and $A/\rho_n \mapsto \{nf_n(w) \mid w \in A\}/v$ constitutes an isomorphism between $C'_{n,\mathcal{R}}$ and $\mathcal{R}P'_n/v$. \square

The $n > 2$ pairs of brackets $p_i, q_i, i < n$ of Δ_n can be coded by two pairs, say b, d and p, q of Δ_2 , via $p_i := bp^i$ and $q_i := q^i d$. This extends to an embedding of P'_n in P'_2 .

2.3 The Tensor Product of \mathcal{A} -Dioids

In a category whose objects have a monoid structure, two morphisms $F_1 : M_1 \rightarrow M$ and $F_2 : M_2 \rightarrow M$ are *relatively commuting*, if for all $m_1 \in M_1$ and $m_2 \in M_2$, $F_1(m_1) \cdot F_2(m_2) = F_2(m_2) \cdot F_1(m_1)$.

The *tensor product of M_1 and M_2* is an object $M_1 \otimes M_2$ with two relatively commuting morphisms $\top_1 : M_1 \rightarrow M_1 \otimes M_2$ and $\top_2 : M_2 \rightarrow M_1 \otimes M_2$ such that for any pair of relatively commuting morphisms $f : M_1 \rightarrow M$ and $g : M_2 \rightarrow M$ there is a unique morphism $h_{f,g} : M_1 \otimes M_2 \rightarrow M$ with $f = h_{f,g} \circ \top_1$ and $g = h_{f,g} \circ \top_2$:

$$\begin{array}{ccccc}
 M_1 & \xrightarrow{\top_1} & M_1 \otimes M_2 & \xleftarrow{\top_2} & M_2 \\
 & \searrow f & \vdots h_{f,g} & \swarrow g & \\
 & & M & &
 \end{array}$$

By the universal property, the tensor product of M_1 and M_2 is unique up to isomorphism.

Example 4. In the category \mathbb{M} of monoids, the tensor product of M_1 and M_2 is the cartesian product $M_1 \times M_2$ with componentwise multiplication, unit $(1, 1)$ and injections defined by $\top_1(a) = (a, 1)$ for $a \in M_1$ and $\top_2(b) = (1, b)$ for $b \in M_2$. The induced morphism $h_{f,g}$ is the one given by $h_{f,g}(a, b) = f(a) \cdot g(b)$. \diamond

Theorem 10 (Hopkins and Leiß (2018)). *A tensor product of \mathcal{A} -dioids D_1 and D_2 exists,*

$$\top_1 : D_1 \longrightarrow D_1 \otimes_{\mathcal{A}} D_2 \longleftarrow D_2 : \top_2,$$

consisting of $D_1 \otimes_{\mathcal{A}} D_2 := \mathcal{A}(M_1 \times M_2)/\equiv$, where $M_i = \widehat{\mathcal{A}}D_i$ is the monoid underlying D_i and \equiv is the \mathcal{A} -congruence generated by the “tensor product relations”

$$\{(\sum A, \sum B)\} = A \times B \quad (A \in \mathcal{A}M_1, B \in \mathcal{A}M_2),$$

together with the embeddings \top_1, \top_2 defined by

$$\top_1(d_1) := \{(d_1, 1)\}/\equiv \text{ for } d_1 \in D_1 \text{ and } \top_2(d_2) := \{(1, d_2)\}/\equiv \text{ for } d_2 \in D_2.$$

The liftings of the embeddings of M_i into $M_1 \times M_2$ map $A \in \mathcal{A}M_1$ to $A \times \{1\} \in \mathcal{A}(M_1 \times M_2)$ and $B \in \mathcal{A}M_2$ to $\{1\} \times B \in \mathcal{A}(M_1 \times M_2)$, hence $A \times B = (A \times \{1\})(\{1\} \times B) \in \mathcal{A}(M_1 \times M_2)$. The tensor product relations are needed to make the $\top_i : D_i \rightarrow D_1 \otimes_{\mathcal{A}} D_2$ be \mathcal{A} -morphisms.

For $U \in \mathcal{A}(M_1 \times M_2)$, we write $[U]$ for U/\equiv . By the definition of \sum on $D_1 \otimes_{\mathcal{A}} D_2$,

$$[U] = \bigsqcup \{ \{(a, b)\} \mid (a, b) \in U \} = \sum \{ \{ \{(a, b)\} \} \mid (a, b) \in U \} = \sum \{ a \otimes b \mid (a, b) \in U \},$$

using $a \otimes b$ for $\top_1(a) \cdot \top_2(b) = [\{(a, b)\}] \in D_1 \otimes_{\mathcal{A}} D_2$. In particular, for $A \in \mathcal{A}M_1, B \in \mathcal{A}M_2$, via $A \times B \in \mathcal{A}(M_1 \times M_2)$ we have, by the tensor product relations,

$$(\sum A) \otimes (\sum B) = \sum \{a \otimes b \mid a \in A, b \in B\}.$$

Since $D_1 \otimes_{\mathcal{A}} D_2$ is the quotient of $\mathcal{A}(M_1 \times M_2)$ by the \mathcal{A} -congruence \equiv , its partial order is the dioid-order on the quotient, i.e. $[U] \leq [V]$ iff $[U \cup V] = [V]$ for $U, V \in \mathcal{A}(M_1 \times M_2)$, which is coarser than the order on $\mathcal{A}(M_1 \times M_2)$ among the representatives U, V .

Proposition 11 (Hopkins and Leiß (2018)). *For monoids M_1 and M_2 , the tensor product of the \mathcal{A} -dioids $\mathcal{A}M_1$ and $\mathcal{A}M_2$ is isomorphic to the \mathcal{A} -dioid $\mathcal{A}(M_1 \times M_2)$:*

$$\mathcal{A}M_1 \otimes_{\mathcal{A}} \mathcal{A}M_2 \simeq \mathcal{A}(M_1 \times M_2).$$

Proof. (Sketch) The isomorphism $\mathcal{A}M_1 \otimes_{\mathcal{A}} \mathcal{A}M_2 \rightarrow \mathcal{A}(M_1 \times M_2)$ maps $[R] = \sum \{A \otimes B \mid (A, B) \in R\} \in \mathcal{A}M_1 \otimes_{\mathcal{A}} \mathcal{A}M_2$ to $\bigcup \{A \times B \mid (A, B) \in R\}^e$, where $R \in \mathcal{A}(\mathcal{A}M_1 \times \mathcal{A}M_2)$; the inverse maps $S \in \mathcal{A}(M_1 \times M_2)$ to $\sum \{\{a\} \otimes \{b\} \mid (a, b) \in S\}$. \square

Since $(A, B) \mapsto A \times B$ is a homomorphism from $\mathcal{A}M \times \mathcal{A}N$ to $\mathcal{A}(M \times N)$, its lifting is a homomorphism from $\mathcal{A}(\mathcal{A}M \times \mathcal{A}N)$ to $\mathcal{A}(\mathcal{A}(M \times N))$, so for $R \in \mathcal{A}(\mathcal{A}M \times \mathcal{A}N)$,

$$S_R := \bigcup \{A \times B \mid (A, B) \in R\} \in \mathcal{A}(M \times N).$$

By the isomorphism, we have $[R] = [R']$ iff $S_R = S_{R'}$. Conversely, for $S \in \mathcal{A}(M \times N)$,

$$R_S := \{(\{a\}, \{b\}) \mid (a, b) \in S\} \in \mathcal{A}(\mathcal{A}M \times \mathcal{A}N),$$

hence $[R_S] = \sum \{\{a\} \otimes \{b\} \mid (a, b) \in S\} \in \mathcal{A}M \otimes_{\mathcal{A}} \mathcal{A}N$. Notice that $S_{(R_S)} = S$ and $[R_{(S_R)}] = [R]$.

We can push this a bit further:

Theorem 12. *Let M be a monoid and N a monoid with annihilating element 0 . Then*

$$\mathcal{A}M \otimes_{\mathcal{A}} (\mathcal{A}N/\mathfrak{v}) \simeq \mathcal{A}(M \times N)/\tilde{\mathfrak{v}} \quad (8)$$

where \mathfrak{v} is the least \mathcal{A} -congruence on $\mathcal{A}N$ containing $(\{0\}, \emptyset)$ and $\tilde{\mathfrak{v}}$ is the least \mathcal{A} -congruence on $\mathcal{A}(M \times N)$ containing $(\{(1, 0)\}, \emptyset)$.

Putting $R_{\mathfrak{v}} := \{(A, B/\mathfrak{v}) \mid (A, B) \in R\}$ for $R \in \mathcal{A}(\mathcal{A}M \times \mathcal{A}N)$, the isomorphism is given by

$$\begin{aligned} [R_{\mathfrak{v}}] &\mapsto (S_R)/\tilde{\mathfrak{v}}, \quad \text{where } S_R := \bigcup \{A \times B \mid (A, B) \in R\} \text{ for } R \in \mathcal{A}(\mathcal{A}M \times \mathcal{A}N), \\ S/\tilde{\mathfrak{v}} &\mapsto [(R_S)_{\mathfrak{v}}], \quad \text{where } R_S := \{(\{m\}, \{n\}) \mid (m, n) \in S\} \text{ for } S \in \mathcal{A}(M \times N). \end{aligned}$$

Proof. Since $B \mapsto B/\mathfrak{v}$ is a surjective \mathcal{A} -morphism from $\mathcal{A}N$ to $\mathcal{A}N/\mathfrak{v}$, each element of $\mathcal{A}(\mathcal{A}M \times (\mathcal{A}N/\mathfrak{v}))$ is of the form $R_{\mathfrak{v}}$ for some $R \in \mathcal{A}(\mathcal{A}M \times \mathcal{A}N)$.

To prove $\mathcal{A}M \otimes_{\mathcal{A}} (\mathcal{A}N/\mathfrak{v}) \simeq \mathcal{A}(M \times N)/\tilde{\mathfrak{v}}$, it is sufficient to show that

$$\top'_1 : \mathcal{A}M \longrightarrow \mathcal{A}(M \times N)/\tilde{\mathfrak{v}} \longleftarrow \mathcal{A}N/\mathfrak{v} : \top'_2$$

form a tensor product of $\mathcal{A}M$ and $\mathcal{A}N/\mathfrak{v}$ in $\mathbb{D}\mathcal{A}$, where \top'_1 and \top'_2 are defined by

$$\top'_1(A) = (A \times \{1\})/\tilde{\mathfrak{v}} \quad \text{and} \quad \top'_2(B/\mathfrak{v}) = (\{1\} \times B)/\tilde{\mathfrak{v}}.$$

This will be done in the Appendix. We here show that the isomorphism $\mathcal{A}M \otimes_{\mathcal{A}} (\mathcal{A}N/\mathfrak{v}) \simeq \mathcal{A}(M \times N)/\tilde{\mathfrak{v}}$ consisting of the \mathcal{A} -morphisms $h_{\top'_1, \top'_2}$ and $h'_{\top'_1, \top'_2}$ induced by the universal

^eThe hint to prove Prop.7 of Leiß and Hopkins (2018) erroneously says that $[R]$ maps to $\{(\sum A, \sum B) \mid (A, B) \in R\}$.

properties of the two tensor product constructions

$$\begin{array}{ccccc}
 \mathcal{A}M & \xrightarrow{\top'_1} & \mathcal{A}(M \times N)/\tilde{\nu} & \xleftarrow{\top'_2} & \mathcal{A}N/\nu \\
 & \searrow \top_1 & \vdots & \swarrow \top_2 & \\
 & & \mathcal{A}M \otimes_{\mathcal{A}} (\mathcal{A}N/\nu) & &
 \end{array}$$

h'_{\top_1, \top_2} (downward arrow from $\mathcal{A}(M \times N)/\tilde{\nu}$ to $\mathcal{A}M \otimes_{\mathcal{A}} (\mathcal{A}N/\nu)$)
 $h^{\top'_1, \top'_2}$ (upward arrow from $\mathcal{A}M \otimes_{\mathcal{A}} (\mathcal{A}N/\nu)$ to $\mathcal{A}(M \times N)/\tilde{\nu}$)

is as claimed in the theorem. The induced \mathcal{A} -morphisms are defined by

$$\begin{aligned}
 h_{\top'_1, \top'_2}([R\nu]) &:= \sum \{ \top'_1(A) \cdot \top'_2(B/\nu) \mid (A, B) \in R \} \\
 &= \sum \{ (A \times \{1\})/\tilde{\nu} \cdot (\{1\} \times B)/\tilde{\nu} \mid (A, B) \in R \} \\
 &= \sum \{ (A \times B)/\tilde{\nu} \mid (A, B) \in R \} \\
 &= (\bigcup \{ A \times B \mid (A, B) \in R \})/\tilde{\nu} = (S_R)/\tilde{\nu}
 \end{aligned}$$

and

$$\begin{aligned}
 h'_{\top_1, \top_2}(S/\tilde{\nu}) &:= \sum \{ \top_1(\{m\}) \cdot \top_2(\{v\}/\nu) \mid (m, v) \in S \} \\
 &= \sum \{ \{m\} \otimes \{v\}/\nu \mid (m, v) \in S \} \\
 &= [(R_S)\nu].
 \end{aligned}$$

Since $S_{(R_S)} = S$, we have

$$(h_{\top'_1, \top'_2} \circ h'_{\top_1, \top_2})(S/\tilde{\nu}) = h_{\top'_1, \top'_2}([(R_S)\nu]) = S_{(R_S)}/\tilde{\nu} = S/\tilde{\nu},$$

and since the tensor product embeddings \top_1, \top_2 are \mathcal{A} -morphisms,

$$\begin{aligned}
 (h'_{\top_1, \top_2} \circ h_{\top'_1, \top'_2})([R\nu]) &= h'_{\top_1, \top_2}(S_R/\tilde{\nu}) = [(R_{(S_R)})\nu] \\
 &= \sum \{ A \otimes B/\nu \mid (A, B) \in R_{(S_R)} \} \\
 &= \sum \{ \{m\} \otimes \{v\}/\nu \mid (m, v) \in S_R \} \\
 &= \sum \{ \{m\} \otimes \{v\}/\nu \mid (m, v) \in A \times B, (A, B) \in R \} \\
 &= \sum \{ A \otimes B/\nu \mid (A, B) \in R \} = [R\nu]. \quad \square
 \end{aligned}$$

With Proposition 9, Theorem 12 provides us with a representation of $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$ as a quotient of a free \mathcal{R} -dioid extension:

Corollary 13. *Let ν be the least \mathcal{R} -congruence on $\mathcal{R}P'_2$ containing $(\{0\}, \emptyset)$ and $\tilde{\nu}$ the least \mathcal{R} -congruence on $\mathcal{R}(M \times P'_2)$ containing $(\{(1, 0)\}, \emptyset)$. Then*

$$\mathcal{R}M \otimes_{\mathcal{R}} C'_2 = \mathcal{R}M \otimes_{\mathcal{R}} (\mathcal{R}P'_2/\nu) \simeq \mathcal{R}(M \times P'_2)/\tilde{\nu}.$$

The *centralizer* $Z_C(D)$ of a set C under an embedding $i: C \rightarrow D$ in an \mathcal{A} -dioid D is the set

$$\{ d \in D \mid d \cdot i(c) = i(c) \cdot d \text{ for each } c \in C \}$$

of all elements of D that commute elementwise with the image of C in D under i .

In the following, we are concerned with $Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$, the centralizer of C'_2 in $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$ under the tensor product embedding $\top_2: C'_2 \rightarrow \mathcal{R}M \otimes_{\mathcal{R}} C'_2$.

Proposition 14. *For each \mathcal{R} -dioid K , $Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$ is an \mathcal{R} -dioid.*

Proof. It is clear that $Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$ is closed under $0, 1, +$ and \cdot , so it is a sub-dioid of $K \otimes_{\mathcal{R}} C'_2$. It is also closed under the restriction of $\Sigma: \mathcal{R}(K \otimes_{\mathcal{R}} C'_2) \rightarrow K \otimes_{\mathcal{R}} C'_2$ to $\mathcal{R}(Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2))$: if $c \in C'_2$ and $U \in \mathcal{R}(Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)) \subseteq \mathcal{R}(K \otimes_{\mathcal{R}} C'_2)$, then since $K \otimes_{\mathcal{R}} C'_2$ is \mathcal{R} -distributive,

$$(1 \otimes c) \sum U = \sum \{ (1 \otimes c)u \mid u \in U \} = \sum \{ u(1 \otimes c) \mid u \in U \} = (\sum U)(1 \otimes c),$$

so $\sum U \in Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$. \square

We will later see that for monoids M , $Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$ is a \mathcal{C} -dioid and isomorphic to $\mathcal{C}M$. The reason to care about this is that it provides us with regular expressions as a notation system for context-free sets. Every $L \in \mathcal{C}M$ is, under $\mathcal{C}M \simeq Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$, an element of the Kleene algebra $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$ and therefore denoted by a regular expression in the generators of $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$.

Example 5. Let $\Delta_2 = \{b, d, p, q\}$ and M be a monoid. For $x, y \in M$, consider $L = \{x^n y^n \mid n \in \mathbb{N}\} \in \mathcal{C}M$ and the regular expression $b(px)^*(yq)^*d$. Interpreted in $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$, elements $m \in M$ mean $\top_1(\{m\}) = \{m\} \otimes \{1\}/v$ and elements $\delta \in \Delta_2$ mean $\top_2(\{\delta\}) = \{1\} \otimes \{\delta\}/v$. By $*$ -continuity and since m and δ commute with each other in $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$, we obtain

$$\begin{aligned} b(px)^*(yq)^*d &= b(\sum \{ (px)^k \mid k \in \mathbb{N} \})(\sum \{ (yq)^n \mid n \in \mathbb{N} \})d \\ &= \sum \{ b(px)^k (yq)^n d \mid k \in \mathbb{N}, n \in \mathbb{N} \} \\ &= \sum \{ x^k y^n b p^k q^n d \mid k \in \mathbb{N}, n \in \mathbb{N} \} \\ &= \sum \{ x^n y^n \mid n \in \mathbb{N} \}, \end{aligned}$$

where in the final step, $b p^k q^n d = \delta_{k,n}$ since $pq = 1 = bd$ and $bq = 0 = pd$. We will see that for any $L \in \mathcal{C}M$, its image $\widehat{L} := \sum \{ \{m\} \otimes 1 \mid m \in L \}$ in $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$ belongs to $Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$. \diamond

To prove that $Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$, together with the embedding \top_1 , is the \mathcal{C} -closure of $\mathcal{R}M$, we want to show that any \mathcal{R} -morphism $f: \mathcal{R}M \rightarrow C$ to a \mathcal{C} -dioid C has a unique extension to a \mathcal{C} -morphism $\tilde{f}: Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2) \rightarrow C$, which can be defined by

$$\tilde{f}(\widehat{L}) = \sum \{ f(\{m\}) \mid m \in L \} \quad \text{for } L \in \mathcal{C}M.$$

At least this works if for each element $e \in Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$ there is a unique $L \in \mathcal{C}M$ with $e = \widehat{L}$. This uniqueness condition will be established in the following two sections: by Theorem 17 of Section 3, there is at most one such $L \in \mathcal{C}M$, by Theorem 26 of Section 4, there is at least one such L .

We remark that the category $\mathbb{D}\mathcal{A}$ has coproducts and free extensions (Hopkins and Leiß (2018)). In particular, the free \mathcal{A} -dioid extension of $\mathcal{A}M$ by the set Δ , $(\mathcal{A}M)[\Delta]$, is isomorphic to $\mathcal{A}(M[\Delta])$, whence we simply write $\mathcal{A}M[\Delta]$ below.

3. The CST for Monoids and its Algebraic Version

We now prove the Chomsky-Schützenberger theorem for arbitrary monoids M by reducing $\mathcal{C}M$ to $\mathcal{R}M$ and the pure Dyck language $D_2 \in \mathcal{C}\Delta_2^*$ and then give an algebraic version that embeds $\mathcal{C}M$ into $Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$. Recall that Δ_2 is an alphabet of two pairs of brackets and $M[\Delta_2]$ the free extension of M by Δ_2 , which is just $(X \cup \Delta_2)^*$ in case $M = X^*$.

3.1 The CST for Monoids

For a finite set X , Dyck's language $D_2(X) \in \mathcal{C}X^*[\Delta_2]$ of balanced strings over $X^*[\Delta_2]$ is the least solution of

$$y \geq 1 + X + yy + p_0yq_0 + p_1yq_1$$

in $\mathcal{P}(X^*[\Delta_2])$. By the classical theorem of Chomsky and Schützenberger (1963),

$$\mathcal{C}X^* \subseteq \{ h_{X^*}(R \cap D_2(X)) \mid R \in \mathcal{R}X^*[\Delta_2] \}.$$

Since, for any monoid M , $\mathcal{C}M$ is defined inductively as an extension of $\mathcal{F}M$ by least solutions (in $\mathcal{P}M$) of polynomial systems with parameters from $\mathcal{C}M$, one can reduce the parameters in the polynomial systems to the base case $\mathcal{F}M$, so that all elements of M that occur in the parameters come from a finite set $X \subseteq M$. Therefore,

$$\mathcal{C}M = \bigcup \{ \mathcal{C}\langle X \rangle \mid X \in \mathcal{F}M \}, \quad \text{likewise} \quad \mathcal{R}M = \bigcup \{ \mathcal{R}\langle X \rangle \mid X \in \mathcal{F}M \},$$

where $\langle X \rangle \subseteq M$ is the submonoid of M generated by X . We identify a system $y_1 \geq p_1, \dots, y_n \geq p_k$ with polynomials $p_i \in \mathcal{C}\langle X \rangle[y_1, \dots, y_k]$ with a context-free grammar with nonterminal symbols y_1, \dots, y_k , start symbol y_1 and terminal symbols from X . Since M resp. $D_2(M)$ belong to $\mathcal{C}M$ resp. $\mathcal{C}M[\Delta_2]$ only if M is finitely generated, we first state the CST for finitely generated monoids:

Theorem 15 (CST for finitely generated monoids). *For any finitely generated monoid M ,*

$$\mathcal{C}M \subseteq \{ h_M(R \cap D_2(M)) \mid R \in \mathcal{R}M[\Delta_2] \}$$

where $D_2(M) \in \mathcal{C}M[\Delta_2]$ is the set of elements of $M[\Delta_2]$ with balanced brackets.

Our proof is essentially a proof of the classical Chomsky-Schützenberger theorem, with a perhaps more intuitive construction of R from L than can be found in textbooks; it may be helpful to first consider Example 6 below. We use the grammatical definitions of $\mathcal{C}M$ and $\mathcal{R}M[\Delta_2]$.

Proof. Let $L_1, \dots, L_k \in \mathcal{C}M$ be the least solution of the grammar $G = \langle y_1 \geq p_1, \dots, y_k \geq p_k \rangle$. We can assume that the parameters in p_1, \dots, p_k are singletons or empty. Suppose there are $n - 1$ occurrences of the variables y_1, \dots, y_k in p_1, \dots, p_k . Let $\Delta_n = \{ \langle 0 \rangle, \dots, \langle n - 1 \rangle, |0\rangle, \dots, |n - 1\rangle \}$ be a set of n of “opening brackets” $\langle i \rangle$ and n “closing brackets” $|i\rangle$, $i < n$, and reserve the brackets $\langle 0 \rangle, |0\rangle$ for later usage. Surrounding the i -th variable occurrence by the i -th bracket pair, for $0 < i < n$, turns G into a grammar $G' = \langle y_1 \geq p'_1, \dots, y_n \geq p'_k \rangle$ with least solution $L'_1, \dots, L'_k \in \mathcal{C}M[\Delta_n]$. For example, a monomial $m_1y_{j,1} \dots m_s y_{j,s} m_{s+1}$ of p_j is turned into a monomial $m_1 \langle i_1 | y_{j,1} | i_1 \rangle \dots m_s \langle i_s | y_{j,s} | i_s \rangle m_{s+1}$ of p'_j . By the choice of p'_1, \dots, p'_k , clearly $L'_j \subseteq D_n(M)$ and $L_j = h_M(L'_j)$ for $1 \leq j \leq k$.

Approximate L'_1 from above by a regular set $R_1 \in \mathcal{R}M[\Delta_n]$ as follows. To each monomial α of p'_j (i.e. grammar rule (y_j, α) or $y_j \rightarrow \alpha$) attach a “follow”- or “continuation”-variable y_j^F and split αy_j^F into right-linear factors $m'y$, where $m' \in M[\Delta_n]$ and $y \in \{y_1, \dots, y_k, y_j^F\}$. Let G^F be the right-linear polynomial system

$$y_1 \geq p''_1, \dots, y_k \geq p''_k, \quad y_1^F \geq p_1^F + 1, \quad y_2^F \geq p_2^F, \dots, y_k^F \geq p_k^F,$$

where the monomials of p''_j are the initial factors $m'y$ of the monomials of $p'_j y_j^F$ and those of p_j^F are the factors $m'y$ that follow the occurrences of y_j in $p'_1 y_1^F, \dots, p'_k y_k^F$. Let R_1 be the first component of the least solution of G^F in $\mathcal{R}M[\Delta_n]$.

Claim 1. For all $1 \leq j \leq k$ and $w \in M[\Delta_n]$, if $y_j \Rightarrow_{G'}^* w$, then $y_j \Rightarrow_{G^F}^* w y_j^F$.

Proof of Claim 1. By induction on the derivation length $r \geq 1$ in $y_j \Rightarrow_{G'}^r w$. If $r = 1$, then $y_j \rightarrow w$ is a rule of G' , hence $y_j \rightarrow w y_j^F$ is a rule of G^F . If $y_j \Rightarrow_{G'}^1 \alpha \Rightarrow_{G'}^r w$, and the rule $y_j \rightarrow \alpha$ of G' is

$y_j \rightarrow m_1 \langle i_1 | y_{j,1} | i_1 \rangle \dots m_s \langle i_s | y_{j,s} | i_s \rangle m_{s+1}$, then G^F has the rules

$$y_j \rightarrow m_1 \langle i_1 | y_{j,1}, \quad y_{j,1}^F \rightarrow |i_1\rangle m_2 \langle i_2 | y_{j,2}, \quad \dots, \quad y_{j,s}^F \rightarrow |i_s\rangle m_{s+1} y_j^F,$$

and so $y_j \Rightarrow_{G^F}^* \alpha$. By the induction hypothesis, we get $\alpha \Rightarrow_{G^F}^* w y_j^F$, hence $y_j \Rightarrow_{G^F}^* w y_j^F$. \triangleleft

Using $y_1^F \geq 1$, Claim 1 implies $L'_1 \subseteq R_1$, hence $L'_1 \subseteq R_1 \cap D_n(M)$.

Claim 2. For all $1 \leq j \leq k$ and $w \in D_n(M)$, if $y_j \Rightarrow_{G^F}^* w y_j^F$, then $y_j \Rightarrow_{G'}^* w$.

Proof of Claim 2. by induction on the derivation length $r \geq 1$ in $y_j \Rightarrow_{G^F}^r w y_j^F$. If $r = 1$, then $y_j \rightarrow w$ is a rule of G' , hence $y_j \Rightarrow_{G'}^* w$. For length $r + 1 > 1$, suppose $y_j \Rightarrow_{G^F}^1 \alpha \Rightarrow_{G^F}^r w y_j^F$. By the choice of $y_j \geq p_j'$ of G^F , there is $m_1 \in M$ such that $\alpha = m_1 \langle i_1 | y_{j,1}$ for some i_1 and $y_{j,1} \in \{y_1, \dots, y_k\}$, or $\alpha = m_1 y_j^F$. The second case cannot occur, since $m_1 y_j^F \Rightarrow_{G^F}^r w y_j^F$ with $r \geq 1$ is impossible: all monomials of p_j^F either begin with a closing bracket $|i\rangle$ for some i , or are 1. In the first case, we have $m_1 \langle i_1 | y_{j,1} \Rightarrow_{G^F}^r w y_j^F$. As $w \in D_n(M)$, there are $u_1, w_1 \in D_n(M)$ such that $w = m_1 \langle i_1 | u_1 | i_1 \rangle w_1$ and $y_{j,1} \Rightarrow_{G^F}^r u_1 | i_1 \rangle w_1 y_j^F$. By the construction of G^F , $|i_1\rangle$ occurs only in rules $y_{j,1}^F \rightarrow |i_1\rangle m_2 y_j^F$ or $y_{j,1}^F \rightarrow |i_1\rangle m_2 \langle i_2 | y_{j,2}$ with $m_2 \in M$ and $y_{j,2} \in \{y_1, \dots, y_k\}$. So we must have

$$y_{j,1} \Rightarrow_{G^F}^{r_1} u_1 y_{j,1}^F \Rightarrow_{G^F}^{r_2} u_1 | i_1 \rangle w_1 y_j^F \quad \text{with } r = r_1 + r_2.$$

Since $r_1 \leq r$, we get $y_{j,1} \Rightarrow_{G'}^* u_1$ by induction. If the part $y_{j,1}^F \Rightarrow_{G^F}^{r_2} | i_1 \rangle w_1 y_j^F$ is

$$y_{j,1}^F \Rightarrow_{G^F} | i_1 \rangle m_2 y_j^F \Rightarrow_{G^F}^{r_2-1} | i_1 \rangle w_1 y_j^F,$$

we must have $r_2 = 1$ and $w_1 = m_2$, so $y_j \rightarrow m_1 \langle i_1 | y_{j,1} | i_1 \rangle m_2$ is a rule of G' , and since $w = m_1 \langle i_1 | u_1 | i_1 \rangle m_2$, the claim $y_j \Rightarrow_{G'}^* w$ is shown. If the part $y_{j,1}^F \Rightarrow_{G^F}^{r_2} | i_1 \rangle w_1 y_j^F$ is

$$y_{j,1}^F \Rightarrow_{G^F} | i_1 \rangle m_2 \langle i_2 | y_{j,2} \Rightarrow_{G^F}^{r_2-1} | i_1 \rangle w_1 y_j^F,$$

there are $u_2, w_2 \in D_n(M)$ such that $w_1 = m_2 \langle i_2 | u_2 | i_2 \rangle w_2$ and $y_{j,2} \Rightarrow_{G^F}^{r_2-1} u_2 | i_2 \rangle w_2$. As $|i_2\rangle$ occurs only in rules $y_{j,2}^F \rightarrow |i_2\rangle m_3 y_j^F$ or $y_{j,2}^F \rightarrow |i_2\rangle m_3 \langle i_3 | y_{j,3}$ with $m_3 \in M$ and $y_{j,3} \in \{y_1, \dots, y_k\}$, we can continue this way and get $1 \leq i_1, \dots, i_s \leq n$, $r_1, \dots, r_s \leq r$, $m_1, \dots, m_{s+1} \in M$ and $u_1, \dots, u_s \in D_n(M)$ such that $w = m_1 \langle i_1 | u_1 | i_1 \rangle \dots m_s \langle i_s | u_s | i_s \rangle m_{s+1}$ and

$$y_j \Rightarrow_{G^F} m_1 \langle i_1 | y_{j,1}, \quad y_{j,1} \Rightarrow_{G^F}^{r_1} u_1 y_{j,1}^F, \quad y_{j,1}^F \Rightarrow_{G^F} | i_1 \rangle m_2 \langle i_2 | y_{j,2}, \quad \dots \\ y_{j,s} \Rightarrow_{G^F}^{r_s} u_s y_{j,s}^F, \quad y_{j,s}^F \Rightarrow_{G^F} | i_s \rangle m_{s+1} y_j^F.$$

By construction of G^F there is a rule $y_j \rightarrow m_1 \langle i_1 | y_{j,1} | i_1 \rangle \dots \langle i_s | y_{j,s} | i_s \rangle m_{s+1}$ in G' , and by induction $y_{j,1} \Rightarrow_{G'}^* u_1, \dots, y_{j,s} \Rightarrow_{G'}^* u_s$. The claim $y_j \Rightarrow_{G'}^* w$ follows. \triangleleft

Using $y_1^F \Rightarrow_{G^F} 1$, Claim 2 implies $R_1 \cap D_n(M) \subseteq L'_1$. So $L'_1 = R_1 \cap D_n(M)$, and $L_1 = h_M(L'_1) = h_M(R_1 \cap D_n(M))$. Finally, $D_n(M)$ can be replaced by $D_2(M)$, since two bracket pairs b, d and p, q can be used to code $\langle i |$ by $b p^i$ and $| i \rangle$ by $q^i d$ for $i < n$. \square

Example 6. Let $a, b, c \in M$ and let $G = \langle y \geq p_y(y, z), z \geq p_z(y, z) \rangle$ be the following grammar

$$y \geq a + b z c, \quad z \geq y y,$$

which defines sets $L_y, L_z \in \mathcal{CM}$. Adding brackets around the three variable occurrences on the right turns G into the grammar $G' = \langle y \geq p'_y, z \geq p'_z \rangle$

$$y \geq a + b \langle 1 | z | 1 \rangle c, \quad z \geq \langle 2 | y | 2 \rangle \langle 3 | y | 3 \rangle,$$

defining $L'_y, L'_z \in \mathcal{CM}[\Delta_3]$. It is clear that $L_y = h_M(L'_y)$ and $L'_y \subseteq D_4(M)$. Now add follow- or continuation-variables to the summands of p'_y resp. p'_z

$$y \geq ay^F + b\langle 1|z|1\rangle cy^F, \quad z \geq \langle 2|y|2\rangle \langle 3|y|3\rangle z^F;$$

now split the monomials of $p'_y y^F$ and $p'_z z^F$ into right-linear factors mx with $m \in M[\Delta_3]$ and $x \in \{y, z, y^F, z^F\}$ to build the right-linear grammar $G^F = \langle y \geq p''_y, z \geq p''_z, y^F \geq p^F_y + 1, z^F \geq p^F_z \rangle$ where p''_y resp. p''_z are the sums of the initial factors from $p'_y y^F$ resp. $p'_z z^F$, and p^F_y resp. p^F_z are the sums of factors following an occurrence of y resp. z in $p'_y y^F$ and $p'_z z^F$:

$$y \geq ay^F + b\langle 1|z, \quad z \geq \langle 2|y, \quad y^F \geq |2\rangle \langle 3|y + |3\rangle z^F + 1, \quad z^F \geq |1\rangle cy^F. \quad (9)$$

We need $y^F \geq 1$, since y is the start symbol of G . Let $R_y \in \mathcal{RM}[\Delta_3]$ be the y -component of the least solution of G^F . Intuitively, $L'_y \subseteq R_y$ since y^F collects what follows *any* occurrence of y . A finite automaton accepting R_y is shown in Figure 2 (with initial node y and accepting node y^F). A regular expression for R_y can be obtained by solving the inequation system in Kleene algebra (i.e. replace $x \geq Ax + B$ with constant A, B by $x = A^*B$) with respect to y . This leads via $z^F = |1\rangle cy^F$,

$$\begin{aligned} y^F &= |2\rangle \langle 3|y + |3\rangle |1\rangle cy^F + 1 \\ &= (|3\rangle |1\rangle c)^* (|2\rangle \langle 3|y + 1) \\ &= (|3\rangle |1\rangle c)^* |2\rangle \langle 3|y + (|3\rangle |1\rangle c)^* \end{aligned}$$

and $z = \langle 2|y$ to the least solution in y by

$$\begin{aligned} y &= ay^F + b\langle 1| \langle 2|y \\ &= a(|3\rangle |1\rangle c)^* |2\rangle \langle 3|y + a(|3\rangle |1\rangle c)^* + b\langle 1| \langle 2|y \\ &= (b\langle 1| \langle 2| + a(|3\rangle |1\rangle c)^* |2\rangle \langle 3|)y + a(|3\rangle |1\rangle c)^* \\ &= (b\langle 1| \langle 2| + a(|3\rangle |1\rangle c)^* |2\rangle \langle 3|)^* a(|3\rangle |1\rangle c)^*. \end{aligned}$$

For example, R_y contains the description $b\langle 1| \langle 2|a|2\rangle \langle 3|a|3\rangle |1\rangle c$ of a parse tree of $baac \in L_y$.

Let us add a preview to the algebraic version of the theorem and its reverse. Those elements of R_y that are not descriptions of parse trees of words in L_y contain (modulo commuting brackets with elements of M) bracket mismatches or have initial closing brackets or final opening brackets, like $w = a|3\rangle |1\rangle c|2\rangle \langle 3|a \in R_y$. With the reserved pair $\langle 0|, |0\rangle$ of brackets, we can go from R_y to $\{\langle 0|\}R_y\{|0\rangle\}$ and thereby turn the latter kind of elements into further ones with bracket mismatches, like $\langle 0|w|0\rangle = \langle 0|a|3\rangle |1\rangle c|2\rangle \langle 3|a|0\rangle$. The modified automaton shown in Figure 3 therefore is a description of L_y : if we only extract sequences without bracket mismatches, we obtain all parse trees of words in L_y . An algorithm to extract only bracket-mismatch-free strings from a similar automaton is given by Hulden (2011) for a CFG-parser based on the Chomsky-Schützenberger theorem. \diamond

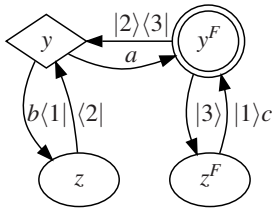


Figure 2. automaton for $R_y \in \mathcal{RM}[\Delta_3]$

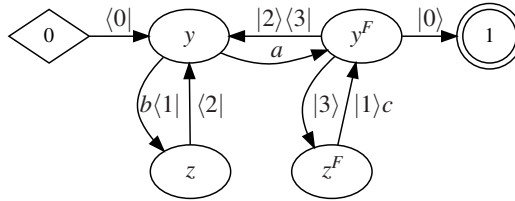


Figure 3. automaton for $L_y \in \mathcal{CM} \simeq \mathcal{Z}_{C_4}(\mathcal{RM} \otimes_{\mathcal{R}} C_4)$

A version without the assumption that M be finitely generated follows:

Corollary 16 (CST for monoids). *Let M be a monoid M and $D_2 \in \mathcal{C}\Delta_2^*$ the pure Dyck language over Δ_2 . For any $L \in \mathcal{C}M$ there is some $R \in \mathcal{R}(M \times \Delta_2^*)$ such that*

$$L = \{ m \mid (m, d) \in R, d \in D_2 \}.$$

Moreover, for $(m, d) \in R$ the normal form $nf_2(d)$ in $P'_2 = Q_2^*P_2^* \cup \{0\}$ of d belongs to $\{0, 1\}$.

Proof. For each $L \in \mathcal{C}M$ there is a finitely generated submonoid $N \subseteq M$ with $L \in \mathcal{C}N$, and by Theorem 15 there is $S \in \mathcal{R}N[\Delta_2]$ such that $L = h_N(S \cap D_2(N))$. The homomorphism $\langle h_N, h_{\Delta_2^*} \rangle : N[\Delta_2] \rightarrow N \times \Delta_2^*$ lifts to a homomorphism $h : \mathcal{R}N[\Delta_2] \rightarrow \mathcal{R}(N \times \Delta_2^*) \subseteq \mathcal{R}(M \times \Delta_2^*)$. Then $R := h(S) \in \mathcal{R}(M \times \Delta_2^*)$ and $L = h_N(S \cap D_2(N)) = \pi_1(R \cap (M \times D_2))$, where $\pi_1 : M \times \Delta_2^* \rightarrow M$ is the first projection.

The normal form $nf_2 : (\Delta_2 \cup \{0\})^* \rightarrow P'_2 = Q_2^*P_2^* \cup \{0\}$ maps those $d \in \Delta_2^*$ with $d \in D_2$ to $1 \in Q_2^*P_2^*$ and those containing a bracket mismatch to 0. To obtain $nf_2(d) \in \{0, 1\}$ for all $(m, d) \in R$, the idea is to wrap all strings of Δ_2^* with a third pair of brackets; this turns strings with normal form 0 or 1 into strings with the same normal form and those with normal form in $Q_2^*P_2^* \setminus \{1\}$ into strings with normal form 0, because of a mismatch with the new wrapping brackets; finally, the three bracket pairs are coded by two.

To carry this out, suppose $\Delta_2 = \{\langle 0 \mid, \mid 0 \rangle, \langle 1 \mid, \mid 1 \rangle\}$. The embedding of Δ_2 in Δ_2^* given by $\langle i \mid \mapsto \langle 0 \mid \langle 1 \mid^{i+1}$ and $\mid i \rangle \mapsto \mid 1 \rangle^{i+1} \mid 0 \rangle$ for $i < 2$ extends to a homomorphism $' : \Delta_2^* \rightarrow \Delta_2^*$, such that $d \in D_2$ iff $d' \in D_2$ for each $d \in \Delta_2^*$. Then $(m, d) \mapsto (m, d')$ also is a homomorphism, which lifts to a homomorphism from $\mathcal{R}(M \times \Delta_2^*)$ to $\mathcal{R}(M \times \Delta_2^*)$. From $R \in \mathcal{R}(M \times \Delta_2^*)$ we therefore obtain $R' := \{ (m, d') \mid (m, d) \in R \} \in \mathcal{R}(M \times \Delta_2^*)$, hence also

$$\tilde{R} := \{ (1, \langle 0 \mid \rangle) \cdot R' \cdot \{ (1, \mid 0 \rangle) \} \in \mathcal{R}(M \times \Delta_2^*).$$

For $d \in \Delta_2^*$ we have $d \in D_2$ iff $d' \in D_2$ iff $\langle 0 \mid d' \mid 0 \rangle \in D_2$, so $nf_2(d) = 1$ iff $nf_2(\langle 0 \mid d' \mid 0 \rangle) = 1$, and

$$L = \{ m \mid (m, d) \in R, d \in D_2 \} = \{ m \mid (m, \tilde{d}) \in \tilde{R}, \tilde{d} \in D_2 \}.$$

Moreover, if $nf_2(d) = 0$, then $nf_2(\langle 0 \mid d' \mid 0 \rangle) = 0$. Finally, if $nf_2(d) \in Q_2^*P_2^* \setminus \{1\}$, then $nf_2(d')$ begins with $\mid 1 \rangle^{i+1} \mid 0 \rangle$ or ends in $\mid 0 \rangle \langle 1 \mid^{i+1}$, for some $i < 2$, so that $nf_2(\langle 0 \mid d' \mid 0 \rangle) = 0$. Hence, with \tilde{R} instead of R both claims in the statement of the corollary hold. \square

3.2 Algebraic Version of the CST for Monoids

In the algebraic formulation, the Chomsky-Schützenberger theorem relates the objects $\mathcal{R}M$ and $\mathcal{C}M$ of the Kleisli subcategories of $\mathbb{D}\mathcal{R}$ and $\mathbb{D}\mathcal{C}$. While in the classical formulation, each $L \in \mathcal{C}M$ has a regular approximation $R \in \mathcal{R}M[\Delta_2]$ such that $L = h_M(R \cap D_2(M))$, one can now perform both the intersection with $D_2(M)$ and the removal of brackets by h_M algebraically in $\mathcal{R}M \otimes_{\mathcal{R}} \mathcal{C}'_2$, and the relation between L and R becomes one of representing the same tensor.

Theorem 17 (Algebraic CST for monoids). *For any monoid M and $L \in \mathcal{C}M$, the elementwise \top_1 -image $\{ \{m\} \otimes 1 \mid m \in L \} \in \mathcal{C}(\mathcal{R}M \otimes_{\mathcal{R}} \mathcal{C}'_2)$ of L has a least upper bound in $\mathcal{R}M \otimes_{\mathcal{R}} \mathcal{C}'_2$,*

$$\sum \{ \{m\} \otimes 1 \mid m \in L \}.$$

In fact, there is an injective map $\hat{\cdot} : \mathcal{C}M \rightarrow Z_{\mathcal{C}'_2}(\mathcal{R}M \otimes_{\mathcal{R}} \mathcal{C}'_2)$ given by

$$L \mapsto \hat{L} := \sum \{ \{m\} \otimes 1 \mid m \in L \}.$$

Proof. By Corollary 16, for $L \in \mathcal{C}M$ there is $S \in \mathcal{R}(M \times \Delta_2^*)$ with $L = \{ m \mid (m, d) \in S, d \in D_2 \}$, and the normal form $nf_2(d)$ of each $d \in \Delta_2^*$ with $(m, d) \in S$ belongs to $\{0, 1\} \subset P'_2$. There is an

isomorphism $\mathcal{R}(M \times \Delta_2^*) \simeq \mathcal{R}M \otimes_{\mathcal{R}} \mathcal{R}\Delta_2^*$ which maps $S \in \mathcal{R}(M \times \Delta_2^*)$ to the congruence class of

$$R_S = \{ (\{m\}, \{d\}) \mid (m, d) \in S \} \in \mathcal{R}(\mathcal{R}M \times \mathcal{R}\Delta_2^*).$$

By definition, using $\{m\} \otimes \{d\} := [\{(\{m\}, \{d\})\}]$, this congruence class is

$$[R_S] = \sum \{ \{m\} \otimes \{d\} \mid (m, d) \in S \} \in \mathcal{R}M \otimes_{\mathcal{R}} \mathcal{R}\Delta_2^*,$$

where \sum is the least upper bound in the tensor product. The canonical map $\cdot/\rho_2 : \mathcal{R}\Delta_2^* \rightarrow C'_2 = \mathcal{R}\Delta_2^*/\rho_2$ is an \mathcal{R} -morphism mapping those $\{d\}$ with $d \in D_2$ to $1 \in C'_2$ and those where d contains a bracket mismatch to $0 \in C'_2$. By the assumption on S , for all $(m, d) \in S$ either $nf_2(d) = 1$, i.e. $d \in D_2$, or $nf_2(d) = 0$, i.e. d contains a bracket mismatch. Hence for all d with $(m, d) \in S$,

$$d \in D_2 \iff \{d\}/\rho_2 = 1, \quad d \notin D_2 \iff \{d\}/\rho_2 = 0.$$

The image of R_S under the homomorphism $h := Id \times \cdot/\rho_2 : \mathcal{R}M \times \mathcal{R}\Delta_2^* \rightarrow \mathcal{R}M \times C'_2$ is

$$\{ (\{m\}, \{d\}/\rho_2) \mid (m, d) \in S \} \in \mathcal{R}(\mathcal{R}M \times C'_2),$$

and therefore, since $\{m\} \otimes 0 = 0$,

$$\begin{aligned} \sum \{ \{m\} \otimes 1 \mid m \in L \} &= \sum \{ \{m\} \otimes \{d\}/\rho_2 \mid (m, d) \in S, d \in D_2 \} \\ &= \sum \{ \{m\} \otimes \{d\}/\rho_2 \mid (m, d) \in S \} \\ &= [\{ (\{m\}, \{d\}/\rho_2) \mid (m, d) \in S \}] \in \mathcal{R}M \otimes_{\mathcal{R}} C'_2. \end{aligned}$$

Since each $\{m\} \otimes 1$ and $\{m\} \otimes 0$ commutes with each $1 \otimes c$ for $c \in C'_2$, we have

$$\widehat{L} = \sum \{ \{m\} \otimes 1 \mid m \in L \} \in Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$$

by the \mathcal{R} -distributivity of $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$.

To show that the map $L \mapsto \widehat{L}$ is injective, we present two relatively commuting \mathcal{R} -morphisms f, g such that the induced \mathcal{R} -morphism $h_{f,g}$ maps \widehat{L} back to a copy of L .

$$\begin{array}{ccc} \mathcal{R}M & \xrightarrow{\top_1} & \mathcal{R}M \otimes_{\mathcal{R}} C'_2 & \xleftarrow{\top_2} & C'_2 \\ & \searrow f & \vdots h_{f,g} & \swarrow g & \\ & & \mathcal{P}M \otimes_{\mathcal{P}} \text{Mat}_{P_2^*, P_2^*}(\mathbb{B}) & & \end{array}$$

Let f be defined by $f(A) = A \otimes I$ for $A \in \mathcal{R}M$, where $I \in \text{Mat}_{P_2^*, P_2^*}(\mathbb{B})$ is the unit matrix. By g we interpret C'_2 in $\text{Mat}_{P_2^*, P_2^*}(\mathbb{B})$, which is isomorphic to the \mathcal{P} -dioid of binary relations over P_2^* , the “stack part” of $P'_2 = (Q_2^* P_2^* \cup \{0\}, \cdot, 1)$. To define g , let $\gamma : \Delta_2^* \rightarrow \text{Mat}_{P_2^*, P_2^*}(\mathbb{B})$ be the homomorphism generated by the relations “push p_i ” and “pop p_i ” on the stack part P_2^* of P'_2 ,

$$\begin{aligned} \gamma(p_i) &:= \{ (u, v) \in P_2^* \times P_2^* \mid u \cdot p_i = v \}, \\ \gamma(q_i) &:= \{ (u, v) \in P_2^* \times P_2^* \mid u \cdot q_i = v \} \end{aligned}$$

for $i < 2$. Clearly, γ extends to an \mathcal{R} -morphism $\gamma^* : \mathcal{R}\Delta_2^* \rightarrow \text{Mat}_{P_2^*, P_2^*}(\mathbb{B})$ by

$$\gamma^*(U) = \sum \{ \gamma(u) \mid u \in U \}.$$

The semiring equations $p_i q_j = \delta_{i,j}$ hold in $\text{Mat}_{P_2^*, P_2^*}(\mathbb{B})$ when 0 and 1 are interpreted by the matrices 0 and I , and p_i and q_j by $\gamma^*(\{p_i\})$ and $\gamma^*(\{q_j\})$, for $i < 2$. Therefore, γ^* is constant on ρ_2 -congruence classes, and hence

$$g(U/\rho_2) = \{1\} \otimes \gamma^*(U), \quad \text{for } U \in \mathcal{R}\Delta_2^*,$$

defines an \mathcal{R} -morphism $g : C'_2 \rightarrow \mathcal{P}M \otimes_{\mathcal{P}} \text{Mat}_{P_2^*, P_2^*}(\mathbb{B})$. Obviously, f and g are relatively commuting.

By the choice of S for $L \in \mathcal{C}M$ we have $L = \{ m \mid (m, d) \in S, d \in D_2 \}$ and

$$\{d\}/\rho_2 = \begin{cases} 1 & d \in \pi_2(S) \cap D_2, \\ 0 & d \in \pi_2(S) \setminus D_2. \end{cases}$$

So $g(\{d\}/\rho_2)$ is the unit matrix for $d \in \pi_2(S) \cap D_2$ and the zero matrix for $d \in \pi_2(S) \setminus D_2$. As $h_{f,g}$ is an \mathcal{R} -morphism and $\{ \{m\} \otimes \{d\}/\rho_2 \mid (m, d) \in S \} \in \mathcal{R}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$, we obtain

$$\begin{aligned} h_{f,g}(\widehat{L}) &= h_{f,g}(\sum \{ \{m\} \otimes 1 \mid m \in L \}) \\ &= h_{f,g}(\sum \{ \{m\} \otimes \{d\}/\rho_2 \mid (m, d) \in S \}) \\ &= \sum \{ h_{f,g}(\{m\} \otimes \{d\}/\rho_2) \mid (m, d) \in S \} \\ &= \sum \{ f(\{m\}) \cdot g(\{d\}/\rho_2) \mid (m, d) \in S \} \\ &= \sum \{ \{m\} \otimes I \mid m \in L \} \\ &= L \otimes I, \end{aligned}$$

using the \mathcal{C} -continuity of the embedding $\top'_1 : \mathcal{P}M \rightarrow \mathcal{P}M \otimes_{\mathcal{P}} \text{Mat}_{P_2^*, P_2^*}(\mathbb{B})$ in the final step. Since \top'_1 is injective, $h_{f,g}$ is essentially an inverse to $L \mapsto \widehat{L}$.^f \square

4. The Reverse CST for Monoids and its Algebraic Version

The “reverse” Chomsky-Schützenberger theorem, i.e. that for finite X ,

$$\{ h_{X^*}(R \cap D_2(X)) \mid R \in \mathcal{R}(X^*[\Delta_2]) \} \subseteq \mathcal{C}X^*,$$

involves showing that $R \cap D_2(X) \in \mathcal{C}(X^*[\Delta_2])$, which is usually done by coupling a finite-state acceptor with a push-down automaton. This construction does not work for an arbitrary finitely generated monoid M , since there is no standard presentation of elements of M as a sequence of elements from a finite generating subset; we therefore give a grammatical proof of $h_M(R \cap D_2(M)) \in \mathcal{C}M$ below. A version of a reverse CST for arbitrary monoids follows.

4.1 The Reverse CST for Monoids

If M is a finitely generated monoid, M belongs to $\mathcal{C}M$ and $D_2(M)$ to $\mathcal{C}(M[\Delta_2])$. So we can formulate the reverse CST for such M as direct generalization from the classical case.

Theorem 18 (Reverse CST for finitely generated monoids). *For any finitely generated monoid M ,*

$$\{ h_M(R \cap D_2(M)) \mid R \in \mathcal{R}(M[\Delta_2]) \} \subseteq \mathcal{C}M.$$

Proof. Since the erasing homomorphism $h_M : M[\Delta_2] \rightarrow M$ lifts to $h_M : \mathcal{C}(M[\Delta_2]) \rightarrow \mathcal{C}M$, it is sufficient to show

$$\{ R \cap D_2(M) \mid R \in \mathcal{R}(M[\Delta_2]) \} \subseteq \mathcal{C}(M[\Delta_2]). \quad (10)$$

Suppose $R \in \mathcal{R}(M[\Delta_2])$. Then R is a component of the least solution \vec{R} of a system

$$y_1 \geq p_1(y_1, \dots, y_n), \dots, y_n \geq p_n(y_1, \dots, y_n)$$

with right-linear polynomials $p_1, \dots, p_n \in \mathcal{F}(M[\Delta_2])[y_1, \dots, y_n]$. By means of additional variables and inequations, we can assume that all parameters from $\mathcal{F}(M[\Delta_2])$ are singletons or the

^fIt can be shown that $\mathcal{P}M \otimes_{\mathcal{P}} \text{Mat}_{P_2^*, P_2^*}(\mathbb{B}) \simeq \text{Mat}_{P_2^*, P_2^*}(\mathcal{P}M)$, so $L \otimes I$ is the diagonal matrix $LI \in \text{Mat}_{P_2^*, P_2^*}(\mathcal{P}M)$.

empty set, so that the system can be written as

$$\begin{aligned} y_1 &\geq w_{1,1}y_1 + \dots + w_{1,n}y_n + w_1, \\ &\vdots \\ y_n &\geq w_{n,1}y_1 + \dots + w_{n,n}y_n + w_n, \end{aligned} \tag{11}$$

where the parameters $w_{i,j}$, w_i are from $M[\Delta_2]$, standing for the singletons $\{w_{i,j}\}$ resp. $\{w_i\}$, or are 0 and stand for \emptyset . As any $w \in M[\Delta_2]$ is an interleaving sequence of products of generators of M and elements of Δ_2 , we can further assume that $w_{i,j} \in X \cup \Delta_2 \cup \{0\}$ and $w_i \in \{0, 1\}$, where X is the finite set of generators of M . Then \bar{R} can be obtained by first taking the least solution of (11) in $\mathcal{R}(X^*[\Delta_2])$ and then interpreting sequences $m_1 \dots m_k \in X^*$ by the product $m_1 \cdot^M \dots \cdot^M m_k \in M$ of their members. Let $D_2(M) \in \mathcal{C}(M[\Delta_2])$ be the likewise interpretation of the Dyck language $D_2(X) \in \mathcal{C}(X^*[\Delta_2])$. To show $R \cap D_2(M) \in \mathcal{C}(M[\Delta_2])$, we construct a context-free grammar G , the least solution of which assigns $R \cap D_2(M)$ to its main variable S . The variables Y of G are S and all $[y, D, z]$, $[y, d, z]$ and $[y, D, 1]$ with $y, z \in \{y_1, \dots, y_n\}$, $d \in \Delta_2$ and auxiliary symbol D . Its inequations $Y \geq \alpha_1 + \dots + \alpha_k$ are obtained by combining all inequations $Y \geq \alpha_1, \dots, Y \geq \alpha_k$ with the same left hand side Y from the following table:

<i>context-free G for $R \cap D_2(X)$</i>	<i>definition (11) of R contains</i>
$S \geq [y, D, z]$	y as main variable
$[y, D, z] \geq m$	$y \geq mz$ with $m \in X$
$[y, d, z] \geq d$	$y \geq dz$ with $d \in \Delta_2$
$[y, D, 1] \geq 1$	$y \geq 1$
$[y, D, y] \geq 1$	
$[y, D, z] \geq [y, D, y'] [y', D, z]$	
$[y, D, z] \geq [y, p_i, y'] [y', D, z'] [z', q_i, z]$	$(i < 2)$

Claim 1. For all $y \in \{y_1, \dots, y_n\}$ and $z \in \{y_1, \dots, y_n\} \cup \{1\}$,

$$[y, D, z] \Rightarrow_G^* w \text{ and } w \in M[\Delta_2] \quad \text{iff} \quad y \Rightarrow_R^* wz \text{ and } w \in D_2(X).$$

Proof of Claim 1. \Rightarrow : This is easily shown by induction on k in $[y, D, z] \Rightarrow_G^k w$.

\Leftarrow : Suppose $y \Rightarrow_R^k wz$ and $w \in D_2(X)$. There is a sequence

$$y_{i_0} \geq w_1 y_{i_1}, \quad y_{i_1} \geq w_2 y_{i_2}, \quad \dots, \quad y_{i_{k-1}} \geq w_k y_{i_k}$$

such that $y = y_{i_0}$, $w = w_1 \dots w_k \in D_2(X)$ and $y_{i_k} = z$. We show by induction on k that

$$[y_{i_0}, D, y_{i_k}] \Rightarrow_G^* w_1 \dots w_k.$$

If $k=0$, then $w=1$, and $[y_{i_0}, D, y_{i_0}] \Rightarrow_G 1$. If $k=1$, then $y_{i_0} \geq w y_{i_1}$ and $w \in X \cup \Delta_2$. From $w \in D_2(X)$ we get $w \in X$, hence $[y_{i_0}, D, y_{i_1}] \Rightarrow_G w$.

Suppose $k > 1$. By the construction of $D_2(X)$, there either is $j < k$ with $u := w_1 \dots w_j, v := w_{j+1} \dots w_k \in D_2(X)$ and $w = uv$, or $w = p_0 u q_0$ or $w = p_1 u q_1$ with $u := w_2 \dots w_{k-1} \in D_2(X)$.

In the first case, we have $y_{i_0} \Rightarrow_R^* u y_{i_j}$ and $y_{i_j} \Rightarrow_R^* v y_{i_k}$ in less than k steps, hence $[y_{i_0}, D, y_{i_j}] \Rightarrow_G^* u$ and $[y_{i_j}, D, y_{i_k}] \Rightarrow_G^* v$ by induction, hence

$$[y_{i_0}, D, y_{i_k}] \Rightarrow_G [y_{i_0}, D, y_{i_j}] [y_{i_j}, D, y_{i_k}] \Rightarrow_G^* uv = w.$$

In the second case, we have $y_{i_1} \Rightarrow_R^* u y_{i_{k-1}}$ in less than k steps, hence $[y_{i_1}, D, y_{i_{k-1}}] \Rightarrow_G^* u$ by induction, hence for $j \in \{0, 1\}$,

$$\begin{aligned} [y_{i_0}, D, y_{i_k}] &\Rightarrow_G [y_{i_0}, p_j, y_{i_1}] [y_{i_1}, D, y_{i_{k-1}}] [y_{i_{k-1}}, q_j, y_{i_k}] \\ &\Rightarrow_G^2 p_j [y_{i_1}, D, y_{i_{k-1}}] q_j \Rightarrow_G^* p_j u q_j = w. \quad \triangleleft \end{aligned}$$

It follows that G defines $R \cap D_2(X)$: If $w \in L(G)$, then $S \Rightarrow_G^* w \in M[\Delta_2]$, so by the claim, $w \in R \cap D_2(X)$. Conversely, if $y \Rightarrow_R^* w z$ and y is the main variable of (11), then $S \Rightarrow_G [y, D, z] \Rightarrow_G^* w$, so $w \in L(G)$. \square

As with the CST, a version of the reverse CST for arbitrary monoids follows:

Corollary 19 (Reverse CST for monoids). *For any monoid M and $R \in \mathcal{R}(M \times \Delta_2^*)$,*

$$\{ m \mid (m, d) \in R, d \in D_2 \} \in \mathcal{C}M.$$

Proof. If $R \in \mathcal{R}(M \times \Delta_2^*)$, there is a finitely generated submonoid $N \subseteq M$ with $R \in \mathcal{R}(N \times \Delta_2^*)$. The homomorphism $\langle h_N, h_{\Delta_2^*} \rangle : N[\Delta_2] \rightarrow N \times \Delta_2^*$ is surjective and lifts to a surjective homomorphism $h : \mathcal{R}N[\Delta_2] \rightarrow \mathcal{R}(N \times \Delta_2^*)$, by Theorem 1. So there is $S \in \mathcal{R}N[\Delta_2]$ with $R = h(S)$. For any $(m, d) \in R$ there is $w \in S$ with $(m, d) = \langle h_N, h_{\Delta_2^*} \rangle(w) = (h_N(w), h_{\Delta_2^*}(w))$. Then $d \in D_2$ iff $w \in D_2(N)$ iff $w \in S \cap D_2(N)$ iff $m \in h_N(S \cap D_2(N))$. Hence

$$\{ m \mid (m, d) \in R, d \in D_2 \} = h_N(S \cap D_2(N)) \in \mathcal{C}N \subseteq \mathcal{C}M,$$

using Theorem 18. \square

Using the first projection $\pi_1 : M \times \Delta_2^* \rightarrow M$, Corollaries 16 and 19 combine to:

$$\mathcal{C}M = \{ \pi_1(R \cap (M \times D_2)) \mid R \in \mathcal{R}(M \times \Delta_2^*) \}, \quad \text{for any monoid } M.$$

We can simplify this further by going from strings in Δ_2^* to their normal forms in the polycyclic monoid $P_2' \simeq (Q_2^* P_2^* \cup \{0\}, \cdot, 1)$ where $u \cdot v = nf_2(uv)$ for $u, v \in Q_2^* P_2^* \cup \{0\}$:[§]

Proposition 20. *For any monoid M ,*

$$\mathcal{C}M = \{ \{ m \mid (m, 1) \in R \} \mid R \in \mathcal{R}(M \times P_2') \}.$$

Proof. Let $\eta : \Delta_2^* \rightarrow (\Delta_2 \cup \{0\})^*$ be the inclusion homomorphism. Clearly $nf_2 : (\Delta_2 \cup \{0\})^* \rightarrow P_2'$ is a homomorphism, and surjective since $0 = nf_2(p_0 q_1)$. Therefore, the lifting of $Id_M \times (nf_2 \circ \eta) : M \times \Delta_2^* \rightarrow M \times P_2'$ to the set level is a surjective homomorphism

$$h : \mathcal{R}(M \times \Delta_2^*) \rightarrow \mathcal{R}(M \times P_2').$$

For $d \in \Delta_2^*$, $nf_2(d) = 1$ iff $d \in D_2$. So if $S \in \mathcal{R}(M \times \Delta_2^*)$ and $R \in \mathcal{R}(M \times P_2')$ with $R = h(S)$, then

$$\{ m \mid (m, 1) \in R \} = \{ m \mid (m, d) \in S, d \in D_2 \} = \pi_1(S \cap (M \times D_2)).$$

Since h is surjective, the claim follows from the above combination of Corollaries 16 and 19. \square

Intuitively, since each $w \in P_2'$ defines a binary transition relation $\{ (u, v) \mid u \cdot w = v \} \subseteq P_2' \times P_2'$, each $(m, w) \in M \times P_2'$ gives a transition \xrightarrow{m}_w on P_2' with output in M , via

$$\xrightarrow{m}_w := \{ (u, m, v) \mid u \cdot w = v \} \subseteq P_2' \times M \times P_2'.$$

[§]The following proposition and properties of P_2' and $\mathcal{R}P_2'$ have been developed independently of the work of Render and Kambites (2009), who mention that the subsets of X^* accepted by P_2' -valence automata over a finite set X coincide with $\mathcal{C}X^*$. This seems closely related to Proposition 20.

If $(m, w) = (m, nf_2(d))$ or $(m, w) = (h_M(\alpha), nf_2(h_{\Delta_2^*}(\alpha)))$, the same relation is obtained from $(m, d) \in M \times \Delta_2^*$ or $\alpha \in M[\Delta_2]$. From $R \in \mathcal{R}(M \times P_2')$ an $S \in \mathcal{R}(M \times \Delta_2^*)$ with $R = h(S)$ as in the proof above can be obtained by induction on the construction of R , using $S = R$ for R of size 0 or 1, except $S = \emptyset$ for $R = \{(m, 0)\}$. Similarly, when M is finitely generated we can obtain $S \in \mathcal{R}M[\Delta_2]$ with $\{m \mid (m, 1) \in R\} = h_M(S \cap D_2(M))$ by induction on the construction of R , using $S = \{mw\}$ for $R = \{(m, w)\}$ in the singleton case, except again $S = \emptyset$ for $R = \{(m, 0)\}$.

Since, by Corollary 13, $\mathcal{R}M \otimes_{\mathcal{R}} C_2'$ is a quotient of $\mathcal{R}(M \times P_2')$, the above proposition can be used below to show $Z_{C_2'}(\mathcal{R}M \otimes_{\mathcal{R}} C_2') \subseteq \mathcal{C}M$.

4.2 Algebraic Version of the Reverse CST for Monoids

To prove an algebraic form of the reverse CST for monoids, i.e. that $Z_{C_2'}(\mathcal{R}M \otimes_{\mathcal{R}} C_2')$ embeds into $\mathcal{C}M$, we need to know which elements of $\mathcal{R}M \otimes_{\mathcal{R}} C_2'$ belong to the centralizer of C_2' . For this it is convenient to represent C_2' by $C_2' = \mathcal{R}P_2'/\nu$ as in Proposition 9 and to first characterize the elements of C_2' that commute with all generators of C_2' in C_2' , i.e.

$$Z_{\Delta_2}(C_2') := \{X/\nu \in \mathcal{R}P_2'/\nu \mid \{\delta\}/\nu \cdot X/\nu = X/\nu \cdot \{\delta\}/\nu \text{ for each } \delta \in \Delta_2\}.$$

Let I be the set of idempotent elements of P_2' , i.e. the set of those $w \in P_2'$ such that $ww = w$.

Proposition 21. *The set of idempotent elements of P_2' is*

$$\{\bar{q}\bar{p} \mid \bar{q} \in Q_2^*, \bar{p} \in P_2^*, \bar{p}\bar{q} = 1\} \cup \{0\}.$$

Proof. Let $I' := \{\bar{q}\bar{p} \mid \bar{q} \in Q_2^*, \bar{p} \in P_2^*, \bar{p}\bar{q} = 1\} \cup \{0\}$. Clearly, $0 \in I$. Suppose $\bar{q} = q_{i_1} \cdots q_{i_k} \in Q_2^*$, $\bar{p} = p_{j_m} \cdots p_{j_1} \in P_2^*$ and $\bar{q}\bar{p} \in I'$. Then $\bar{p}\bar{q} = 1$, so $\bar{q}\bar{p} \cdot \bar{q}\bar{p} = \bar{q}(\bar{p}\bar{q})\bar{p} = \bar{q}\bar{p}$, so $\bar{q}\bar{p} \in I$.

Suppose $w \in P_2' = Q_2^*P_2^* \cup \{0\}$ is idempotent. As $0 \in I'$, we may assume that $w = \bar{q}\bar{p}$ with $\bar{q} = q_{i_1} \cdots q_{i_k} \in Q_2^*$ and $\bar{p} = p_{j_m} \cdots p_{j_1} \in P_2^*$ for some $k, m \geq 0$. Suppose $k > m$. Since $ww = w \neq 0$, we have $p_{j_1}q_{i_1} = 1, \dots, p_{j_m}q_{i_m} = 1$ and

$$\bar{q}\bar{p}\bar{q}\bar{p} = \bar{q}p_{j_m} \cdots p_{j_1}q_{i_1} \cdots q_{i_k}\bar{p} = \bar{q}q_{i_{m+1}} \cdots q_{i_k}\bar{p} = \bar{q}\bar{p},$$

which is impossible. So $k \leq m$, and by symmetry, $k = m$. It follows that $\bar{p}\bar{q} = 1$, hence $\bar{q}\bar{p} \in I'$. \square

Proposition 22. *If $R \subseteq P_2'$ satisfies $\{\delta\}R \setminus \{0\} = R\{\delta\} \setminus \{0\}$, for all $\delta \in \Delta_2$, then $R \subseteq I$. If additionally $R \not\subseteq \{0, 1\}$, then $I_0 \subseteq R$ or $I_1 \subseteq R$, where*

$$I_j := \{\bar{q}q_j p_j \bar{p} \mid \bar{q} \in Q_2^*, \bar{p} \in P_2^*, \bar{p}\bar{q} = 1\} \cup \{1\} \quad \text{for } j \in \{0, 1\}.$$

Proof. Clearly, if $R \subseteq \{0, 1\}$, then $R \subseteq I$. So suppose $w \in R \setminus \{0, 1\}$. By symmetry, we may assume $w \in Q_2^+P_2^+$, say $w = q_0\bar{q}\bar{p}'$ for some $\bar{q} \in Q_2^*$ and $\bar{p}' \in P_2^*$. Then

$$0 \neq \bar{q}\bar{p}' = p_0q_0\bar{q}\bar{p}' \in p_0R \setminus \{0\} \subseteq Rp_0 \setminus \{0\},$$

so there is $w' \in R$ such that $\bar{q}\bar{p}' = w'p_0$. It follows that $\bar{p}' = \bar{p}p_0$ for some $\bar{p} \in P_2^*$, so $w = q_0\bar{q}\bar{p}p_0$. By induction on the length of $w' = \bar{q}\bar{p} \in R \setminus \{0\}$, it follows that \bar{p} is the inverse of \bar{q} , so $w' \in I$ and $w \in I$. Hence $R \subseteq I$. We have also seen that when $q_0w p_0 \in R$, then $w \in R$; in particular, if $R \not\subseteq \{0\}$, then $1 \in R$.

Finally, suppose $w \in R \setminus \{0, 1\}$. By the above argument, we may assume $w = q_j p_j$ with $j \in \{0, 1\}$. For each $i \in \{0, 1\}$, $q_i w \in q_i R \setminus \{0\} \subseteq R q_i \setminus \{0\}$, hence there is $w' \in R$ with $q_i w = w' q_i$, and since $w \in Q_2^+P_2^+$, we must have $w' = q_i w p_i$. By induction, it follows that $\{\bar{q}w\bar{p} \mid \bar{q} \in Q_2^*, \bar{p} \in P_2^*, \bar{p}\bar{q} = 1\} \subseteq R$, so $I_j \subseteq R$. \square

We next want to see that the infinite sets R with $I_j \subseteq R \subseteq I = I_0 \cup I_1 \cup \{0\}$ of Proposition 22 are not regular, i.e. do not belong to $\mathcal{R}P_2'$. In order to show, say, $I \notin \mathcal{R}P_2'$, we want to apply a pumping

lemma, representing I as the homomorphic image S/ρ_2 of some regular set S of the free monoid $(\Delta_2 \cup \{0\})^*$ by the monoid congruence

$$\rho_2 = \langle \{ p_i q_j = \delta_{i,j} \mid i, j < 2 \} \cup \{ x0 = 0 = 0x \mid x \in \Delta_2 \cup \{0\} \} \rangle.$$

But if $w = uv \in S$ is a “long” word with a splitting $w = xyzv$ with $xy^kzv \in S$ for all $k \geq 0$, the images $nf_2(xy^kzv)$ of the “pumped strings” xy^kzv need not be outside of I : for example, if $w = u.v = qpqbdaq.ppp$ with $nf_2(w) = qqppp \in I$ and $u = x.y.z = qp.qb.dq$, then for all $k \neq 1$, $nf_2(xy^kzv) = 0 \in I$, since from $pd = 0 = bq$ we get $nf_2(xz) = nf_2(qpdq) = 0$ and $nf_2(y^2) = nf_2(qbqb) = 0$. So a little care is needed to transfer the pumping of S to a pumping of $nf_2(S) = S/\rho_2$.

Proposition 23. *If $R \in \mathcal{RP}'_2$ and $R \subseteq I$, then R is finite.*

Proof. Let $\Delta = \Delta_2 \cup \{0\}$. The polycyclic monoid P'_2 is the quotient of the free monoid Δ^* by the monoid congruence ρ_2 above. Since $\cdot/\rho_2 : \Delta^* \rightarrow P'_2$ is a surjective homomorphism, every $R \in \mathcal{RP}'_2$ is the image under \cdot/ρ_2 of some $S \in \mathcal{R}\Delta^*$, i.e. $R = nf_2(S) = \{ nf_2(w) \mid w \in S \}$.

We first show that we may assume $R \subseteq S$. Let

$$\mathcal{A} = (A, \langle \xrightarrow{x} \mid x \in \Delta \cup \{1\} \rangle, I^{\mathcal{A}}, F^{\mathcal{A}})$$

be a finite automaton over Δ that recognizes S , with transition relations $\xrightarrow{x} \subseteq A \times A$ for $x \in \Delta$ and identity relation $\xrightarrow{1} \subseteq A \times A$, and sets $I^{\mathcal{A}} \subseteq A$ of initial and $F^{\mathcal{A}} \subseteq A$ of final states. Let

$$\mathcal{A}' = (A, \langle \xrightarrow{x'} \mid x \in \Delta \cup \{1\} \rangle, I^{\mathcal{A}'}, F^{\mathcal{A}'})$$

be the automaton obtained from \mathcal{A} by “path-compression”, i.e. the relations $\xrightarrow{x'} \supseteq \xrightarrow{x}$ are obtained as follows: for edges $s \xrightarrow{p_i} t \xrightarrow{q_i} u$, $i < 2$, add an edge $s \xrightarrow{1} u$, for edges $s \xrightarrow{x} t \xrightarrow{1} u$ or $s \xrightarrow{1} t \xrightarrow{x} u$, add an edge $s \xrightarrow{x} u$. Likewise, for edges $s \xrightarrow{p_i} t \xrightarrow{q_{1-i}} u$, $i < 2$, add an edge $s \xrightarrow{0} u$, and for edges $s \xrightarrow{x} t \xrightarrow{0} u$ or $s \xrightarrow{0} t \xrightarrow{x} u$ add $s \xrightarrow{0} u$. Let $S' \in \mathcal{R}\Delta^*$ be the set recognized by \mathcal{A}' . As each accepting path of \mathcal{A}' is the compression of an accepting path of \mathcal{A} , we have $S \subseteq S'$, and as their labellings have the same normal form, $nf_2(S') = nf_2(S) = R$. Since for each $\vec{q}\vec{p} \in R$ there is some $w \in S$ with $nf_2(w) = \vec{q}\vec{p}$, there is an accepting path in \mathcal{A}' labelled by $\vec{q}\vec{p}$, likewise for $0 \in R$, so $R \subseteq S'$. Henceforth, we assume $S = S'$ and $\mathcal{A} = \mathcal{A}'$.

We can now show that R cannot be infinite, by applying the pumping lemma. Let m be the number of states of an automaton \mathcal{A} recognizing a set $S \in \mathcal{R}\Delta^*$ with $nf_2(S) = R \subseteq S$. Suppose $R \subseteq I$ is infinite. Then there is $\vec{q}\vec{p} \in R \subseteq S$ with $|\vec{q}| > m$, $\vec{q} \in Q_2^*$, $\vec{p} \in P_2^*$ and $\vec{p}\vec{q} = 1$. Since \mathcal{A} has an accepting path labelled $\vec{q}\vec{p}$ and $m > |\vec{q}|$, by the pumping lemma for S there is a splitting $\vec{q} = xyz$ with $x, y, z \in Q_2^*$ with $0 < |y| \leq m$ such that $xy^kz\vec{p} \in S$ for all $k \geq 0$. But since $xz \in Q_2^*$, $\vec{p} \in P_2^*$ and $xz\vec{p} \in S$, we have $nf_2(xz\vec{p}) = xz\vec{p}$ (and $\neq 0$), which is impossible since $|xz| < |\vec{q}| = |\vec{p}|$ and $nf_2(S) = R \subseteq I$. It follows that R cannot be infinite. \square

When using $C'_2 \simeq \mathcal{RP}'_2/\nu$ according to Proposition 9, we write $0'$ and $1'$ for the neutral elements of $+$ and \cdot in C'_2 , respectively, to distinguish them from the elements 0 and 1 of P'_2 . The following lemmata characterize the elements of the centralizer of C'_2 in C'_2 and in $\mathcal{RM} \otimes_{\mathcal{R}} C'_2$, respectively:

Lemma 24. $Z_{C'_2}(C'_2) = \{0', 1'\} = Z_{\Delta_2}(C'_2)$.

Proof. $\{0', 1'\} \subseteq Z_{C'_2}(C'_2) \subseteq Z_{P'_2}(C'_2) \subseteq Z_{\Delta_2}(C'_2)$: Clearly, $0' = \emptyset/\nu = \{0\}/\nu$ and $1' = \{1\}/\nu = \{0, 1\}/\nu$ commute with R/ν for every $R \in \mathcal{RP}'_2$. The other two inclusions follow from the representations of $w \in P'_2$ and $\delta \in \Delta_2$ in C'_2 by $\{w\}/\nu$ and $\{\delta\}/\nu$.

$Z_{\Delta_2}(C'_2) \subseteq \{0', 1'\}$: Suppose $R \in \mathcal{RP}'_2$ and R/ν commutes with $\{\delta\}/\nu$ for every $\delta \in \Delta_2$. Then, by Proposition 9, $\{\delta\}R \setminus \{0\} = R\{\delta\} \setminus \{0\}$ for every $\delta \in \Delta_2$, so $R \subseteq I$ by Proposition 22, then

R is finite by Proposition 23, and so by Proposition 22 again, $R \subseteq \{0, 1\}$. It follows that $R/v = \emptyset/v = \{0\}/v = 0'$ or $R/v = \{1\}/v = \{0, 1\}/v = 1'$, hence $Z_{\Delta_2}(C'_2) \subseteq \{0', 1'\}$. \square

Lemma 25. $Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2) = \{ [R] \mid R \in \mathcal{R}(\mathcal{R}M \times C'_2), R \subseteq \mathcal{R}M \times \{0', 1'\} \}$.

Proof. \supseteq : For each $c \in C'_2$, $cb = bc$ for $b \in \{0', 1'\}$, so by \mathcal{R} -distributivity,

$$\begin{aligned} (\{1\} \otimes c)[R] &= (\{1\} \otimes c) \sum \{ A \otimes b \mid (A, b) \in R \} \\ &= \left(\sum \{ A \otimes b \mid (A, b) \in R \} \right) (\{1\} \otimes c) = [R](\{1\} \otimes c). \end{aligned}$$

\subseteq : Every element of $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$ is of the form $[R_v]$ with $R \in \mathcal{R}(\mathcal{R}M \times \mathcal{R}P'_2)$ and $R_v = \{ (A, B/v) \mid (A, B) \in R \}$. The isomorphism

$$\mathcal{R}M \otimes_{\mathcal{R}} C'_2 \simeq \mathcal{R}(M \times P'_2)/\tilde{v}$$

of Corollary 13 maps $[R_v]$ to $(S_R)/\tilde{v}$, where $S_R = \bigcup \{ A \times B \mid (A, B) \in R \}$. Suppose $[R_v] \in Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$. Then for every $w \in P'_2 \setminus \{0\}$, $[R_v]$ commutes with $[T_v]$ for $T = \{(\{1\}, \{w\})\}$, so the image $(S_R)/\tilde{v}$ of $[R_v]$ commutes with the image $(S_T)/\tilde{v} = \{(1, w)\}/\tilde{v}$ of $[T_v]$. Therefore, $(S_R)/\tilde{v} \in Z_{P'_2}(\mathcal{R}(M \times P'_2)/\tilde{v})$. In particular, $(S_R)/\tilde{v} \in Z_{\Delta_2}(\mathcal{R}(M \times P'_2)/\tilde{v})$, which is equivalent to

$$\{(1, \delta)\} S_R \setminus (M \times \{0\}) = S_R \{(1, \delta)\} \setminus (M \times \{0\}), \quad \text{for all } \delta \in \Delta_2.$$

Let $X \in \mathcal{R}P'_2$ be the second projection of $S_R \in \mathcal{R}(M \times P'_2)$. Then

$$\{\delta\} X \setminus \{0\} = X \{\delta\} \setminus \{0\}, \quad \text{for all } \delta \in \Delta_2,$$

i.e. $X/v \in Z_{\Delta_2}(C'_2)$, and by the proof of Lemma 24, $X \subseteq \{0, 1\}$, hence $S_R \subseteq M \times \{0, 1\}$. So for $(A, B) \in R$, $B \subseteq \{0, 1\}$, which implies $R_v \subseteq \mathcal{R}M \times \{0', 1'\}$. \square

We can now prove the algebraic form of the reverse CST for monoids:

Theorem 26 (Algebraic reverse CST for monoids). *For any monoid M and $R \in \mathcal{R}(\mathcal{R}M \times C'_2)$,*

$$\bigcup \{ A \mid (A, 1') \in R \} \in \mathcal{C}M.$$

In fact, there is an injective, monotone map $\vee : Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2) \rightarrow \mathcal{C}M$ defined by

$$[R] \mapsto [R]^\vee := \bigcup \{ A \mid (A, 1') \in R \} \quad \text{for } R \in \mathcal{R}(\mathcal{R}M \times C'_2).$$

Proof. We switch the notation and use $[R_v]$ for the $[R]$ of the statement above, where

$$R_v = \{ (A, B/v) \mid (A, B) \in R \} \in \mathcal{R}(\mathcal{R}M \times C'_2)$$

for some $R \in \mathcal{R}(\mathcal{R}M \times \mathcal{R}P'_2)$. For these R , put

$$L_R := \bigcup \{ A \mid (A, B) \in R, 1 \in B \}.$$

By Lemma 25, if $[R_v] \in Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$, then $R_v \subseteq \mathcal{R}M \times \{0', 1'\}$ and $R \subseteq \mathcal{R}M \times \mathcal{P}\{0, 1\}$.

Claim 1. The map $\cdot^\vee : Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2) \rightarrow \mathcal{P}M$, given by

$$[R_v]^\vee = \bigcup \{ A \mid (A, 1') \in R_v \}$$

for $R \in \mathcal{R}(\mathcal{R}M \times \mathcal{R}P'_2)$, is well-defined and has values in $\mathcal{C}M$.

Proof of Claim 1. Clearly, $\bigcup \{ A \mid (A, 1') \in R_v \} = L_R$. Suppose there is $T \in \mathcal{R}(\mathcal{R}M \times \mathcal{R}P'_2)$ with $[R_v] = [T_v]$. By the isomorphism $[R_v] \mapsto (S_R)/\tilde{v}$ of Corollary 13, $(S_R)/\tilde{v} = (S_T)/\tilde{v}$, hence

$$S_R \setminus (M \times \{0\}) = S_T \setminus (M \times \{0\}).$$

Therefore, using the fact that $S_R, T_R \subseteq M \times \{0, 1\}$,

$$\begin{aligned} L_R &= \bigcup \{A \mid (A, B) \in R, 1 \in B\} \\ &= \{m \mid (A, B) \in R, (m, 1) \in A \times B\} \\ &= \{m \mid (m, 1) \in S_R\} \\ &= \{m \mid (m, w) \in S_R \setminus (M \times \{0\}), w \in P'_2\} \\ &= \{m \mid (m, w) \in S_T \setminus (M \times \{0\}), w \in P'_2\} \\ &= L_T. \end{aligned}$$

It follows that $\bigcup \{A \mid (A, 1') \in R_V\} = L_R = L_T = \bigcup \{A \mid (A, 1') \in T_V\}$, and \cdot^\vee is well defined. Since $L_R = \{m \mid (m, 1) \in S_R\}$ for $S_R \in \mathcal{R}(M \times P'_2)$, we have $L_R \in \mathcal{C}M$ by Proposition 20. \triangleleft

Claim 2. The map $\cdot^\vee : Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2) \rightarrow \mathcal{C}M$ is injective.

Proof of Claim 2. We show that the map $\widehat{\cdot} : \mathcal{C}M \rightarrow Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$ of Theorem 17 is an inverse to \cdot^\vee . It maps $[R_V]^\vee = L_R \in \mathcal{C}M$ back to $[R_V]$, because

$$\begin{aligned} \widehat{[R_V]^\vee} &= \widehat{L_R} = \sum \{ \{m\} \otimes 1' \mid m \in L_R \} \\ &= \sum \{ A \otimes 1' \mid (A, B) \in R, 1 \in B \} \\ &= \sum \{ A \otimes 1' \mid (A, 1') \in R_V \} \\ &= [R_V] \end{aligned}$$

since $R_V \subseteq \mathcal{R}(\mathcal{R}M \times \{0', 1'\})$ and $A \otimes 0' = 0$. \triangleleft

Claim 3. The map $\cdot^\vee : Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2) \rightarrow \mathcal{C}M$ is monotone.

Proof of Claim 3. Suppose $R, S \in \mathcal{R}(\mathcal{R}M \times \mathcal{R}P'_2)$ such that $[R_V], [S_V] \in Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$ and $[R_V] \leq [S_V]$. By the definition of \leq on $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$, $[R_V \cup S_V] = [S_V]$, hence with Claim 1,

$$[R_V]^\vee \subseteq \bigcup \{A \mid (A, 1') \in R_V \cup S_V\} = \bigcup \{A \mid (A, 1') \in S_V\} = [S_V]^\vee,$$

so \cdot^\vee is monotone. \square

5. Algebraic Representation of the \mathcal{C} -Closure $\mathcal{C}M$ of $\mathcal{R}M$

We can now prove our first main result, which combines the algebraic Chomsky-Schützenberger theorem for monoids M and its reverse, Theorems 17 and 26, to an algebraic representation of the μ -continuous Chomsky algebra $\mathcal{C}M$ in the $*$ -continuous Kleene algebra $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$:

Theorem 27. For any monoid M , $Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$ is a \mathcal{C} -doid, and the maps

$$\widehat{\cdot} : \mathcal{C}M \rightleftarrows Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2) : \cdot^\vee$$

form a \mathcal{C} -isomorphism, where for $L \in \mathcal{C}M$ and $R \in \mathcal{R}(\mathcal{R}M \times C'_2)$ with $[R] \in Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$,

$$\widehat{L} := \sum \{ \{m\} \otimes 1' \mid m \in L \} \quad \text{and} \quad [R]^\vee := \bigcup \{A \mid (A, 1') \in R\}.$$

Proof. We use $D := Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$ to simplify the notation.

Claim 1. The maps $\cdot^\vee : D \rightarrow \mathcal{C}M$ and $\widehat{\cdot} : \mathcal{C}M \rightarrow D$ are inverse to each other.

Proof of Claim 1. In the proof of Theorem 17 we already showed that $\widehat{\cdot}$ is an inverse of \cdot^\vee . For $L \in \mathcal{CM}$ there is $R \in \mathcal{R}(\mathcal{RM} \times \mathcal{RP}'_2)$ with $R_\vee \subseteq \mathcal{RM} \times \{0', 1'\}$ and $\widehat{L} = [R_\vee]$, hence

$$(\widehat{L})^\vee = (\sum \{ \{m\} \otimes 1' \mid m \in L \})^\vee = [R_\vee]^\vee = \bigcup \{ A \mid (A, 1') \in R_\vee \} = L_R.$$

By applying $\widehat{\cdot}$, we get $\widehat{L} = \widehat{L}_R$, and since $\widehat{\cdot}$ is injective by Theorem 17, $(\widehat{L})^\vee = L_R = L$. \triangleleft

It remains to be shown that the bijection between \mathcal{CM} and D given by the maps $\widehat{\cdot}$ and \cdot^\vee is a \mathcal{C} -isomorphism.

Claim 2. The maps $\widehat{\cdot} : \mathcal{CM} \rightleftharpoons D : \cdot^\vee$ are monotone homomorphisms.

Proof of Claim 2. Obviously, $\widehat{\cdot}$ is monotone; the monotonicity of \cdot^\vee is noted in Theorem 17.

1. The map $\cdot^\vee : D \rightarrow \mathcal{CM}$ is a homomorphism: first,

$$1^\vee = (\{1\} \otimes 1')^\vee = [\{\{1\}, 1'\}]^\vee = \{1\},$$

and, second, for $[R], [S] \in D$ represented by $R, S \in \mathcal{R}(\mathcal{RM} \times \mathcal{C}'_2)$ with $R, S \subseteq \mathcal{RM} \times \{0', 1'\}$,

$$\begin{aligned} ([R] \cdot [S])^\vee &= [RS]^\vee \\ &= \bigcup \{ C \mid (C, 1') \in RS \} \\ &= \bigcup \{ AB \mid (A, 1') \in R, (B, 1') \in S \} \\ &= \bigcup \{ A \mid (A, 1') \in R \} \cdot \bigcup \{ B \mid (B, 1') \in S \} \\ &= [R]^\vee \cdot [S]^\vee, \end{aligned}$$

because $(C, 1') = (A, a)(B, b)$ iff $AB = C$ and $a = 1' = b$, since $a, b \in \{0', 1'\}$.

2. The map $\widehat{\cdot} : \mathcal{CM} \rightarrow D$ is a homomorphism: clearly $\widehat{\{1\}} = \{1\} \otimes 1' = 1$, and for $L_1, L_2 \in \mathcal{CM}$, since \cdot^\vee is a homomorphism and inverse to $\widehat{\cdot}$ by Claim 1, we have

$$(\widehat{L}_1 \cdot \widehat{L}_2)^\vee = (\widehat{L}_1)^\vee \cdot (\widehat{L}_2)^\vee = L_1 \cdot L_2 = \widehat{L_1 \cdot L_2}^\vee;$$

by the injectivity of \cdot^\vee , we get $\widehat{L}_1 \cdot \widehat{L}_2 = \widehat{L_1 \cdot L_2}$. \triangleleft

Claim 3. D is a \mathcal{C} -dioid.

Proof of Claim 3. By Theorem 17, for $L \in \mathcal{CM}$, $\widehat{L} = \sum \{ \{m\} \otimes 1' \mid m \in L \}$ is the least upper bound of the image $\widehat{f}(L)$ of L under the homomorphism $f : \mathcal{M} \rightarrow D$ given by $f(m) = \{m\} \otimes 1'$. By the previous claims, $\widehat{\cdot} : \mathcal{CM} \rightarrow D$ is a surjective homomorphism. By Proposition 14, D is an \mathcal{R} -dioid, hence a dioid, and so by Lemma 3, D is a \mathcal{C} -dioid. \triangleleft

Claim 4. $\widehat{\cdot} : \mathcal{CM} \rightarrow D$ and $\cdot^\vee : D \rightarrow \mathcal{CM}$ are \mathcal{C} -morphisms.

Proof of Claim 4. Since $\widehat{\cdot}$ is a homomorphism, so is $\mathcal{C}(\widehat{\cdot}) : \mathcal{CCM} \rightarrow \mathcal{CD}$. Hence for $U \in \mathcal{CCM}$, $\{\widehat{L} \mid L \in U\} \in \mathcal{CD}$, and by the proof of Lemma 3, its least upper bound is $\sum \{\widehat{L} \mid L \in U\} = \widehat{\bigcup U}$. Since $\widehat{\cdot}$ is a monotone homomorphism, this shows that $\widehat{\cdot}$ is a \mathcal{C} -morphism.

For \cdot^\vee , for each $V \in \mathcal{CD}$ there is $U \in \mathcal{CCM}$ such that $V = \{\widehat{L} \mid L \in U\}$ and $\sum V = \widehat{\bigcup U}$. Hence, since $\widehat{\cdot} : \mathcal{CM} \rightleftharpoons D : \cdot^\vee$ is a bijection, $(\widehat{L})^\vee = L$ for each $L \in \mathcal{CM}$. With $\bigcup U \in \mathcal{CM}$, it follows that

$$(\sum V)^\vee = (\widehat{\bigcup U})^\vee = \bigcup U = \bigcup \{\widehat{L}^\vee \mid L \in U\} = \bigcup \{v^\vee \mid v \in V\}.$$

As \cdot^\vee is a monotone homomorphism, this shows that it is a \mathcal{C} -morphism. \triangleleft \square

Before considering an algebraic representation of the fixed-point closure of an arbitrary \mathcal{R} -dioid K , we remark that the case $K = \mathcal{R}M$ treated so far is sufficient to provide a semantics to regular-expression tools for context-free languages, like the one proposed by Hopkins (2007).

The results proven for $n=2$ hold as well for $n > 2$, which is convenient to treat $\langle 0 |, |0 \rangle$ as a fresh, unused pair of brackets. If X is a finite set disjoint from Δ_n , all regular expressions over $X \cup \Delta_n$ have an interpretation in $\mathcal{R}X^* \otimes_{\mathcal{R}} C'_n$. Using the tensor product embeddings, atoms $x \in X$ are interpreted by $\{x\} \otimes 1$ and atoms $\delta \in \Delta_n$ by $\{1\} \otimes \{\delta\}/\rho_n$, as in Example 5. A subset of the regular expressions over $X \cup \Delta_n$ that are sufficient to name all context-free sets of $\mathcal{C}X^* \simeq Z_{C'_n}(\mathcal{R}X^* \otimes_{\mathcal{R}} C'_n)$ is easily selected as follows.

Corollary 28. *For $n > 2$, an element $e \in \mathcal{R}X^* \otimes_{\mathcal{R}} C'_n$ belongs to $Z_{C'_n}(\mathcal{R}X^* \otimes_{\mathcal{R}} C'_n)$ iff e is the value of a regular expression $\langle 0|r|0 \rangle$ over $X \cup \Delta_n$ where $\langle 0 |$ and $|0 \rangle$ do not occur in r .*

Proof. \Rightarrow : Suppose $e \in Z_{C'_n}(\mathcal{R}X^* \otimes_{\mathcal{R}} C'_n)$. By Theorem 27, there is $L \in \mathcal{C}X^*$ with $L = e^\vee$. Let

$$y_1 \geq p_1, \dots, y_k \geq p_k$$

be a context-free grammar for L , i.e. L is the first component L_1 of its least solution $L_1, \dots, L_k \in \mathcal{C}X^*$. Construct the right-linear grammar

$$y_1 \geq p_1'', \dots, p_k \geq p_k'', \quad y_1^F \geq p_1^F + 1, \quad y_2^F \geq p_2^F, \dots, y_k^F \geq p_k^F$$

as in the proof of Theorem 15. We can assume that $\langle 1 |, \dots, \langle n |, |1 \rangle, \dots, |n \rangle$ are sufficient to wrap each occurrence of a variable y_1, \dots, y_k in p_1, \dots, p_k by a different bracket pair $\langle i |, |i \rangle$, otherwise use the two pairs $\langle 1 |, |1 \rangle$ and $\langle 2 |, |2 \rangle$ to encode new bracket pairs. Let r be the regular expression over $X \cup \Delta_n$ for the least solution in y_1 of the right-linear grammar; r does not contain $\langle 0 |$ and $|0 \rangle$. The corresponding regular set $R \in \mathcal{R}(X^*[\Delta_n])$ satisfies $L = h_{X^*}(R \cap D_n(X))$, and the value of $\langle 0|r|0 \rangle$ in $\mathcal{R}X^* \otimes_{\mathcal{R}} C'_2$ is $e = \widehat{L} \in Z_{C'_n}(\mathcal{R}X^* \otimes_{\mathcal{R}} C'_n)$. We can see $\langle 0|r|0 \rangle$ as the solution term for \widehat{y} of the right-linear grammar

$$\widehat{y} \geq \langle 0|y_1, \quad y_1 \geq p_1'', \dots, y_k'' \geq p_k'', \quad y_1^F \geq p_1^F + |0 \rangle, \quad y_2^F \geq p_2^F, \dots, y_k^F \geq p_k^F.$$

(Cf. the automata in Example 6.)

\Leftarrow : Let r be a regular expression over $X \cup \Delta_n$ not containing $\langle 0 |, |0 \rangle$. By Corollary 13,

$$\mathcal{R}M \otimes_{\mathcal{R}} C'_2 \simeq \mathcal{R}X^* \otimes_{\mathcal{R}} (\mathcal{R}P'_n/\tilde{\nu}) \simeq \mathcal{R}(X^* \times P'_n)/\tilde{\nu},$$

where $\tilde{\nu}$ is the \mathcal{R} -congruence generated by $\{(1, 0)\} = \emptyset$. So the value of r in $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$ can be represented by $T/\tilde{\nu}$ with $T \in \mathcal{R}(X^* \times P'_n)$. By the choice of r , for no $(m, w) \in T$ do $\langle 0 |$ or $|0 \rangle$ occur in its second component $w \in P'_n = Q_n^* P'_n \cup \{0\}$. So the value of $\langle 0|r|0 \rangle$ is $S/\tilde{\nu}$ for $S = \{(1, \langle 0 |)\} T \{(1, |0 \rangle)\}$, and $S \subseteq X^* \times \{0, 1\}$ due to bracket mismatches in the second components. The isomorphisms of Theorem 12 and Proposition 9 map $S/\tilde{\nu}$ to some $[R] \in \mathcal{R}M \otimes_{\mathcal{R}} C'_2$ with $R \subseteq \mathcal{R}X^* \times \{0', 1'\}$. By Lemma 25, the value $[R]$ of $\langle 0|r|0 \rangle$ belongs to the centralizer of C'_n . \square

Notice that regular combinations of expressions of the above form $\langle 0|r|0 \rangle$ are no longer of this form, but of course also denote sets in $\mathcal{C}X^*$.

There is another way to find, for $L \in \mathcal{C}X^*$, a regular expression r such that $\langle 0|r|0 \rangle$ evaluates to \widehat{L} in $\mathcal{R}M \otimes_{\mathcal{R}} C'_n$, which is perhaps more appealing than the one in the proof above: a non-deterministic bottom-up, top-down or Earley parser (cf. Sikkel (1998)) for a context-free grammar $G = (X, Y, P, S)$ for L is an iterative program, i.e. a regular expression r over an alphabet of basic commands. Suppose that Δ_n contains, besides $\langle 0 |$ and $|0 \rangle$, an opening bracket $\langle z |$ and a closing bracket $|z \rangle$ for each $z \in X \cup Y$. The bottom-up parser, for example, can then be expressed by

$$r = \left(\sum \{ \text{shift}_x \mid x \in X \} + \sum \{ \text{reduce}_{(A, \alpha)} \mid (A, \alpha) \in P \} \right)^*$$

with $\text{shift}_x = x \langle x |$ and $\text{reduce}_{(A, \alpha)} = |\alpha \rangle \langle A |$, where $|z_1 \dots z_k \rangle = |z_k \rangle \dots |z_1 \rangle$ for $z_i \in X \cup Y$. Then $\widehat{L} = \sum \{ \{m\} \otimes 1 \mid m \in L \} = \langle 0|r|S \rangle |0 \rangle$. If G is the grammar of Example 6, the word *baac* is

accepted by the shift-reduce sequence

$$\text{shift}_b; \text{shift}_a; \text{reduce}_{(y,a)}; \text{shift}_a; \text{reduce}_{(y,a)}; \text{reduce}_{(z,yy)}; \text{shift}_c; \text{reduce}_{(y,bzc)};$$

its value in $\mathcal{R}X^* \otimes_{\mathcal{R}} C'_n$ is $\{baac\} \otimes \{y\}$, and since y was the start symbol S , $\{baac\} \otimes 1 \leq \langle 0|r|S|0 \rangle = \widehat{L}$. This indicates that $\mathcal{R}X^* \otimes_{\mathcal{R}} C'_2$ may be useful to study parsing algebraically.

6. Algebraic Representation of the \mathcal{C} -Closure of an Arbitrary \mathcal{R} -Dioid K

We can now generalize Theorem 27, the algebraic representation of the \mathcal{C} -closure of \mathcal{R} -dioids $\mathcal{R}M$ with monoid M to an algebraic construction of the \mathcal{C} -closure $Q_{\mathcal{R}}^{\mathcal{C}}: \mathbb{D}\mathcal{R} \rightarrow \mathbb{D}\mathcal{C}$ on the Eilenberg-Moore category $\mathbb{D}\mathcal{R}$ of all \mathcal{R} -dioids. We need a series of steps.

Lemma 29. *For any \mathcal{R} -dioid K ,*

$$\mathcal{C}K = \{ \{ m \mid (m, 1) \in R \} \mid R \in \mathcal{R}(K \times C'_2), R \subseteq K \times \{0, 1\} \}.$$

Proof. \subseteq : For $\Delta_2 = \{p_0, q_0, p_1, q_1\}$, let ρ_2 be the \mathcal{R} -congruence on $\mathcal{R}\Delta_2^*$ generated by the semiring equations

$$p_i q_j = \delta_{i,j}, \quad 0 \leq i, j \leq 1,$$

and $\cdot/\rho_2: \mathcal{R}\Delta_2^* \rightarrow C'_2 = \mathcal{R}\Delta_2^*/\rho_2$ the canonical \mathcal{R} -morphism. By Corollary 16 there is a set $S \in \mathcal{R}(M \times \Delta_2^*)$ such that

$$L = \{ m \mid (m, d) \in S, d \in D_2 \},$$

and for each $(m, d) \in S$,

$$\{d\}/\rho_2 = \begin{cases} 1 & d \in D_2, \\ 0 & d \notin D_2. \end{cases} \quad (12)$$

The isomorphism $\mathcal{R}(K \times \Delta_2^*) \simeq \mathcal{R}K \otimes_{\mathcal{R}} \mathcal{R}\Delta_2^*$ of Proposition 11 maps S to $[R_S]$, where

$$R_S = \{ (\{m\}, \{d\}) \mid (m, d) \in S \} \in \mathcal{R}(\mathcal{R}K \times \mathcal{R}\Delta_2^*).$$

The lifting of the homomorphism $\Sigma \times (\cdot/\rho_2): \mathcal{R}K \times \mathcal{R}\Delta_2^* \rightarrow K \times C'_2$, maps R_S to

$$R := \{ (m, \{d\}/\rho_2) \mid (m, d) \in S \} \in \mathcal{R}(K \times C'_2).$$

By (12), R is a subset of $K \times \{0, 1\}$ such that

$$L = \{ m \mid (m, d) \in S, d \in D_2 \} = \{ m \mid (m, 1) \in R \}.$$

\supseteq : Suppose $R \in \mathcal{R}(K \times C'_2)$ and $R \subseteq K \times \{0, 1\}$. Then for $R' = \{ (\{m\}, c) \mid (m, c) \in R \}$, we have $R' \in \mathcal{R}(\mathcal{R}K \times C'_2)$ and $R' \subseteq \mathcal{R}K \times \{0, 1\}$, so $[R'] \in Z_{C'_2}(\mathcal{R}K \otimes_{\mathcal{R}} C'_2)$ by Lemma 25, hence

$$\{ m \mid (m, 1) \in R \} = \bigcup \{ A \mid (A, 1) \in R' \} = [R']^{\vee} \in \mathcal{C}K.$$

by Theorem 26. \square

Since for $L \in \mathcal{C}K$, $\widetilde{\top}_1(L) = \{ m \otimes 1 \mid m \in L \} \in \mathcal{C}(Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2))$, the next lemma is a step towards the \mathcal{C} -completeness of $Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$.

Lemma 30. *For each \mathcal{R} -dioid K , there is a surjective homomorphism*

$$\widehat{\cdot}: \mathcal{C}K \rightarrow Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2),$$

that assigns to each $L \in \mathcal{C}K$ a least upper bound of $\widetilde{\top}_1(L)$ in $K \otimes_{\mathcal{R}} C'_2$,

$$\widehat{L} = \sum \widetilde{\top}_1(L) = \sum \{ m \otimes 1 \mid m \in L \}.$$

Proof. For each $L \in \mathcal{C}K$ there is, by Lemma 29, a set $R \in \mathcal{R}(K \times C'_2)$ with $R \subseteq K \times \{0, 1\}$ and $L = \{ m \mid (m, 1) \in R \}$. As $m \otimes 0 = 0$ for all $m \in K$, it follows that

$$[R] = \sum \{ m \otimes d \mid (m, d) \in R \} = \sum \{ m \otimes 1 \mid (m, 1) \in R \},$$

showing that $\widetilde{\top}_1(L) = \{ m \otimes 1 \mid m \in L \}$ has a least upper bound in $K \otimes_{\mathcal{R}} C'_2$, namely

$$\widehat{L} := \sum \{ m \otimes 1 \mid m \in L \} = [R].$$

Since all $m \otimes d$ with $(m, d) \in R$ belong to $Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$, we have $[R] \in Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$ by Proposition 14. This defines a map $\widehat{\cdot}: \mathcal{C}K \rightarrow Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$.

The map $\widehat{\cdot}$ is a homomorphism: $\widehat{\{1\}} = \sum \{1 \otimes 1\} = 1 \otimes 1 = 1$ and for $L_1, L_2 \in \mathcal{C}K$, by Lemma 29 there are $R_1, R_2 \in \mathcal{R}(K \times C'_2)$ such that $R_i \subseteq K \times \{0, 1\}$ and $L_i = \{ m \mid (m, 1) \in R_i \}$ for $i = 1, 2$. Therefore,

$$\begin{aligned} \widehat{L_1} \cdot \widehat{L_2} &= [R_1][R_2] = [R_1 R_2] \\ &= [\{ (m_1 m_2, 1) \mid (m_1, 1) \in R_1, (m_2, 1) \in R_2 \}] \\ &= [\{ (m_1 m_2, 1) \mid m_1 \in L_1, m_2 \in L_2 \}] \\ &= \sum \{ m \otimes 1 \mid m \in L_1 L_2 \} \\ &= \widehat{L_1 L_2}, \end{aligned}$$

as summands $m_1 m_2 \otimes 0 = 0$ in $[R_1 R_2]$ can be ignored.

We still have to show that $\widehat{\cdot}$ is surjective. We use Σ and \otimes for the given operations of $K \otimes_{\mathcal{R}} C'_2$, Σ' and \otimes' for those of $\mathcal{R}K \otimes_{\mathcal{R}} C'_2$. By the universal property of $\mathcal{R}K \otimes_{\mathcal{R}} C'_2$, the given $\Sigma: \mathcal{R}K \rightarrow K$ induces an \mathcal{R} -morphism $h = h_{f,g}$ for $f = \top_1 \circ \Sigma$ and $g = \top_2$ such that $f = h \circ \top'_1$ and $g = h \circ \top'_2$:

$$\begin{array}{ccccc} K & \xrightarrow{\top_1} & K \otimes_{\mathcal{R}} C'_2 & & \\ \Sigma \uparrow & & \uparrow h & & \\ \mathcal{R}K & \xrightarrow{\quad} & \mathcal{R}K \otimes_{\mathcal{R}} C'_2 & \xleftarrow{\quad} & C'_2 \\ & \xrightarrow{\top'_1} & & \xleftarrow{\top'_2} & \end{array}$$

Since h is a homomorphism and $h(\top'_2(c)) = \top_1(c)$ for $c \in C'_2$, h preserves commutativity relations with C'_2 , so its restriction is an \mathcal{R} -morphism

$$h: Z_{C'_2}(\mathcal{R}K \otimes_{\mathcal{R}} C'_2) \rightarrow Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2).$$

Clearly, $h: \mathcal{R}K \otimes_{\mathcal{R}} C'_2 \rightarrow K \otimes_{\mathcal{R}} C'_2$ is surjective: if $R \in \mathcal{R}(K \times C'_2)$ we have

$$\begin{aligned} [R] &= \sum \{ m \otimes c \mid (m, c) \in R \} \\ &= \sum \{ (\sum \{m\}) \otimes c \mid (m, c) \in R \} \\ &= h(\sum' \{ \{m\} \otimes' c \mid (m, c) \in R \}) \\ &= h([R']) \end{aligned}$$

for $R' = \{ (\{m\}, c) \mid (m, c) \in R \} \in \mathcal{R}(\mathcal{R}K \times C'_2)$. And if $[R] \in Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$, then $[R'] \in Z_{C'_2}(\mathcal{R}K \otimes_{\mathcal{R}} C'_2)$, so the restriction $h: Z_{C'_2}(\mathcal{R}K \otimes_{\mathcal{R}} C'_2) \rightarrow Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$ also is surjective.

Now take $[R] \in Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ with $R \in \mathcal{R}(K \times C_2')$, and let $R' = \{(\{m\}, c) \mid (m, c) \in R\}$. By Theorem 27, for $[R'] \in Z_{C_2'}(\mathcal{R}K \otimes_{\mathcal{R}} C_2')$ there is $L \in \mathcal{C}K$ such that

$$[R'] = \sum' \widehat{\top}_1'(L) = \sum' \{ \{m\} \otimes' 1 \mid m \in L \}.$$

We claim that $[R] = \widehat{L} = \sum \{ m \otimes 1 \mid m \in L \}$. By Lemma 29, there is $S \in \mathcal{R}(K \times C_2')$ with $S \subseteq K \times \{0, 1\}$ and $L = \{m \mid (m, 1) \in S\}$. Then $S' = \{(\{m\}, c) \mid (m, c) \in S\} \in \mathcal{R}(\mathcal{R}K \times C_2')$ and

$$[S'] = \sum' \{ \{m\} \otimes' 1 \mid m \in L \} = [R'],$$

hence $[R] = h([R']) = h([S']) = [S] = \sum \{ m \otimes c \mid (m, c) \in S \} = \sum \widehat{\top}_1(L) = \widehat{L}$. \square

In contrast to Theorem 17, the map $\widehat{\cdot} : \mathcal{C}K \rightarrow Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ need not be injective:

Example 7. Take $K = \mathcal{R}M$ for $M = \{a, b\}^*$. For $L \in K$, $\{ \{m\} \mid m \in L \} \in \mathcal{R}(\mathcal{R}M)$, and since $L \mapsto L \otimes 1$ is the \mathcal{R} -morphism $\top_1 : K \rightarrow K \otimes_{\mathcal{R}} C_2'$, we have

$$L \otimes 1 = (\bigcup \{ \{m\} \mid m \in L \}) \otimes 1 = \sum \{ \{m\} \otimes 1 \mid m \in L \}.$$

It follows that for $U \in \mathcal{C}K$,

$$\widehat{U} = \sum \{ L \otimes 1 \mid L \in U \} = \sum \{ \{m\} \otimes 1 \mid m \in \bigcup U \}.$$

Consider the following two polynomial systems over K :

$$\begin{aligned} x &\geq \{ \{1, ab\} \} \cup x \cup \{ \{a\} \} x \{ \{b\} \} \cup \{ \{a^2\} \} x \{ \{b^2\} \}, \\ x &\geq \{ \{1, a^2b^2\}, \{ab, a^3b^3\} \} \cup x \cup \{ \{a^2\} \} x \{ \{b^2\} \} \end{aligned}$$

The least solutions $U_1, U_2 \in \mathcal{C}K$ of these are, respectively,

$$U_1 = \{ \{a^n b^n, a^{n+1} b^{n+1}\} \mid n \in \mathbb{N} \} \quad \text{and} \quad U_2 = \{ \{a^n b^n, a^{n+2} b^{n+2}\} \mid n \in \mathbb{N} \}.$$

But while $U_1 \neq U_2$, we have $\bigcup U_1 = \bigcup U_2 = \{a^n b^n \mid n \in \mathbb{N}\}$, hence $\widehat{U}_1 = \widehat{U}_2$. \diamond

Lemma 31. For each \mathcal{R} -dioid K , $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ is a \mathcal{C} -dioid. Moreover,

$$Z_{C_2'}(K \otimes_{\mathcal{R}} C_2') = \{ [R] \mid R \in \mathcal{R}(K \times C_2'), R \subseteq K \times \{0, 1\} \}.$$

Proof. By Lemma 30, the map $\widehat{\cdot} : \mathcal{C}K \rightarrow Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ is a surjective homomorphism that assigns to $L \in \mathcal{C}K$ the least upper bound of the image of L under the homomorphism $\top_1 : K \rightarrow K \otimes_{\mathcal{R}} C_2'$, $\widehat{L} = \sum \widehat{\top}_1(L)$. By Lemma 3, $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ is a \mathcal{C} -dioid, i.e. there is a \mathcal{C} -distributive least-upper-bound operator

$$\sum : \mathcal{C}(Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')) \rightarrow Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'),$$

where for $V \in \mathcal{C}(Z_{C_2'}(K \otimes_{\mathcal{R}} C_2'))$, $\sum V = \widehat{\bigcup U}$ for any $U \in \mathcal{C}\mathcal{C}K$ such that $V = \{ \widehat{L} \mid L \in U \}$.

For \supseteq of the equation, suppose $R \in \mathcal{R}(K \times C_2')$ and $R \subseteq K \times \{0, 1\}$. By \supseteq of Lemma 29,

$$L := \{ m \mid (m, 1) \in R \} \in \mathcal{C}K.$$

But then $[R] = \widehat{L} \in Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ by Lemma 30. The reverse inclusion \subseteq follows from the surjectivity of $\widehat{\cdot}$ and \subseteq of Lemma 29, as shown at the beginning of the proof of Lemma 30. \square

Before we can present the second main result, that for any \mathcal{R} -dioid K , $Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ is the \mathcal{C} -closure of K , we have to consider a special case^h:

^hThe next two proofs elaborate proofs of Hopkins (2006), Section 3.6

Theorem 32. Let C be a \mathcal{C} -dioid and K its restriction to an \mathcal{R} -dioid. The \mathcal{R} -morphism $\top_1 : K \rightarrow K \otimes_{\mathcal{R}} C'_2$ is a \mathcal{C} -isomorphism

$$C \xrightarrow{\cong} Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2).$$

Proof. By Lemma 31, $D := Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$ is a \mathcal{C} -dioid, so let $\Sigma : \mathcal{C}D \rightarrow D$ be its supremum operator. Since C and K have the same multiplicative monoid, $\mathcal{C}C = \mathcal{C}K$, so let $\Sigma' : \mathcal{C}K \rightarrow K$ be the supremum operator of the \mathcal{C} -dioid C . We show that the \mathcal{R} -morphism \top_1 of the tensor product

$$\top_1 : K \longrightarrow K \otimes_{\mathcal{R}} C'_2 \longleftarrow C'_2 : \top_2,$$

is a surjective \mathcal{C} -morphism $\top_1 : C \rightarrow D$, and its inverse $m \otimes 1 \mapsto m$ is a \mathcal{C} -morphism $j : D \rightarrow C$.

With $\eta : K \rightarrow \mathcal{C}K$, $\eta(m) = \{m\}$ for $m \in K$, we have

$$\top_1(m) = m \otimes 1 = \sum \{m \otimes 1\} = \sum \widetilde{\top_1}(\{m\}) = (\Sigma \circ \widetilde{\top_1} \circ \eta)(m),$$

and consider the following diagram:

$$\begin{array}{ccc} K & \xrightleftharpoons{j} & D \subseteq K \otimes_{\mathcal{R}} C'_2 \\ & \searrow \eta & \nearrow \widehat{\cdot} \\ & \mathcal{C}K & \xrightarrow{\widetilde{\top_1}} \mathcal{C}D \\ & \swarrow \Sigma' & \searrow \Sigma \end{array}$$

To show that $\top_1 : K \rightarrow D$ is a \mathcal{C} -morphism, we must show that for every $L \in \mathcal{C}K$,

$$(\Sigma' L) \otimes 1 = \sum \{m \otimes 1 \mid m \in L\}. \quad (13)$$

Clearly, $(\Sigma' L) \otimes 1$ is an upper bound of $\{m \otimes 1 \mid m \in L\} = \widetilde{\top_1}(L)$, so

$$\widehat{L} := \sum \{m \otimes 1 \mid m \in L\} \leq (\Sigma' L) \otimes 1.$$

For the reverse inequality $\widehat{L} \geq (\Sigma' L) \otimes 1$, recall that $C'_2 = \mathcal{R}\Delta_2^*/\rho_2$ embeds into the polycyclic \mathcal{C} -dioid $C'_{2,\mathcal{C}} = \mathcal{C}\Delta_2^*/\rho_2$ of Section 2.2. Consider the tensor product

$$\top'_1 : C \longrightarrow C \otimes_{\mathcal{C}} C'_{2,\mathcal{C}} \longleftarrow C'_{2,\mathcal{C}} : \top'_2$$

in the category of \mathcal{C} -dioids. Since \top'_1 is a \mathcal{C} -morphism,

$$(\Sigma' L) \otimes 1' = \top'_1(\Sigma' L) = \sum' \widetilde{\top'_1}(L) = \sum' \{m \otimes 1' \mid m \in L\},$$

where we write $1'$ for the unit in $C'_{2,\mathcal{C}}$, $\top'_1(c) = c \otimes 1'$ for $c \in C$, and on the right, Σ' also for the supremum operator of $C \otimes_{\mathcal{C}} C'_{2,\mathcal{C}}$. Combined with the identity from K to C , the embedding of C'_2 into $C'_{2,\mathcal{C}}$ gives a monotone embedding \mathcal{R} -morphism $h : K \otimes_{\mathcal{R}} C'_2 \rightarrow C \otimes_{\mathcal{C}} C'_{2,\mathcal{C}}$. Therefore,

$$\begin{aligned} h((\Sigma' L) \otimes 1) &= (\Sigma' L) \otimes 1' \\ &= \sum' \{m \otimes 1' \mid m \in L\} \\ &= \sum' \{h(m \otimes 1) \mid m \in L\} \\ &= \sum' \widetilde{h}(\widetilde{\top_1}(L)) \\ &\leq h(\widehat{L}), \end{aligned}$$

where the inequality holds since \widehat{L} is an upper bound of $\widetilde{\top_1}(L)$. Together with $\widehat{L} \leq (\Sigma' L) \otimes 1$, this gives $h((\Sigma' L) \otimes 1) = h(\widehat{L})$, and as h is injective, (13). So $\top_1 : K \rightarrow D$ is a \mathcal{C} -morphism.

It follows that $\widehat{\cdot} = \top_1 \circ \Sigma' : \mathcal{C}K \rightarrow D$ is a \mathcal{C} -morphism, because $\Sigma' : \mathcal{C}C \rightarrow C$ and $\top_1 : C \rightarrow D$ are \mathcal{C} -morphisms. Since $\widehat{\cdot} : \mathcal{C}K \rightarrow D$ is surjective, so is $\top_1 : K \rightarrow D$, by (13), hence

$$D = \{ m \otimes 1 \mid m \in K \}.$$

It remains to be shown that its inverse $j : D \rightarrow K$, $m \otimes 1 \mapsto m$, is a \mathcal{C} -morphism. The surjective homomorphism $\widehat{\cdot} : \mathcal{C}K \rightarrow D$ lifts to a surjective homomorphism from $\mathcal{C}\mathcal{C}K$ to $\mathcal{C}D$. Hence for $V \in \mathcal{C}D$ there is $U \in \mathcal{C}\mathcal{C}K$ with $V = \{\widehat{L} \mid L \in U\}$, and since $\bigcup U \in \mathcal{C}K$ and $\widehat{\cdot}$ is a \mathcal{C} -morphism,

$$\sum V = \sum \{\widehat{L} \mid L \in U\} = \widehat{\bigcup U} = (\Sigma' \bigcup U) \otimes 1$$

by (13). By applying j and using (13) for each $L \in U$,

$$\begin{aligned} j(\sum V) &= \Sigma' \bigcup U \\ &= \Sigma' \{ \Sigma' L \mid L \in U \} \\ &= \Sigma' \{ j((\Sigma' L) \otimes 1) \mid L \in U \} \\ &= \Sigma' \{ j(\widehat{L}) \mid L \in U \} \\ &= \Sigma' \{ j(v) \mid v \in V \}. \end{aligned}$$

Therefore, $j : D \rightarrow K$ is \mathcal{C} -continuous, i.e. preserves suprema of \mathcal{C} -sets. \square

Under the assumptions of Theorem 32, $\{ (L, \{\Sigma' L\}) \mid L \in \mathcal{C}K \} \subseteq \ker(\widehat{\cdot})$, because by (13)

$$\widehat{L} = (\Sigma' L) \otimes 1 = \sum \{ m \otimes 1 \mid m \in \{\Sigma' L\} \} = \widehat{\{\Sigma' L\}}.$$

We can now prove our second main result, which will imply that $\mathcal{C}K / \ker(\widehat{\cdot}) \simeq Q_{\mathcal{R}}^{\mathcal{C}}(K)$:

Theorem 33 (Algebraic representation of \mathcal{C} -closure). *For any \mathcal{R} -doid K there is a \mathcal{C} -isomorphism*

$$Q_{\mathcal{R}}^{\mathcal{C}}(K) \simeq Z_{C_2'}(K \otimes_{\mathcal{R}} C_2').$$

Proof. It is sufficient to prove that $D := Z_{C_2'}(K \otimes_{\mathcal{R}} C_2')$ is a \mathcal{C} -completion of K . Let $\eta : K \rightarrow D$ be the embedding obtained from $\top_1 : K \rightarrow K \otimes_{\mathcal{R}} C_2'$,

$$\eta(m) := \top_1(m) = m \otimes 1 \in D \quad \text{for } m \in K,$$

and $f : K \rightarrow C$ an \mathcal{R} -morphism to a \mathcal{C} -doid C . Define $\bar{f} : D \rightarrow C$ by

$$\bar{f}(\widehat{L}) := \sum \{ f(m) \mid m \in L \} = \sum \tilde{f}(L) \quad \text{for } L \in \mathcal{C}K.$$

We prove later that \bar{f} is well-defined and first show the remaining properties.

$$f = \bar{f} \circ \eta: \text{ for } m \in K, \bar{f}(\eta(m)) = \bar{f}(\sum \{m \otimes 1\}) = \bar{f}(\widehat{\{m\}}) = \sum \{f(m)\} = f(m).$$

\bar{f} is a monotone homomorphism: $\bar{f}(1) = \bar{f}(\widehat{\{1\}}) = \sum \{f(1)\} = f(1) = 1$, and since $\widehat{\cdot} : \mathcal{C}K \rightarrow D$ is a homomorphism and $\sum \circ \bar{f} : \mathcal{C}K \rightarrow C$ is \mathcal{C} -distributive,

$$\bar{f}(\widehat{L} \cdot \widehat{L}') = \bar{f}(\widehat{LL'}) = \sum \tilde{f}(LL') = \sum \tilde{f}(L) \sum \tilde{f}(L') = \bar{f}(\widehat{L}) \bar{f}(\widehat{L'}).$$

\bar{f} is \mathcal{C} -continuous: Suppose $V \in \mathcal{C}D$. Since $\widehat{\cdot} : \mathcal{C}K \rightarrow D$ is a surjective homomorphism, so is its lifting to $\mathcal{C}\mathcal{C}K \rightarrow \mathcal{C}D$, hence there is $U \in \mathcal{C}\mathcal{C}K$ such that $V = \{\widehat{L} \mid L \in U\} \in \mathcal{C}D$. Therefore,

$$\begin{aligned} \bar{f}(\sum V) &= \bar{f}(\sum \{m \otimes 1 \mid m \in \bigcup U\}) \\ &= \bar{f}(\widehat{\bigcup U}) \\ &= \sum \tilde{f}(\bigcup U) \\ &= \sum \bigcup \{\tilde{f}(L) \mid L \in U\} \\ &= \sum \{\tilde{f}(\widehat{L}) \mid L \in U\}. \\ &= \sum \{\tilde{f}(v) \mid v \in V\}. \end{aligned}$$

\bar{f} is well-defined: Suppose $L, L' \in \mathcal{C}K$ and $\widehat{L} = \widehat{L}'$. We must show

$$\sum \tilde{f}(L) = \sum \tilde{f}(L'). \quad (14)$$

Let $K' = \mathcal{Q}_{\mathcal{C}}^{\mathcal{R}}(C)$ be the restriction of C to an \mathcal{R} -dioid,

$$\top'_1 : K' \longrightarrow K' \otimes_{\mathcal{R}} C'_2 \longleftarrow C'_2 : \top'_2$$

its tensor product with C'_2 in $\mathbb{D}\mathcal{R}$, and $D' := Z_{C'_2}(K' \otimes_{\mathcal{R}} C'_2)$. By Theorem 32, $\top'_1 : C \rightarrow D'$ is a \mathcal{C} -isomorphism, with inverse $j : D' \rightarrow C$. We write $\top'_1(c) = c \otimes 1$ for $c \in K'$ and \sum' for the least-upper-bound-operator of D' or $K' \otimes_{\mathcal{R}} C'_2$.

Consider the commuting \mathcal{R} -morphisms $f' = \top'_1 \circ f$ and $g' = \top'_2$ shown in

$$\begin{array}{ccccc} K' & \xrightarrow{\top'_1} & K' \otimes_{\mathcal{R}} C'_2 & & \\ \uparrow f & \nearrow f' & \uparrow h_{f',g'} & \nwarrow g' & \\ K & \xrightarrow{\top_1} & K \otimes_{\mathcal{R}} C'_2 & \xleftarrow{\top_2} & C'_2 \end{array}$$

Let $h_{f',g'}$ be the \mathcal{R} -morphism with $f' = h_{f',g'} \circ \top_1$ and $g' = h_{f',g'} \circ \top_2$. For $R \in \mathcal{R}(K \times C'_2)$, put $R_f = \{(f(m), c) \mid (m, c) \in R\} \in \mathcal{R}(K' \times C'_2)$. Then

$$\begin{aligned} h_{f',g'}([R]) &= h_{f',g'}(\sum \{m \otimes c \mid (m, c) \in R\}) \\ &= \sum \{f'(m)g'(c) \mid (m, c) \in R\} \\ &= \sum \{(f(m) \otimes 1)(1' \otimes c) \mid (m, c) \in R\} \\ &= \sum \{f(m) \otimes c \mid (m, c) \in R\} \\ &= [R_f]. \end{aligned}$$

The restriction of $h_{f',g'}$ to $D \subseteq K \otimes_{\mathcal{R}} C'_2$ clearly is an \mathcal{R} -morphism $h' : D \rightarrow D'$. The situation is visualized by the following diagram:

$$\begin{array}{ccc}
 C & \xleftarrow{j} & D' \\
 \uparrow f & \swarrow \Sigma & \uparrow \widehat{\cdot}' \\
 & \mathcal{C}K' & \\
 & \uparrow \widetilde{f} & \\
 & \mathcal{C}K & \\
 \uparrow \eta_K & \searrow \widehat{\cdot} & \\
 K & \xrightarrow{\tau_1} & D
 \end{array}
 \quad
 \begin{array}{c}
 \subseteq K' \otimes_{\mathcal{R}} C'_2 \\
 \uparrow \\
 \subseteq K \otimes_{\mathcal{R}} C'_2
 \end{array}$$

By Lemma 29, there are $R, R' \in \mathcal{R}(K \times C'_2)$ with $R, R' \subseteq K \times \{0, 1\}$ and $L = \{m \mid (m, 1) \in R\}$, $L' = \{m \mid (m, 1) \in R'\}$, so

$$[R] = \widehat{L} = \widehat{L}' = [R'].$$

By an application of $h_{f',g'}$, $[R_f] = [R'_f]$. Using $\widetilde{f}(L) \in \mathcal{C}K'$, we get

$$\begin{aligned}
 [R_f] &= \widetilde{f}(L)' \\
 &= \sum' \{ f(m) \otimes' c \mid (m, c) \in R \} \\
 &= \sum' \{ f(m) \otimes' 1 \mid (m, 1) \in R \} && (R \subseteq K \times \{0, 1\}) \\
 &= \sum' \{ f(m) \otimes' 1 \mid m \in L \} && (L = \{m \mid (m, 1) \in R\}) \\
 &= (\sum' \widetilde{f}(L)) \otimes' 1 && (\text{by (13) on } D').
 \end{aligned}$$

Likewise, $[R'_f] = \widetilde{f}(L')' = (\sum \widetilde{f}(L')) \otimes' 1$, and so

$$(\sum \widetilde{f}(L)) \otimes' 1 = [R_f] = [R'_f] = (\sum \widetilde{f}(L')) \otimes' 1.$$

(Hence $h' \circ \widehat{\cdot} = \widehat{\cdot}' \circ \widetilde{f}$ is a \mathcal{C} -morphism.) An application of j gives (14).

\widetilde{f} is the unique \mathcal{C} -morphism $h : D \rightarrow C$ with $f = h \circ \eta$: Suppose h is such a \mathcal{C} -morphism and e an element of D . By Lemma 30 there is $L \in \mathcal{C}K$ such that $e = \widehat{L} = \sum \{ m \otimes 1 \mid m \in L \}$, hence

$$\begin{aligned}
 h(e) &= h(\widehat{L}) \\
 &= h(\sum \{ m \otimes 1 \mid m \in L \}) \\
 &= \sum \{ h(m \otimes 1) \mid m \in L \} \\
 &= \sum \{ h(\eta(m)) \mid m \in L \} \\
 &= \sum \{ f(m) \mid m \in L \} \\
 &= \widetilde{f}(\widehat{L}) = \widetilde{f}(e).
 \end{aligned}$$

Therefore, $h = \widetilde{f}$. □

Corollary 34. For any \mathcal{R} -dioid K , the \mathcal{C} -closure of K is $\mathcal{C}K/\ker(\widehat{\cdot})$.

Proof. By Theorem 33, $D := Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$ is the \mathcal{C} -closure of K , so we must show that $\mathcal{C}K/\ker(\widehat{\cdot})$ is a \mathcal{C} -dioid and there is a \mathcal{C} -isomorphism between D and $\mathcal{C}K/\ker(\widehat{\cdot})$. By Lemma 30, $\widehat{\cdot} : \mathcal{C}K \rightarrow D$ is a monotone and surjective homomorphism between \mathcal{C} -dioids. It is a \mathcal{C} -morphism, because $\widehat{\bigcup U} = \sum\{\widehat{L} \mid L \in U\}$ for $U \in \mathcal{C}\mathcal{C}K$. Hence its kernel $\ker(\widehat{\cdot})$ is a \mathcal{C} -congruence on $\mathcal{C}K$, and so $\mathcal{C}K/\ker(\widehat{\cdot})$ is a \mathcal{C} -dioid. Let us verify that the bijection between $\mathcal{C}K/\ker(\widehat{\cdot})$ and D is a \mathcal{C} -isomorphism. In the one direction, the mapping $f : \mathcal{C}K/\ker(\widehat{\cdot}) \rightarrow D$, well-defined by $f(L/\ker(\widehat{\cdot})) = \widehat{L}$ for $L \in \mathcal{C}K$, is a \mathcal{C} -morphism, since for $V \in \mathcal{C}(\mathcal{C}K/\ker(\widehat{\cdot}))$ there is $U \in \mathcal{C}\mathcal{C}K$ such that $V = \{L/\ker(\widehat{\cdot}) \mid L \in U\}$, and by the definition of \sum on quotients, $\sum V = (\bigcup U)/\ker(\widehat{\cdot})$, so $f(\sum V) = \widehat{\bigcup U} = \sum\{\widehat{L} \mid L \in U\} = \sum \tilde{f}(V)$. In the other direction, by Lemma 31 and Lemma 29, elements of D are congruence classes $[R]$ of sets $R \in \mathcal{R}(K \times C'_2)$ with $R \subseteq K \times \{0, 1\}$, such that $[R] = \widehat{L}_R$ for $L_R := \{m \mid (m, 1) \in R\} \in \mathcal{C}K$. Hence, a map $g : D \rightarrow \mathcal{C}K/\ker(\widehat{\cdot})$ is well-defined by

$$g([R]) = L_R/\ker(\widehat{\cdot}), \quad \text{for } R \in \mathcal{R}(K \times C'_2) \text{ with } R \subseteq K \times \{0, 1\},$$

and obviously g and f are inverses of each other. To see that g is a \mathcal{C} -morphism, suppose $V \in \mathcal{C}D$. There is $U \in \mathcal{C}\mathcal{C}K$ such that $V = \{\widehat{L} \mid L \in U\}$, and $\sum V = \widehat{\bigcup U}$. There is $R \in \mathcal{R}(K \times C'_2)$ with $R \subseteq K \times \{0, 1\}$ and $\widehat{\bigcup U} = [R] = \widehat{L}_R$, and so

$$g(\sum V) = L_R/\ker(\widehat{\cdot}) = (\bigcup U)/\ker(\widehat{\cdot}) = \sum\{L/\ker(\widehat{\cdot}) \mid L \in U\} = \sum \tilde{g}(V),$$

using suitable $R_L \in \mathcal{R}(K \times C'_2)$ with $R_L \subseteq K \times \{0, 1\}$ and $\widehat{L} = [R_L]$ for the last step. \square

7. Conclusion

The algebraic abstraction of the regular subsets $\mathcal{R}M$ and the context-free subsets $\mathcal{C}M$ of a monoid M are the categories of \mathcal{R} -dioids or $*$ -continuous Kleene algebras and \mathcal{C} -dioids or μ -continuous Chomsky algebras. Just as $\mathcal{C}M$ is the closure of $\mathcal{R}M$ under a well-behaved least fixed-point operator μ , each \mathcal{R} -dioid K has a closure \overline{K} as \mathcal{C} -dioid, briefly called its fixed-point closure. We have shown that \overline{K} can be represented in the product $K \otimes_{\mathcal{R}} C'_2$ of K with C'_2 in the category of \mathcal{R} -dioids, where the polycyclic \mathcal{R} -dioid C'_2 on two generators is a quotient of the regular sets $\mathcal{R}\Delta_2^*$ over an alphabet $\Delta_2 = \{b, d, p, q\}$ of two pairs of brackets, b, d and p, q . Namely, \overline{K} is isomorphic to the centralizer $Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$ of C'_2 in $K \otimes_{\mathcal{R}} C'_2$, which, intuitively, consists of the sums over regular sets of elements $m \otimes c$ where $m \in K$ and $c \in C'_2$ is restricted to be 0 or 1.

This representation theorem is an algebraic form and categorical generalization of the classical Chomsky-Schützenberger theorem (and its reverse) of formal language theory. The latter reduces the set $\mathcal{C}X^*$ of context-free languages over a finite alphabet X to the set $\mathcal{R}(X \cup \Delta_2)^*$ of regular languages over $X \cup \Delta_2$, by means of intersection with the context-free Dyck language $D_2(X) \in \mathcal{C}(X \cup \Delta_2)^*$ of balanced bracketed strings and the bracket-erasing homomorphism from $(X \cup \Delta_2)^*$ to X^* . In the special case $K = \mathcal{R}X^*$ with $\overline{K} = \mathcal{C}X^*$, the representation theorem intuitively splits bracketed strings $u \in (X \cup \Delta_2)^*$ to pairs $(w, t) \in X^* \times \Delta_2^*$ and performs the balance-checking and bracket-erasure within C'_2 by algebraically reducing the second component of $\{w\} \otimes \{t\} \in K \otimes_{\mathcal{R}} C'_2$ to 1 or 0, depending on whether t belongs to the pure Dyck language $D_2 \in \mathcal{C}\Delta_2^*$ or not. The underlying part of deriving from $L \in \mathcal{C}X^*$ a suitable regular set R of strings $u \in (X \cup \Delta_2)^*$ is as in the classical Chomsky-Schützenberger theorem (cf. Theorem 15). A finite automaton for $R \subseteq (X \cup \Delta_2)^*$ represents L in the sense that the words $u \in R$ evaluate in $\mathcal{R}X^* \otimes_{\mathcal{R}} C'_2$ to 0 or to the elements $\{w\} \otimes 1$ with $w = h_{X^*}(u) \in L$ (cf. Figure 3).

The algebra $\mathcal{R}X^* \otimes_{\mathcal{R}} C'_2$ is hoped to turn out useful to construct CFG-parsers for context-free languages over X , for example by simplifying the calculation of item-sets of LR-parsers.

Some remarks on the difference between using the bra-ket \mathcal{R} -dioid C_2 and the polycyclic \mathcal{R} -dioid C'_2 are in order. Theorem 5 of Hopkins and Leiß (2018) announced that $\mathcal{C}M \simeq Z_{C_2}(\mathcal{R}M \otimes_{\mathcal{R}} C_2)$ for monoids M , indicating that “the equation $db + qp = 1$ not present in C'_2 is needed to encode stack operations in C_2 ”. We here have shown that one can actually do without this equation and prove the stronger result $\mathcal{C}M \simeq Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$. One may obtain C_2 as the quotient $C'_2 / \langle db + qp = 1 \rangle$ of C'_2 by the \mathcal{R} -congruence generated by the completeness equation. The quotient map is not injective, but its extension to $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$ apparently does not identify different members of $Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2)$, so the announced claim with C_2 should follow.

The motivation for writing this article came from an effort to understand Hopkins’ unpublished proof that the fixed-point closure \bar{K} of an \mathcal{R} -dioid K can be algebraically represented by $Z_{C_2}(K \otimes_{\mathcal{R}} C_2)$. An inspection of the proof showed that the completeness assumption $db + qp = 1$ of C_2 was not used for the analog $\bar{K} \subseteq Z_{C_2}(K \otimes_{\mathcal{R}} C_2)$ of the Chomsky-Schützenberger theorem. For the reverse $Z_{C_2}(K \otimes_{\mathcal{R}} C_2) \subseteq \bar{K}$, the assumption is used to show that $\hat{\cdot} : \mathcal{C}K \rightarrow Z_{C_2}(K \otimes_{\mathcal{R}} C_2)$ given by $\hat{L} = \sum \{ m \otimes 1 \mid m \in L \}$ is surjective. The surjectivity follows from an interesting automata-theoretic Normal Form Theorem for elements of $K \otimes_{\mathcal{R}} C_2$, which implicitly assumes that $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2)$ is an \mathcal{R} -dioid and shortly is as follows. For each element $e = [R] \in K \otimes_{\mathcal{R}} C_2$, by induction on the construction of $R \in \mathcal{R}(K \times C_2)$ there is an automaton $\langle S, A, F \rangle$ such that e is the language accepted by the automaton, i.e. $e = SA^*F = \sum \{ SA^kF \mid k \in \mathbb{N} \}$, where for some n , $S \in \{0, 1\}^{1 \times n}$ and $F \in \{0, 1\}^{n \times 1}$ code the sets of initial and final states and A is a transition matrix $U + X + V \in Mat_{n,n}(K \otimes_{\mathcal{R}} C_2)$ with $X \in K^{n \times n}$, $U \in \{b, p, 0\}^{n \times n}$, $V \in \{d, q, 0\}^{n \times n}$. It is then shown that $N = b(Up + X + qV)d$ is a least solution of $y \geq (UyV + X)^*$ in $Mat_{n,n}(K \otimes_{\mathcal{R}} C_2)$, a version of Dyck’s language, and $N \in (Z_{C_2}(K \otimes_{\mathcal{R}} C_2))^{n \times n}$. Using the completeness assumption $db + qp = 1$, Hopkins proves

$$e = S(NV)^*N(UV)^*F,$$

which simplifies to $e = SNF$ for $e \in Z_{C_2}(K \otimes_{\mathcal{R}} C_2)$. Notice that the Normal Form Theorem provides us with a description both of elements in the centralizer of C_2 and of arbitrary elements of $K \otimes_{\mathcal{R}} C_2$, while the proof of $\bar{K} = Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$ in Section 6 above only gives us a description of elements of the centralizer of C'_2 as equivalence classes $[R]$ of $R \in \mathcal{R}(K \times C'_2)$ with $R \subseteq K \times \{0, 1\}$. In Section 6, the surjectivity of $\hat{\cdot} : \mathcal{C}K \rightarrow Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$ is reduced to that of $\hat{\cdot} : \mathcal{C}K \rightarrow Z_{C'_2}(\mathcal{R}K \otimes_{\mathcal{R}} C'_2)$, which in turn is shown by giving an inverse. However, it is an interesting question whether the Normal Form Theorem also holds for $K \otimes_{\mathcal{R}} C'_2$; Hopkins (personal communication, 2021) has found a partial positive answer. A least solution N of $y \geq (UyV + X)^*$ in $Mat_{n,n}(Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2))$ exists, since $Z_{C'_2}(K \otimes_{\mathcal{R}} C'_2)$ is a \mathcal{C} -dioid and this category is closed under $n \times n$ -matrix semirings.

Another difference of the proofs is that in the case $\mathcal{R}M \otimes_{\mathcal{R}} C'_2$ we can replace the tensor product by a quotient (Corollary 13) and so, strictly speaking, need the tensor product only to represent the \mathcal{C} -closure of arbitrary \mathcal{R} -dioids K in $K \otimes_{\mathcal{R}} C'_2$.

A minor question is whether our proof for the special case $K = \mathcal{R}M$ can be simplified. Since it is well-known that $\mathcal{C}X^*$ is related to the polycyclic monoids P'_n , Propositions 20-23 might be implicit in existing literature, such as the one referred to by Render and Kambites (2009). Moreover, it seems inelegant to first show that the maps $\hat{\cdot} : \mathcal{C}M \rightleftharpoons Z_{C'_2}(\mathcal{R}M \otimes_{\mathcal{R}} C'_2) : \cdot^{\vee}$ form a bijection and then use this to show that they are \mathcal{C} -morphisms.

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Competing interests

The author declares none.

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8. Appendix

The postponed part of the proof of Theorem 12 is as follows.

Proof. To prove $\mathcal{A}M \otimes_{\mathcal{A}} (\mathcal{A}N/\nu) \simeq \mathcal{A}(M \times N)/\tilde{\nu}$, it is sufficient to show that

$$\top'_1 : \mathcal{A}M \longrightarrow \mathcal{A}(M \times N)/\tilde{\nu} \longleftarrow \mathcal{A}N/\nu : \top'_2$$

form a tensor product of $\mathcal{A}M$ and $\mathcal{A}N/\nu$ in $\mathbb{D}\mathcal{A}$, where \top'_1 and \top'_2 are defined by

$$\top'_1(A) = (A \times \{1\})/\tilde{\nu} \quad \text{and} \quad \top'_2(B/\nu) = (\{1\} \times B)/\tilde{\nu}.$$

\top'_1 is an \mathcal{A} -morphism, since $A \mapsto A \times \{1\}$ and the quotient map $\cdot/\tilde{\nu}$ are. To see that \top'_2 is well-defined, we first check that

- (1) For $B, B' \in \mathcal{A}N$, $B/\nu = B'/\nu \iff B \setminus \{0\} = B' \setminus \{0\}$,
- (2) for $S, S' \in \mathcal{A}(M \times N)$, $S/\tilde{\nu} = S'/\tilde{\nu} \iff S \setminus (M \times \{0\}) = S' \setminus (M \times \{0\})$.

Concerning (1), since $\{0\}/\nu = \emptyset/\nu$ is the least element of $\mathcal{A}N/\nu$, for $B \in \mathcal{A}N$ we have

$$B/\nu = \sum \{ \{n\}/\nu \mid n \in B \} = \sum \{ \{n\}/\nu \mid n \in B \setminus \{0\} \}.$$

Hence if $B, B' \in \mathcal{A}N$ with $B \setminus \{0\} = B' \setminus \{0\}$, then $B/\nu = B'/\nu$. For the converse, one shows that

$$\nu_0 := \{ (B, B') \in \mathcal{A}N \times \mathcal{A}N \mid B \setminus \{0\} = B' \setminus \{0\} \}$$

is an \mathcal{A} -congruence on $\mathcal{A}N$ containing $(\{0\}, \emptyset)$, and obviously $\nu \subseteq \nu_0$.

For (2), notice that for each $m \in M$,

$$\{(m, 0)\}/\tilde{\nu} = \{(m, 1)\}/\tilde{\nu} \cdot \{(1, 0)\}/\tilde{\nu} = \{(m, 1)\}/\tilde{\nu} \cdot \emptyset/\tilde{\nu} = \emptyset/\tilde{\nu}.$$

Therefore, similar arguments as for (1) apply with $S \setminus (M \times \{0\})$ instead of $B \setminus \{0\}$.

By (1) and (2), it follows that \top_2 is well-defined, since

$$\begin{aligned} B/\nu = B'/\nu &\iff B \setminus \{0\} = B' \setminus \{0\} \\ &\iff (\{1\} \times B) \setminus (M \times \{0\}) = (\{1\} \times B') \setminus (M \times \{0\}) \\ &\iff (\{1\} \times B)/\tilde{\nu} = (\{1\} \times B')/\tilde{\nu}. \end{aligned}$$

\top'_2 is an \mathcal{A} -morphism: it is clearly a monotone homomorphism, and if $U \in \mathcal{A}(\mathcal{A}N/\nu)$, there is $U' \in \mathcal{A}\mathcal{A}N$ such that $U = \{ B/\nu \mid B \in U' \}$, hence

$$\begin{aligned} \top'_2(\sum U) &= \top'_2(\sum \{ B/\nu \mid B \in U' \}) = \top_2((\bigcup U')/\nu) \\ &= (\{1\} \times (\bigcup U'))/\tilde{\nu} = (\bigcup \{ \{1\} \times B \mid B \in U' \})/\tilde{\nu} \\ &= \sum \{ (\{1\} \times B)/\tilde{\nu} \mid B \in U' \} = \sum \{ \top'_2(B/\nu) \mid B/\nu \in U \}. \end{aligned}$$

Clearly, \top'_1 and \top'_2 are relatively commuting.

To show the universal property of a tensor product, let f, g be relatively commuting \mathcal{A} -morphisms to an \mathcal{A} -dioid D as shown:

$$\begin{array}{ccccc}
 \mathcal{A}M & \xrightarrow{\top'_1} & \mathcal{A}(M \times N)/\tilde{\nu} & \xleftarrow{\top'_2} & \mathcal{A}N/\nu \\
 & \searrow f & \vdots h'_{f,g} & \swarrow g & \\
 & & D & &
 \end{array}$$

We define an \mathcal{A} -morphism $h'_{f,g} : \mathcal{A}(M \times N)/\tilde{\nu} \rightarrow D$ by

$$h'_{f,g}(S/\tilde{\nu}) = \sum \{ f(\{m\}) \cdot g(\{n\}/\nu) \mid (m, n) \in S \}, \quad S \in \mathcal{A}(M \times N).$$

- $h'_{f,g}$ is well-defined: Since f and g are relatively commuting, $(m, n) \mapsto f(\{m\}) \cdot g(\{n\}/\nu)$ defines a homomorphism from $M \times N$ to D , so its lifting maps $S \in \mathcal{A}(M \times N)$ to an \mathcal{A} -set $\{ f(\{m\}) \cdot g(\{n\}/\nu) \mid (m, n) \in S \} \in \mathcal{A}D$, and the \sum exists. Since g is a semiring-morphism, for $(m, 0) \in S$ we have

$$f(\{m\}) \cdot g(\{0\}/\nu) = f(\{m\}) \cdot g(0^{\mathcal{A}N/\nu}) = f(\{m\}) \cdot 0^D = 0^D,$$

whence the value of $h'_{f,g}(S/\tilde{\nu})$ depends on $S \setminus (M \times \{0\})$ only, i.e. on $S/\tilde{\nu}$ by (2).

- $h'_{f,g}$ is a homomorphism: clearly

$$h'_{f,g}(\{(1, 1)\}/\tilde{\nu}) = \sum \{ f(\{1\}) \cdot g(\{1\}/\nu) \} = f(\{1\}) \cdot g(\{1\}/\nu) = 1^D.$$

Since f and g are relatively commuting, for $S, S' \in \mathcal{A}(M \times N)$ the set $\{ f(\{m\}) \cdot g(\{n\}/\nu) \mid (m, n) \in SS' \}$ is the product in $\mathcal{A}D$ of

$$\{ f(\{m\}) \cdot g(\{n\}/\nu) \mid (m, n) \in S \} \text{ and } \{ f(\{m\}) \cdot g(\{n\}/\nu) \mid (m, n) \in S' \},$$

and since $\sum : \mathcal{A}D \rightarrow D$ is \mathcal{A} -distributive, this gives

$$\begin{aligned}
 h'_{f,g}(S/\tilde{\nu} \cdot S'/\tilde{\nu}) &= h'_{f,g}(SS'/\tilde{\nu}) \\
 &= \sum \{ f(\{m\}) \cdot g(\{n\}/\nu) \mid (m, n) \in SS' \}, \\
 &= h'_{f,g}(S/\tilde{\nu}) \cdot h'_{f,g}(S'/\tilde{\nu}).
 \end{aligned}$$

- $h'_{f,g}$ is an \mathcal{A} -morphism: If $V \in \mathcal{A}(\mathcal{A}(M \times N)/\tilde{\nu})$, there is $U \in \mathcal{A}(\mathcal{A}(M \times N))$ such that $V = U/\tilde{\nu} := \{ S/\tilde{\nu} \mid S \in U \}$, and $\sum V = (\bigcup U)/\tilde{\nu}$. So

$$\begin{aligned}
 h'_{f,g}(\sum V) &= h'_{f,g}((\bigcup U)/\tilde{\nu}) \\
 &= \sum \{ f(\{m\}) \cdot g(\{n\}/\nu) \mid (m, n) \in \bigcup U \}, \\
 &= \sum \{ \sum \{ f(\{m\}) \cdot g(\{n\}/\nu) \mid (m, n) \in S \} \mid S \in U \}, \\
 &= \sum \{ h'_{f,g}(S/\tilde{\nu}) \mid S \in U \} = \sum (\mathcal{A}h'_{f,g})(V),
 \end{aligned}$$

showing that $h'_{f,g}$ is \mathcal{A} -continuous; this implies that it is monotone.

- $f = h'_{f,g} \circ \top'_1$ and $g = h'_{f,g} \circ \top'_2$: for $A \in \mathcal{A}M$ and $B \in \mathcal{A}N$,

$$\begin{aligned}
h'_{f,g}(\top'_1(A)) &= h'_{f,g}((A \times \{1\})/\tilde{\nu}) \\
&= \sum \{ f(\{m\}) \cdot g(\{n\}/\nu) \mid (m, n) \in A \times \{1\} \} \\
&= \sum \{ f(\{m\}) \cdot g(\{1\}/\nu) \mid m \in A \} \\
&= \sum \{ f(\{m\}) \mid m \in A \} \\
&= f(\bigcup \{ \{m\} \mid m \in A \}) \\
&= f(A),
\end{aligned}$$

$$\begin{aligned}
h'_{f,g}(\top'_2(B/\nu)) &= h'_{f,g}((\{1\} \times B)/\tilde{\nu}) \\
&= \sum \{ f(\{m\}) \cdot g(\{n\}/\nu) \mid (m, n) \in \{1\} \times B \} \\
&= \sum \{ f(\{1\}) \cdot g(\{n\}/\nu) \mid n \in B \} \\
&= \sum \{ g(\{n\}/\nu) \mid n \in B \} \\
&= g(\sum \{ \{n\}/\nu \mid n \in B \}) \\
&= g(\bigcup \{ \{n\} \mid n \in B \}/\nu) \\
&= g(B/\nu).
\end{aligned}$$

- $h'_{f,g}$ is the only \mathcal{A} -morphism h' with $f = h' \circ \top'_1$ and $g = h' \circ \top'_2$: Suppose $h' : \mathcal{A}(M \times N)/\tilde{\nu} \rightarrow D$ is such an \mathcal{A} -morphism. Then

$$\begin{aligned}
h'_{f,g}(S/\tilde{\nu}) &= \sum \{ f(\{m\}) \cdot g(\{n\}/\nu) \mid (m, n) \in S \} \\
&= \sum \{ h'(\top'_1(\{m\})) \cdot h'(\top'_2(\{n\}/\nu)) \mid (m, n) \in S \} \\
&= \sum \{ h'(\top'_1(\{m\}) \cdot \top'_2(\{n\}/\nu)) \mid (m, n) \in S \} \\
&= h'(\sum \{ \top'_1(\{m\}) \cdot \top'_2(\{n\}/\nu) \mid (m, n) \in S \}) \\
&= h'(\sum \{ (\{m\} \times \{1\})/\tilde{\nu} \cdot (\{1\} \times \{n\})/\tilde{\nu} \mid (m, n) \in S \}) \\
&= h'(\sum \{ (\{m\} \times \{n\})/\tilde{\nu} \mid (m, n) \in S \}) \\
&= h'(\sum \{ \{(m, n)\}/\tilde{\nu} \mid (m, n) \in S \}) \\
&= h'((\bigcup \{ \{(m, n)\} \mid (m, n) \in S \})/\tilde{\nu}) \\
&= h'(S/\tilde{\nu}). \quad \square
\end{aligned}$$