Module theory over nonunital rings

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§**1. Motivation.** For any unital ring R, let $\mathrm{Mod}(R)$ denote the category of (left) unitary R-modules, i.e. satisfying $1m = m$ for all elements m. This category is important in many areas of mathematics such as ring theory, representation theory, and homological algebra. The purpose of this paper is to extend the category $\mathrm{Mod}(R)$ to nonunital rings A. In general there are several interesting ways to do this, but in the idempotent case: $A = A^2$ they agree, yielding a nice abelian category $\mathcal{M}(A)$, which gives rise to a theory of Morita equivalence for idempotent rings extending the usual Morita theory in the unital case.

Let A be a nonunital ring, and let $\tilde{A}$ denote the unital ring $\mathbf{Z} \oplus A$ obtained by adjoining an identity. A module over A is the same thing as a unitary $\tilde{A}$-module, and so the category of A-modules can be identified with $\mathrm{Mod}(\tilde{A})$. This category is too big for our purposes. For example, suppose A happens to be unital, and let e denote the identity of A to distinguish it from the canonical identity 1 in $\tilde{A}$. Then

$$\mathrm{Mod}(\tilde{A}) = \mathrm{Mod}(A) \times \mathrm{Mod}(\mathbf{Z})$$

since any A-module has a canonical splitting $M = eM \oplus (1-e)M$ into a unitary A-module and a module killed by A. This indicates that for a general A we need to replace $\mathrm{Mod}(\tilde{A})$ by either a suitable subcategory or quotient category in order to generalize $\mathrm{Mod}(A)$ when A is unital. Both methods will be used and shown to coincide for idempotent rings.

Nonunital rings A occurring in practice are often algebras over some field k. It is natural in this context to restrict attention to A-modules which have a compatible k-module structure, i.e. which are unitary modules over the k-algebra $k \oplus A$ obtained by adjoining an identity to A. In order to handle this situation, we construct our module category for a nonunital ring A starting with $\mathrm{Mod}(R)$, where R is any unital ring containing A as ideal. We will show later that the resulting theory is independent of the choice of R.

§**2. Nil modules and firm modules.** We fix a unital ring R and an ideal A in R. Unless stated otherwise, the terms module and right module will mean objects of $\mathrm{Mod}(R)$ and $\mathrm{Mod}(R^{op})$ respectively, where R^{op} means R equipped with the opposite multiplication.

(2.1) **Definition** A module M such that $A^n M = 0$ for some integer $n \geq 1$ will be called an A-*nil module*, or simply a nil module when the ideal is clear from the context. Let $\mathcal{N}(R, A)$, or simply $\mathcal{N}$, denote the full subcategory of nil modules in $\mathrm{Mod}(R)$.

A module map will be called a *nil isomorphism* when its kernel and cokernel are nil modules.

For example, the canonical multiplication map

$$(2.2) \qquad \mu : A \otimes_R M \to M, \qquad a \otimes m \mapsto am$$

is a nil isomorphism, because its kernel and cokernel are killed by A. This is clear for the cokernel M/AM, and if $k = \sum a_i \otimes m_i$ belongs to the kernel, then $ak = \sum aa_i \otimes m_i = a \otimes \sum a_i m_i = 0$.

(2.3) **Definition** By an *A-firm module*, or simply firm module when the ideal is clear, we mean a module M such that the map μ is an isomorphism. Let $\mathcal{F} = \mathcal{F}(R, A)$ be the full subcategory of firm modules in $\mathrm{Mod}(R)$.

The short exact sequence $A \to R \to R/A$ yields the exact sequence

$$0 \to \mathrm{Tor}_1^R(R/A, M) \to A \otimes_R M \xrightarrow{\mu} M \to M/AM \to 0$$

hence M is firm iff $\mathrm{Tor}_j^R(R/A, M) = 0$ for $j = 0, 1$. In particular, a flat module F is firm iff $F = AF$.

We next construct modules which are both firm and flat by an iteration method.

(2.4) **Lemma** *If M satisfies $M = AM$, then there exists a firm flat module F mapping onto M.*

Proof. Assuming $M = AM$, we choose a surjection $g : P \to M$ with P projective. The restriction $AP \to M$ of g is surjective, hence there is a map $f : P \to P$ such that $gf = g$ and $f(P) \subset AP$. Let F be the colimit of the system $(P_n)_{n \geq 0}$, where $P_n = P$ and $P_n \to P_{n+1}$ is the map f for all n. Since F is a filtered colimit of projectives, it is flat. Also $F = AF$, since f maps P_n into AP_{n+1}, and hence F is firm. Finally, there is a surjection $F \to M$ given by $g : P_n \to M$ for all n. $\square$

(2.5) In general the module F we have constructed may not be improved to a firm projective module, since the only such module may be zero. For example, if R is a local ring with maximal ideal A, then by a theorem of Kaplansky [K] any projective R-module P is free, hence $P < AP$ unless $P = 0$. We note that such examples can be found with A idempotent: $A = A^2$, namely, rings of germs of continuous functions and nondiscrete valuation rings.

Our next result characterizes firm modules M by means of the functor $- \otimes_R M : V \mapsto V \otimes_R M$ from right modules to abelian groups. By abuse of language we say that a functor T *inverts* a morphism u when $T(u)$ is an isomorphism.

(2.6) **Proposition** *The following properties are equivalent for a module M :*
(a) *M is firm.*

(b) M *is the cokernel of a map between firm flat modules.*
(c) *The functor* $-\otimes_R M : \mathrm{Mod}(R^{op}) \to \mathrm{Ab}$ *inverts* A^{op}*-nil isomorphisms.*

Proof. (a) $\Rightarrow$ (b). Assuming M is firm, we know that $M = AM$ and $\mathrm{Tor}_1^R(R/A, M) = 0$. By the lemma there is a firm flat module F_0 mapping onto M. If M_1 is the kernel, then $\mathrm{Tor}_1^R(R/A, M) \xrightarrow{\sim} M_1/AM_1$, so $M_1 = AM_1$ and there is a firm flat module F_1 mapping onto M_1, proving (b).

(b) $\Rightarrow$ (c). Let M be the cokernel of a map $F_1 \to F_0$ of firm flat modules. By right exactness of tensor product, $-\otimes_R M$ is the cokernel of the induced map of functors from $-\otimes_R F_1$ to $-\otimes_R F_0$. It thus suffices to show for F firm flat that $-\otimes_R F$ inverts nil isomorphisms between right modules. This follows from the exactness of this functor and the fact that $VA = 0$ and $F = AF$ imply $V \otimes_R F = 0$.

Finally, (c) $\Rightarrow$ (a) by applying $-\otimes_R M$ to the right module nil isomorphism $A \subset R$. □

§3. Some abelian category theory. This section reviews some basic ideas pertaining to the quotient abelian category $\mathcal{A}/\mathcal{S}$ of an abelian category by a Serre subcategory, namely, perpendicular categories and (co-)localizing subcategories [GL]. These will be applied when $\mathcal{A} = \mathrm{Mod}(R)$ and $\mathcal{S}$ is either the category of A-nil modules, or the category of A-torsion modules to be introduced later.

Let $\mathcal{A}$ be an abelian category and let $\mathcal{S}$ be a full subcategory, equivalently, a class of objects in $\mathcal{A}$. The *left perpendicular category* ${}^{\perp}\mathcal{S}$ is defined to be the full subcategory of $\mathcal{A}$ consisting of objects satisfying the following conditions.

(3.1) **Proposition** *The following are equivalent for an object* M *of* $\mathcal{A}$:
(a) $\mathrm{Ext}^j(M, N) = 0$ *for* $j = 0, 1$ *and any* N *in* $\mathcal{S}$.
(b) $\mathrm{Hom}(M, -)$ *inverts any map whose kernel and cokernel are (isomorphic to objects) in* $\mathcal{S}$.

Proof. We use the long exact sequence

$$0 \to \mathrm{Hom}(M, N') \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(M, N'') \to \mathrm{Ext}^1(M, N') \to$$

arising from a short exact sequence $N' \to N \to N''$. This shows that $\mathrm{Hom}(M, -)$ inverts any epimorphism $N \to N''$ with kernel in $\mathcal{S}$, and also any monomorphism $N' \to N$ with cokernel in $\mathcal{S}$. Thus (a) implies (b).

Next assume (b) holds, and let N be in $\mathcal{S}$. Applying (b) to the projection $N \oplus M \to M$ we see that $\mathrm{Hom}(M, N) = 0$. If x is an element of $\mathrm{Ext}^1(M, N)$, there is a corresponding extension $N \to E \to M$ such that in the long exact sequence

$$\to \mathrm{Hom}(M, E) \to \mathrm{Hom}(M, M) \xrightarrow{\delta} \mathrm{Ext}^1(M, N) \to$$

one has $\delta(1_M) = x$. Applying (b) to the map $E \to M$ we find $x = 0$, showing that $\mathrm{Ext}^1(M, N) = 0$. Thus (b) implies (a). □

(3.2) Suppose now that $\mathcal{S}$ is a *Serre subcategory* of $\mathcal{A}$, i.e. nonempty and such that if $M' \to M \to M''$ is a short exact sequence in $\mathcal{A}$, then M is in $\mathcal{S}$ iff both M' and M'' are in $\mathcal{S}$.

Let $\mathcal{A}/\mathcal{S}$ be the corresponding quotient abelian category. It has the same objects as $\mathcal{A}$, and its maps are obtained from the maps in $\mathcal{A}$ by formal inverting all $\mathcal{S}$-isomorphisms (i.e. those maps whose kernel and cokernel are in $\mathcal{S}$). There is thus a canonical functor $j^* : \mathcal{A} \to \mathcal{A}/\mathcal{S}$, which is universal for functors defined on $\mathcal{A}$ that invert $\mathcal{S}$-isomorphisms. The functor j^* is exact and $j^*(M) \simeq 0$ iff M is in $\mathcal{S}$. Consequently any map inverted by j^* is an $\mathcal{S}$-isomorphism, since j^* kills its kernel and cokernel.

(3.3) **Proposition** *If M is in ${}^\perp\mathcal{S}$, then*

$$\mathrm{Hom}_{\mathcal{A}}(M,N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(j^*M, j^*N)$$

for all N in $\mathcal{A}$. In particular, denoting by $\iota : {}^\perp\mathcal{S} \to \mathcal{A}$ the inclusion functor, the functor $j^*\iota : {}^\perp\mathcal{S} \to \mathcal{A}/\mathcal{S}$ is fully faithful.

Proof. We use the following description of maps in $\mathcal{A}/\mathcal{S}$ as a filtered colimit:

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(j^*M, j^*N) = \mathrm{colim}\, \mathrm{Hom}_{\mathcal{A}}(M', N'') \tag{3.4}$$

where M' runs over subobjects of M and N'' runs over quotient objects of N such that canonical maps $M' \to M$ and $N \to N''$ are $\mathcal{S}$-isomorphisms. If M is in ${}^\perp\mathcal{S}$, then applying condition (b) above to these $\mathcal{S}$-isomorphisms, we see that $M' = M$ and $\mathrm{Hom}_{\mathcal{A}}(M,N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(M', N'')$ for all M', N'. Since the colimit is taken over a directed set, the desired result follows. □

(3.5) **Proposition** *The functor $j^*\iota : {}^\perp\mathcal{S} \to \mathcal{A}/\mathcal{S}$ is an equivalence of categories iff for every M in $\mathcal{A}$ there is an $\mathcal{S}$-isomorphism $M_\# \to M$ with $M_\#$ in ${}^\perp\mathcal{S}$.*

Proof. $\Leftarrow$ Given any object j^*M in $\mathcal{A}/\mathcal{S}$, we have $j^*\iota(M_\#) = j^*(M_\#) \xrightarrow{\sim} j^*M$, since j^* inverts $\mathcal{S}$-isomorphisms. This shows $j^*\iota$ is essentially surjective, and since it is fully faithful by the preceding proposition, it is an equivalence.

$\Rightarrow$ If $j^*\iota$ is an equivalence, then for any M in $\mathcal{A}$ there is an $M_\#$ in ${}^\perp\mathcal{S}$ and an isomorphism $j^*(M_\#) \xrightarrow{\sim} j^*M$. By (3.3) this isomorphism comes from a unique map $M_\# \to M$ which must be an $\mathcal{S}$-isomorphism, since it is inverted by j^*. □

(3.6) **Definition** When the conditions of (3.5) hold, $\mathcal{S}$ is called a *colocalizing* subcategory of $\mathcal{A}$. One has the following additional facts in this situation.

- *The quasi-inverse for $j^*\iota$ is given by $j^*M \mapsto M_\#$. Consequently $M_\#$ is determined up to canonical isomorphism and is a functor of M inverting $\mathcal{S}$-isomorphisms.*

• $\iota : {}^{\perp}\mathcal{S} \to \mathcal{A}$ *admits a right adjoint* $M \mapsto M_{\#}$ *such that the adjunction map* $\iota(M_{\#}) \to M$ *is an* $\mathcal{S}$*-isomorphism for all* M. Indeed, by (3.3) we have

$$\mathrm{Hom}_{{}^{\perp}\mathcal{S}}(L, M_{\#}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(\iota(L), M)$$

for L in ${}^{\perp}\mathcal{S}$.

• j^* *admits a left adjoint* $j_! : j^*M \mapsto M_{\#}$. This is clear from

$$\mathrm{Hom}_{\mathcal{A}}(M_{\#}, N) = \mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(j^*(M_{\#}), j^*N) = \mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(j^*M, j^*N).$$

Conversely one can show that if either ι or j^* admits such an adjoint, then $\mathcal{S}$ is colocalizing.

(3.7) We briefly mention the more familiar dual version of the preceding. One has the right perpendicular category $\mathcal{S}^{\perp}$ consisting of M such that $\mathrm{Hom}(-, M)$ inverts $\mathcal{S}$-isomorphisms. The functor $\mathcal{S}^{\perp} \to \mathcal{A}/\mathcal{S}$ induced by j^* is always fully faithful, and it is an equivalence iff for every M there is an $\mathcal{S}$-isomorphism $M \to M^{\#}$ with $M^{\#}$ in $\mathcal{S}^{\perp}$. In this case $\mathcal{S}$ is called a *localizing* subcategory.

§**4.** We now return to our unital ring R and ideal A and apply the preceding discussion to the abelian category $\mathrm{Mod}(R)$ and the Serre subcategory $\mathcal{N} = \bigcup_n \mathrm{Mod}(R/A^n)$ of nil modules. We write $\mathcal{M} = \mathcal{M}(R, A)$ for the corresponding quotient abelian category of $\mathrm{Mod}(R)$.

The next result identifies the firm module category $\mathcal{F}$ with the left perpendicular categories of both $\mathcal{N}$ and its subcategory $\mathrm{Mod}(R/A)$. This gives a characterization of firm modules quite different from (2.6).

(4.1) **Proposition** *The following are equivalent for a module* M :
(a) M *is firm.*
(b) $\mathrm{Ext}^j_R(M, N) = 0$ *for* $j = 0, 1$ *and any nil module* N (*resp. any module* N *such that* $AN = 0$).
(c) $\mathrm{Hom}_R(M, -)$ *inverts nil isomorphisms.*

Proof. The two parts of (b) are equivalent because any nil module is a finite iterated extension of modules killed by A. The conditions (b) and (c) are equivalent by (3.1). The equivalence of (a) and (b) will be proved using the spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p_{R/A}(\mathrm{Tor}^R_q(R/A, M), N) \Rightarrow \mathrm{Ext}^n_R(M, N)$$

where N is any R/A-module. When M is firm the Tor groups vanish in degrees $0, 1$, hence so does the abutment. Conversely, assume the abutment has this vanishing property for all N, and take N to be any injective R/A-module. The spectral sequence degenerates showing that the only maps from the Tor groups in degrees $0, 1$ to such an N are zero. Thus these Tor groups vanish, and M is firm. □

Since $\mathcal{F} = {}^{\perp}\mathcal{N}$, we know by (3.3) that

$$\mathrm{Hom}_R(M, N) = \mathrm{Hom}_{\mathcal{M}}(j^*M, j^*N)$$

when M is firm, and that $j^*\iota : \mathcal{F} \subset \mathrm{Mod}(R) \to \mathcal{M}$ is fully faithful.

(4.2) **Proposition** *Colimits exist in $\mathcal{F}$. The functors $\iota : \mathcal{F} \to \mathrm{Mod}(R)$ and $j^*\iota : \mathcal{F} \to \mathcal{M}$ respect colimits.*

Proof. Let $k \mapsto M_k$ be a functor from a small category to $\mathcal{F}$, and let M denote its colimit in $\mathrm{Mod}(R)$. Then

$$\mathrm{Hom}_R(M, N) = \lim_k \mathrm{Hom}_R(M_k, N)$$

where the functor of N on the right inverts nil isomorphisms, hence M is firm. Restricting N to be in $\mathcal{F}$, we see that colimits exist in $\mathcal{F}$ and that ι respects colimits. Then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}}(j^*M, j^*N) &= \mathrm{Hom}_R(M, N) \\ &= \lim_k \mathrm{Hom}_R(M_k, N) \\ &= \lim_k \mathrm{Hom}_{\mathcal{M}}(j^*M_k, j^*N) \end{aligned}$$

shows that the functor $j^*\iota$ respects colimits. □

We are going to show in the idempotent case $A = A^2$ that $j^*\iota : \mathcal{F} \to \mathcal{M}$ is an equivalence of categories, in other words, the subcategory of nil modules is colocalizing.

We introduce the notation: $A^{(2)} = A \otimes_R A$, and more generally $A^{(n)}$ for the n-fold tensor product of the R-bimodule A.

(4.3) **Proposition** *Assume $A = A^2$. Then $A \otimes_R -$ inverts any surjective nil isomorphism, and $A^{(2)} \otimes_R -$ inverts any nil isomorphism.*

Proof. Note that when $A = A^2$ we have $\mathcal{N} = \mathrm{Mod}(R/A)$, hence the first assertion follows from right exactness of tensor product, and the fact that $A \otimes_R N = 0$ when $A = A^2$ and $AN = 0$. It is then clear that $A^{(2)} \otimes_R - = A \otimes_R A \otimes_R -$ inverts surjective nil isomorphisms, so it remains to see that this functor inverts any injection $i : M' \to M$ with cokernel killed by A. We have a commutative square

$$\begin{array}{ccc} A \otimes_R M' & \xrightarrow{1 \otimes i} & A \otimes_R M \\ \downarrow & & \downarrow \\ M' & \xrightarrow{i} & M \end{array}$$

where the top arrow is surjective, and the vertical arrows are the multiplication maps μ. The kernel of $1 \otimes i$ is killed by A, since this is true for the μ map for M' and i is injective. Thus $A \otimes_R -$ inverts the surjective nil isomorphism $1 \otimes i$, so $A^{(2)} \otimes_R -$ inverts i. □

(4.4) **Proposition** *Assume $A = A^2$. If $M = AM$, then $A \otimes_R M$ is firm. Morever, $A^{(2)} \otimes_R M$ is firm for any M.*

Proof. If $AM = M$, then $\mu : A\otimes_R M \to M$ is a surjective nil isomorphism, so by the preceding result the map $A^{(2)}\otimes_R M \to A\otimes_R M$, $a_1 \otimes a_2 \otimes m \mapsto a_1 \otimes a_2 m$, is an isomorphism. Since $a_1 \otimes a_2 m = a_1 a_2 \otimes m$, it follows that $A\otimes_R M$ is firm. The second assertion follows from the first and fact that $M' = A\otimes_R M$ satisfies $AM' = M'$ when $A = A^2$. □

Finally, the composition $A^{(2)}\otimes_R M \to A\otimes_R M \to M$ of μ maps is a nil isomorphism for any module M. Thus when $A = A^2$, the second condition of (3.5) is satisfied with $M_{\#} = A^{(2)}\otimes_R M$, and we have established the following.

(4.5) **Theorem** *If M is firm, then* $\mathrm{Hom}_R(M,N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{M}}(j^*M, j^*N)$ *for any module N. In particular, the functor $\mathcal{F} = {}^{\perp}\mathcal{N} \to \mathcal{M}$ induced by j^* is always fully faithful. When $A = A^2$ it is an equivalence of categories, and the quasi-inverse functor is $j^*M \mapsto A^{(2)}\otimes_R M$.*

(4.6) This theorem yields immediately that when A is idempotent, the category $\mathcal{F}$ of firm modules is an abelian category. It is easy to see that a sequence $0 \to M' \to M \to M'' \to 0$ of firm modules is exact in the abelian category $\mathcal{F}$ iff it is right exact in $\mathrm{Mod}(R)$ and the kernel of $M' \to M$ is a nil module. More generally, an arbitrary sequence of firm modules is exact in $\mathcal{F}$ iff it is a complex and the homology groups of this complex in $\mathrm{Mod}(R)$ are nil modules.

(4.7) If two ideals A, A' give rise to the same adic topology in the sense that $A \supset A'^n$ and $A' \supset A^n$ for some $n \geq 1$, then clearly their associated nil module categories are the same, and similarly for firm modules since $\mathcal{F} = {}^{\perp}\mathcal{N}$. Consequently, the case where A is *essentially idempotent*, i.e. $A^n = A^{n+1}$ for some $n \geq 1$, reduces to the case of the idempotent ideal A^n.

§**5. Closed modules.** We discuss next a dual version of firm modules, based on the functor $\mathrm{Hom}_R(A, -)$ instead of $A\otimes_R -$.

For any module M we have a canonical map

$$\mu' : M \to \mathrm{Hom}_R(A, M), \qquad \mu'(m) = (a \mapsto am) \tag{5.1}$$

in $\mathrm{Mod}(R)$, which is a nil isomorphism, since its kernel and cokernel are killed by A. This is clear for the kernel, and if $f \in \mathrm{Hom}_R(A, M)$, then we have $(a'f)(a) = f(aa') = af(a')$, i.e. $a'f = \mu'(f(a'))$, showing that the cokernel of μ' is killed by A.

(5.2) **Definition** We say that M is A*-closed*, or simply closed when the ideal is understood, when the map μ' is an isomorphism. Let $\mathcal{C} = \mathcal{C}(R, A)$ be the full subcategory of closed modules in $\mathrm{Mod}(R)$.

Applying $\mathrm{Hom}_R(-, M)$ to the short exact sequence $A \to R \to R/A$ yields the exact sequence

$$0 \to {}_AM \to M \xrightarrow{\mu'} \mathrm{Hom}_R(A,M)M \to \mathrm{Ext}^1_R(R/A,M) \to 0$$

where ${}_AM$ means $\mathrm{Hom}_R(R/A,M) = \{m \in M | Am = 0\}$. Thus M is closed iff $\mathrm{Ext}^j_R(R/A,M) = 0$ for $j = 0,1$. In particular, an injective module Q is closed iff ${}_AQ = 0$.

Our next result, the analogue of (2.6), identifies $\mathcal{C}$ with the right perpendicular category $\mathcal{N}^\perp$.

(5.3) **Proposition** *The following conditions are equivalent:*
(a) *M is closed.*
(b) *M is the kernel of a map $Q^0 \to Q^1$ between closed injective modules.*
(c) $\mathrm{Hom}_R(-,M)$ *inverts nil isomorphisms.*

This is proved in virtually the same way as (2.6), using the following analogue of (2.4).

(5.4) **Lemma** *If ${}_AM = 0$, then M can be embedded in a closed injective module Q.*

Indeed, if Q is the injective hull of M, then ${}_AQ \cap M = {}_AM = 0$, so ${}_AQ = 0$, as Q is an essential extension of M.

Since $\mathcal{C} = \mathcal{N}^\perp$, we know by the dual version of (3.3) that

$$\mathrm{Hom}_R(N,M) = \mathrm{Hom}_{\mathcal{M}}(j^*N, j^*M)$$

when M is closed, and that $j^*\iota : \mathcal{C} \subset \mathrm{Mod}(R) \to \mathcal{M}$ is fully faithful. The following is analogous to (4.2) and has the same sort of proof.

(5.5) **Proposition** *Limits exist in $\mathcal{C}$. The inclusion functor $\iota : \mathcal{C} \to \mathrm{Mod}(R)$ and $j^*\iota : \mathcal{F} \to \mathcal{M}$ respects colimits.*

We consider next the analogue of Theorem (4.5). For any module M put $M^\# = \mathrm{Hom}_R(A^{(2)}, M)$. There is a nil isomorphism

$$M \to \mathrm{Hom}_R(A,M) \to \mathrm{Hom}_R(A, \mathrm{Hom}_R(A,M)) = M^\#$$

given by the composition of two μ' maps (5.1). We have the adjoint functor relation

$$\mathrm{Hom}_R(A^{(2)} \otimes_R N, M) = \mathrm{Hom}_R(N, M^\#).$$

Assuming $A = A^2$, the left hand side considered as a functor of N inverts nil isomorphisms by (4.3), hence $M^\#$ is closed by (5.3). Thus $\mathcal{N}$ is localizing, and we obtain the following.

(5.6) **Theorem** *If M is closed, then* $\mathrm{Hom}_R(N, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{M}}(j^*N, j^*M)$ *for any module N. In particular, the functor $\mathcal{C} = \mathcal{N}^{\perp} \to \mathcal{M}$ induced by j^* is always fully faithful. When $A = A^2$ it is an equivalence of categories, and the quasi-inverse functor is* $j^*M \mapsto \mathrm{Hom}_R(A^{(2)}, M)$.

§6. **$\mathcal{M}$ in the idempotent case.** In this section we assume $A = A^2$, and we discuss properties of the quotient abelian category $\mathcal{M}$. We have seen that there are equivalences $\mathcal{F} \xrightarrow{\sim} \mathcal{M}$ and $\mathcal{C} \xrightarrow{\sim} \mathcal{M}$ induced by j^*. It is natural to regard $\mathcal{M}$ as more fundamental than the other two, since it contains them as full subcategories in the general case. In the idempotent case $\mathcal{F}$ and $\mathcal{C}$ then provide alternative pictures of the basic category $\mathcal{M}$.

In the firm picture the functor j^* and its left adjoint $j_! : j^*M \mapsto A^{(2)} \otimes_R M$ become $M \mapsto A^{(2)} \otimes_R M$ from $\mathrm{Mod}(R)$ to $\mathcal{F}$ and the inclusion in the opposite direction. In the closed picture j^* and its right adjoint j_* become $M \mapsto \mathrm{Hom}_R(A^{(2)}, M)$ from $\mathrm{Mod}(R)$ to $\mathcal{C}$ and the inclusion.

(6.1) The firm picture is convenient for describing colimits in $\mathcal{M}$, while limits are treated better using closed modules. Combining (4.2) with the equivalence $\mathcal{F} \xrightarrow{\sim} \mathcal{M}$, we see that colimits in $\mathcal{M}$ exist, and they are calculated as usual module colimits in the equivalent category $\mathcal{F}$. Similarly limits exist in $\mathcal{M}$, and they are calculated as usual module limits in the closed picture.

The next result shows that the firm (resp. closed) picture is convenient for describing projective (resp. injective) objects in $\mathcal{M}$.

(6.2) **Proposition** *The functor j^* induces an equivalence of categories between firm projective modules and projective objects in $\mathcal{M}$, and between closed injective modules and injective objects in $\mathcal{M}$.*

Consequently, $\mathcal{M}$ has enough injectives by (5.4), but it may not have enough projectives by (2.5).

Proof. Up to isomorphism any object of $\mathcal{M}$ has the form j^*M with M firm, and we have

$$\mathrm{Hom}_R(M, N) = \mathrm{Hom}_{\mathcal{M}}(j^*M, j^*N).$$

Assuming j^*M is projective, the right hand side is an exact functor of N, so M is a projective module. For the converse we use the fact that any short exact sequence in $\mathcal{M}$ is isomorphic to one lifting to a short exact sequence in $\mathrm{Mod}(R)$. It follows that when M is projective, the right hand side is an exact functor of j^*N, and so j^*M is a projective object of $\mathcal{M}$. The injective case is handled similarly. □

(6.3) Roos [R] has characterized abelian categories equivalent to $\mathcal{M}(R, A)$ for some unital ring and idempotent ideal as abelian categories $\mathcal{A}$ having a generator and satisfying the axioms AB4* and AB6 of Grothendieck [Gr]. These axioms mean exactly that $\mathcal{A}$ has the following properties:

- Sums and products exist in $\mathcal{A}$.
- The product of a family of epimorphisms is an epimorphism.
- For any M in $\mathcal{A}$, index set J, and family indexed by $j \in J$ of (increasing) directed sets $\{M_{jk}\}_{k \in K_j}$ of subobjects of M, the canonical map

$$\bigcup_u \bigcap_{j \in J} M_{ju(j)} \xrightarrow{\sim} \bigcap_{j \in J} \bigcup_{k \in K_j} M_{jk}$$

where u runs over $\prod_{j \in J} K_j$, is an isomorphism.

(6.4) We have seen that $\mathrm{Mod}(R/A)$ for an idempotent ideal A is a bilocalizing (i.e. both colocalizing and localizing) subcategory of $\mathrm{Mod}(R)$. In fact, we obtain in this way a one-one correspondences between idempotent ideals of R and such subcategories $\mathcal{S}$.

To see this, we note first that because $\mathcal{S}$ is colocalizing, it is a Serre subcategory closed under products. Indeed, the canonical functor $j^* : \mathrm{Mod}(R) \to \mathrm{Mod}(R)/\mathcal{S}$ admits a left adjoint $j_!$, hence j^* respects limits, and so $\mathcal{S}$, the subcategory of modules killed by j^*, is closed under products.

Since the direct sum for modules is a submodule of the product, $\mathcal{S}$ is closed under direct sums, and hence it is determined by the family of all cyclic modules R/l which are in $\mathcal{S}$. The product of these cyclic modules contains R/A as a submodule, where A is the intersection of these left ideals l. Because $\mathcal{S}$ is closed under products, R/A is in $\mathcal{S}$, and it follows easily that $\mathcal{S}$ is the full subcategory of modules satisfying $AM = 0$. Then A is an ideal, as A kills R/A, and A is idempotent since R/A^2, being an extension of R/A by A/A^2, is in $\mathcal{S}$. Thus $\mathcal{S}$ has the desired form. This argument proves the following.

(6.5) **Proposition** *A Serre subcategory of* $\mathrm{Mod}(R)$ *is bilocalizing iff it is colocalizing iff it is closed under products. There is a one-one correspondence between these subcategories and idempotent ideals given by* $A \mapsto \mathrm{Mod}(R/A)$.

§7. **A-torsion modules.** In this section and the next we discuss what can be said in general about firm and closed modules with respect to an ideal A. For example, the category of closed modules is always abelian, in fact, it is the Grothendieck category arising naturally from a torsion theory on $\mathrm{Mod}(R)$ associated to A. This material is not essential for the rest of the paper, which concerns the idempotent case we have already treated.

We recall that a torsion theory on $\mathrm{Mod}(R)$ may be defined as a Serre subcategory $\mathcal{S}$ which is closed under direct sums in $\mathrm{Mod}(R)$. Modules in $\mathcal{S}$ are the torsion modules for the torsion theory, and a module is torsion-free when the only torsion submodule is zero. Such a subcategory $\mathcal{S}$ is the same as a localizing subcategory. The quotient abelian category $\mathrm{Mod}(R)/\mathcal{S}$ is a Grothendieck category (i.e. having a generator and exact filtered colimits), and in particular it has enough injectives. Furthermore, every Grothendieck category arises in this way by the Gabriel-Popescu theorem.

(7.1) **Definition** Let $\mathcal{T} = \mathcal{T}(R, A)$ be the smallest Serre subcategory of $\mathrm{Mod}(R)$ closed under direct sums and containing R/A, hence all nil modules. A module in $\mathcal{T}$ will be called an A-*torsion* module, or simply a torsion module when the ideal is clear.

If A is essentially idempotent: $A^n = A^{n+1}$ for some $n \geq 1$, then $\mathcal{T} = \mathcal{N} = \mathrm{Mod}(R/A^n)$ is the nil module category for the idempotent ideal A^n, and we have the bilocalizing situation discussed already. On the contrary, if $A^n > A^{n+1}$ for all n, then $\bigoplus_n R/A^n$ is a torsion module and not a nil module, so $\mathcal{N} < \mathcal{T}$ in this case.

Our next result gives some interesting descriptions of torsion modules.

(7.2) **Proposition** *M is torsion iff the following equivalent conditions hold:*
(a) *For any submodule $M' < M$, we have ${}_A(M/M') \neq 0$.*
(b) *There exists a (weakly) increasing filtration of M by submodules M_α for $\alpha \leq \gamma$, where γ is an ordinal, such that $M_0 = 0$ and $M_\gamma = M$, such that $M_{\alpha+1}/M_\alpha$ is killed by A for any $\alpha < \gamma$, and such that $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for any limit ordinal $\alpha \leq \gamma$.*
(c) *$V \otimes_R M = 0$ for all right modules V such that $V = VA$.*
(d) *(T-nilpotence condition) For any $m \in M$ and sequence $a_1, a_2, \ldots$ in A, there is an n such that $a_n \cdots a_1 m = 0$.*

Proof. We first show these conditions are equivalent.

(a) $\Rightarrow$ (b). Consider the canonical filtration constructed using transfinite induction such that $M_0 = 0$, $M_{\alpha+1}/M_\alpha = {}_A(M/M_\alpha)$, and such that M_α is the union of M_β for $\beta < \alpha$ if α is a limit ordinal. Assuming (a) holds, this is a strictly increasing filtration until one has $M_\alpha = M$; such a point must be reached by cardinality reasons.

(b) $\Rightarrow$ (c). We take a filtration as in (b) and prove $V \otimes_R M_\alpha = 0$ by transfinite induction. As $V = AV$, the functor $V \otimes_R -$ kills $M_{\alpha+1}/M_\alpha$. The induction step from α to $\alpha+1$ and the step to a limit ordinal follow from the fact that this functor is right exact and respects colimits.

(c) $\Rightarrow$ (d). Given $a_1, a_2, \ldots$ in A, then

$$F = \operatorname{colim}\left(R \xrightarrow{a_1\cdot} R \xrightarrow{a_2\cdot} \cdots\right) \tag{7.3}$$

is a flat right module such that $F = FA$. We have

$$F \otimes_R M = \operatorname{colim}\left(M \xrightarrow{a_1\cdot} M \xrightarrow{a_2\cdot} \cdots\right).$$

By (c) this vanishes, which implies for any $m \in M$ that $a_n \cdots a_1 m = 0$ for some n, proving (d).

(d) $\Rightarrow$ (a). Assume (a) false, i.e. for some $M' < M$ we have ${}_A(M/M') = 0$. The T-nilpotence condition for M implies the same condition holds for any quotient module, so we can assume ${}_AM = 0$. Choose $m \neq 0$ in M. As $Am \neq 0$ there is an $a_1 \in A$ such that $a_1 m \neq 0$. Then as $Aa_1 m \neq 0$ there is an $a_2 \in A$ such that $a_2 a_1 m \neq 0$. Repeating yields a sequence in A showing (d) is false.

Finally we show these conditions are equivalent to M being torsion. Let $\mathcal{T}'$ be the full subcategory of modules satisfying these conditions. Using the T-nilpotence condition (d), one can check that $\mathcal{T}'$ is a Serre subcategory closed under direct sums in $\mathrm{Mod}(R)$ and containing R/A. This also follows from the fact that $\mathcal{T}'$ consists of those modules killed by the exact functors $F \otimes_R -$ for a family of firm flat right modules F, namely the ones constructed above from sequences in A. Consequently $\mathcal{T}'$ contains the smallest such Serre subcategory $\mathcal{T}$.

On the other hand, since $\mathcal{T}$ is closed under quotient modules and direct sums, any module M has a largest torsion submodule M_t. Moreover we have ${}_A(M/M_t) = 0$, otherwise we could enlarge M_t using the fact that $\mathcal{T}$ is closed under extensions and contains all R/A modules. Consequently, if M is in $\mathcal{T}'$, then (a) implies $M = M_t$ is in $\mathcal{T}$. Thus $\mathcal{T} = \mathcal{T}'$, completing the proof of the proposition. □

(7.4) **Corollary** *Any nonzero torsion module M satisfies ${}_AM \neq 0$. Consequently, a module N is torsion-free iff ${}_AN = 0$.*

(7.5) **Corollary** *One has $\mathcal{T} = \mathrm{Mod}(R)$ iff A is left T-nilpotent, i.e. for every sequence $a_1, a_2, \ldots$ in A, there exists n such that $a_n \cdots a_1 = 0$.*

We now relate torsion modules to closed modules introduced in §5. The next result improves the property (c) of (5.3) for closed modules and shows that $\mathcal{C} = \mathcal{T}^\perp$.

(7.6) **Lemma** *If M is closed, then $\mathrm{Hom}_R(-, M)$ inverts torsion isomorphisms.*

Proof. If Q is a closed injective module, i.e. satisfying ${}_AQ = 0$, then the modules killed by $\mathrm{Hom}_R(-, Q)$ form a Serre subcategory closed under direct sums and containing R/A. Hence this functor kills all torsion modules, and as it is exact, it inverts torsion isomorphisms. By (5.3) any closed module M is the kernel of a map $Q^0 \to Q^1$ between closed injective modules, hence $\mathrm{Hom}_R(-, M)$ inverts torsion isomorphisms. □

We next show that $\mathcal{T}$ is localizing, i.e. for any module M there is a torsion isomorphism $M \to M^\#$ with $M^\#$ in $\mathcal{T}^\perp = \mathcal{C}$. This follows from general theory [Ga], but the argument merits a brief description. Using (5.4) we embed M modulo its largest torsion submodule into a closed injective Q^0, and then embed the cokernel of $M \to Q^0$ modulo its largest torsion submodule into a closed injective Q^1. Then $M^\# = \mathrm{Ker}(Q^0 \to Q^1)$ is closed by (5.3), and the obvious map $M \to M^\#$ is a torsion isomorphism.

Let $\mathcal{M}^t = \mathcal{M}^t(R, A)$ denote the quotient abelian category $\mathrm{Mod}(R)/\mathcal{T}$ and j^* the canonical functor $\mathrm{Mod}(R) \to \mathcal{M}^t$. This preceding discussion yields the following, showing that closed modules always form an abelian category, in fact a Grothendieck category.

(7.7) **Proposition** *One has an equivalence of categories $\mathcal{C} = \mathcal{T}^\perp \xrightarrow{\sim} \mathcal{M}^t$ induced by*

j^*.

We remark that aspects of the closed picture: $\mathcal{C} \overset{\sim}{\rightarrow} \mathcal{M}$ in the idempotent case hold in general with $\mathcal{M}^t$ in place of $\mathcal{M}$. Thus injectives objects of $\mathcal{C} \overset{\sim}{\rightarrow} \mathcal{M}^t$ are closed injective modules, and limits in $\mathcal{C}$ are calculated as usual module limits.

Unless A is essentially idempotent, $\mathcal{M}^t$ is a strictly smaller quotient category of $\mathcal{M}$. For example, if A is T-nilpotent but not nilpotent, then $\mathcal{M}^t = 0$ by (7.5) and $\mathcal{M} \neq 0$. Thus, although $\mathcal{M}^t$ has nice properties, it contains less information about A than $\mathcal{M}$.

§8. Firm modules and $\mathcal{M}^t$.

(8.1) Although the canonical functor $\mathcal{F} \to \mathcal{M}$ is fully faithful, this need not be true when $\mathcal{M}$ is replaced by $\mathcal{M}^t$. For example, let A be a ring which is left T-nilpotent but not right T-nilpotent, and let R be a unital ring containing A as ideal, e.g. $R = \tilde{A}$. The standard example of such an A is infinite strictly upper triangular matrices with finite support over a unital ring. Then $\mathcal{M}^t = 0$ by (7.5), but $\mathcal{F} \neq 0$, since there is a sequence a_n such that $a_1 \cdots a_n \neq 0$ for all n, and hence the colimit of

$$R \xrightarrow{\cdot a_1} R \xrightarrow{\cdot a_2}$$

is a nonzero firm flat module.

On the other hand suppose A is right T-nilpotent but not left T-nilpotent. Then $\mathcal{M}^t \neq 0$ and every right module is torsion, i.e. in $\mathcal{T}(R^{op}, A^{op})$. The latter implies $\mathcal{F} = 0$, because if M nonzero and firm, then $R \otimes_R M \neq 0$ contradicts R being a torsion right module by condition (c) of (7.2).

This shows that in general there is no relation between $\mathcal{M}^t$ and the firm module category $\mathcal{F}$. However, there is a close relation given by tensor product between $\mathcal{M}^t$ and the firm right module category $\mathcal{F}(R^{op}, A^{op})$, which we now discuss.

(8.2) **Definition** A functor from $\mathrm{Mod}(R)$ to the category Ab of abelian groups is said to be *right continuous* when it respects colimits, equivalently, when it respects direct sums (hence is additive) and is right exact.

We recall that there is an equivalence of categories

$$\mathrm{Mod}(R^{op}) \simeq \mathrm{rtcontfun}(\mathrm{Mod}(R), \mathrm{Ab}) \tag{8.3}$$

between right modules and right continuous functors given by $V \mapsto V \otimes_R -$ and $F \mapsto F(R)$. We are going to derive an analogous equivalence with $\mathrm{Mod}(R)$ replaced by the quotient category $\mathcal{M}^t$.

(8.4) **Lemma** *The functor*

$$\mathrm{rtcontfun}(\mathcal{M}^t, \mathrm{Ab}) \to \mathrm{rtcontfun}(\mathrm{Mod}(R), \mathrm{Ab}), \qquad G \mapsto Gj^*$$

gives an isomorphism between the former category and the full subcategory of the latter consisting of functors inverting torsion isomorphisms. Furthermore G is exact iff Gj^* is exact.

Proof. Let $G : \mathcal{M}^t \to \mathrm{Ab}$ be a functor. We check that G is right continuous (resp. exact) iff the same is true for Gj^*. The only if part is clear, as j^* is right continuous and exact. Since any family of objects in $\mathcal{M}^t$ lifts to a family in $\mathrm{Mod}(R)$, it follows that Gj^* respects direct sums if G does. Similarly one sees that Gj^* is right exact (resp. exact) if the same holds for G, using the fact that any short exact sequence in $\mathcal{M}^t$ is isomorphic to the image of a short exact sequence in $\mathrm{Mod}(R)$.

Since $\mathcal{M}^t$ is obtained from $\mathrm{Mod}(R)$ by formally inverting torsion isomorphisms, $G \mapsto Gj^*$ gives a one-one correspondence between functors defined on $\mathcal{M}^t$ and functors inverting torsion isomorphisms defined on $\mathrm{Mod}(R)$. Moreover, using the fact that any map in $\mathcal{M}^t$ is a composition of maps coming from $\mathrm{Mod}(R)$ and inverses of such maps, we have a one-one correspondence between maps of functors $G \to G'$ and $Gj^* \to G'j^*$. This proves the lemma. □

We will need the following the improvement of condition (c) of (2.6).

(8.5) **Lemma** *If V is firm, then $V \otimes_R -$ inverts torsion isomorphisms.*

Proof. Using the fact that a firm module is the cokernel of a map of firm flat right modules and right exactness of tensor product we reduce to the case where V is a firm flat right module. Then $V \otimes_R -$ is exact and kills torsion modules by (7.2), (c), hence this functor inverts torsion isomorphisms. □

(8.6) **Proposition** *One has an equivalence of categories*

$$\mathcal{F}(R^{op}, A^{op}) \simeq \mathrm{rtcontfun}(\mathcal{M}^t(R, A), \mathrm{Ab}).$$

sending V to the unique functor G such that $Gj^* = V \otimes_R -$, and G to $G(j^*R)$. Furthermore, under this equivalence firm flat right modules correspond to exact right continuous functors.

Proof. The second lemma shows that $\mathcal{F}(R^{op}, A^{op})$ is the full subcategory of $\mathrm{Mod}(R^{op})$ consisting of V such that $V \otimes_R -$ inverts torsion isomorphisms. The desired equivalence is obtained by restricting the equivalence (8.3) to this full subcategory and the one described in (8.4). □

(8.7) We conclude this section with some unsolved problems. The following questions have affirmative answers for A idempotent, and it would be interesting to know the answers in the general case.

- Is the category of firm modules always abelian?
- Can (8.6) be turned around to recover $\mathcal{M}^t$ as right continuous functors on firm

right modules, i.e. is the functor

$$\mathcal{M}^t \to \mathrm{rtcontfun}(\mathcal{F}(R^{op}, A^{op}), \mathrm{Ab})$$

given by tensor product an equivalence? This functor is faithful, because the firm flat right modules (7.3) constructed from sequences detect nontorsion modules.

§9. **Independence of the embedding into a unital ring.** In this section we show that the categories $\mathcal{M}, \mathcal{M}^t, \mathcal{F}, \mathcal{C}$ associated to the pair (R, A) depend up to equivalence only on the (nonunital) ring A.

We consider the canonical unital ring homomorphism $\tilde{A} \to R$ extending the inclusion $A \subset R$.

(9.1) **Proposition** *One has an equivalence of categories $\mathcal{M}(\tilde{A}, A) \simeq \mathcal{M}(R, A)$ induced by extension and restriction of scalars with respect to the canonical homomorphism $\tilde{A} \to R$. The same is true for the $\mathcal{M}^t$ categories.*

Proof. Let $F : M \mapsto R \otimes_{\tilde{A}} M$ be the extension of scalars functor from $\mathrm{Mod}(\tilde{A})$ to $\mathrm{Mod}(R)$ associated to this homomorphism. Although F need not be exact, it is exact modulo $\mathcal{N}(R, A)$. Namely, a short exact sequence $M' \to M \to M''$ yields an exact sequence

$$\mathrm{Tor}_1^{\tilde{A}}(R, M'') \to R \otimes_{\tilde{A}} M' \to R \otimes_{\tilde{A}} M \to R \otimes_{\tilde{A}} M'' \to 0$$

where the Tor group is killed by A, since left multiplication by a on R factors through $\tilde{A}$. We note that if M is killed by A, then so is $F(M) = R \otimes_{\tilde{A}} M$, since $a(r \otimes m) = 1 \otimes (ar)m$. Consequently, $j^*F : \mathrm{Mod}(\tilde{A}) \to \mathcal{M}(R, A)$ is exact and kills $\mathcal{N}(\tilde{A}, A)$, so there is a unique functor $\bar{F} : \mathcal{M}(\tilde{A}, A) \to \mathcal{M}(R, A)$ such that $\bar{F}j^* = j^*F$.

Next, let G be the restriction of scalars functor from $\mathrm{Mod}(R)$ to $\mathrm{Mod}(\tilde{A})$. Then G is exact, and it carries $\mathcal{N}(R, A)$ into $\mathcal{N}(\tilde{A}, A)$, so we have a unique $\bar{G} : \mathcal{M}(R, A) \to \mathcal{M}(\tilde{A})$ satisfying $\bar{G}j^* = j^*G$.

The functors F and G are naturally adjoint, where the canonical adjunction maps $\alpha : FG \to 1$ and $\beta : 1 \to GF$ are given by

$$\alpha : R \otimes_{\tilde{A}} N \to N, \qquad r \otimes n \mapsto rn$$

$$\beta : M \mapsto R \otimes_{\tilde{A}} M, \qquad m \mapsto 1_R \otimes m.$$

it is easily seen that α is surjective with kernel killed by A, hence α is a nil isomorphism for any R-module N. Moreover, from the commutative diagram

$$\begin{tikzcd} M \arrow[r, "\beta"] \arrow[d, "a\cdot"'] & R \otimes_{\tilde{A}} M \arrow[d, "a\cdot"] \arrow[dl, "u"'] \\ M \arrow[r, "\beta"] & R \otimes_{\tilde{A}} M \end{tikzcd}$$

where $u(r \otimes m) = (ar)m$, we see that multiplication by a kills the kernel and cokernel of β. Thus β is a nil isomorphism for any $\tilde{A}$-module M, We recall that $\mathcal{M}(R,A)$ has the same objects as $\mathrm{Mod}(R)$ and that every map in the quotient category is a composition of maps coming from module maps and inverses of such maps. It follows that $\alpha : FG \to 1$ determines a unique map of functors $\bar{\alpha}$ from $\bar{F}\bar{G}$ to the identity functor of $\mathcal{M}(R,A)$ satisfying $\bar{\alpha}.j^* = j^*.\alpha$. Similarly $\beta : 1 \to GF$ induces a map $\bar{\beta}$ from the identity functor of $\mathcal{M}(\tilde{A},A)$ to $\bar{G}\bar{F}$. By the preceding paragraph $\bar{\alpha}$ and $\bar{\beta}$ are isomorphisms. This proves the assertion concerning the $\mathcal{M}$ categories. The same arguments work in the $\mathcal{M}^t$ case, once we know that F and G carry $\mathcal{T}(\tilde{A},A)$ and $\mathcal{T}(R,A)$ into each other, and this can be proved using the filtrations given by (7.2), (b). $\square$

The next result shows that various module categories associated to the pair (R,A) depend only on the ring A.

(9.2) **Proposition** *Restricting scalars from R to $\tilde{A}$ gives a one-one correspondence between firm module structures on any abelian group for the pairs (R,A) and $(\tilde{A},A)$. Consequently this functor induces an isomorphism (in particular an equivalence) between the categories of firm modules for the pairs (R,A) and $(\tilde{A},A)$. The same is true for firm flat modules, closed modules, and closed injective modules.*

Proof. Suppose given W in $\mathrm{Mod}(R^{op})$ and M in $\mathrm{Mod}(R)$. If $AM = M$, then we have a canonical isomorphism

$$W \otimes_{\tilde{A}} M \xrightarrow{\sim} W \otimes_R M, \qquad w \otimes_{\tilde{A}} m \mapsto w \otimes_R m. \tag{9.3}$$

Indeed, $wr \otimes_{\tilde{A}} am = wra \otimes_{\tilde{A}} m = w \otimes_{\tilde{A}} ram$ shows that $w \otimes_{\tilde{A}} m$ is R-bilinear, hence we have a map in the opposite direction such that $w \otimes_R m \mapsto w \otimes_{\tilde{A}} m$. In particular $A \otimes_{\tilde{A}} M \xrightarrow{\sim} A \otimes_R M$, so M is in $\mathcal{F}(R,A)$ iff M is in $\mathcal{F}(\tilde{A},A)$.

On the other hand, suppose given M in $\mathcal{F}(\tilde{A},A)$, so that $A \otimes_{\tilde{A}} M \xrightarrow{\sim} M$. The source of this isomorphism has an R-module structure given by $r(a \otimes m) = ra \otimes m$, and so M has an R-module structure given by $r(am) = (ra)m$. It follows that M has a unique R-module structure extending its $\tilde{A}$-module structure. By the preceding paragraph M is in $\mathcal{F}(R,A)$, which proves the part of the proposition concerning firm modules.

When M is $\tilde{A}$-flat, the left side of the isomorphism (9.3) is an exact functor of the right R-module W, hence M is R-flat. Conversely, assume M is flat over R and let $V' \to V \to V''$ be any short exact sequence of $\tilde{A}^{op}$-modules. We have an exact sequence of R^{op}-modules

$$\mathrm{Tor}_1^{\tilde{A}}(V'', R) \to V' \otimes_{\tilde{A}} R \to V \otimes_{\tilde{A}} R$$

where the Tor group is A^{op}-nil, because right multiplication by a on R factors through $\tilde{A}$. Applying the exact functor $- \otimes_R M$, we see that $V' \otimes_{\tilde{A}} M \to V' \otimes_{\tilde{A}} M$ is injective, whence M is flat over $\tilde{A}$. This proves the assertion about firm flat modules.

Similar proofs can be given for closed modules and closed injective modules, e.g.

one replaces (9.3) by

$$\operatorname{Hom}_R(N, M) \xrightarrow{\sim} \operatorname{Hom}_{\tilde{A}}(N, M)$$

if N, M are R-modules and ${}_AM = 0$. However, the essential result that the closed module categories for (R, A) and $(\tilde{A}, A)$ are equivalent can be obtained immediately from the equivalence $\mathcal{C} \xrightarrow{\sim} \mathcal{M}^t$ and the $\mathcal{M}^t$ part of (9.1). The equivalence of the closed injective module categories then follows, since these modules are the injective objects in $\mathcal{C}$. $\square$

(9.4) **Corollary** *If the ideal A is unital as a ring, then the categories $\mathcal{M}, \mathcal{M}^t, \mathcal{F}, \mathcal{C}$ associated to the pair (R, A) are all equivalent to* $\operatorname{Mod}(A)$.

Indeed, these categories agree up to equivalence for (R, A) and (A, A).

We conclude this section by deriving an unpublished result of Wodzicki which completes his treatment of universal flatness [W].

(9.5) **Corollary** *Let A be a ring such that $A = A^2$. Then for every embedding of A as an ideal in a unital ring R it is true that A is a flat R-module, provided that this is true for some embedding of A in a unital ring.*

Indeed, the firm flat part of part of (9.2) says that A is R-flat iff it is $\tilde{A}$-flat, because once A is known to be flat the firmness follows from $A = A^2$. Wodzicki's proof is based on the linear equations criterion for flatness, using $A = A^2$ to replace equations with coefficients in R by equations with coefficients in A.

§10. Reduced modules and the Jacobson radical. In this section we examine another type of module that is in a sense intermediate between firm and closed module. These modules are suggested by the colimit formula (3.4) for maps in $\mathcal{M}$, which yields immediately the formula

(10.1) $$\operatorname{Hom}_R(M, N) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{M}}(j^*M, j^*N), \quad \text{if } M = AM \text{ and } {}_AN = 0.$$

(10.2) **Definition** We say that a module M is *reduced* when ${}_AM = 0$ and $M/AM = 0$, i.e. when M has no nil submodule or quotient module other than 0. Let $\mathcal{R} = \mathcal{R}(R, A)$ be the full subcategory of reduced modules in $\operatorname{Mod}(R)$.

(10.3) **Proposition** *The canonical functor $\mathcal{R} \to \mathcal{M}$ is fully faithful. When A is idempotent, this functor is an equivalence of categories.*

Proof. The first assertion is clear from (10.1). To prove the second, it suffices to show the functor is essentially surjective. Given j^*M in $\mathcal{M}$, let $M' = M/{}_AM$ and $M'' = AM'$. Since $A = A^2$ by hypothesis, we have ${}_AM' = 0$ and $M'' = AM''$. Also ${}_AM'' = 0$ as $M'' \subset M$, so M'' is reduced. Since there are nil isomorphisms $M \to M' \supset M''$, we have $j^*M \simeq j^*M''$. $\square$

The following analogue of (9.2) shows that the category $\mathcal{R}(R, A)$ depends only on the ring A.

(10.4) **Proposition** *Restricting scalars from R to $\tilde{A}$ yields a one-one correspondence between reduced module structures on any abelian group for the pairs (R, A) and $(\tilde{A}, A)$.*

Proof. Given an R-module M, it is clear that M in $\mathcal{R}(R, A)$ iff after restricting scalars M is in $\mathcal{R}(\tilde{A}, A)$. Thus it suffices to show any M in $\mathcal{R}(\tilde{A}, A)$ has a unique R-module structure extending the $\tilde{A}$-module structure. The uniqueness follows from the relation $r(am) = (ra)m$. The existence follows from the fact that M is the image of the map

$$A \otimes_{\tilde{A}} M \to \mathrm{Hom}_{\tilde{A}}(A, M), \qquad a \otimes m \mapsto (a' \mapsto a'am)$$

which is naturally an R-module map. □

(10.5) One reason for introducing reduced modules is that they arise in connection with simple modules and the Jacobson radical.

Let M be a simple (unitary) R-module. Then the submodules AM and ${}_AM$ must be 0 or M, hence M is either killed by A, or we have $AM = M$ and ${}_AM = 0$. Thus a simple module is either killed by A or is reduced.

Next, let M be an object of $\mathcal{R}(R, A)$, equivalently by (10.4), an object of $\mathcal{R}(\tilde{A}, A)$. Clearly M simple over $\tilde{A}$ implies M simple over R. Conversely, assume M simple over R, and let M' be a $\tilde{A}$-submodule of M. Then AM' is an R-submodule, so either $AM' = 0$, whence $M' = 0$ as ${}_AM = 0$, or $AM' = M$, whence $M' = M$. Thus M is simple over $\tilde{A}$. This shows that reduced simple modules for (R, A) and $(\tilde{A}, A)$ are the same in the sense that the analogue of (10.4) holds.

Let us define the Jacobson radical $J(A)$ to be the ideal of $a \in A$ such that $aM = 0$ for all reduced simple modules M for (R, A). The foregoing shows that $J(A)$ depends only on the ring A, so $J(A)$ is well-defined.

Taking A to be the unit ideal R of R, we see that $J(R)$ for a unital ring R is the ideal of $r \in R$ such that $rM = 0$ for all simple R-modules M.

We note that simple $\tilde{A}$-modules are either reduced simple modules for $(\tilde{A}, A)$ or simple $\mathbf{Z}$-modules killed by A. Since $J(\mathbf{Z}) = 0$, it follows that $J(\tilde{A}) = J(A)$.

Now $A \cap J(R)$ is the ideal of $a \in A$ killing all reduced simple modules for (R, A), since the other simple R-modules are killed by A. Thus we have the relation $J(A) = A \cap J(R)$ whenever A is an ideal in a unital ring R. Applying this in the case $R = \tilde{B}$, we obtain Jacobson's theorem [J]: $J(A) = A \cap J(B)$, whenever A is an ideal in a ring B.

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