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NOTE

Edge Characterization Using Normalized Edge Detector

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The behavior of the normalized gradient of the Gaussian edge operator is analyzed over many scales in one and two dimensions. A knowledge of the changes that occur over scale in the output of the operator and the physical conditions that cause these changes is essential for the proper interpretation and application of the results. The behavior of several edge models and combinations of edges is examined. As a result it is shown that the slope of an edge can be estimated very accurately using one small scale. By following the rate of change in the output of the operator as scale changes, an optimal scale can be determined for estimating the width and total contrast of the edge. Results on real images are shown and it is demonstrated that the information obtained by these methods can be used to characterize edge points.

1. INTRODUCTION

While detecting the location of edges is an important step in early image processing, information about the *shapes* of edge profiles is also important in determining the types of physical event creating the edges and in matching problems such as stereo and motion. The types of edge profiles created by different types of events, such as occlusion, convex or concave fold edges, shadows, surface markings, etc. have been discussed by a number of authors [2, 6, 8]. Since Canny's work [4], the edge operator of choice has been the gradient of the Gaussian. He has shown that this approach is very close to an "optimal" edge detector. The equation of the Gaussian is

$$g(x_1, \ldots, x_n, \sigma) = (\sqrt{2\pi}\,\sigma)^{-n} \exp\left(-\frac{(x_1^2 + \cdots + x_n^2)}{2\sigma^2}\right) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-x_i^2}{2\sigma^2}\right),$$

where n is the dimension (usually 1 or 2) and σ is the standard deviation.

Because it is very difficult to identify one scale, or value of σ, which gives the best noise suppression without smoothing out significant features, multiple scales are often used [3, 10, 11, 14]. These authors have combined the information at the different scales to determine which edges were significant. Hildreth [7] used multiple scales for another purpose, to compute characteristics of an edge. This method used the slope of the zero-crossing, a third derivative, at two scales to estimate the slope and width of a ramp, but did not consider interaction with nearby edges. In this paper we demonstrate how use of the gradient of the Gaussian at multiple scales can yield information about edge profiles and the type of interaction with neighboring edges.

When an image smoothed at *one* scale is considered, operations such as nonmaxima suppression and zero-crossing detection are concerned only with the comparative magnitudes of the gradient at different points, or with points where the Laplacian of the Gaussian has zero values. Thus the normalizing factor of the n-dimensional Gaussian $(\sqrt{2\pi}\,\sigma)^{-n}$, is often omitted or replaced by a more convenient scaling factor; e.g., [7, 12, 13]. However, when more than one scale is examined, the choice of factor is important. Clark [5] shows that the contrast of a zero crossing caused by a true edge decreases as σ increases, while that of a zero crossing caused by a gradient minimum increases. The magnitude of the gradient of the Gaussian is an acceptable contrast function; thus it satisfies this condition. However, omitting the $(\sqrt{2\pi}\,\sigma)^{-n}$ term gives a function for which this result does not hold. For this function the magnitude of the response increases and decreases as a result of edge profile and edge interaction. Korn [9] suggests using a two-dimensional gradient of the Gaussian operator which has been normalized by multiplying by a factor of $\sqrt{2\pi}\,\sigma$. He defines the scale of

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an edge to be the scale at which the magnitude of the gradient vector obtained with this operator first reaches its maximum value.

In this paper we separate the two-dimensional operator into the product of a one-dimensional normalized gradient of the Gaussian operator and a one-dimensional Gaussian. We analyze the behavior of the magnitude of the response to the normalized gradient operator as scale changes. This operator is of interest because the maximum response of an ideal step edge is constant for all nonzero values of σ, as shown in Section 3.1. This provides a basis for comparing edge responses at different scales, and makes certain types of information about the edges more accessible. In Section 2 we define the normalized gradient of the Gaussian operator; then in Section 3 we discuss the behavior of idealized edges and combinations of edges under the operator and define the circumstances under which the gradient magnitude will increase or decrease. We also show that with the information derived, a single small scale is sufficient to determine the slope of a ramp edge, and that for isolated edges, the stepsize and width of the edge can be determined by the behavior of the gradient. Further, the stepsize of an edge undergoing interaction with its neighbors, as well as information about the relative parity of the neighbors, can be estimated using the methods developed in this paper. In Section 4 we present simulations of the operator applied to ideal edges and also demonstrate how the theoretical results of Section 3 can be applied to obtain information about edges in real images, while Section 5 presents conclusions and areas for further work.

2. THE NORMALIZED OPERATOR

In the following discussions the prime, ′, always indicates the derivative with respect to x. The derivative of the one-dimensional Gaussian is $g'(x, \sigma) = (-x/\sqrt{2\pi}\,\sigma^3) \exp(-x^2/2\sigma^2)$. The area between the curve and the x-axis is in two parts; that for $x < 0$ is above the axis, while that for $x > 0$ is below. Multiplying the one-dimensional gradient of the Gaussian by the factor $\sqrt{2\pi}\,\sigma$ makes the area of both regions under the curve constant and equal to one. The product of the Gaussian with this normalized gradient operator gives a two-dimensional gradient operator, $\sqrt{2\pi}\,\sigma g'(x, \sigma)g(y, \sigma)$, which gives the gradient in the x-direction and has volume under the surface of 1 for the negative and positive parts. The gradient in the y direction is computed similarly. When two-dimensional gradient magnitudes are discussed in this paper, the sums of the squares of the gradients in the x and the y directions are intended. Since the gradient operator is separable, we can examine the normalized one-dimensional derivative operator to determine the behavior of cross sections of edges in two dimensions. The normalized gradient operator is defined as $G'(x, \sigma) = \sqrt{2\pi}\,\sigma g'(x, \sigma) = (-x/\sigma^2) \exp(-x^2/2\sigma^2)$ and for consistency, $G(x, \sigma) = \sqrt{2\pi}\,\sigma g(x, \sigma) = \exp(-x^2/2\sigma^2)$. Note that for a given $\sigma \neq 0$, the maximum value of $G(x, \sigma)$ is 1 and occurs at $x = 0$. When $\sigma = 0$ and $x \neq 0$, $G(x, 0) = 0$. When $x = 0$, $\lim_{\sigma\to 0} G(0, \sigma) = 1$.

3. IDEAL EDGE MODELS

In this section we examine the behavior of certain ideal edges as the normalized edge operator is applied.

3.1. *Step Edge*

The step edge with stepsize or contrast c at $x = 0$ is represented by the equation

$$U(x) = \begin{cases} 0 & \text{if } x < 0 \\ c & \text{otherwise.} \end{cases}$$

Convolving a step with G' gives $\mathcal{U}'(x, \sigma) = cU(x) * G'(x, \sigma) = cU'(x) * G(x, \sigma) = cG(x, \sigma)$, since $U'(x)$ is the impulse function and convolution with it gives the original function. By the note at the end of the previous section, $\mathcal{U}'(0, \sigma) = c$. The response is independent of σ and its value is the stepsize.

3.2. *Ramp*

A ramp edge is represented by the equation

$$r(x) = \begin{cases} 0 & \text{if } x < 0 \\ mx & \text{if } 0 \le x \le w \\ mw & \text{if } x > w, \end{cases}$$

where m is the slope of the ramp and w is its width. Convolving $r'(x)$ with the normalized Gaussian gives $R'(x, \sigma) = r'(x) * G(x, \sigma) = m \int_{x-w}^{x} G(u, \sigma)\, du$. The integral has its largest value at $x = w/2$, the ramp's midpoint: $R'(w/2, \sigma) = m \int_{-w/2}^{w/2} G(u, \sigma)\, du$. When σ is small enough that most of the support of the Gaussian falls inside the interval $(-w/2, w/2)$, the value of this integral is approximately $m\sqrt{2\pi}\,\sigma$. As σ increases, the value of the gradient increases linearly with σ until σ becomes large enough for the ends of the ramp to be included in the support of the operator. Since 98% of the support of the Gaussian falls in a 5σ interval, the linear behavior is apparent when 5σ is less than the width of the ramp.

Further, $\lim_{\sigma\to\infty} R'(w/2, \sigma) = mw$, which is the total contrast. At $x = w/2$, R' is monotonic with respect to σ. Hence, if a ramp is isolated, the gradient magnitude increases linearly with σ until the ends of the ramp begin to influence the value; then the rate of increase will slow, but the value will continue to increase, approaching the limit $|mw|$. When the behavior of an edge over scale does not display this behavior, it indicates interaction with a nearby edge. In the following sections we see that a neighboring edge of opposite parity causes values to decrease as σ increases, while a neighbor having the same parity causes values to increase at a faster rate.

3.3. *Staircase and Pulse*

The staircase having steps of the same parity at $x = -a$ and $x = a$ and relative heights b $(0 \le b \le 1)$ is given by the equation

$$s_a(x) = U(x + a) + bU(x - a).$$

Convolving with the normalized gradient of Gaussian gives

$$S'_a(x, \sigma) = G(x + a, \sigma) + bG(x - a, \sigma).$$

When σ is small, there is no interaction between the two edges: $S'_a(a, \sigma) \approx b$ and $S'_a(-a, \sigma) \approx 1$. Thus the individual edges behave like isolated step edges. When σ is greater than 0, a valley or phantom edge exists between the two step edges. As σ increases the two edges move together, and the weaker edge and the phantom edge join and disappear, while the stronger one remains [13]. For large values of σ, $\lim_{\sigma \to \infty} S'_a(x, \sigma) = 1 + b$. Thus for large σ, the staircase appears like a single step having stepsize equal to the sum of the separate stepsizes, and its location is $x = a(b - 1)/(b + 1)$ [15]. This behavior is demonstrated for a sample edge in Fig. 1.

A pulse is defined as two neighboring steps of opposite parity. The + in the staircase equations is replaced with a −. Again, for small values of σ the two edges behave as step edges. As the edges begin to interact, they move apart. When σ becomes large, $\lim_{\sigma \to \infty} P'_a(x, \sigma) = 1 - b$; the pulse appears as a single step edge having contrast equal to the difference of the two steps. The stronger edge will approach the location $x = a(b + 1)/(b - 1)$ [15]. The weaker edge approaches the horizontal parabola $x = -\sigma^2 \ln b/2a$ as σ increases [15]. But $P'_a(-\sigma^2 \ln b/2a, \sigma) = 0$; the gradient value of the weak edge approaches 0. This behavior is demonstrated for a sample edge in Fig. 2.

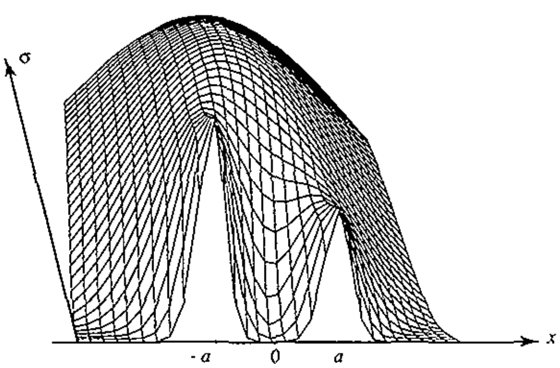

FIG. 1. Three-dimensional plot of gradient magnitude for staircase having $b = 0.5$.

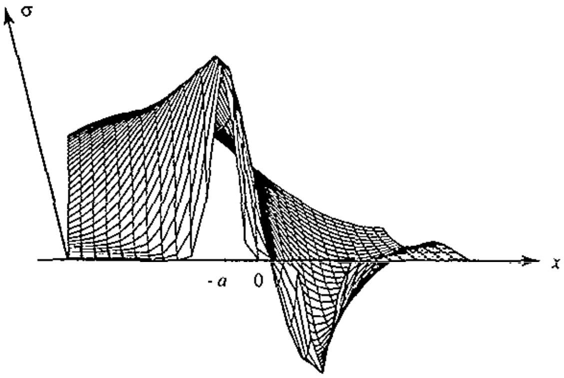

FIG. 2. Three-dimensional plot of gradient magnitude for pulse having $b = 0.5$.

This behavior can be summarized in the following theorem:

CONSERVATION OF CONTRAST. *If two adjacent step edges have contrasts b and c, then the sum of the gradient maxima is b + c when the image is smoothed with the normalized gradient of the Gaussian operator having* $\sigma = 0$. *The gradient maximum for the stronger edge also approaches b + c as* $\sigma \to \infty$. *If b and c have the same sign then the maximum for the weaker edge will disappear for some value of* σ; *if b = c the two maxima will combine to become one. If b and c have opposite signs, then the gradient maximum of the weaker edge approaches* 0. *When* $b = -c$ *the gradient maxima for both edges approach* 0.

It is interesting that although this principle holds for $\sigma = 0$ and ∞, experimental results indicate that it does not hold for intermediate values. As can be seen in Tables 2b and 2c, after the weaker edge has disappeared (in the case of a staircase), or become very small (in the case of a pulse), the magnitude of the stronger edge continues to change, approaching the limiting value. Thus an edge near a stronger one continues to influence the response of the stronger one even at scales at which it cannot itself be detected.

3.4. *Ramp Staircase and Ramp Pulse*

Just as most isolated real edges are approximated more closely by a ramp than by a step, adjacent pairs of edges more closely resemble ramp pairs. We call these pairs ramp staircase and ramp pulse. The equations of a ramp staircase and a ramp pulse with the centers of the ramps at $x = a$ and $x = -a$ are given by

$$rs(x) = r(x + a + w/2) \pm r(x - a + w/2).$$

After convolving with the normalized gradient of Gaussian, the equations are

$$RS'(x, \sigma) = m \int_{x+a-w/2}^{x+a+w/2} G(u, \sigma)\, du$$

$$\pm m \int_{x-a-w/2}^{x-a+w/2} G(u, \sigma)\, du$$

These equations display complex behavior. Each of the individual terms increases, as isolated ramps do, for smaller values of σ. However, edge interaction at larger values of σ causes behavior similar to step edge pairs. In the case of the ramp staircase, the two terms will combine, causing $\partial RS'/\partial\sigma$ at the maxima to increase as the interaction begins. In the case of the ramp pulse, the maximum values of RS' will increase, then decrease. Both of these behaviors can be detected, as shown by the simulations in Section 4.1.

The behavior of the edge models discussed in this section is summarized in Table 1.

4. SIMULATIONS AND EXPERIMENTAL RESULTS

Simulations were performed on a number of synthetic one-dimensional images of the ideal edges examined theoretically. The normalized gradient operator was also applied to real two-dimensional images. For these the slopes of the edges were estimated and the edges were characterized by the behavior of the gradient maxima at different scales.

4.1. *Simulations*

Table 2 details the behavior of examples of the ideal edges described above as the normalized gradient of Gaussian operator is applied. Figure 3 shows graphs of the edges used in the simulation. Values of the gradient for $\sigma = 1$ appear lower than expected due to the fact that the discrete gradient mask does not accurately approximate the continuous one. When a location in the table is given as a closed interval, then the entire interval has the same gradient value.

The first two sections of Table 2a give ramps having the same slope and different widths, to demonstrate how the interaction with the ends of the ramp develop. The estimate of the slope for the narrower ramp is beginning to be affected when $\sigma = w/5 = 2$ with less than 2% of the Gaussian outside the ramp. Using this information, it can be seen that an estimate of the width of an isolated ramp is given by finding the value of σ at which the slope first decreases, then using the formula $w = 5\sigma$. Note that the estimate of the slope for the ramps in Tables 2a, 2d, and 2e, where $w = 10$, are very close to the actual values when $\sigma = 1, 2$, then begin to decrease. The third column in Table 2a is for a ramp with $m = 10$ in the center, steepest portion of the ramp and width approximately 20.

TABLE 1
Summary of Gradient Behavior

Edge type	Max magnitude of normalized gradient: When $\sigma = 0$	Max magnitude of normalized gradient: When $\sigma = \infty$	Location of edge as σ increases	Change in grad. mag. as σ increases
Step	Contrast	Contrast	At step	Constant
Ramp	$m\sqrt{2\pi}\ \sigma = 0$	Contrast	Middle of ramp	Up
Staircase (strong)	Contrast	Sum of contrasts $(1 + b)$	At step, moving to $a(b - 1)/(b + 1)$	Up
Staircase (weak)	Contrast	—	Moves toward 0, then disappears	Up
Pulse (strong)	Contrast	Difference of contrasts, $(1 - b)$	At step, moving to $a(b + 1)/(b - 1)$	Down
Pulse (weak)	Contrast	0	At step, moving to $\sigma^2 \ln b/2a$	Down
Ramp staircase	$m\sqrt{2\pi}\ \sigma = 0$	Sum of contrasts	Same as staircase, a is ramp mid pt.	Up
Ramp pulse	$m\sqrt{2\pi}\ \sigma = 0$	Difference of contrasts	Same as pulse, a is ramp mid pt.	Up then down

Note. In formulas, σ is the scale, m is slope of ramp edges. Step and ramp edges are at 0, pairs of edges in staircases and pulses are at a, $-a$, with the step at $-a$ having contrast 1, that at a having contrast b, $0 < b \le 1$.

TABLE 2
Results of Applying Normalized Gradient to Ideal Edges

a

Ramp, $m = 10$, $w = 20$				Ramp, $m = 10$, $w = 10$				Smoothed ramp, $m = 10$, $w = 20$			
σ	Location	Gradient	m	σ	Loc	Gradient	m	σ	Location	Gradient	m
1	[−7,7]	25	9.97	1	[−2,2]	25	9.97	1	[−4,4]	25	9.97
2	[−3,3]	50	9.99	2	0	49.6	9.89	2	0	50.1	9.99
4	0	98	9.82	4	0	78.9	7.87	4	0	95.6	9.53
8	0	157	7.85	8	0	93.5	4.66	8	0	148.9	7.43
20	0	189	3.77	20	0	97.4	1.94	20	0	185.7	3.70

b

Pulse, location −5, 5								
Stepsize = 100				Stepsizes = 100, −60				
σ	Locations		Gradient	σ	Locations		Gradient	
1	[−6,−5]	[5,6]	±91.2	1	[−6,−5]	[5,6]	91.2	−54.7
2	[−6,−5]	[5,6]	±97.85	2	[−6,−5]	[5,6]	97.9	−58.7
4	−6	6	±97.66	4	−6	6	98.2	−58.0
16	−17	17	±39.8	16	−12	30	58.6	−12.7
20	−21	21	±32.2	30	−17	45	47.5	−.9

c

Staircase, location −5, 5								
Stepsize = 50				Stepsizes = 40, 60				
σ	Locations		Gradient	σ	Locations		Gradient	
1	[−6,−5]	[5,6]	45.6	1	[−6,−5]	[5,6]	36.5	54.7
2	[−6,−5]	[5,6]	48.9	2	[−6,−5]	[5,6]	39.1	58.7
4	−5	5	51.0	4	−5	5	41.4	60.6
8	0	0	78.5	8	—	2	—	79.9
20	0	0	94.7	20	—	1	—	94.9

d

Ramp pulse, loc −10, 10, $m = 10$ contrast = 100, −100				
σ	Locations		Gradient	m
1	[−12,−8]	[8,12]	25	9.97
2	−10	10	49.6	9.89
4	−10	10	78.9	7.87
8	−11	11	88.6	4.42
10	−12	12	84.6	3.37
20	−21	21	54.2	1.08

e

Ramp staircase, loc −10, 10, $m = 5$ contrast each = 50					
σ	Locations		Gradient	Δgrad/$\Delta\sigma$	m
1	[−12,−8]	[8,12]	12.5	—	4.99
2	−10	10	24.8	12.3	4.94
4	−10	10	39.5	7.3	3.94
8	−8	8	50.3	2.7	2.51
12	0	0	70.2	4.9	2.33
16	0	0	80.9	2.7	2.02

The slope estimates for small σ are the same as for the ramp, but begin to decrease sooner.

Table 2b gives results for a pulse with equal and unequal stepsizes. The decreases in the gradient value can be seen as the two steps of opposite parity interact. The difference of the two gradient values in the case where stepsizes are unequal is near 40 for small values of σ where there is little interaction. However, for larger values of σ, the difference of the two becomes greater than 40. When the smaller edge has almost disappeared, the gradient magnitude of the stronger edge continues to decrease. This phenomenon of an undetectable edge continuing to affect a nearby edge can be seen more strikingly in (c), a staircase with equal and unequal steps. The weaker edge combines with the phantom edge between $\sigma = 4$ and $\sigma = 8$ and disappears as a gradient maximum, but is still present as an inflection point in the gradient magnitude graph. For larger values of σ the magnitude of the stronger edge continues to increase and the edge continues to move.

In Table 2d, a pulse composed of ramps rather than steps is given. The gradient value begins low, then when $\sigma = 8$ a maximum estimate of contrast, 88.6, is reached. A more accurate value for contrast could be obtained by first estimating the width of the ramp by the method described above. The value of σ at which the slope estimate begins to decrease is 2. Multiplying by 5 gives an estimated width of 10. This is multiplied by the largest slope estimate (9.97) to give a very good estimate of 99.7 for contrast. Finally, Table 2e gives a staircase composed of ramps rather than steps. Of special interest here is the rate of change of gradient value. Between $\sigma = 8$ and $\sigma = 12$, the rate increases. For an isolated ramp edge the rate of increase should be a decreasing function. The increase indicates that there is interaction with an edge of the same parity. In this case the best estimate of contrast is given by the gradient value at $\sigma = 8$ and is 50.3. Higher scales cause the two edges to be combined.

4.2. *Analysis of Real Images*

Because edges in real images often appear like ramp edges or smoothed ramp edges, applying the smallest practical Gaussian to the image gives an estimate of the slope of the edges. Larger scales should be used to distinguish significant edge points from noise and fine texture.

In order to estimate stepsize, the behavior of edges is analyzed as follows:

1. When a point is identified as an edge at the present scale, search in the direction of the gradient for an edge at the next larger scale.

2. If the direction of the new edge is different or the unnormalized magnitude increases, assume that it is *not* the same edge as the one at the present scale.

3. If an edge is found with similar direction, it is assumed to be the same edge.

   (a) If the magnitude is the same or smaller at the next scale, choose the present scale (the closest neighbor must have opposite parity).

   (b) If the rate of change of magnitude is larger at the next scale, choose the present scale (the closest neighbor must have the same parity).

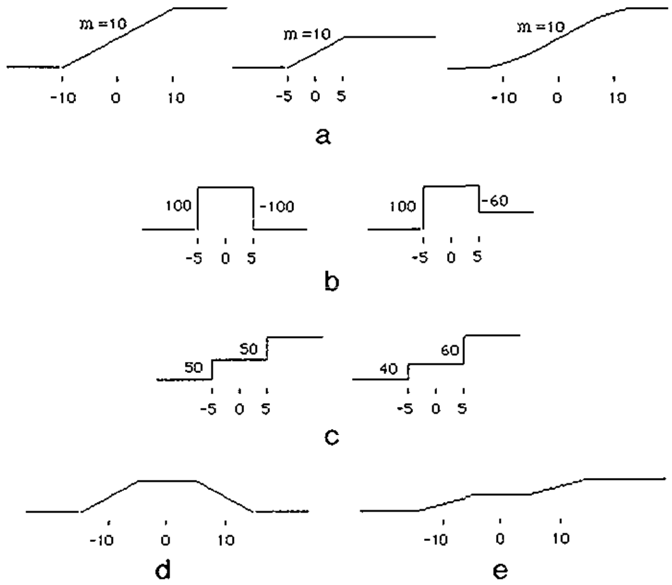

FIG. 3. Graphs of edges in Table 2.

   (c) If the present scale is the largest scale being considered, choose the present scale (the edge is a wide ramp).

   (d) Otherwise, go to the next larger scale.

Edges having small scale are those having nearby edges, which includes much of the noise, and those having a small width. Diffuse edges, for example those due to illumination gradients or shadows, have larger scale.

Whenever multiple scales are to be used, two issues which must be addressed are what the interval separating scales ($\Delta\sigma$) should be and what range of scales should be used. We did not find much benefit from using values of σ below 1. Most of the edges had a scale of 4 or smaller in the images tested, thus this was the largest value used. If too large a value of $\Delta\sigma$ is used, events of interest may be missed. Thus, $\Delta\sigma$ performs as a threshold even though it is not normally identified as such. Its value should be large enough to smooth small irregularities in the shape of a ramp, but not so large as to miss significant edge interactions. If $\Delta\sigma$ is 1, then most edge points do not move more than 1 pixel [1, 15].

An example is given of applying these tests to a real image. In Fig. 4, the original picture is (a), the slope is given in (b), the stepsize in (c) and the scale at which the stepsize was estimated is (d). In the scale image, the smallest scale is brightest. The image is 128×128 pixels. The values of σ used were 1, 2, 3, and 4.

In the upper right part of Fig. 4 there is a shadow edge. Its slope is smaller than the object edge casting the shadow, but its stepsize is close to that of the object. Note also that the slope of the rather isolated shadow

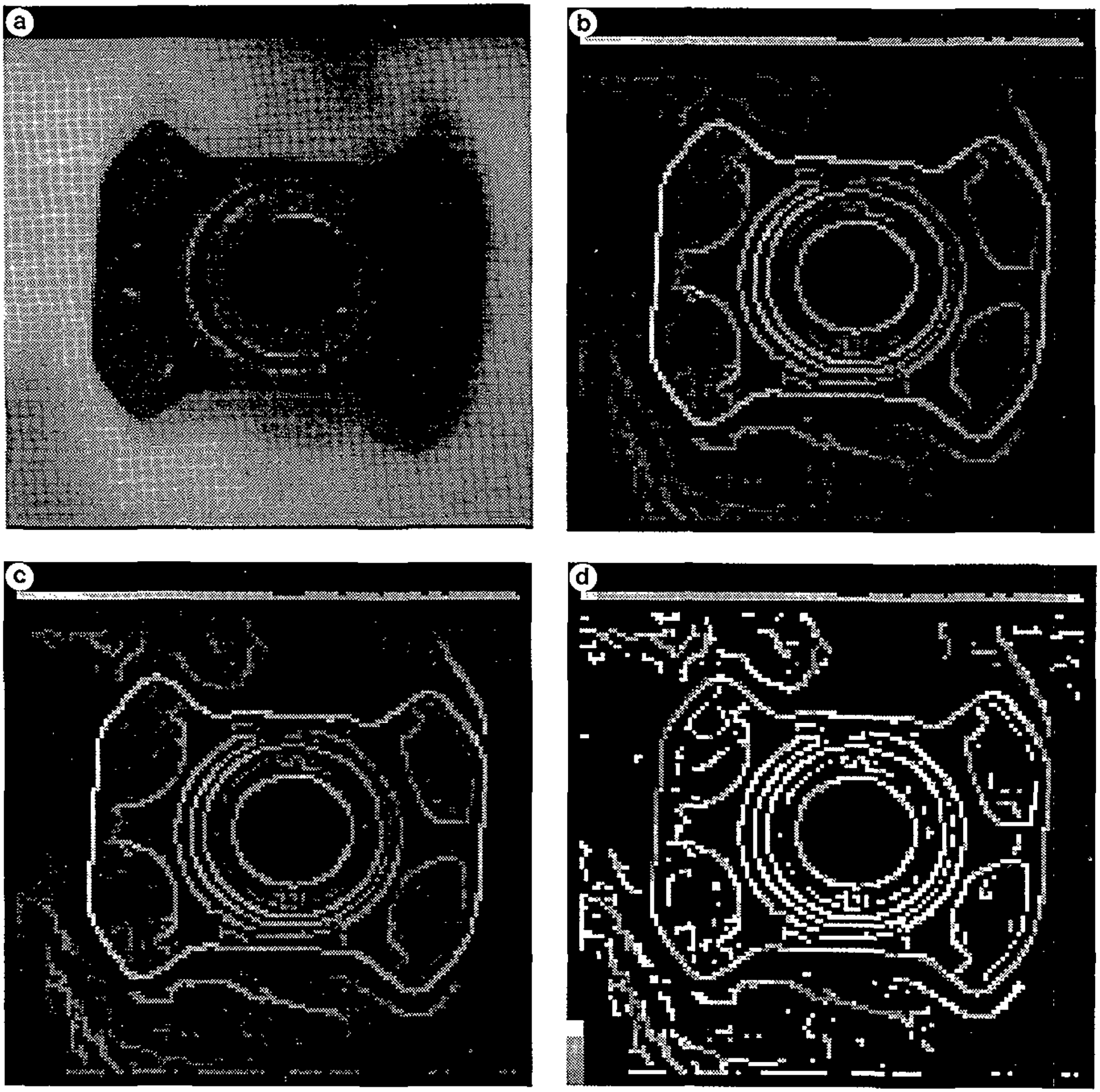

FIG. 4. Part image: (a) original image, (b) slope, (c) stepsize, (d) scale for estimating stepsize.

edge in the middle bottom is small, while the scale chosen to estimate its size is large due to its isolation and width.

5. CONCLUSION

A normalized Gaussian operator, which can be separated into normalized functions in the x and y directions, was presented. The behavior of the gradient values of step and ramp edges and combinations of pairs of these idealized edges obtained using this operator was analyzed. It was shown that the slope of a ramp edge could be estimated using the gradient operator at a single small scale and that the contrast and width of a ramp could also be estimated using only gradient information. In addition, the relative parity and the distance of the nearest neighbor can be estimated. Thus the gradient is shown to contain much more information than is typically used in gradient based edge detectors. Simulations were performed on ideal edges and edge pairs to demonstrate the actual behavior of the operator for intermediate values. The operator was also applied to real images, and slope, contrast, and scale obtained at gradient maxima points were displayed.

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