Variational Principles Cheat Sheet

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1 Principle of Virtual Work

Let F be the forces acting on a system with configuration q. The principle of virtual work states that the system is in equilibrium if and only if

$$F \cdot \delta q = 0$$

where δq are reversible, kinematically admissible variations.

Reference: Lanczos, C. [1986], *Variational Principles of Mechanics*, pp 75-76.

2 d'Alembert's Principle

also known as the **Lagrange-d'Alembert** *principle*

Let F^e be the effective forces on a system, i.e., external forces plus the force of inertia. Then d'Alembert's principle states that

$$F^e \cdot \delta q = 0$$

at any point in time.

Reference: Lanczos, C. [1986], *Variational Principles of Mechanics*, pp 88-90.

3 Gauss' Principle of Least Constraint

Let F_i, m_i, and A_i be the forces, masses, and accelerations of the N particles of a system. Gauss' principle states that the trajectory taken by this system is given by the minimum of the quantity

$$Z = \sum_{i=1}^{N} \frac{1}{2m_i} (F_i - m_i A_i)^2$$

among all paths satisying given kinematical constraints.

Reference: Lanczos, C. [1986], *Variational Principles of Mechanics*, pp 106-110.

4 Hamilton's Principle

Let L be the Lagrangian of a system; Hamilton's principle states that the physical trajectory taken by that system satisfies

$$\delta \int_{t_1}^{t_2} L dt = 0$$

for any pair of times t_1, t_2, where variations are taken with respect to q and are fixed at the endpoints.

4.1 Euler-Lagrange Equations

Hamilton's principle is extremized by curves satisfying

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q},$$

which are called the *Euler-Lagrange equations*.

Reference: Lanczos, C. [1986], *Variational Principles of Mechanics*, pp 111-115.

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5 Hamilton's Phase Space Principle

Let H be the Hamiltonian of a system with configuration q and (generalized) momentum p. Then the trajectory of the system satisfies

$$\delta \int_{t_1}^{t_2} p\dot{q} - H \, dt = 0$$

where the δq are fixed at endpoints.

5.1 Hamilton's Equations

Hamilton's principle is equivalent to *Hamilton's equations of motion*

$$\dot{q} = \frac{\partial H}{\partial p},$$

$$\dot{p} = -\frac{\partial H}{\partial q}.$$

Reference: Marsden, J. E. and T. S. Ratiu [1999], *Introduction to Mechanics and Symmetry*, pp. 224-225.

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6 Hamilton-Pontryagin Principle

Let q, v, and p be the configuration, velocity, and momentum of a system, respectively. The Hamilton-Pontryagin principle states that the trajectory of the system satisfies

$$\delta \int_{t_1}^{t_2} L(q, v, t) + p(\dot{q} - v) \, dt = 0$$

where variations are taken with respect to q, v, and p, but only the endpoints of q are fixed.

6.1 Implicit Euler-Lagrange Equations

The Hamilton-Pontryagin principle imples that the following relationships hold:

$$\dot{p} = \frac{\partial L}{\partial q},$$

$$p = \frac{\partial L}{\partial v},$$

$$\dot{q} = v.$$

Reference: Yoshimura, H. and J. E. Marsden [2006], Dirac Structures and Lagrangian Mechanics Part II: Variational Structures, *J. Geom and Physics* **57**, 209-250.

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7 Lagrange-d'Alembert Principle

Let F be external forces on a system with configuration q and Lagrangian L. The Lagrange-d'Alembert Principle states that the trajectory of the system will satisfy

$$\int_{t_1}^{t_2} L dt + \int_{t_1}^{t_2} F \cdot \delta q \, dt = 0$$

where variations are taken with respect to q and fixed at the endpoints.

7.1 Forced Euler-Lagrange Equations

The equations of motion extremizing the Lagrange-d'Alembert principle are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F.$$

Reference: Bullo, F. and A. D. Lewis [2005], *Geometric Control of Mechanical Systems*, 193-194.

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8 Lagrange-d'Alembert-Pontryagin Principle

also known as the **d'Alembert-Pontryagin** *or* **Pontryagin-d'Alembert** *principle.*

Let q, v, p, and F be the configuration, velocity, momentum, and external forces of a system, respectively. The trajectory satisfies

$$\delta \int_{t_1}^{t_2} L(q, v, t) + p(\dot{q} - v) dt + \int_{t_1}^{t_2} F \cdot \delta q \, dt = 0$$

for variations of q that vanish at endpoints.

Reference: Yoshimura, H. and J. E. Marsden [2006], Dirac Structures and Lagrangian Mechanics Part II: Variational Structures, *J. Geom and Physics* **57**, 209-250.

8.1 Reduced Form

Consider a system whose configuration space G is a Lie group with Lie algebra $\mathfrak{g}$ and Lie coalgebra $\mathfrak{g}^*$, and whose Lagrangian L is left-invariant. Let $g \in G$, $\xi \in \mathfrak{g}$, $\mu \in \mathfrak{g}^*$, and $f \in \mathfrak{g}^*$ be the configuration, body-frame velocity, body-frame angular momentum, and body-fixed forces, respectively. Further, let $\ell(\xi) = L(e, \xi)$. Then the reduced Lagrange-d'Alembert-Pontryagin principle says that the trajectory of the system will satisfy

$$\delta \int_{t_1}^{t_2} \ell(\xi) + \langle \mu, g^{-1}\dot{g} - \xi \rangle dt + \int_{t_1}^{t_2} \langle f, \delta g \rangle dt = 0$$

where variations are taken with respect to g and are fixed at the endpoints. (Note that in the absence of contraints, variations δg are no different from variations δe.)