A combinatorial-topological shape category for polygraphs

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Underlying several advances in the theory of polygraphs, or computads [Str76, Bur93] — what has become a unifying framework for higher-dimensional rewriting [LM09, Mim14, GM16] — there is an analogy with CW complexes: like their topological counterparts, polygraphs are built by progressively adding cells of increasing dimension, pasted along their boundary. For example, in the folk model structure on $\omega\mathbf{Cat}$ of [LMW10], polygraphs have the same role, as cofibrant objects, that CW complexes have in the classical model structure on **Top**.

Whereas in point-set topology the pasting of cells is specified by a point-set map, in the standard theory of polygraphs the same information is supplied through the algebra of strict ω-categories. Unfortunately, this carries over to polygraphs some well-known technical issues of ω-categories, relative to higher-dimensional cells with degenerate boundaries, which become problematic from dimension 3 onwards. In particular,

- the category of polygraphs fails to be a Grothendieck topos [MZ08, Che12], what is commonly considered a benchmark for a good category of spaces [Law92], and
- it lacks a geometric realisation functor with the properties that the analogy would suggest [Sim09, Theorem 4.4.2].

The first issue can be addressed by changing the algebra of pasting in a suitable way, as showed by Batanin [Bat98]; however, for a theory that should serve as a foundation for higher-dimensional algebra, this has the troubling effect that its basic objects become reliant on an external higher-algebraic formalism.

In [Had17b], I furthermore suggested that tensor products and quotients of polygraphs, modelled on topological operations, can be used to introduce a feature of compositionality into categorical universal algebra and rewriting. The original framework, however, suffered from the usual ω-categorical degeneracies,

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affecting my "smash product" construction, and from the difficulty of computing tensor products: the simplest way, introduced by Steiner [Ste04], still relies on covering a polygraph with suitably loop-free polygraphs, and taking a detour through the formalism of augmented directed complexes.

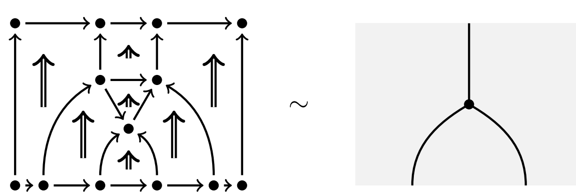
The problems are in fact related: if polygraphs formed a presheaf category $[\mathbf{S}^{\mathrm{op}}, \mathbf{Set}]$ on some shape category $\mathbf{S}$, and $\mathbf{S}$ had (easily computable) tensor products, we could canonically define a monoidal biclosed structure on $[\mathbf{S}^{\mathrm{op}}, \mathbf{Set}]$ by Day convolution [Day70]. This leads to the question: is there a restriction on the shapes of cells of polygraphs that is "harmless enough", and combinatorial in nature, yet produces a shape category with the desired properties?

Since I am interested in modelling higher-dimensional string diagrams, and especially comparing the result with the low-dimensional algebra of *Globular* [BKV16], my notion of "harmless enough" forbids any upper bound on the number of inputs or outputs of a cell. This excludes basically all shape categories in use for higher categories, with the possible exception of Batanin cells [MZ01], but including cubes, the one that is closed under tensor products [AABS02].

Several "strongly loop-free" classes of shapes, considered at various points in the literature [Joh89, KV91, Ste04] are also too restrictive, for they bar the shapes of Frobenius and adjunction axioms [Pow91], both motivating examples for diagrammatic reasoning.

My approach is to take the analogy with CW complexes one step further, by restricting to those cell shapes whose input and output k-boundary, for all k, is *homeomorphic* (through the geometric realisation) to a topological k-disk, if possible without any further restriction. In the lowest-dimensional non-trivial example, the allowed atomic 2-dimensional cell shapes are those with a sequence of n input 1-cells and m output 1-cells, for $n, m > 0$; only the cases $n = 0$ and $m = 0$ are barred.

Superficially, this cannot model 0-ary operations in diagrammatic algebra. In fact we can still directly interpret any string diagram in the presence of suitably defined weak unit cells [Sim09, JK07], with the understanding that "regions of space" are interpreted as weak unit cells, that is, string diagrams are "pasting diagrams filled up with unit cells":



What follows is a report of my progress; all proofs will be published in my thesis [Had17a], which will be made available before the workshop. The definitions are based on ideas of poset topology [Wac06, Koz08], and in particular the characterisation of incidence posets of regular CW complexes in [Bjö84]. First, I recall some standard poset terminology.

Definition 1. Let X be a finite poset with order relation $<$. For all elements $x, y \in X$, y *covers* x if $x < y$ and, for all $y' \in X$, if $x < y' \leq y$, then $y' = y$.

The directed graph HX with X as set of vertices, and an edge $c_{y,x} : y \to x$ for all pairs y, x such that y covers x, is called the *Hasse diagram* of X.

Let $X_\bot$ be X extended with a least element $\bot$; X is *graded* if, for all $x \in X$, all paths from x to $\bot$ in the Hasse diagram $HX_\bot$ have the same length. In this case, if n is the length of paths from x to $\bot$, let $\dim(x) := n - 1$, the *dimension* of x, and $X_n := \{x \in X \mid \dim(x) = n\}$.

A subset U of a poset X is *closed* when, for all $x, y \in X$, $y \in U$ and $x \leq y$ implies $x \in U$. Given any subset U of X, its *closure* is the closed subset $\mathrm{cl}(U) := \{x \in X \mid \exists y \in U\, x \leq y\}$. For all $x \in X$, let $U_x := \mathrm{cl}\{x\}$.

If X is graded, a closed $U \subseteq X$ is *pure* if $U = \mathrm{cl}(U \cap X_n)$; in that case, let $\dim(U) := n$.

Definition 2. Let X be a finite poset. An *orientation* on X is a labelling of edges of HX with elements of $\{+, -\}$, that is, a function $o : HX_1 \to \{+, -\}$, where HX_1 is the set of edges of HX. The orientation extends to $X_\bot$ by $o(c_{x,\bot}) := +$ for all x of dimension 0. An finite poset with an orientation is an *oriented poset.*

Suppose X is graded and oriented, and $U \subseteq X$ is a pure subset with $\dim(U) = n$. For $\alpha \in \{+, -\}$, let

$$\begin{aligned}\Delta^\alpha U := \{x \in U \mid\ & \dim(x) = n - 1 \text{ and, for all } y \in U, \\ & \text{if } y \text{ covers } x, \text{ then } o(c_{y,x}) = \alpha\},\end{aligned}$$

and $\partial^\alpha U := \mathrm{cl}(\Delta^\alpha U)$, $\partial U := \partial^+ U \cup \partial^- U$.

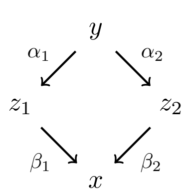
An oriented graded poset is essentially what Steiner called a directed precomplex [Ste93]. Elements x of the poset with $\dim(x) = n$ correspond to n-dimensional cells, and if y covers x, and $o(c_{y,x}) = +$ (respectively, $-$), then x is in the output (respectively, input) boundary of y.

The conditions involved in the combinatorial characterisation of incidence posets of regular CW complexes [Bjö84, Proposition 4.5] are

1. *thinness*, a local condition which essentially imposes that cells be manifold-like, and
2. a version of *shellability* , a global condition, preventing cells from having globally non-spherical (for example, toroidal) boundaries.

The first has the following oriented analogue.

Definition 3. An oriented graded poset X is *thin* if all intervals $[x, y]$ of length 2 in $X_\bot$ are of the form



in the labelled Hasse diagram $HX_\bot$, where $\alpha_1\beta_1 = -\alpha_2\beta_2$, with sign multiplication defined in the usual way: $++, -- := +$, and $+-, -+ := -$.

Shellability, on the other hand, can be reimagined in the oriented context as a kind of sequential, pairwise composability of cells in the boundary of another cell.

Definition 4. Let X be an oriented thin poset. The class of *globes* in X is defined inductively on dimension and number of maximal elements, as follows. For all $x \in X$, $\dim(x) = 0$, $\{x\}$ is a 0-dimensional globe.

For all $x \in X$, $\dim(x) = n > 0$, U_x is an *atomic* n-globe if $\partial^\alpha U_x$ is an $(n-1)$-dimensional globe, $\alpha \in \{+,-\}$.

Given two pure, n-dimensional $U, U' \subseteq X$, U and U' are *mergeable* if

1. $U \cap U' = \partial^\alpha U \cap \partial^{-\alpha} U'$ for some $\alpha \in \{+,-\}$;
2. $U \cap U'$ is an $(n-1)$-globe;
3. $\partial^\beta (U \cup U')$ is an $(n-1)$-globe, for $\beta \in \{+,-\}$.

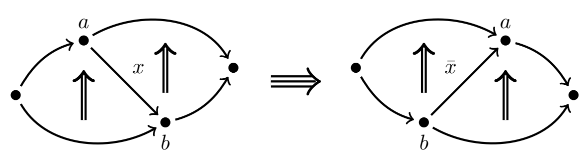
Then, a pure n-dimensional U is an n-globe if it is atomic, or if there exists a non-trivial bi-partition $\{x_{1,1}, \ldots, x_{1,p}\}$, $\{x_{2,1}, \ldots, x_{2,q}\}$ of its n-dimensional elements such that

$$U_1 := \mathrm{cl}\{x_{1,1}, \ldots, x_{1,p}\} \quad \text{and} \quad U_2 := \mathrm{cl}\{x_{2,1}, \ldots, x_{2,q}\}$$

are mergeable n-globes.

Definition 5. An oriented thin poset X is a *globular poset* if, for all $x \in X$, U_x is a globe.

Example 6. The following pasting diagram does not correspond to a parity complex in the sense of [Str91], nor to a pasting scheme with no direct loops in the sense of [Joh89], due to the presence of the loop $(a, x, b, \bar{x}, a)$:



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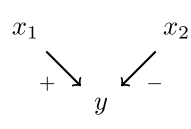
However, it does correspond to a valid globular poset (an atomic 3-globe).

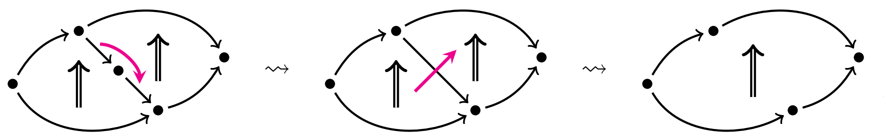
The following notion of composition is implied in the definition of globe. Suppose that there are n-dimensional elements x_1 and x_2 with the following property: $U_{x_1} \cap U_{x_2} = U_y$ for some $(n-1)$-dimensional y, only covered by x_1 and x_2, and



in the labelled Hasse diagram of X. Let X' be the poset obtained from X by identifying the elements x_1, x_2, and y; we say X' is obtained from X by a *simple merger*.

If X is thin, X' inherits an orientation that makes it thin, and with the right choice of elements, if X is globular, so is X'. Induction on sequences of simple mergers is the main technique used in proving most of the following statements.

Example 7. The following is a sequence of simple mergers on a 2-globe, depicted by pasting diagrams, the coloured arrow pointing from x_1 to x_2:



Theorem 8. *Let X be an n-globe, $n > 1$. Then, for $\alpha = \{+, -\}$,*

$$\partial^\alpha(\partial^+ X) = \partial^\alpha(\partial^- X).$$

To my knowledge, this is the first definition of a type of "pasting presentation" that does not assume Theorem 8 as an axiom, in one form or another.

Posets have a standard notion of geometric realisation $X \mapsto |X|$, composing the simplicial nerve of a poset with the geometric realisation of a simplicial set; we can apply it to oriented posets, simply forgetting the orientation.

Theorem 9. *Let X be an n-globe. Then $|X|$ is homeomorphic to an n-disk, and $|\partial X|$ is homeomorphic to an $(n-1)$-sphere.*

Corollary 10. *Let X be a globular poset. Then the underlying poset of X is the incidence poset of a regular CW complex.*

The proofs are based on the fact that simple mergers of globular posets induce homeomorphisms of geometric realisations.

Definition 11. Let X, Y be oriented posets. The *tensor product* $X \otimes Y$ of X and Y is the graded poset $X \times Y$, oriented as follows: write $x \otimes y$ for an element (x, y) of $X \times Y$; then, for all x' covered by x in X, y' covered by y in Y, let

$$\begin{aligned} o(c_{x\otimes y, x'\otimes y}) &:= o_X(c_{x,x'}), \\ o(c_{x\otimes y, x\otimes y'}) &:= (-1)^{\dim(x)} o_Y(c_{y,y'}). \end{aligned}$$

Theorem 12. *Let X, Y be oriented posets. Then,*

1. *if X, Y are thin, $X \otimes Y$ is thin;*
2. *if X, Y are globular posets, $X \otimes Y$ is a globular poset.*

There may be other interesting notions of morphism of globular posets, but so far I have only considered the category $\mathbf{GlobPos}_\subset$ whose morphisms are closed embeddings of the underlying posets that also preserve the orientation. Tensor products induce a monoidal structure on $\mathbf{GlobPos}_\subset$.

Proposition 13. *There is a monoidal functor* $D : \mathbf{GlobPos}_{\subset} \to \mathbf{ADC}$*, where* $\mathbf{ADC}$ *is the category of augmented directed complexes of [Ste04]. For all globular posets* X*,* DX *has a unital basis whose elements are the elements of* X*.*

In fact, I conjecture that any globular poset X is a directed complex in the sense of [Ste93]; a proof would involve connecting the "simple merger" composition to ω-categorical algebra. This would also imply that a globular poset presents an ω-category generated by its elements: this is is an important open problem, that I am still investigating.

Since atomic globes, in particular, are closed under tensor products, they form a suitable class of shapes by the criteria discussed earlier.

Definition 14. Let $\mathbf{RG}$ be a skeleton of the full subcategory of $\mathbf{GlobPos}_{\subset}$ whose objects are atomic globes. A *regular polygraph* X is a presheaf $X : \mathbf{RG}^{\mathrm{op}} \to \mathbf{Set}$. A *map* $f : X \to Y$ of regular polygraphs is a morphism of presheaves.

The *tensor product* $X \otimes Y$ of two regular polygraphs X, Y is their Day convolution with respect to the tensor product of globular posets. The tensor product defines a monoidal (in fact, monoidal biclosed) structure on the category **RPol** of regular polygraphs and maps.

Remark 15. The definition of **RG** as a skeleton is not very satisfactory; ideally, we would want an inductive enumeration of the isomorphism classes, akin to the definition of opetopes in terms of zoom complexes [KJBM10].

The shape category **RG** contains the category **G** of globes as a full subcategory; as a consequence, any regular polygraph X restricts to a globular set GX. Similarly to the opetopic definition of weak higher categories, one can impose on X various "representability" conditions, in the sense of Hermida [Her00], inducing coherent higher algebraic structure on GX.

At the moment, only the low-dimensional cases are fully worked out. In particular, there is a notion of 0-representability, producing a certain type of equivalence 1-cells, which subsumes the algebraic notions of

1. Saavedra unit [Koc08], for regular 2-polygraphs with an algebraic composition of 2-cells, and

2. Joyal-Kock weak unit [JK13], for regular 3-polygraphs with an algebraic composition of 2-cells and 3-cells.

The first case, combined with an analogous notion of 1-representability, suffices to reconstruct the full algebraic theory of bicategories. The main ideas involved seem to generalise, and the theory in arbitrary dimensions is under development.

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