Approximate Counting and Sampling

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Simons Bootcamp

The Amazing Collaborators

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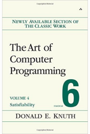
Special shout out to Mate Soos, maintainer of ApproxMC and UniGen

The Tale of Triumph of SAT Solvers

Modern SAT solvers are able to deal routinely with practical problems that involve millions of variables, although such problems were regarded as hopeless just a few years ago. (Donald Knuth, 2016)



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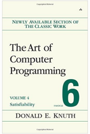

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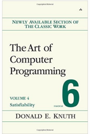

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Now that SAT is "easy", it is time to look beyond satisfiability

Counting and Sampling

- Given
  - Boolean variables $X_1, X_2, \cdots X_n$
  - Formula F over $X_1, X_2, \cdots X_n$
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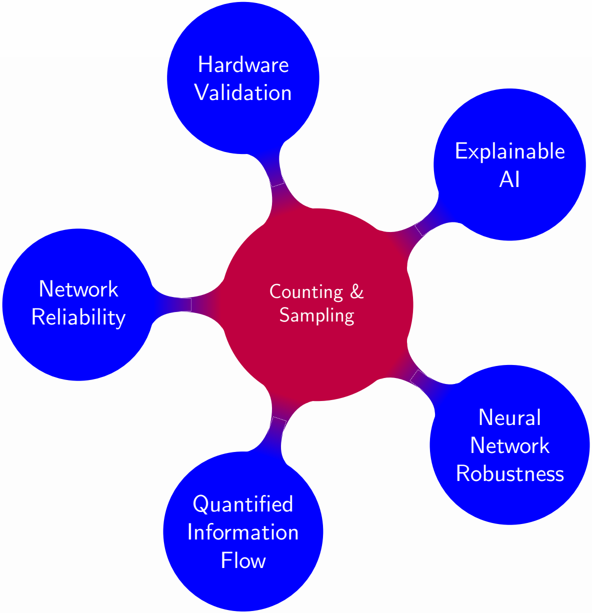
- Given
  - $F := (X_1 \vee X_2)$
- $\mathsf{Sol}(F) = \{(0,1),(1,0),(1,1)\}$
- $|\mathsf{Sol}(F)| = 3$

Applications across Computer Science



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Obs 3 Memoryfulness

- Incremental Solving: Often easier to solve F followed by G if we G can be written as $G = F \wedge H$
- If $F \rightarrow C$ then $(F \wedge H) \implies C$

Constrained Counting

Constrained Counting Hashing Framework

The Rise of Hashing-based Approach: Promise of Scalability and Guarantees
(S83,GSS06,GHSS07,CMV13b,EGSS13b,CMV14,CDR15,CMV16,ZCSE16,AD16 KM18,ATD18,SM19,ABM20,SGM20)

Today's Menu

Constrained Counting

Constrained Sampling

Hashing Framework

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How many people in Berkeley like coffee?

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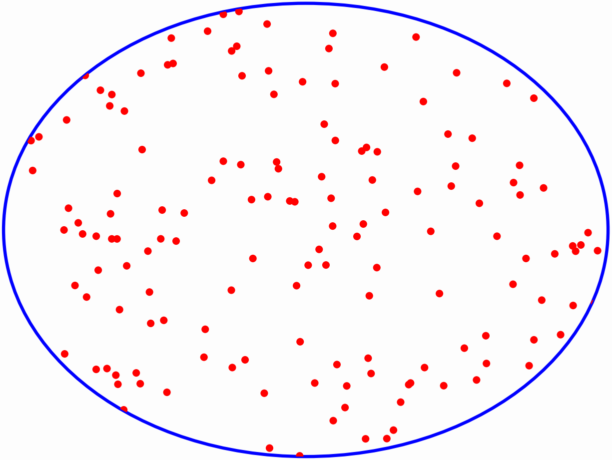
  - Potentially 2^n queries

Can we do with lesser # of SAT queries – $\mathcal{O}(n)$ or $\mathcal{O}(\log n)$?

As Simple as Counting Dots



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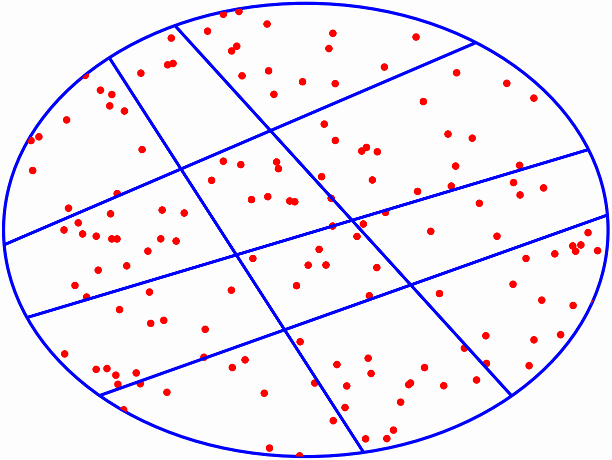
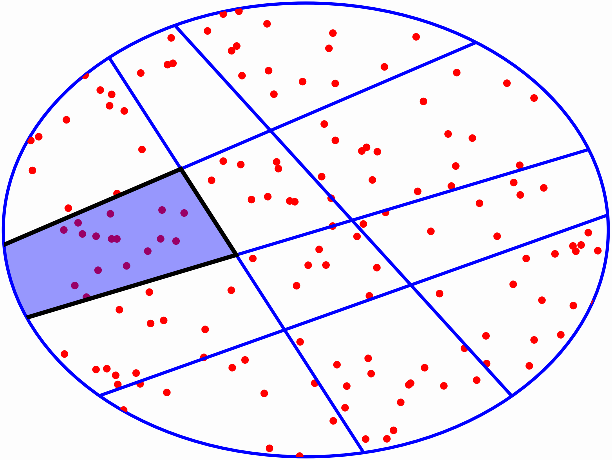


As Simple as Counting Dots

Pick a random cell



Estimate = Number of solutions in a cell × Number of cells

Challenges

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- Solutions in a cell α: $\mathsf{Sol}(F) \cap \{y \mid h(y) = \alpha\}$
- Deterministic h unlikely to work
- Choose h randomly from a large family H of hash functions

  Universal Hashing (Carter and Wegman 1977)

2-wise independent Hashing

- Let H be family of 2-wise independent hash functions mapping $\{0,1\}^n$ to $\{0,1\}^m$

$$\forall y_1, y_2 \in \{0,1\}^n, \alpha_1, \alpha_2 \in \{0,1\}^m, h \xleftarrow{R} H$$

$$\Pr[h(y_1) = \alpha_1] = \Pr[h(y_2) = \alpha_2] = \left(\frac{1}{2^m}\right)$$

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- The power of 2-wise independentity
  - Z be the number of solutions in a randomly chosen cell
  - $\mathsf{E}[Z] = \frac{|\mathsf{Sol}(F)|}{2^m}$
  - $\sigma^2[Z] \leq \mathsf{E}[Z]$

2-wise independent Hash Functions

- Variables: $X_1, X_2, \cdots X_n$
- To construct $h : \{0,1\}^n \rightarrow \{0,1\}^m$, choose m random XORs
- Pick every X_i with prob. $\frac{1}{2}$ and XOR them
  - $X_1 \oplus X_3 \oplus X_6 \cdots \oplus X_{n-2}$
  - Expected size of each XOR: $\frac{n}{2}$

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- Solutions in a cell: $F \wedge Q_1 \cdots \wedge Q_m$

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- Solutions in a cell: $F \wedge Q_1 \cdots \wedge Q_m$
- Performance of state of the art SAT solvers degrade with increase in the size of XORs (SAT Solvers != SAT oracles)

Improved 2-wise Independent Hash Functions

- Not all variables are required to specify solution space of F
  - $F := X_3 \iff (X_1 \vee X_2)$
  - X_1 and X_2 uniquely determines rest of the variables (i.e., X_3)
- Formally: if I is independent support, then $\forall \sigma_1, \sigma_2 \in \mathsf{Sol}(F)$, if σ_1 and σ_2 agree on I then $\sigma_1 = \sigma_2$
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Algorithmic procedure to determine I?

- FP^{NP} procedure via reduction to Minimal Unsatisfiable Subset
- Two orders of magnitude runtime improvement

(IMMV; CP15, Constraints16)

Handling CNF+XOR Formulas

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- CNF + Sparse XORs are still CNF+XOR formulas.
- Translating XORs to CNF and performing CDCL is not sufficient
  - XORs can be solved by Gaussian elimination
- CryptoMiniSAT: Solver designed to perform CDCL and Gaussian Elimination in tandem (SNC09; SM19, SGM20)
- BIRD (Blast, Inprocess, Recover, and Detach): Tighter integration

Challenges

Challenge 1 How to partition into roughly equal small cells of solutions without knowing the distribution of solutions?

- Independent Support-based XORs
- Specialized CNF Solvers

Challenge 2 How many cells?

Challenge 3 What is exactly a *small cell* ?

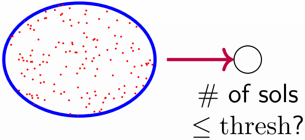
Challenge 2: How many cells?

- We want to partition into 2^{m^*} cells such that $2^{m^*} = \frac{|\mathsf{Sol}(F)|}{\mathrm{thresh}}$

Challenge 2: How many cells?

- We want to partition into 2^{m^*} cells such that $2^{m^*} = \frac{|\mathsf{Sol}(F)|}{\mathrm{thresh}}$
  - Check for every $m = 0, 1, \cdots n$ if the number of solutions $\leq \mathrm{thresh}$

ApproxMC



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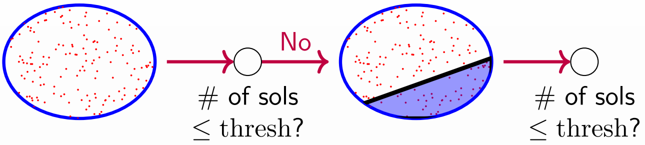

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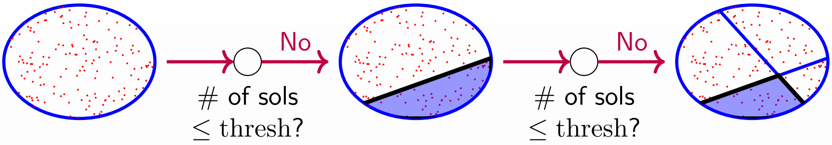

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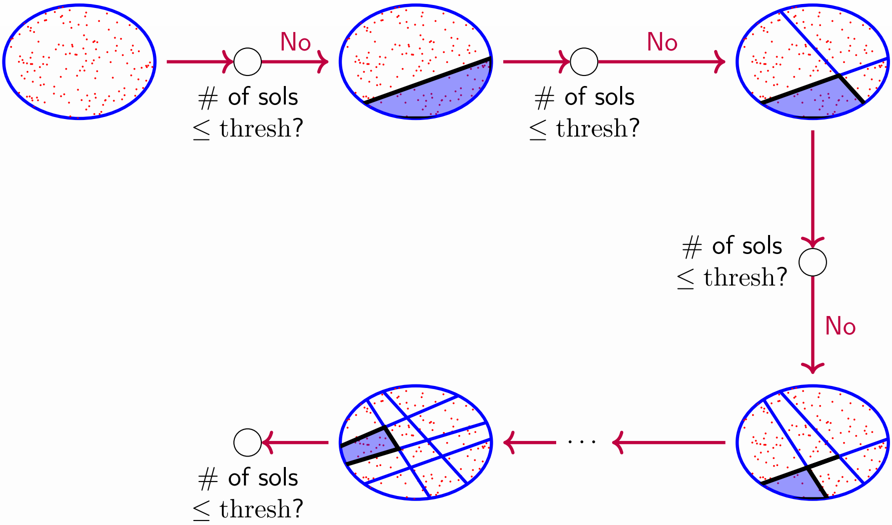
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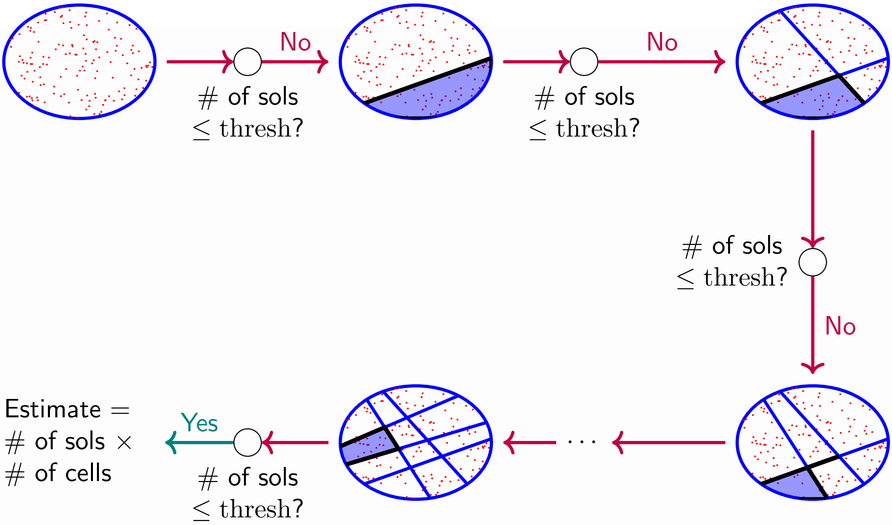
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ApproxMC

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  - $\cdots$
  - Query n: Is $\#(F \wedge Q_1 \wedge Q_2 \cdots \wedge Q_n) \leq \text{thresh}$
- Stop at the first m where Query m returns YES and return estimate as $\#(F \wedge Q_1 \wedge Q_2 \cdots \wedge Q_m) \times 2^m$
- Observation: $\#(F \wedge Q_1 \cdots \wedge Q_i \wedge Q_{i+1}) \leq \#(F \wedge Q_1 \cdots \wedge Q_i)$
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- Will this work? Will the "m" where we stop be close to m^*?

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  - Query 1: Is $\#(F \wedge Q_1) \leq \mathrm{thresh}$
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  - $\cdots$
  - Query n: Is $\#(F \wedge Q_1 \wedge Q_2 \cdots \wedge Q_n) \leq \mathrm{thresh}$
- Stop at the first m where Query m returns YES and return estimate as $\#(F \wedge Q_1 \wedge Q_2 \cdots \wedge Q_m) \times 2^m$
- Observation: $\#(F \wedge Q_1 \cdots \wedge Q_i \wedge Q_{i+1}) \leq \#(F \wedge Q_1 \cdots \wedge Q_i)$
  - If Query i returns YES, then Query $i+1$ must return YES
  - Logarithmic search (# of SAT calls: $\mathcal{O}(\log n)$)
  - Incremental Search
- Will this work? Will the "m" where we stop be close to m^*?
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  - Independence crucial to analysis (Stockmeyer 1983, $\cdots$)

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  - Independence crucial to analysis (Stockmeyer 1983, $\cdots$)
  - Key Insight: The probability of making a bad choice of Q_i is very small for $i \ll m^*$

(CMV, IJCAI16)

Taming the Curse of Dependence

Let $2^{m^*} = \frac{|\mathsf{Sol}(F)|}{\mathrm{thresh}}$ ($m^* = \log(\frac{|\mathsf{Sol}(F)|}{\mathrm{thresh}})$)

Lemma (1)

ApproxMC terminates with $m \in \{m^* - 1, m^*\}$ with probability ≥ 0.8

Lemma (2)

For $m \in \{m^* - 1, m^*\}$, estimate obtained from a randomly picked cell lies within a tolerance of ε of $|\mathsf{Sol}(F)|$ with probability ≥ 0.8

Repeat $\mathcal{O}(\log(1/\delta))$ times and return the median

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- A cell is small cell if it has $\approx$ thresh solutions.
- Approach 1: thresh $= \mathrm{constant} \rightarrow$ 4-factor approximation
  - From 4 to 2-factor

    Let $G = F_1 \wedge F_2$ (i.e., two identical copies of F)

$$\frac{|\mathsf{Sol}(G)|}{4} \leq C \leq 4 \cdot |\mathsf{Sol}(G)| \implies \frac{|\mathsf{Sol}(F)|}{2} \leq \sqrt{C} \leq 2 \cdot |\mathsf{Sol}(F)|$$

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Techniques based on thresh $= \mathcal{O}(\frac{1}{\varepsilon^2})$, despite worse complexity, e.g., ApproxMC scale significantly better than those based on thresh $= \mathrm{constant}$.

The performance of SAT solvers depend on the formulas

ApproxMC

Theorem (Correctness)

$$\Pr\left[\frac{|\mathsf{Sol}(F)|}{1+\varepsilon} \leq \mathit{ApproxMC}(F,\varepsilon,\delta) \leq |\mathsf{Sol}(F)|(1+\varepsilon)\right] \geq 1-\delta$$

Theorem (Complexity)

ApproxMC(F,ε,δ) *makes* $\mathcal{O}(\frac{\log n \log(\frac{1}{\delta})}{\varepsilon^2})$ *calls to SAT oracle.*

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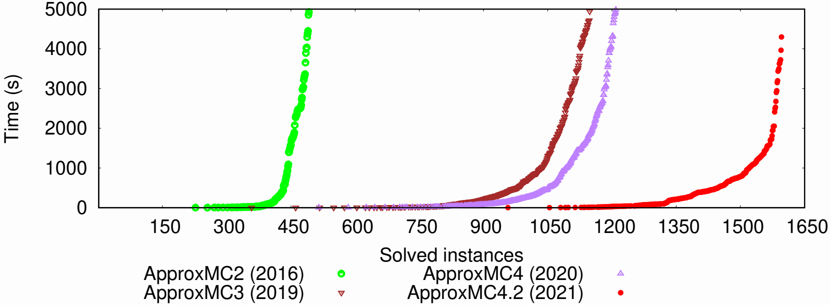
Theorem (FPRAS for DNF; (MSV, FSTTCS 17; CP 18, IJCAI-19))

If F is a DNF formula, then ApproxMC is FPRAS – different from the Monte-Carlo based FPRAS for DNF (Karp, Luby 1983)

Improvements Over the Years



Today's Menu

Constrained Counting✓ Hashing Framework✓

Constrained Sampling

Constrained Sampling

- Given:
  - Set of Constraints F over variables $X_1, X_2, \cdots X_n$
- Uniform Sampler

$$\forall y \in \mathsf{Sol}(F), \Pr[\text{y is output}] = \frac{1}{|\mathsf{Sol}(F)|}$$

- Almost-Uniform Sampler

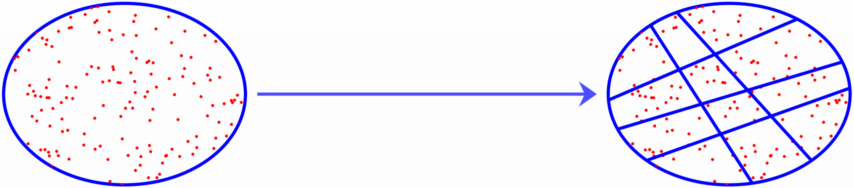
$$\forall y \in \mathsf{Sol}(F), \frac{1}{(1+\varepsilon)|\mathsf{Sol}(F)|} \leq \Pr[\text{y is output}] \leq \frac{(1+\varepsilon)}{|\mathsf{Sol}(F)|}$$

- Approximate counting and almost-uniform sampling are inter-reducible (Jerrum, Valiant and Vazirani, 1986)

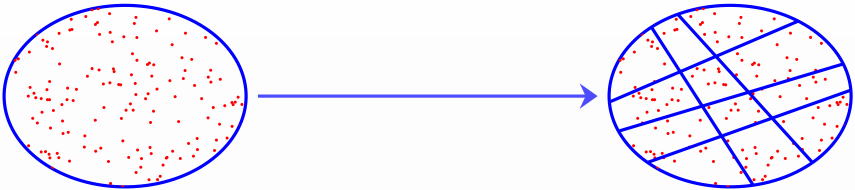
Close Cousins: Counting and Sampling

- Approximate counting and almost-uniform sampling are inter-reducible (Jerrum, Valiant and Vazirani, 1986)
- Is the reduction efficient?
  - Almost-uniform sampler (JVV) require linear number of approximate counting calls

Key Ideas



- Check if a randomly picked cell is *small*
  - If yes, pick a solution randomly from randomly picked cell



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  - $\tilde{m} = \log \frac{C}{\text{thresh}}$
  - Check for $m = \tilde{m}-1, \tilde{m}, \tilde{m}+1$ if a randomly chosen cell is *small*
- Pr[y is output] = Pr[y is chosen] Pr[Cell is small | y is in cell]
- The conditioning in Pr[Cell is small | y is in cell] leads to requirement of 3-wise independence of 2-wise independence.

(CMV14, CFMSV14, CFMSV15,SGM20)

Theoretical Guarantees

Theorem (Almost-Uniformity)

$$\forall y \in \mathsf{Sol}(F),\ \frac{1}{(1+\varepsilon)|\mathsf{Sol}(F)|} \leq \Pr[y \textit{ is output}] \leq \frac{1+\varepsilon}{|\mathsf{Sol}(F)|}$$

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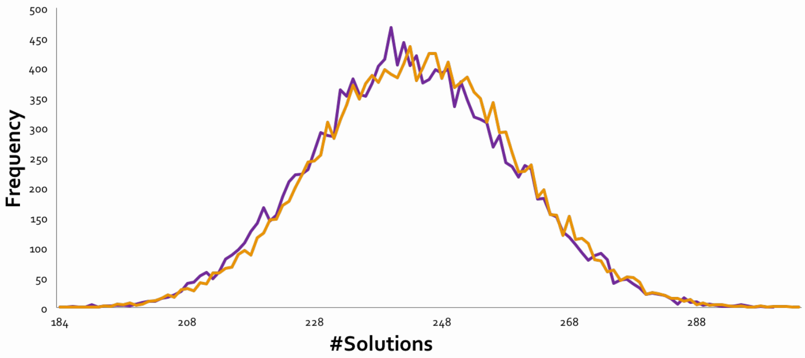
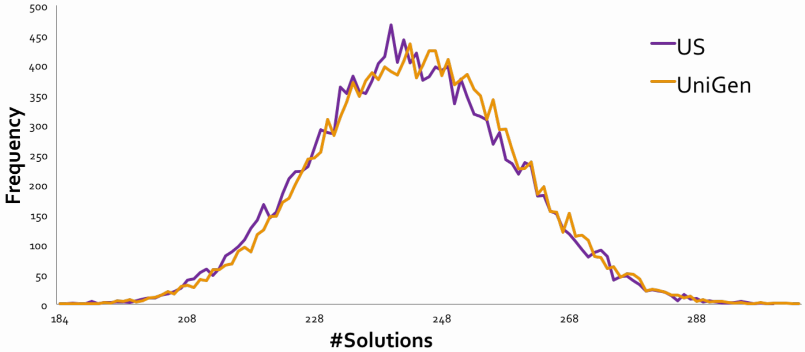
Random XORs are 3-wise independent

Quiz Time: Uniformity



- Benchmark: case110.cnf; #var: 287; #clauses: 1263
- Total Runs: 4×10^6; Total Solutions : 16384

Statistically Indistinguishable



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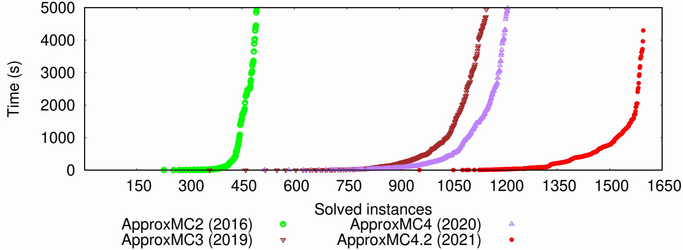
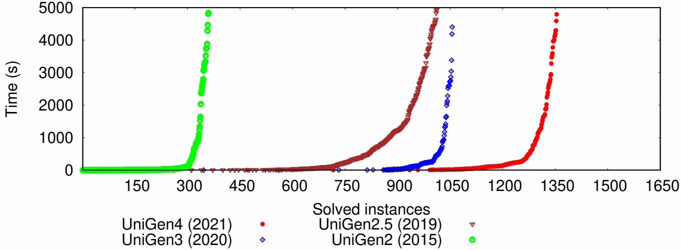
Now that SAT is "easy", it is time to look beyond satisfiability

Improvements Over the Years





Enabling “Beyond NP” Revolution

Challenge Problems

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Civil Engineering Reliability for Los Angeles Transmission Grid

Security Leakage Measurement for C++ program with 1K lines

Hardware Verification Handling SMT formulas with 10K nodes

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Technical Directions

- Tighter integration between solvers and algorithms
- Handling weighted distributions: Connections to theory of integration
- Verification of sampling and counting

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Questions?

Reliability of Critical Infrastructure Networks

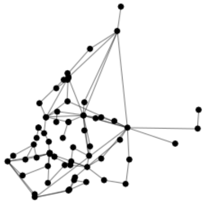

Figure: Plantersville, SC

- $G = (V, E)$; source node: s and terminal node t
- failure probability $g : E \to [0, 1]$
- Compute Pr[s and t are disconnected]?

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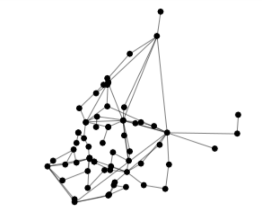

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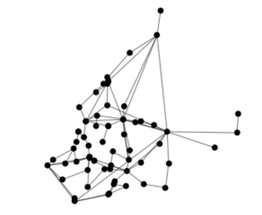

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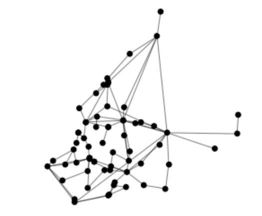

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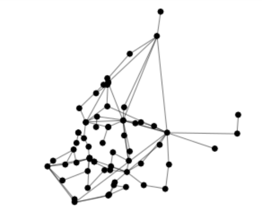

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Constrained Counting

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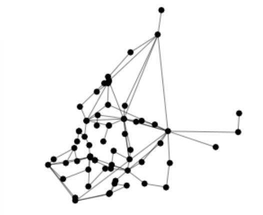
(DMPV, AAAI 17, ICASP-13, RESS 2019)

Reliability of Critical Infrastructure Networks



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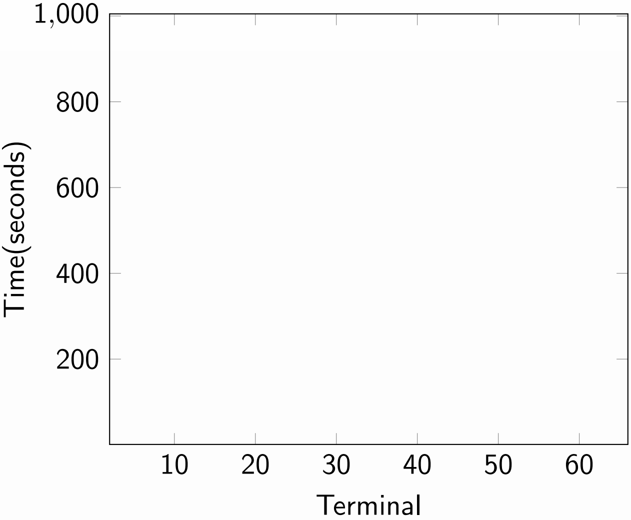

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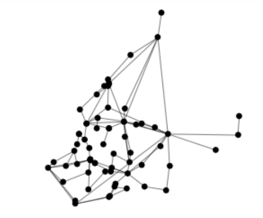
(DMPV, AAAI17)



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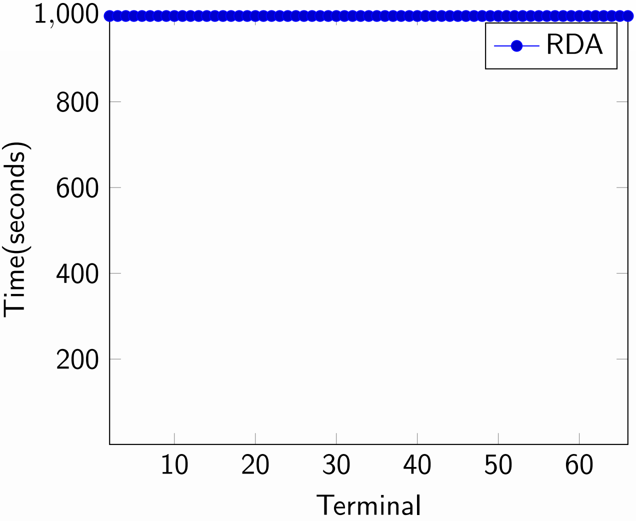

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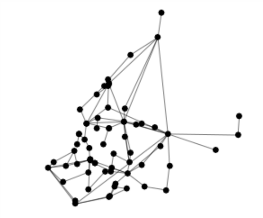

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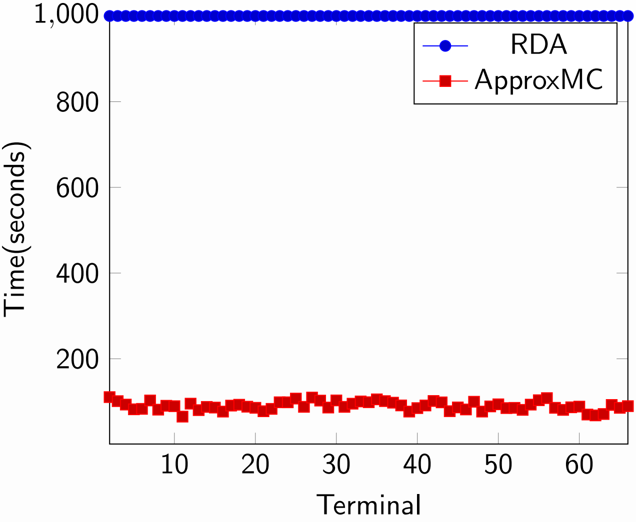

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