The Rise of Approximate Model Counting: Beyond Classical Theory and Practice of SAT

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Beyond Satisfiability

The Amazing Collaborators

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Special shout out to Mate Soos, **the** maintainer of ApproxMC and UniGen

Counting

- Given
  - Boolean variables $X_1, X_2, \cdots X_n$
  - Formula F over $X_1, X_2, \cdots X_n$
- $\mathsf{Sol}(F) = \{$ solutions of F $\}$

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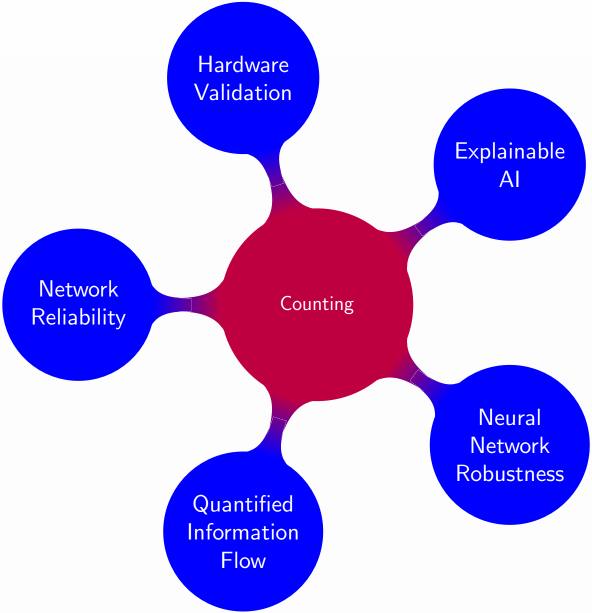
- Given $F := (X_1 \vee X_2)$
- $\mathsf{Sol}(F) = \{(0,1),(1,0),(1,1)\}$
- $|\mathsf{Sol}(F)| = 3$

Applications across Computer Science



Through the Lens of SAT Revolution

Obs 1 SAT Oracle $\neq$ NP Oracle

- Returns UNSAT with a proof
- Return a satisfying assignment if satisfiable

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Obs 3 Memoryfulness

- Incremental Solving: Often easier to solve F followed by G if we G can be written as $G = F \wedge H$
- If $F \rightarrow C$ then $(F \wedge H) \implies C$

SAT Oracle vs NP Oracle vs SAT Solver

ThreshSAT(F, thresh): Does F has $\leq$ thresh solutions?
BoundedSAT(F, thresh): $|\mathrm{Sol}(F)|$ If F has $\leq$ thresh solutions, else $\perp$?

- NP Oracle
  - ThreshSAT: #Queries: 1 Size: $|F| \cdot$ thresh
  - BoundedSAT: #Queries: thresh Size: $|F| \cdot$ thresh

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  - ThreshSAT: #Queries: thresh Size: $|F| + n \cdot \text{thresh}$
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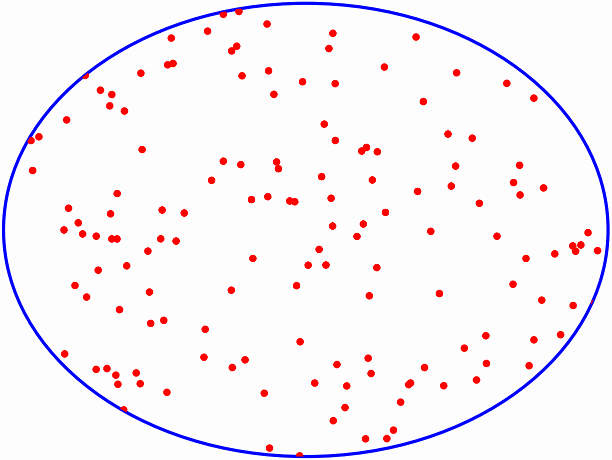
Both ThreshSAT and BoundedSAT have *same complexity*!

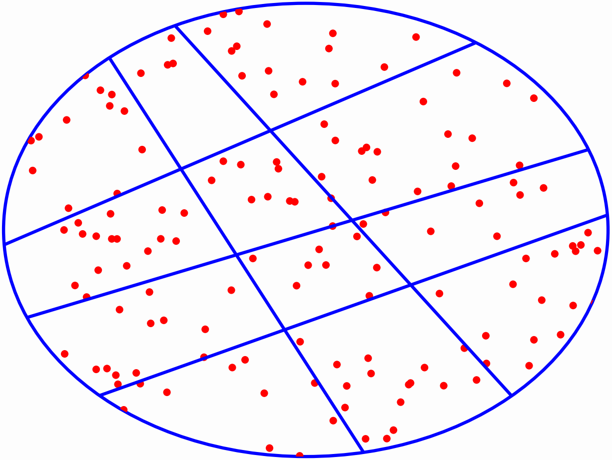
So What Makes Hashing-based Techniques Work?

- Algorithmic
  - From Stockmeyer to ApproxMC
  - The Boon of Dependence
  - Sparse XORs
- System: Efficient CNF+XOR Solving (Soos' possible talk in SAT Seminar?)
- Conceptual
  - Independent Support
  - Projection

The Rise of Hashing-based Approach: Promise of Scalability and Guarantees
(S83,GSS06,GHSS07,CMV13b,EGSS13b,CMV14,CDR15,CMV16,ZCSE16,AD16, KM18,ATD18,SM19,ABM20,SGM20)

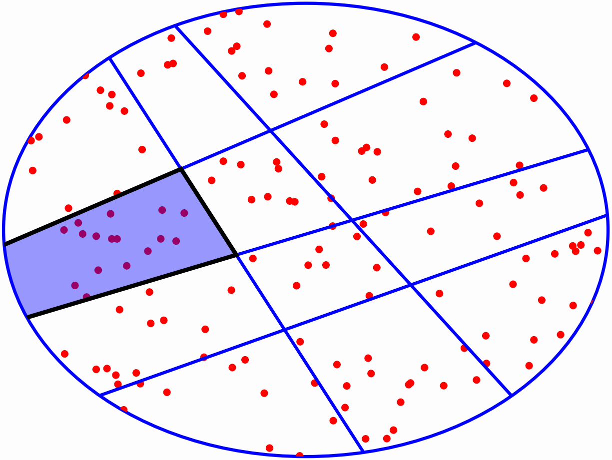
As Simple as Counting Dots





As Simple as Counting Dots

Pick a random cell



Estimate = Number of solutions in a cell $\times$ Number of cells

2-wise independent Hashing

- Let H be family of 2-wise independent hash functions mapping $\{0,1\}^n$ to $\{0,1\}^m$

$$\forall y_1, y_2 \in \{0,1\}^n, \alpha_1, \alpha_2 \in \{0,1\}^m, h \xleftarrow{R} H$$
$$\Pr[h(y_1) = \alpha_1] = \Pr[h(y_2) = \alpha_2] = \left(\frac{1}{2^m}\right)$$

$$\Pr[h(y_1) = \alpha_1 \wedge h(y_2) = \alpha_2] = \left(\frac{1}{2^m}\right)^2$$

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- The power of 2-wise independentity
  - Z be the number of solutions in a randomly chosen cell
  - $\mathsf{E}[Z] = \frac{|\mathsf{Sol}(F)|}{2^m}$
  - $\sigma^2[Z] \leq \mathsf{E}[Z]$

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- $\Pr\left[\frac{\mathsf{E}[Z]}{1+\varepsilon} \leq Z \leq \mathsf{E}[Z](1+\varepsilon)\right] \geq 1 - \frac{1}{(\frac{\varepsilon}{1+\varepsilon})^2(\mathsf{E}[Z])}$

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- $\mathsf{E}[Z] = c(\frac{1+\varepsilon}{\varepsilon})^2$ provides $1 - \frac{1}{c}$ lower bound

2-wise independent Hash Functions

- Variables: $X_1, X_2, \cdots X_n$
- To construct $h : \{0, 1\}^n \to \{0, 1\}^m$, choose m random XORs
- Pick every X_i with prob. $\frac{1}{2}$ and XOR them
  - $X_1 \oplus X_3 \oplus X_6 \cdots \oplus X_{n-2}$
  - Expected size of each XOR: $\frac{n}{2}$

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  - Expected size of each XOR: $\frac{n}{2}$
- To choose $\alpha \in \{0,1\}^m$, set every XOR equation to 0 or 1 randomly

$$X_1 \oplus X_3 \oplus X_6 \cdots \oplus X_{n-2} = 0 \quad (Q_1)$$
$$X_2 \oplus X_5 \oplus X_6 \cdots \oplus X_{n-1} = 1 \quad (Q_2)$$
$$\cdots \quad (\cdots)$$
$$X_1 \oplus X_2 \oplus X_5 \cdots \oplus X_{n-2} = 1 \quad (Q_m)$$

- Solutions in a cell: $F \wedge Q_1 \cdots \wedge Q_m$

- $(1+\varepsilon, \delta)$-Approximation

$$\Pr\left[\frac{|\mathsf{Sol}(F)|}{1+\varepsilon} \leq ApproxCount(F, \varepsilon, \delta) \leq |\mathsf{Sol}(F)|(1+\varepsilon)\right] \geq 1-\delta$$

- $(1+\varepsilon, \delta)$-Approximation

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- Constant Factor Approximation: $(4,\delta)$

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- From 4 to 2-factor

  Let $G = F(X_1) \wedge F(X_2)$ (i.e., two identical copies of F)

$$\frac{|\mathsf{Sol}(G)|}{4} \leq C \leq 4\cdot|\mathsf{Sol}(G)| \implies \frac{|\mathsf{Sol}(F)|}{2} \leq \sqrt{C} \leq 2\cdot|\mathsf{Sol}(F)|$$

“Stockmeyer’s Approach”

Constant Factor Suffices

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- From 4 to $(1+\varepsilon)$-factor
  Construct $G = F(X_1) \wedge F(X_2) \cdots F(X_{\frac{1}{\varepsilon}})$ And then we can take $\frac{1}{\varepsilon}$-root

- aComp(F, k)
  - If $|\text{Sol}(F)| \geq 2^{k+1}$, then aComp($F, k$) returns YES whp
  - If $|\text{Sol}(F)| < 2^{k}$, then aComp(F, k) returns NO whp

"Stockmeyer's Approach" S83, JVV86, BP95

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- Counter(F)
  - Invoke aComp(F, k) for $k = 0, 1, \ldots n$
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- aComp(F, k)
  - Call $\mathcal{O}(\log\log n)$ calls to ThreshSAT($F \wedge Q_1 \wedge \ldots \wedge Q_{k-5}, 48$) and return the median.

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S83, JVV86, BP95

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- For $(1 + \varepsilon)$-approx, invoke Counter on $F(X_1) \wedge F(X_2) \cdots F(X_{\frac{1}{\varepsilon}})$ and return $\frac{1}{\varepsilon}$-root   Too large queries   thresh $= \frac{9}{\varepsilon^2}$
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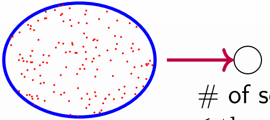
  - Dependence to avoid union bounds

ApproxMC



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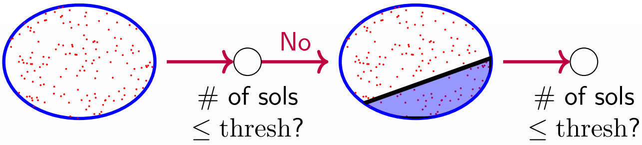

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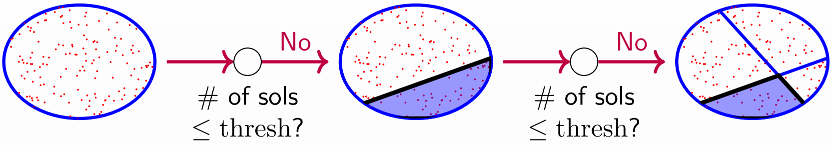

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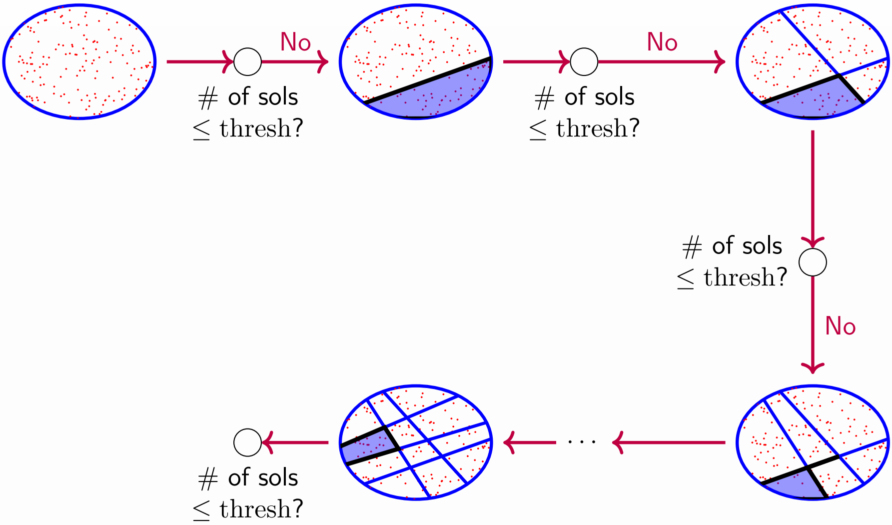
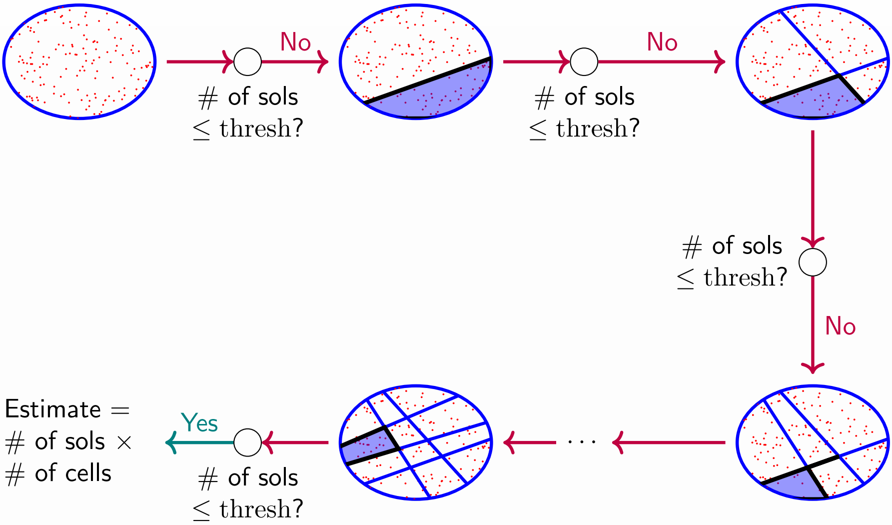
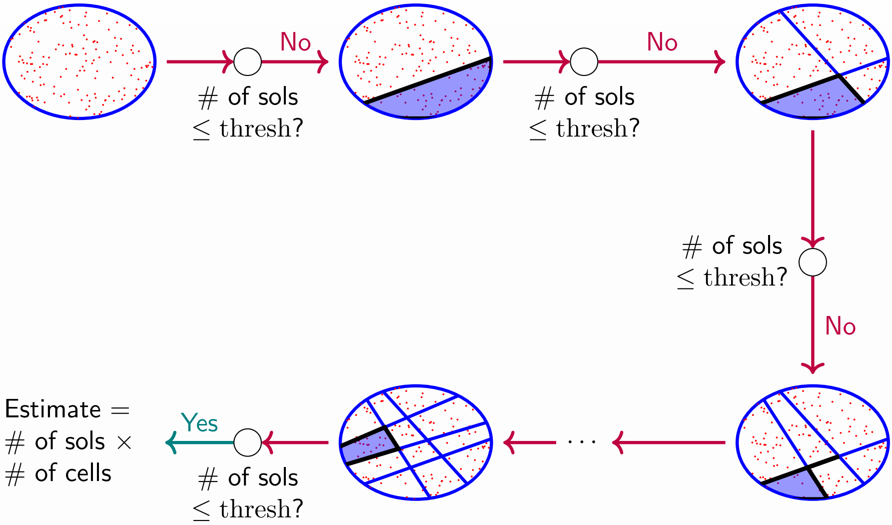

No
# of sols
≤ thresh?
No
# of sols
≤ thresh?
# of sols
≤ thresh?
No
· · ·
# of sols
≤ thresh?

ApproxMC



No
# of sols
≤ thresh?
No
# of sols
≤ thresh?
# of sols
≤ thresh?
No
...
Yes
# of sols
≤ thresh?
Estimate =
# of sols ×
# of cells

ApproxMC



Repeat $\mathcal{O}(\log(1/\delta))$ times and return the median

ApproxMC

- We want to partition into 2^{m^*} cells such that $2^{m^*} = \frac{|\mathsf{Sol}(F)|}{\mathrm{thresh}}$
  - Query 1: Is $\#(F \wedge Q_1) \leq \mathrm{thresh}$
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  - $\cdots$
  - Query n: Is $\#(F \wedge Q_1 \wedge Q_2 \cdots \wedge Q_n) \leq \mathrm{thresh}$
- Stop at the first m where Query m returns YES and return estimate as BoundedSAT$(F \wedge Q_1 \wedge Q_2 \cdots \wedge Q_m, \mathrm{thresh}) \times 2^m$
- Observation: $\#(F \wedge Q_1 \cdots \wedge Q_i \wedge Q_{i+1}) \leq \#(F \wedge Q_1 \cdots \wedge Q_i)$
  - If Query i returns YES, then Query $i+1$ must return YES

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  - $E_i : \left|\#(F \wedge Q_1 \wedge Q_2 \cdots \wedge Q_i) - \frac{|\mathsf{Sol}(F)|}{2^i}\right| \geq (1+\varepsilon)\frac{|\mathsf{Sol}(F)|}{2^i}$

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  - (Loosely), $E_i \subseteq E_{i+1}$ for $i < m * -2$
  - (Loosely), $E_j \supseteq E_{j+1}$ for $j > m^* + 1$

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- Stop at the first m where Query m returns YES and return estimate as BoundedSAT$(F \wedge Q_1 \wedge Q_2 \cdots \wedge Q_m, \text{thresh}) \times 2^m$
- Observation: $\#(F \wedge Q_1 \cdots \wedge Q_i \wedge Q_{i+1}) \leq \#(F \wedge Q_1 \cdots \wedge Q_i)$
  - If Query i returns YES, then Query $i+1$ must return YES
  - Logarithmic search (# of SAT calls: $\mathcal{O}(\log n)$)
  - Incremental Search
- The Boon of Dependence
  - $E_i : \left|\#(F \wedge Q_1 \wedge Q_2 \cdots \wedge Q_i) - \frac{|\mathsf{Sol}(F)|}{2^i}\right| \geq (1+\varepsilon)\frac{|\mathsf{Sol}(F)|}{2^i}$
  - (Loosely), $E_i \subseteq E_{i+1}$ for $i < m * -2$
  - (Loosely), $E_j \supseteq E_{j+1}$ for $j > m^* + 1$
  - $\Pr[\text{Error}] = \Pr[E_{m^*-2}] + \Pr[E_{m^*-1}] + \Pr[E_{m^*}] + \Pr[E_{m^*+1}]$

ApproxMC

Theorem (Correctness)

$$\Pr\left[\frac{|\mathsf{Sol}(F)|}{1+\varepsilon} \leq \mathit{ApproxMC}(F,\varepsilon,\delta) \leq |\mathsf{Sol}(F)|(1+\varepsilon)\right] \geq 1-\delta$$

Theorem (Complexity)

ApproxMC(F,ε,δ) *makes* $\mathcal{O}(\frac{\log n \log(\frac{1}{\delta})}{\varepsilon^2})$ *calls to SAT oracle.*

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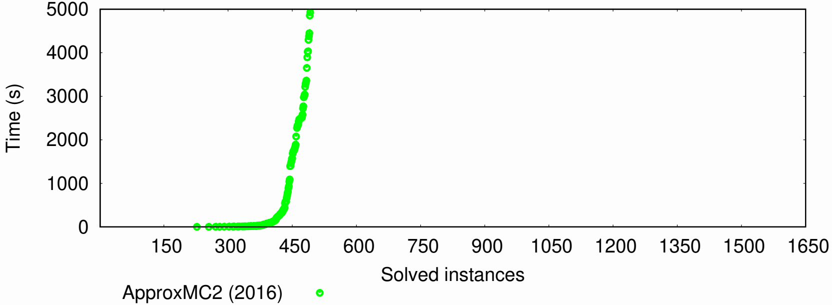
Theorem (FPRAS for DNF; (MSV, FSTTCS 17; CP 18, IJCAI-19))

If F is a DNF formula, then ApproxMC is FPRAS – different from the Monte-Carlo based FPRAS for DNF (Karp, Luby 1983)

A Practical Counter



ApproxMC1 (without dependence) was 10-100$\times$ slower.

The Hope of Short XORs

- If we pick every variable X_i with probability p .
  - Expected Size of each XOR: np
  - $\mathsf{E}[Z_m] = \frac{|\mathsf{Sol}(F)|}{2^m}$
  - $\sigma^2[Z_m] \leq \mathsf{E}[Z_m] + \sum_{\sigma_1 \in \mathsf{Sol}(F)} \sum_{\substack{\sigma_2 \in \mathsf{Sol}(F) \\ w = d(\sigma_1, \sigma_2)}} r(w, m)$
    - where, $r(w, m) = \left(\left(\frac{1}{2} + \frac{(1-2p)^w}{2} \right)^m - \frac{1}{2^m} \right)$
  - For $p = \frac{1}{2}$, we have $\frac{\sigma^2[Z_m]}{\mathsf{E}[Z_m]} \leq 1$
- Earlier Attempts (GSS07,EGSS14,ZCSE16,AD17,ATD18)
  - $\sum_{\sigma_1 \in \mathsf{Sol}(F)} \sum_{\substack{\sigma_2 \in \mathsf{Sol}(F) \\ w = d(\sigma_1, \sigma_2)}} r(w, m) \leq \sum_{\sigma_1 \in \mathsf{Sol}(F)} \sum_{w=0}^{n} \binom{n}{w} r(w, m)$
  - $\binom{n}{w}$ grows very fast with n, so could not upper bound $\frac{\sigma^2[Z_m]}{\mathsf{E}[Z_m]}$
  - The weak bounds lead to significant slowdown: typically $100\times$ to $1000\times$ factor of slowdown! (ATD18,ABM20)

- $\sum_{\sigma_1 \in \mathrm{Sol}(F)} \sum_{\substack{\sigma_2 \in \mathrm{Sol}(F) \\ w = d(\sigma_1, \sigma_2)}} r(w, m) = \sum_{w=0}^{n} C_F(w) r(w, m)$
- $C_F(w) = |\{\sigma_1, \sigma_2 \in \mathrm{Sol}(F) \mid d(\sigma_1, \sigma_2) = w\}|$

The Power of Isoperimetric Inequalities

- $\sum_{\sigma_1 \in \mathrm{Sol}(F)} \sum_{\substack{\sigma_2 \in \mathrm{Sol}(F) \\ w = d(\sigma_1, \sigma_2)}} r(w, m) = \sum_{w=0}^{n} C_F(w) r(w, m)$
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- Isoperimetric Inequalities! (Rashtchian and Raynaud 2019)

Lemma

$$\sum_{w=0}^{n} C_F(w) r(w, m) \leq \sum_{w=0}^{n} \binom{8e\sqrt{n \cdot \ell}}{w} r(w, m) \text{ where } \ell = \log |\mathrm{Sol}(F)|$$

- $\frac{\binom{n}{w}}{\binom{8e\sqrt{n \cdot \ell}}{w}} \approx \left(\frac{n}{\ell}\right)^{\frac{w}{2}}$

From Linear to Logarithmic Size XORs

Theorem (Informal)

For all q, k, $|\mathrm{Sol}(F)| \leq k \cdot 2^m$, $p = \mathcal{O}(\frac{\log m}{m})$ *we have*

$$\frac{\sigma^2[Z_m]}{\mathrm{E}[Z_m]} \leq q (\text{a constant})$$

Recall, average size of XORs: $n \cdot p$
Improvement of p *from* $\frac{m/2}{m}$ *to* $\frac{\log m}{m}$

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Challenge: No meaningful bounds on $|\mathrm{Sol}(F)|$

- $\Pr[\mathrm{Error}] = \Pr[E_{m^*-2}] + \Pr[E_{m^*-1}] + \Pr[E_{m^*}] + \Pr[E_{m^*+1}]$
- Key Insight: When adding m-th XOR, theoretical analysis only requires $\frac{\sigma^2[Z_m]}{\mathrm{E}[Z_m]} \leq q$ whenever $|\mathrm{Sol}(F)| \leq \mathrm{thresh} \cdot 2^m$

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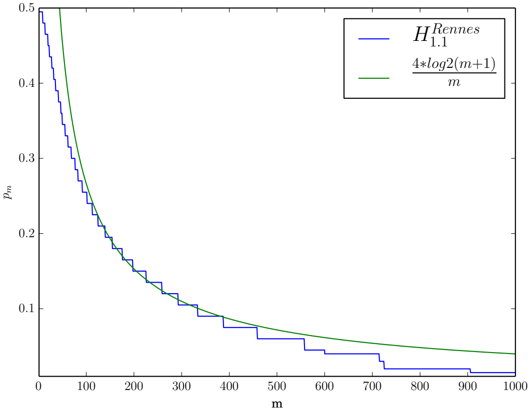
- Add m-th XOR with $p_m = \mathcal{O}(\frac{\log m}{m})$

Sparse Hash Functions



$H_{1.1}^{Rennes}$: Sparse hash functions that guarantee $q = 1.1$

Sparse XORs

Benchmark	Vars	$\log_2$(Count)	ApproxMC	ApproxMC+Sparse	Speedup
03B-4	27966	28.55	983.72	1548.96	0.64
squaring23	710	23.11	0.66	1.21	0.55
case144	765	82.07	102.65	202.06	0.51
modexp8-4-6	83953	32.13	788.23	920.34	0.86
min-28s	3933	459.23	48.63	35.83	1.36
s9234a_7_4	6313	246.0	4.77	2.45	1.95
min-8	1545	284.78	8.86	4.59	1.93
s13207a_7_4	9386	699.0	34.94	17.05	2.05
min-16	3065	539.88	33.67	16.61	2.03
90-15-4-q	1065	839.25	273.1	135.75	2.01
s35932_15_7	17918	1761.0	–	72.32	–
s38417_3_2	25528	1663.02	–	71.04	–
75-10-8-q	460	360.13	–	4850.28	–
90-15-8-q	1065	840.0	–	3717.05	–

Remember; thresh $= \mathcal{O}(\frac{\sigma^2[Z_m]}{\mathsf{E}[Z_m]} \cdot \frac{1}{\varepsilon^2})$

$\frac{\sigma^2[Z_m]}{\mathsf{E}[Z_m]} \leq 1$ for 2-wise independent; $\frac{\sigma^2[Z_m]}{\mathsf{E}[Z_m]} \leq q = 1.1$ for $H_{1.1}^{Rennes}$.

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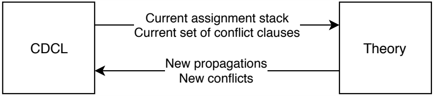
CDCL(T)

For theories that are not efficiently simulated by CDCL

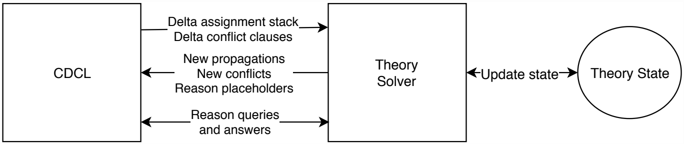
- T is the theory, e.g.:
  - Gauss-Jordan Elimination [SoosNohlCastelluccia'2010]
  - Pseudo-Boolean Reasoning [ChaiKuehlmann'2006]
  - Symmetric Explanation Learning [DevriendtBogaertsBruynooghe'2017]
- Theory is run side-by-side to the CDCL algorithm
- **Propagate** values implied by Theory given current assignment stack of CDCL
- **Conflict** if Theory implies 1=0 given current assignment stack of CDCL
- Theory must give reason for propagations&conflicts



CDCL(T) Cont.

Optimizations:

- Should only send delta of assignment stack + conflict clauses
  - Variables assigned (decisions + propagations)
  - Variables unassigned (backtracking, restarting)
  - New conflict clauses
- Theory only needs to compute delta relative to old state
- Theory can give placeholders for reasons
  - If reason is needed during conflict generation, Theory is queried
  - Called "lazy" (vs "greedy") interpolant generation



CDCL(T) Gauss-Jordan Elimination: Ingredients

What components do we need?

- **Extractor for XOR constraints**: XORs may be encoded as CNF
- **Delta update mechanism** for row-echelon form matrix:
  - how to handle when variable is set
  - how to handle when variable is unset
- **Efficient data structures** to allow for quick updates
- **Reason generation**

CDCL(T) Gauss-Jordan Elimination: Extraction

$$l_1 \oplus l_2 \oplus l_3 = 1 \quad \Leftrightarrow \quad \begin{array}{l} l_1 \vee l_2 \vee l_3 \wedge \\ \bar{l}_1 \vee \bar{l}_2 \vee l_3 \wedge \\ \bar{l}_1 \vee l_2 \vee \bar{l}_3 \wedge \\ l_1 \vee \bar{l}_2 \vee \bar{l}_3 \wedge \end{array}$$

$$l_1 \oplus l_2 \oplus l_3 = 1 \quad \leftarrow \quad \begin{array}{l} l_1 \vee l_2 \vee \wedge \\ \bar{l}_1 \vee \bar{l}_2 \vee l_3 \wedge \\ \bar{l}_1 \vee l_2 \vee \bar{l}_3 \wedge \\ l_1 \vee \bar{l}_2 \vee \bar{l}_3 \wedge \end{array}$$

- Missing literals only mean something stronger than XOR
- XOR is still implied and should be detected

CDCL(T) Gauss-Jordan Elimination: 2-variable watchlist

Let's use a 2-variable watch scheme [HanJiang2012]:

- If 2 or more variables are unset in XOR constraint, it cannot propagate or conflict
- If 1 variable is unset, it must propagate
- If 0 variable is unset, it is either satisfied or is in conflict

Watching for propagation and to perform GJE

- For every row (of XOR), there is a *pivot* variable (among the two variables watching the row)
- A variable is pivot for at most one row.

CDCL(T) GJE: Reason Clauses and Backtracking

What combination of XOR constraints gave us the propagation?

- Each row is a combination of input XOR constraints
- It is guaranteed to propagate/conflict under current variable assignment

During backtracking:

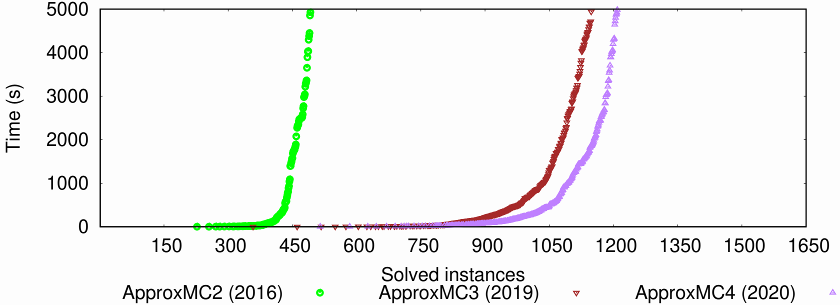
- All previous invariants still hold
- If the column (variable) was pivot for a row, it still is
- Both watches of the row are still good and in the watchlists
- Matrix looks differently than when we last had this assignment... is that a problem?
- No! Observe: new matrix could have been reached from the starting position, pivoting differently(!)

CDCL(T) Gauss-Jordan Elimination: Recap

Let's recap! What was hard:

- Extracting XOR constraints
- Keeping CDCL and GJ in sync:
  - Fast update for variable setting (propagation)
  - Fast update for backtracking (conflict)
- Reason clause generation

Improvements Over the Years



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Improved 2-wise Independent Hash Functions

- Not all variables are required to specify solution space of F
  - $F := X_3 \iff (X_1 \vee X_2)$
  - X_1 and X_2 uniquely determines rest of the variables (i.e., X_3)
- Formally: if I is independent support, then $\forall \sigma_1, \sigma_2 \in \mathsf{Sol}(F)$, if σ_1 and σ_2 agree on I then $\sigma_1 = \sigma_2$
  - $\{X_1, X_2\}$ is independent support but $\{X_1, X_3\}$ is not

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  - $\{X_1, X_2\}$ is independent support but $\{X_1, X_3\}$ is not
- Random XORs need to be constructed only over I (CMV DAC14)
- Typically I is 1-2 orders of magnitude smaller than X
- Auxiliary variables introduced during encoding phase are *dependent* (Tseitin 1968)

Algorithmic procedure to determine I?

Determining Independent Support

Independent Support: I Defined Variables: $X \setminus I$

- If I is independent support and x_n is defined in terms of $I \setminus \{x_n\}$, then $I \setminus \{x_n\}$ is independent support.

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- If I is independent support and x_n is defined in terms of $I \setminus \{x_n\}$, then $I \setminus \{x_n\}$ is independent support.
- Padao's Theorem [1901] x_n is defined in terms of I if and only if

$$F(X) \wedge F(Y) \wedge \bigwedge_{x_i \in I} (x_i = y_i) \implies (x_n = y_n) \text{ is VALID}$$

$$\text{i.e., } F(X) \wedge F(Y) \wedge \bigwedge_{x_i \in I} (x_i = y_i) \wedge x_n \wedge \neg y_n \text{ is UNSAT}$$

- So iterative procedure with initial $I = X$ and remove x_i from I if x_i is defined in terms of $I \setminus \{x_i\}$
- $\mathcal{O}(n)$ SAT calls

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Projected Counting

- Given F on $X \cup Y$, count number of solutions of $\exists Y F(X, Y)$
- Let $X = x_1$; $Y = y_1$; $F = (x_1 \vee y_1)$.
- So $\mathrm{Sol}(\exists Y F(X, Y)) = \{(x_1 = 0), (x_1 = 1)\}$
- Therefore, $|\mathrm{Sol}(\exists Y F(X, Y))| = 2$
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- Approach 1: Perform quantifier elimination
- Approach 2: ApproxMC with minor changes
  - XORs over X and also enumerate solutions over X.
  - ProjThresh(F, X, thresh):
    #Queries: thresh Size: $|F| + \text{thresh} * |X|$ for SAT Oracle
- Usage of $\exists$ can lead to exponentially succinct formulas

Reliability of Critical Infrastructure Networks

- $G = (N, E)$; source node: s and terminal node t
- (wlog) every edge fails with prob $\frac{1}{2}$
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- Variables for Nodes: $P_N = \{p_u\}_{u \in N}$ and Edges: $\{p_e\}_{e \in E}$
- Consider $e = (u, v)$: $p_u \wedge e_{u,v} \rightarrow p_v$
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- Count $\exists P_N(\varphi)$: Projected Counting

So What Makes Hashing-based Techniques Work?

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The Rise of Hashing-based Approach: Promise of Scalability and Guarantees
(S83,GSS06,GHSS07,CMV13b,EGSS13b,CMV14,CDR15,CMV16,ZCSE16,AD16, KM18,ATD18,SM19,ABM20,SGM20)

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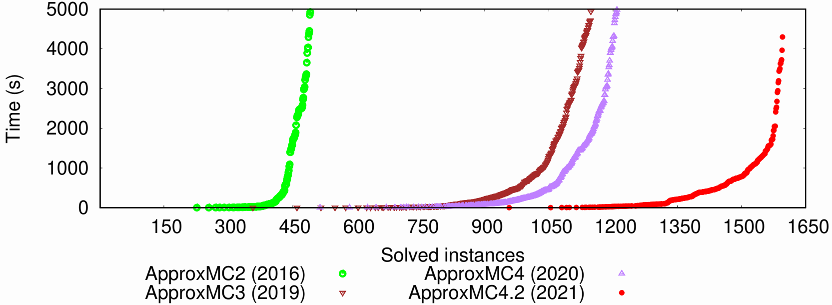


Reliability of Critical Infrastructure Networks

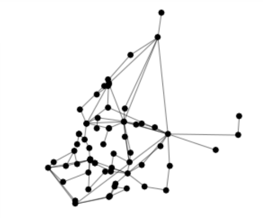
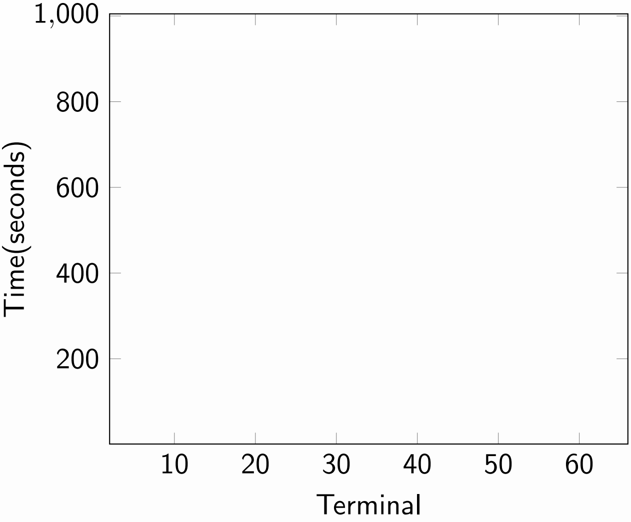


Figure: Plantersville, SC

- $G = (V, E)$; source node: s
- Compute Pr[t is disconnected]?

Timeout = 1000 seconds



(DMPV, AAAI17)

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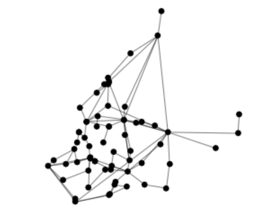

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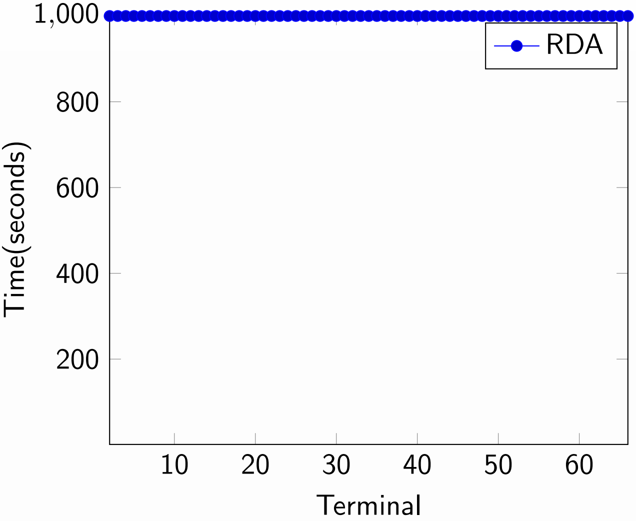

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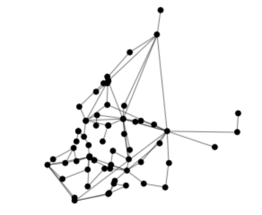

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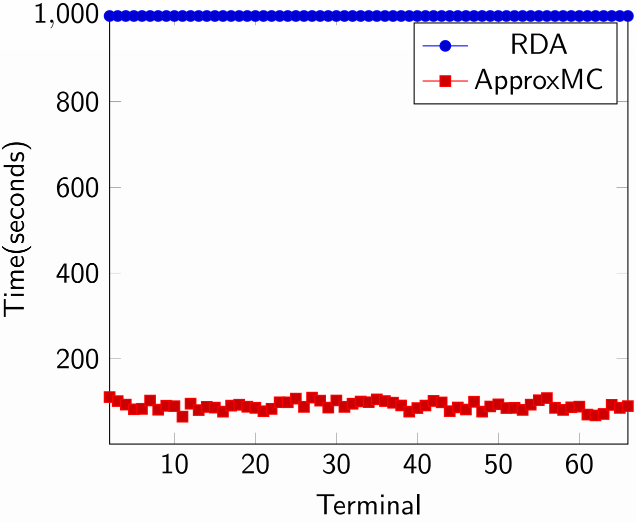

(DMPV, AAAI17)

Where do we go from here?

Algorithmic	• Design of instance-dependent sparse XORs • Can we prove accuracy observed in practice?
System	• Better system for Sparse XORs • Hybrid Counter to exploit complimentary exact and approximate counting
Coneptual	• Independent support is model counting preserving but approximation would suffice • Proof of correctness