Constrained Counting and Sampling: Bridging the Gap between Theory and Practice

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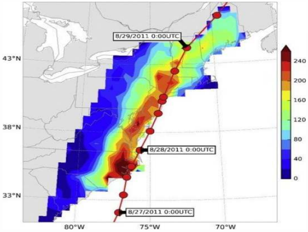
- Data
- Uncertainty



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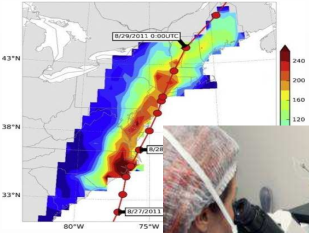

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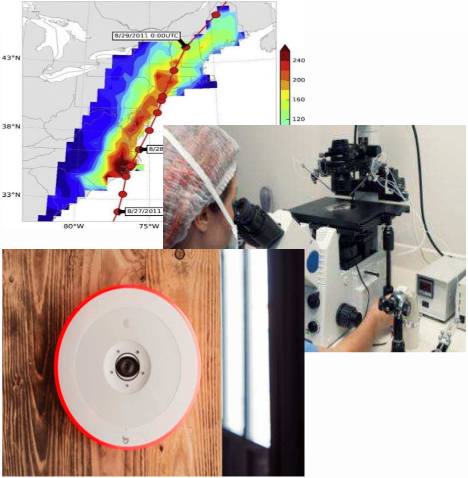

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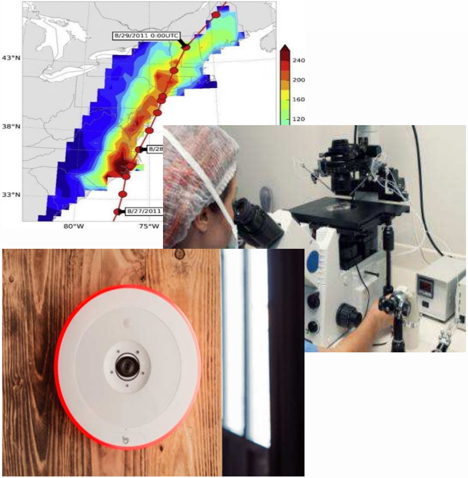

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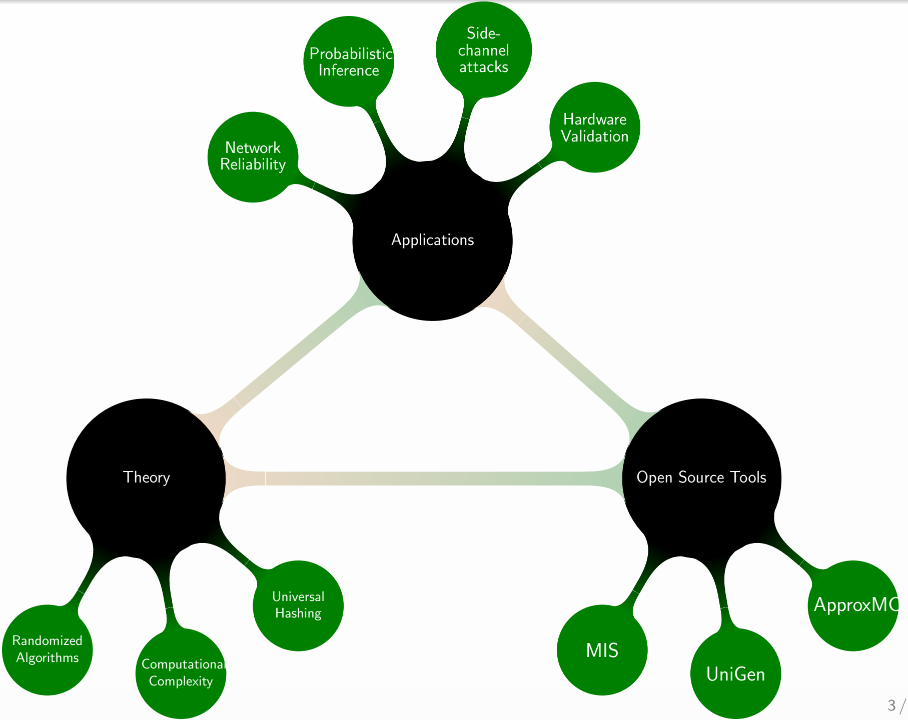
- Scalable
- Guarantees of Accuracy


Applications
Network Reliability
Probabilistic Inference
Side-channel attacks
Hardware Validation
Theory
Randomized Algorithms
Computational Complexity
Universal Hashing
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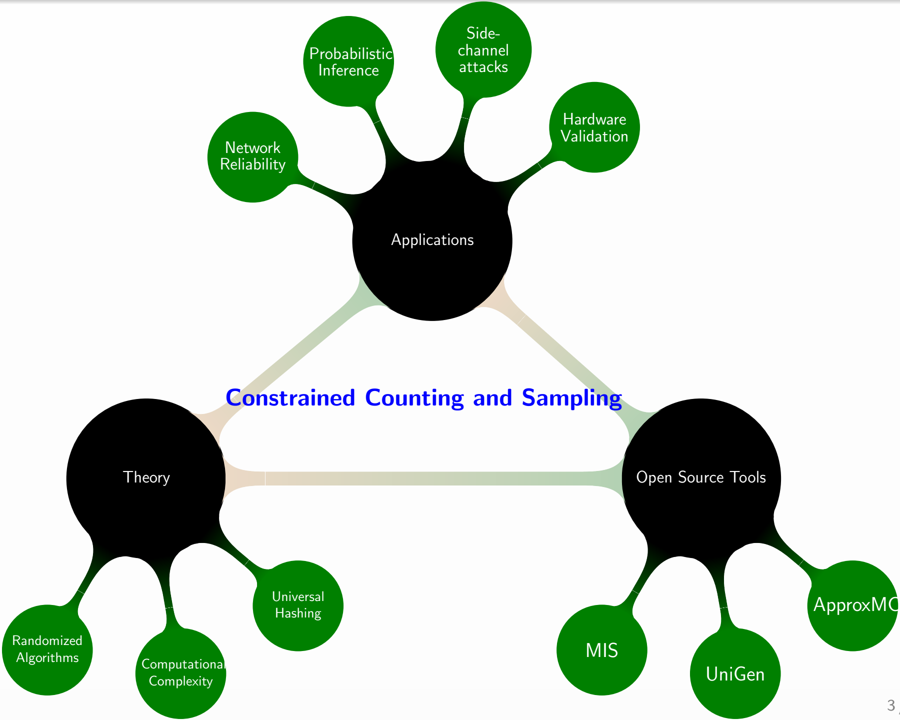

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A Tale of Constraints

Boolean Satisfiability (SAT): Given a Boolean expression, using "and" ($\wedge$), "or" ($\vee$), and "not" ($\neg$) is there a solution, i.e., an assignment of 0's and 1's to the variables that makes the expression equal 1?

Example: $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee \neg x_3)$

$x_1 = 1, x_2 = 1, x_3 = 1$

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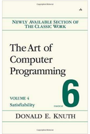
Ernst Schroder, 1841-1902: "Getting a handle on the consequences of any premises, or at least the fastest method for obtaining these consequences, seems to me to be one of the noblest, if not the ultimate goal of mathematics and logic."

Cook, 1971; Levin, 1973: SAT is NP-complete

Modern SAT solvers are able to deal routinely with practical problems that involve many thousands of variables, although such problems were regarded as hopeless just a few years ago. (Donald Knuth, 2016)



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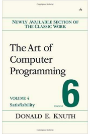

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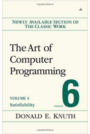

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Now that SAT is "easy", it is time to look beyond satisfiability

Constrained Counting and Sampling

- Given
  - Boolean variables $X_1, X_2, \cdots X_n$
  - Formula F over $X_1, X_2, \cdots X_n$

- $\mathsf{Sol}(F) = \{$ solutions of F $\}$

- Constrained Counting: Determine $|\mathsf{Sol}(F)|$
- Constrained Sampling: Randomly sample from $\mathsf{Sol}(F)$ such that $\Pr[\text{y is sampled}] = \frac{1}{|\mathsf{Sol}(F)|}$

Constrained Counting and Sampling

- Given
  - Boolean variables $X_1, X_2, \cdots X_n$
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- $F := (X_1 \vee X_2)$;
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Today's Menu

Network Reliability

Probabilistic Inference

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Constrained Counting

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Hashing Framework

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Can we reliably predict the effect of natural disasters on critical infrastructure such as power grids?

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Can we predict likelihood of a region facing blackout?

Plantersville, SC

- $G = (V, E)$; source node: s and terminal node t
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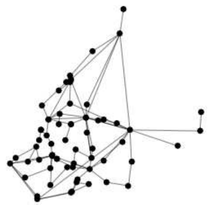

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Constrained Counting

(D**M**PV, AAAI 17)

Probabilistic Models

Patient	Cough	Smoker	Asthma
Alice	1	0	0
Bob	0	0	1
Randee	1	0	0
Tova	1	1	1
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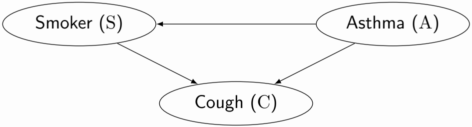
$\Longrightarrow$



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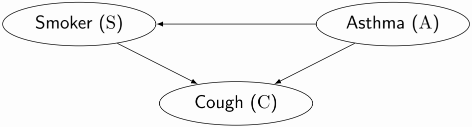

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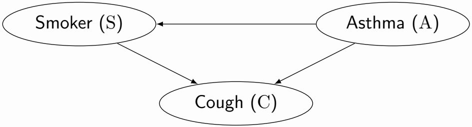

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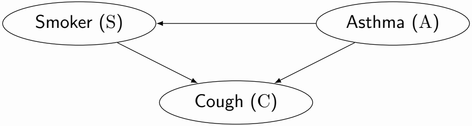

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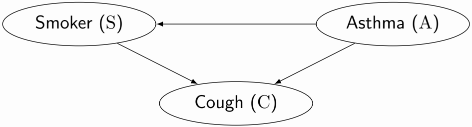

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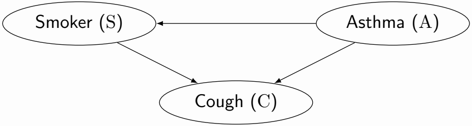

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$\boxed{\text{Constrained Counting}}$

Prior Work

Strong guarantees but poor scalability

- Exact counters (Birnbaum and Lozinskii 1999, Jr. and Schrag 1997, Sang et al. 2004, Thurley 2006)
- Hashing-based approach (Stockmeyer 1983, Jerrum Valiant and Vazirani 1986)

Weak guarantees but impressive scalability

- Bounding counters (Gomes et al. 2007,Kroc, Sabharwal, and Selman 2008, Gomes, Sabharwal, and Selman 2006, Kroc, Sabharwal, and Selman 2008)
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How to bridge this gap?

- ExactCount(F, W): Compute $W(F)$?
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- ApproxCount$(F, W, \varepsilon, \delta)$: Compute C such that

$$\Pr[\frac{W(F)}{1+\varepsilon} \leq C \leq W(F)(1+\varepsilon)] \geq 1-\delta$$

From Weighted to Unweighted Counting

Boolean Formula F and weight function $W : \{0,1\}^n \rightarrow \mathbb{Q}^{\geq 0}$

Boolean Formula F'

$$W(F) = c(W) \times |\mathsf{Sol}(F')|$$

- Key Idea: Encode weight function as a set of constraints

(CF**M**V, IJCAI15)

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How do we estimate $|\mathsf{Sol}(F')|$?

Counting in Singapore

How many people in Singapore like coffee?

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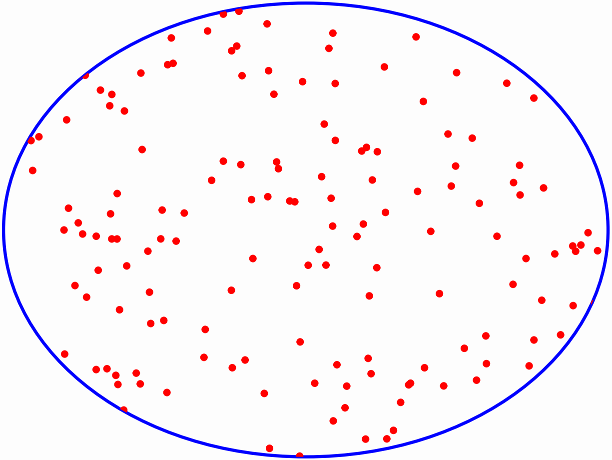
  - Potentially 2^n queries

Can we do with lesser # of SAT queries – $\mathcal{O}(n)$ or $\mathcal{O}(\log n)$?

As Simple as Counting Dots



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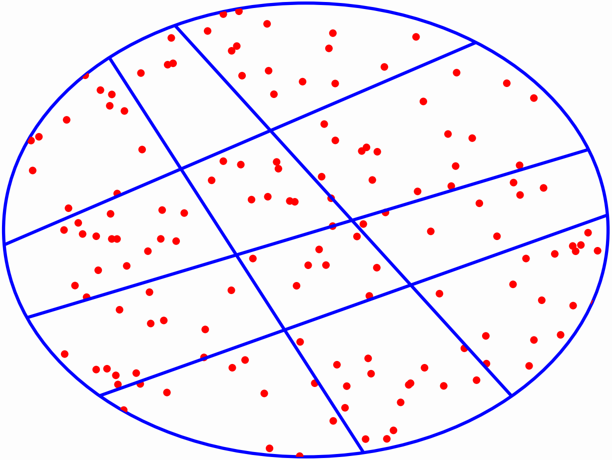
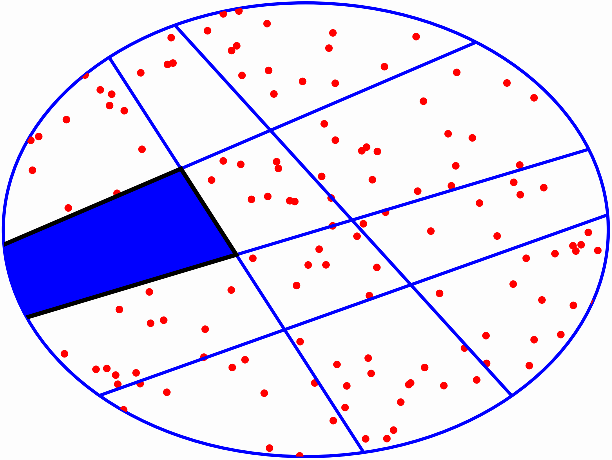


Pick a random cell



Estimate $=$ Number of solutions in a cell $\times$ Number of cells

Challenge 1 How to partition into roughly equal small cells of solutions without knowing the distribution of solutions?

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Challenge 2 How many cells?

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- Choose h randomly from a large family H of hash functions

  $\boxed{\text{Universal Hashing (Carter and Wegman 1977)}}$

2-Universal Hashing

- Let H be family of 2-universal hash functions mapping $\{0,1\}^n$ to $\{0,1\}^m$

$$\forall y_1, y_2 \in \{0,1\}^n, \alpha_1, \alpha_2 \in \{0,1\}^m, h \xleftarrow{R} H$$
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- The power of 2-universality
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  - $\sigma^2[Z] \leq \mathsf{E}[Z]$

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- Performance of state of the art SAT solvers degrade with increase in the size of XORs (SAT Solvers != SAT oracles)
- Two orders of magnitude reduction in the size of XORs by embedding formula into smaller dimension ("Independent Support")

(IMMV CP15, Best Student Paper) (IMMV Constraints16, Invited Paper)

Challenge 1 How to partition into roughly equal small cells of solutions without knowing the distribution of solutions?

- Independent Support-based 2-Universal Hash Functions

Challenge 2 How many cells?

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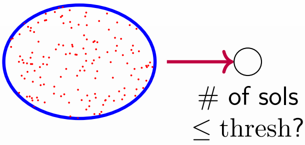
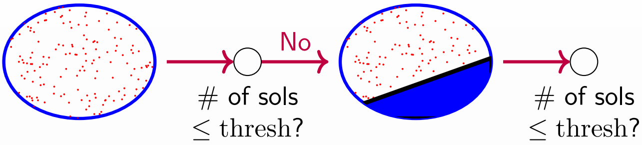
  - Check for every $m = 0, 1, \cdots n$ if the number of solutions $\leq \mathrm{thresh}$

ApproxMC(F, ε, δ)

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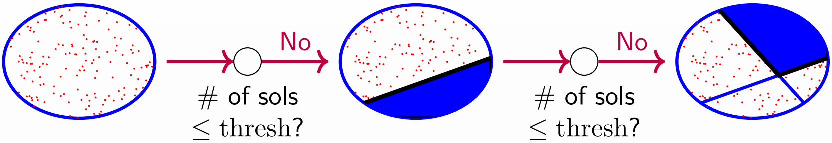

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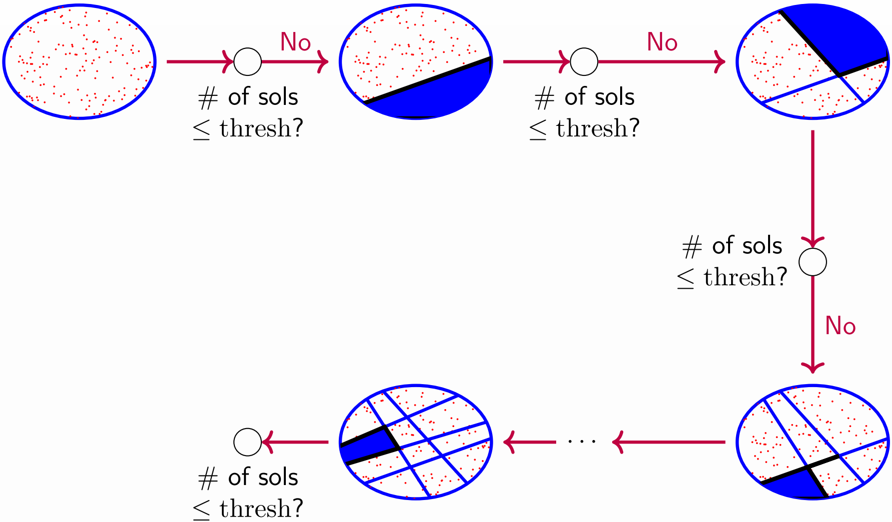

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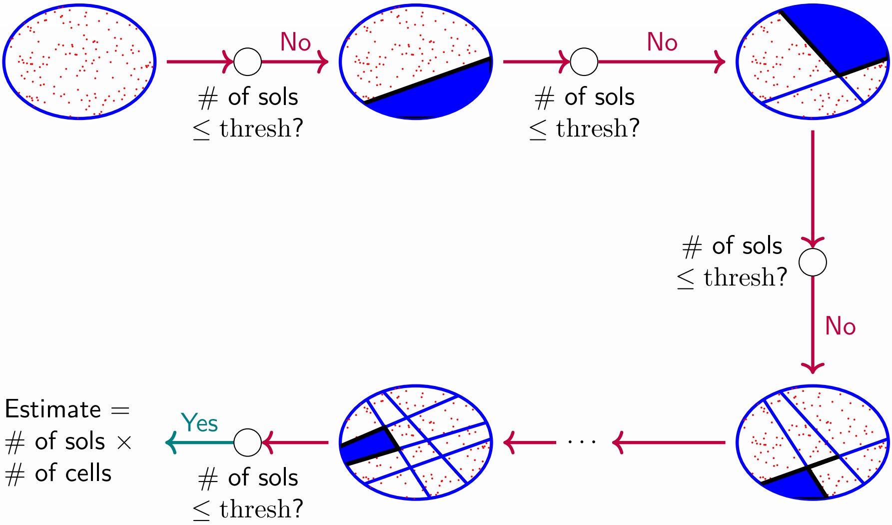

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  - Key Insight: The probability of making a bad choice of Q_i is very small for $i \ll m^*$

(CMV, IJCAI16)

Let $2^{m^*} = \frac{|\mathsf{Sol}(F)|}{\text{thresh}}$ $(m^* = \log(\frac{|\mathsf{Sol}(F)|}{\text{thresh}}))$

Lemma (1)

ApproxMC (F, ε, δ) *terminates with* $m \in \{m^* - 1, m^*\}$ *with probability* $\geq$ *0.8*

Lemma (2)

For $m \in \{m^* - 1, m^*\}$*, estimate obtained from a randomly picked cell lies within a tolerance of* ε *of* $|\mathsf{Sol}(F)|$ *with probability* $\geq$ *0.8*

ApproxMC(F, ε, δ)

Theorem (Correctness)

$$\Pr\left[\frac{|\mathsf{Sol}(F)|}{1+\varepsilon} \leq ApproxMC(F, \varepsilon, \delta) \leq |\mathsf{Sol}(F)|(1+\varepsilon)\right] \geq 1-\delta$$

Theorem (Complexity)

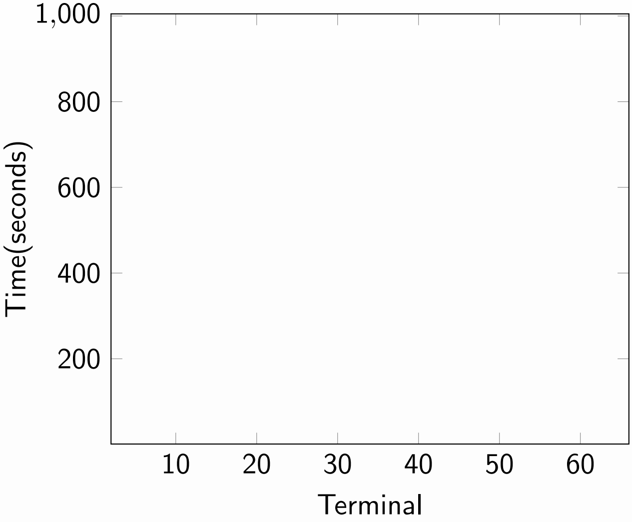
$ApproxMC(F, \varepsilon, \delta)$ makes $\mathcal{O}(\frac{\log n \log(\frac{1}{\delta})}{\varepsilon^2})$ calls to SAT oracle.

- *Prior work required $\mathcal{O}(\frac{\boldsymbol{n} \log \boldsymbol{n} \log(\frac{1}{\delta})}{\varepsilon})$ calls to SAT oracle (Stockmeyer 1983)*

Plantersville, SC

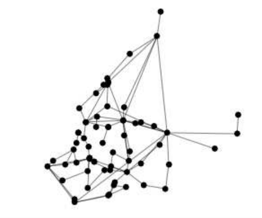
- $G = (V, E)$; source node: s
- Compute Pr[t is disconnected]?

Timeout = 1000 seconds



(DMPV, AAAI17)

Reliability of Critical Infrastructure Networks



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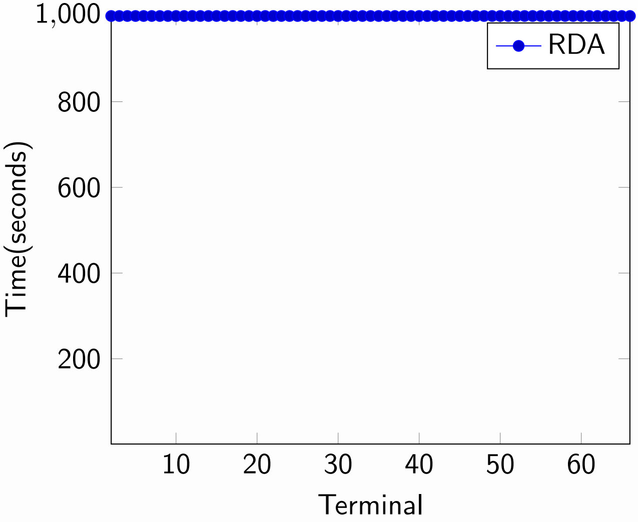

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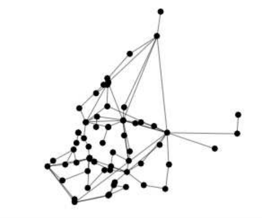

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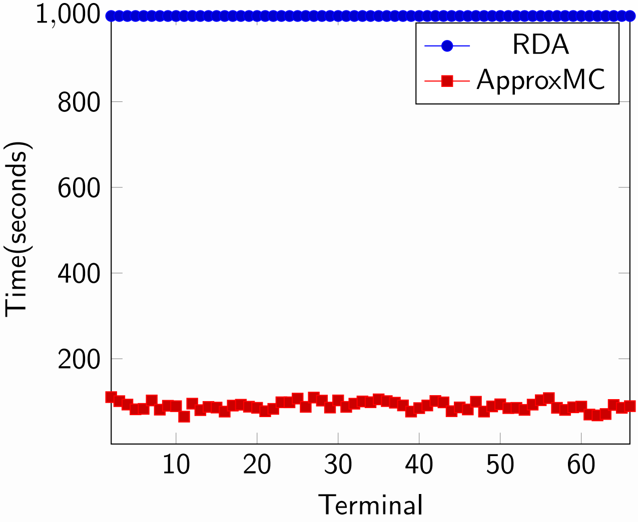

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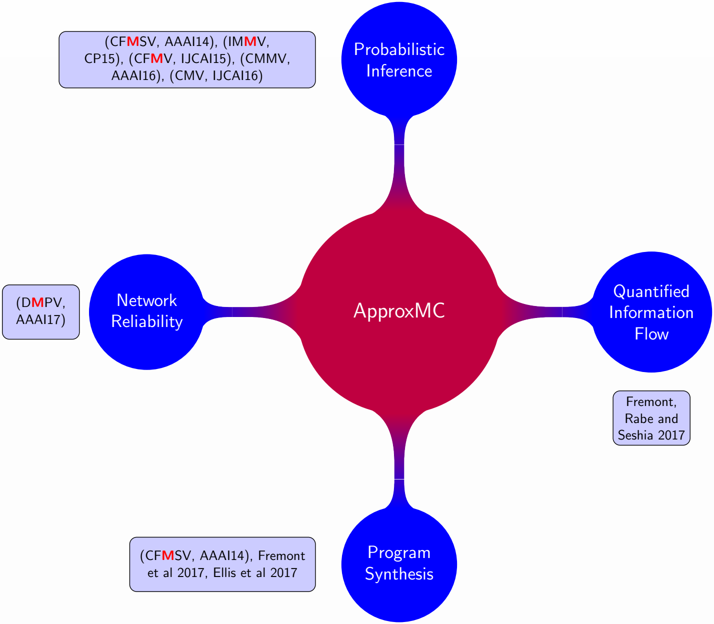
Beyond Network Reliability



Network Reliability

Probabilistic Inference

Constrained Counting

Network Reliability

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Hashing Framework

Network Reliability

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Hardware Validation

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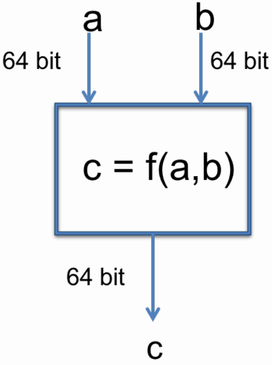
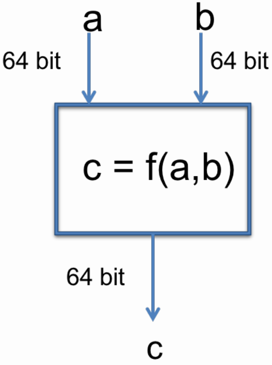


- Design is simulated with test vectors (values of *a* and *b*)
- Results from simulation compared to intended results



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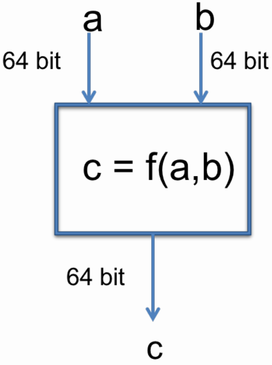

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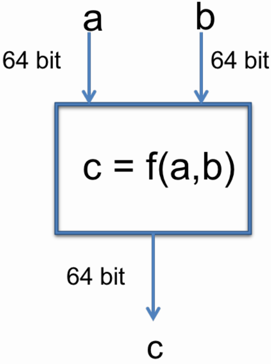
- Use constraints to represent *interesting* verification scenarios



Constraints

- Designers:
  - $a +_{64} 11 * 32b = 12$
  - $a <_{64} (b >> 4)$
- Past Experience:
  - $40 <_{64} 34 + a <_{64} 5050$
  - $120 <_{64} b <_{64} 230$
- Users:
  - $232 * 32a +_{64} b! = 1100$
  - $1020 <_{64} (b /_{64} 2) +_{64} a <_{64} 2200$

Test vectors: random solutions of constraints

Constrained Sampling

- Given:
  - Set of Constraints F over variables $X_1, X_2, \cdots X_n$
- Uniform Sampler

$$\forall y \in \mathsf{Sol}(F), \Pr[\text{y is output}] = \frac{1}{|\mathsf{Sol}(F)|}$$

- Almost-Uniform Sampler

$$\forall y \in \mathsf{Sol}(F), \frac{1}{(1+\varepsilon)|\mathsf{Sol}(F)|} \leq \Pr[\text{y is output}] \leq \frac{(1+\varepsilon)}{|\mathsf{Sol}(F)|}$$

Close Cousins: Counting and Sampling

- Approximate counting and almost-uniform sampling are inter-reducible (Jerrum, Valiant and Vazirani, 1986)
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- Is the reduction efficient?
  - Almost-uniform sampler (JVV) require linear number of approximate counting calls

Prior Work

Strong guarantees but poor scalability

- Polynomial calls to NP oracle (Bellare, Goldreich and Petrank,2000)
- BDD-based techniques (Yuan et al 1999, Yuan et al 2004, Kukula and Shiple 2000)
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Weak guarantees but impressive scalability

- Randomization in SAT solvers (Moskewicz 2001, Nadel 2011)
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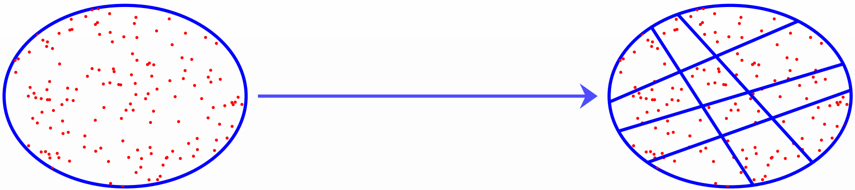
How to bridge this gap?

Key Ideas



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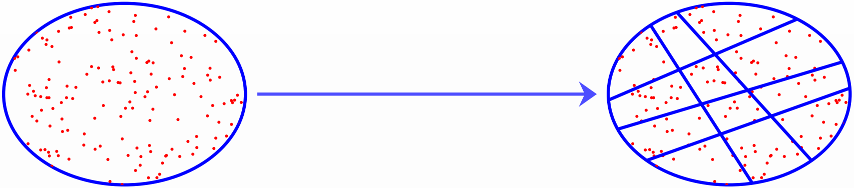

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  - Not just a practical hack required non-trivial proof

(C**M**V, CAV13)
(C**M**V, DAC14)
(CF**M**SV, TACAS15)

Theoretical Guarantees

Theorem (Almost-Uniformity)

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- *JVV (Jerrum, Valiant and Vazirani 1986) makes n calls*

Three Orders of Improvement

	Relative Runtime
SAT Solver	1
Desired Uniform Generator	10

(C**M**V, CAV13)
(C**M**V, DAC14)
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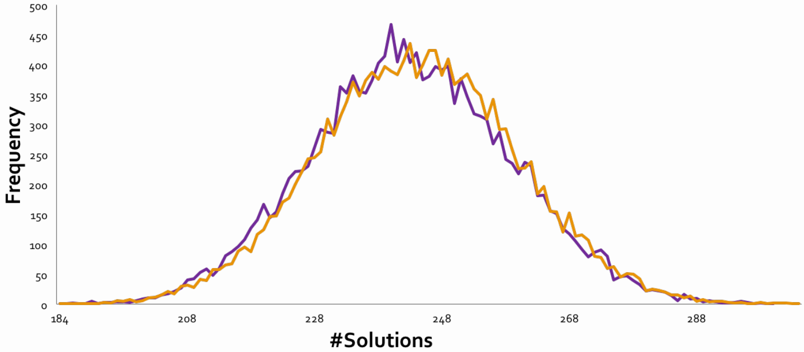
Closer to technical transfer

Uniformity



- Benchmark: case110.cnf; #var: 287; #clauses: 1263
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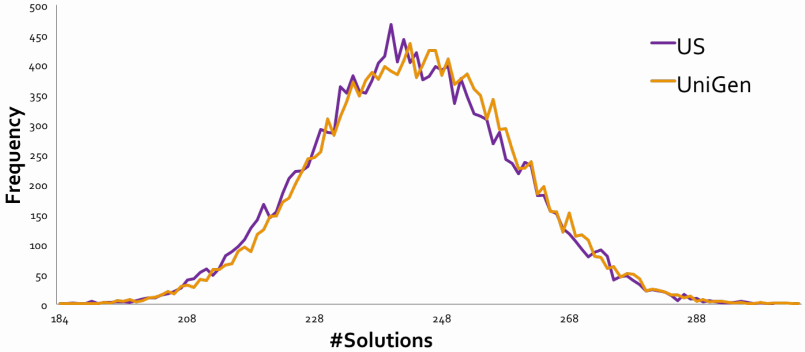
Statistically Indistinguishable



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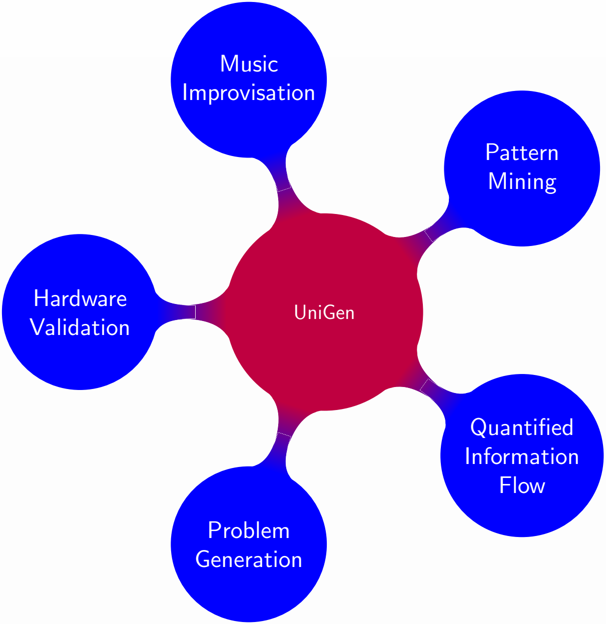
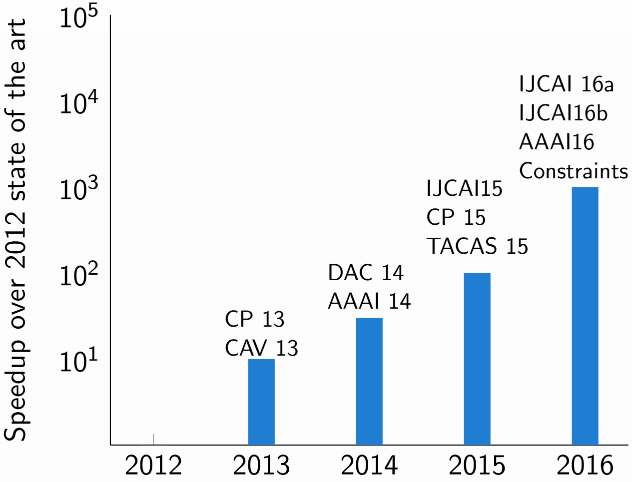
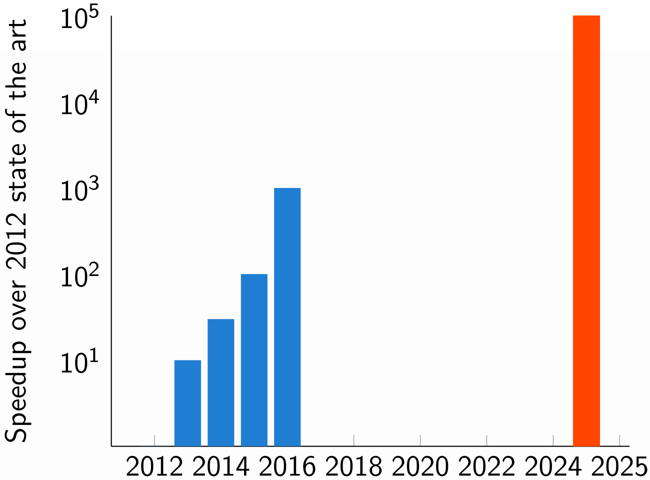
Usages of Open Source Tool: UniGen




Speedup over 2012 state of the art
10^5
10^4
10^3
10^2
10^1
CP 13
CAV 13
DAC 14
AAAI 14
IJCAI15
CP 15
TACAS 15
IJCAI 16a
IJCAI16b
AAAI16
Constraints
2012
2013
2014
2015
2016



Requires combinations of ideas from theory, statistics and systems

Mission 2025: Constrained Counting and Sampling Revolution

- Tighter integration between solvers and algorithms

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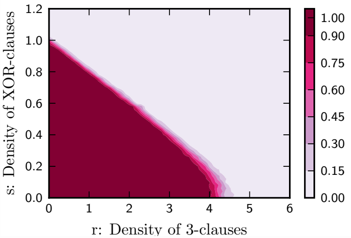
- Exploring solution space structure of CNF+XOR formulas

(D**M**V, IJCAI16),



Mission 2025: Constrained Counting and Sampling Revolution

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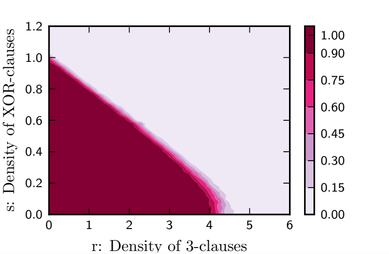
(D**M**V, IJCAI16),



- Beyond Boolean variables – without *bit blasting*

Mission 2025: Constrained Counting and Sampling Revolution

Challenge Problems

Mission 2025: Constrained Counting and Sampling Revolution

Challenge Problems

Civil Engineering Reliability for Los Angeles Transmission Grid

Mission 2025: Constrained Counting and Sampling Revolution

Challenge Problems

Civil Engineering Reliability for Los Angeles Transmission Grid

Privacy Leakage Measurement for C++ program with 100 lines

Mission 2025: Constrained Counting and Sampling Revolution

Challenge Problems

Civil Engineering	Reliability for Los Angeles Transmission Grid
Privacy	Leakage Measurement for C++ program with 100 lines
Artificial Intelligence	Inference for Bayesian network with 1K nodes

Mission 2025: Constrained Counting and Sampling Revolution

Challenge Problems

Civil Engineering Reliability for Los Angeles Transmission Grid

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The Potential of Hashing-based Framework

Machine Learning Probabilistic programming

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Challenge Problems

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Theory Classification of Approximate counting complexity

Mission 2025: Constrained Counting and Sampling Revolution

Challenge Problems

Civil Engineering	Reliability for Los Angeles Transmission Grid
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The Potential of Hashing-based Framework

Machine Learning	Probabilistic programming
Theory	Classification of Approximate counting complexity
Databases	Streaming algorithms

Collaborators



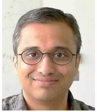


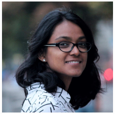








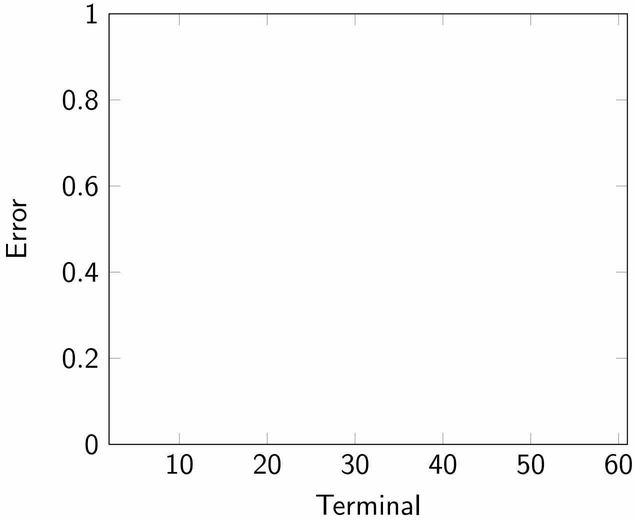




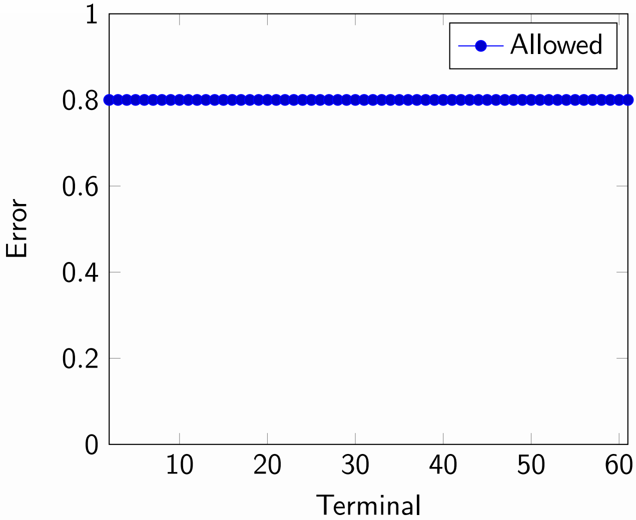
Part I

Backup

Highly Accurate Estimates



Highly Accurate Estimates



Highly Accurate Estimates