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Source: *Mathematics Magazine*, Vol. 77, No. 1 (Feb., 2004), pp. 3-14
Published by: Taylor & Francis, Ltd. on behalf of the Mathematical Association of America
Stable URL: https://www.jstor.org/stable/3219226
Accessed: 03-10-2024 04:06 UTC

Emil Post and His Anticipation of Gödel and Turing

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Emil Post is known to specialists in mathematical logic for several ideas in logic and computability theory: the structure theory of recursively enumerable sets, degrees of unsolvability, and the Post "correspondence problem." However, he *should* be known to a much wider audience. In the 1920s he discovered the incompleteness and unsolvability theorems that later made Gödel and Turing famous. Post missed out on the credit because he failed to publish his results soon enough, or in enough detail. His achievements were known to most of his contemporaries in logic, but this was seldom acknowledged in print, and he now seems to be slipping into oblivion. Recent comprehensive publications, such as Gödel's collected works and the popular history of computation by Martin Davis [**3**] contain only a few words about Post, mostly in footnotes.

In this article I hope to redress the balance a little by telling Post's side of the story and presenting the gist of his ideas. This is not merely to give Post his due; it gives the opportunity to present Post's approach to Gödel's incompleteness theorem, which is not only more general than Gödel's but also simpler. As well as this, Post drew some nontechnical conclusions from the incompleteness theorem—about the interplay between symbolism, meaning, and understanding—that deserve wide circulation in mathematics classrooms.

Post's life and career

Post's life occupied roughly the first half of the 20th century. Here is a brief summary of the main events.

1897 February 11: born Augustów, Poland.

1904 May: emigrated to New York.

1917 B.S. from City College.

1920 Ph.D. from Columbia.

1921 Decidability and completeness of propositional logic in *Amer. J. Math.* Foresaw undecidability and incompleteness of general formal systems.

1936 Independent discovery of Turing machines in *J. Symb. Logic.*

1938 October 28: met with Gödel to outline his discoveries.

1941 Submitted his "Account of an Anticipation" to *Amer. J. Math.*

1944 Paper on recursively enumerable sets in *Bull. Amer. Math. Soc.*

1947 Proved unsolvability of word problem for semigroups in *J. Symb. Logic.*

1954 Died in New York.

I shall elaborate on his discoveries, particularly the unpublished ones, below. But first it is important to appreciate the personal background of his work. Post's life was in some ways a typical immigrant success story: His family brought him to New York as a child, he studied and worked hard and, with the help of a supportive wife and daughter, obtained a position at City College of New York and some renown in his field of research. However, life was tougher for Post than this brief outline would suggest.

When quite young he lost his left arm in an accident, and this ended his early dream of a career in astronomy. Around the age of 13, Post wrote to several observatories asking whether his disability would prevent his becoming an astronomer. Harvard College Observatory thought not, but the head of the U.S. Naval Observatory replied that it would, because "the use of both hands is necessary in all the work of this observatory." Post apparently took his cue from the latter, gave up on astronomy, and concentrated on mathematics instead.

He attended Townsend Harris High School and City College in New York, obtaining a B.S. in mathematics in 1917. As an undergraduate he did original work in analysis which was eventually published in 1930. It includes a result on the Laplace transform now known as the Post-Widder inversion formula. From 1917 to 1920, Post was a graduate student in mathematical logic at Columbia. Part of his thesis, in which he proves the completeness and consistency of the propositional calculus of Whitehead and Russell's *Principia Mathematica*, was published in the *American Journal of Mathematics* [**8**].

In 1920–1921 he held a post-doctoral fellowship at Princeton. During this time he tried to analyze the whole *Principia*, with a view to proving its completeness and consistency as he had done for propositional calculus. This was the most ambitious project possible, because the axioms of *Principia* were thought to imply all theorems of mathematics. Nevertheless, Post made some progress: He showed that all theorems of *Principia* (and probably of any conceivable symbolic logic) could be derived by simple systems of rules he called *normal systems*. At first this looked like a great step forward. But as he struggled to analyze even the simplest normal systems, Post realized that the situation was the opposite of what he had first thought: instead of simplifying *Principia*, he had merely distilled its complexity into a smaller system.

Sometime in 1921, as he later claimed, he caught a glimpse of the true situation:

- Normal systems can simulate any symbolic logic, indeed any mechanical system for deriving theorems.
- This means, however, that all such systems can be mechanically listed, and the diagonal argument then shows that the general problem of deciding whether a given theorem is produced by a given system is unsolvable.
- It follows, in turn, that no consistent mechanical system can produce all theorems.

I shall explain these discoveries of Post in more detail below. They include (in different form) the discoveries of Turing on the nature of computability and unsolvability, and Gödel's theorem on the incompleteness of formal systems for mathematics.

In 1921, Post suffered an attack of manic-depressive illness (as bipolar disorder was known at the time), and his work was disrupted at the height of his creative fever. The condition recurred quite frequently during his life, necessitating hospitalization and preventing Post from obtaining an academic job until 1935. To avert the manic episodes, Post would give himself two problems to work on, switching *off* the one that was going well when he found himself becoming too excited. This did not always work, however, and Post often received the electroshock treatment that was thought effective in those days. (His death from a heart attack at the early age of 57 occurred shortly after one such treatment.)

In 1935, Post gained a foothold in academia with a position at City College of New York. The teaching load was 16 hours per week, and all faculty shared a single large office, so Post did most of his research at home, where his daughter Phyllis was required not to disturb him and his wife Gertrude handled all day-to-day concerns. As Phyllis later wrote (quoted by Davis [**2**]):

> My father was a genius; my mother was a saint ... the buffer in daily life that permitted my father to devote his attention to mathematics (as well as to his varied interests in contemporary world affairs). Would he have accomplished so much without her? I, for one, don't think so.

By this time Post had seen two of his greatest ideas rediscovered by others. In 1931 Gödel published his incompleteness theorem, and in 1935 Church stated *Church's thesis*, which proposes a definition of computability and implies the existence of unsolvable problems. Church's definition of computability was not immediately convincing (at least not to Gödel), and some equivalent definitions were proposed soon after. The one that convinced Gödel was Turing's [**14**], now known as the *Turing machine*. Post's normal systems, another equivalent of the computability concept, were still unpublished. But this time Post had a little luck. Independently of Turing, and at the same time, he had reformulated his concept of computation—and had found a concept virtually identical with Turing's! It was published in a short paper [**9**] in the 1936 *Journal of Symbolic Logic*, with a note from Church affirming its independence from Turing's work.

This gave Post some recognition, but he was still in Turing's shadow. Turing had written a fuller paper, with clearer motivation and striking theorems on the existence of a universal machine and unsolvable problems. The world knew that Post had also found the definition of computation, but did not know that he had already seen the *consequences* of such a definition in 1921. In 1938, he met Gödel and tried to tell him his story. Perhaps the excitement was too much for Post, because he seems to have feared that he had not made a good impression. The next day, October 29, 1938, he sent Gödel a postcard that reads as follows:

> I am afraid that I took advantage of you on this, I hope but our first meeting. But for fifteen years I had carried around the thought of astounding the mathematical world with my unorthodox ideas, and meeting the man chiefly responsible for the vanishing of that dream rather carried me away.
>
> Since you seemed interested in my way of arriving at these new developments perhaps Church can show you a long letter I wrote to him about them. As for any claims I might make perhaps the best I can say is that I would have have *proved* Gödel's theorem in 1921—had I been Gödel.

After a couple more letters from Post, Gödel replied. He courteously assured Post that he had not regarded Post's claims as egotistical, and that he found Post's approach interesting, but he did not take the matter any further.

In 1941, Post made another attempt to tell his story, in a long and rambling paper "Absolutely unsolvable problems and relatively undecidable propositions—an account of an anticipation" submitted to the *American Journal of Mathematics*. The stream-of-consciousness style of parts of the paper and lack of formal detail made it unpublishable in such a journal, though Post received a sympathetic reply from the editor, Hermann Weyl. On March 2, 1942, Weyl wrote

> ... I have little doubt that twenty years ago your work, partly because of its revolutionary character, did not find its true recognition. However, we cannot turn the clock back ... and the American Journal is not the place for historical accounts ... (Personally, you may be comforted by the certainty that most of

the leading logicians, at least in this country, know in a general way of your anticipation.)

Despite these setbacks Post continued his research. In fact his most influential work was yet to come. In 1943, he was invited to address the American Mathematical Society, and his writeup of the talk [**11**] introduced his groundbreaking theory of *recursively enumerable sets*. Among other things, this paper sets out his approach to Gödel's theorem, which is perhaps ultimate in both simplicity and generality. This was followed in 1945 by a short paper [**12**], which introduces the "Post correspondence problem," an unsolvable problem with many applications in the theory of computation. The correspondence problem can be viewed as a problem about free semigroups, and in 1947, Post showed the unsolvability of an even more fundamental problem about semigroups—the *word problem* [**13**].

The unsolvability of this problem is the first link in a chain between logic and group theory and topology. The chain was completed by Novikov [**7**] in 1955, who proved the unsolvability of the word problem for groups, by Markov [**6**] in 1958, who deduced from it the unsolvability of the homeomorphism problem for compact manifolds, and by Higman [**5**] in 1961, who showed that "computability" in groups is equivalent to the classical concept of finite generation.

Thus Post should be celebrated, not only for his fundamental work in logic, but also for constructing a bridge between logic and classical mathematics. Few people today cross that bridge, but perhaps if Post's work were better known, more would be encouraged to make the journey.

Formal systems

In the late 19th century several new branches of mathematics emerged from problems in the foundations of algebra, geometry, and analysis. The rise of new algebraic systems, noneuclidean geometry, and with them the need for new foundations of analysis, created the demand for greater clarity in both the subject matter and methods of mathematics. This led to:

1. Symbolic logic, where all concepts of logic were expressed by symbols and deduction was reduced to the process of applying *rules of inference*.
2. Set theory, in which all mathematical concepts were defined in terms of *sets* and the relations of *membership* and *equality*.
3. Axiomatics, in which theorems in each branch of mathematics were deduced from appropriate *axioms*.

Around 1900, these branches merged in the concept of a *formal system*, a symbolic language capable of expressing all mathematical concepts, together with a set of propositions (axioms) from which theorems could be derived by specific rules of inference. The definitive formal system of the early 20th century was the *Principia Mathematica* of Whitehead and Russell [**15**].

The main aims of *Principia Mathematica* were *rigor* and *completeness*. The symbolic language, together with an explicit statement of all rules of inference, allows theorems to be derived only if they are logical consequences of the axioms. It is impossible for unconscious assumptions to sneak in by seeming "obvious." In fact, all deductions in the *Principia* system can in principle be carried out *without knowing the meaning of the symbols*, since the rules of inference are pure symbol manipulations. Such deductions can be carried out by a machine, although this was not the intention of *Principia*, since suitable machines did not exist when it was written. The intention was to ensure rigor by keeping out unconscious assumptions, and in these terms *Principia* was a complete success.

As for completeness, the three massive volumes of *Principia* were a "proof by intimidation" that all the mathematics then in existence was deducible from the *Principia* axioms, but no more than that. It was not actually known whether *Principia* was even *logically* complete, that is, capable of deriving all valid principles of logic. In 1930, Gödel proved its logical completeness, but soon after he proved its *mathematical* incompleteness. We are now getting ahead of our story, but the underlying reason for Gödel's incompleteness theorem can be stated here: the weakness of *Principia* (and all similar systems) is its very objectivity. Since *Principia* can be described with complete precision, *it is itself a mathematical object*, which can be reasoned about. A simple but ingenious argument then shows that *Principia* cannot prove all facts about itself, and hence it is mathematically incomplete.

Post's program

Post began his research in mathematical logic by proving the completeness and consistency of *propositional logic*. This logic has symbols for the words *or* and *not*—today the symbols $\vee$ and $\neg$ are commonly used—and variables $P, Q, R, \ldots$ for arbitrary propositions. For example, $P \vee Q$ denotes "P or Q", and $(\neg P) \vee Q$ denotes "(not P) or Q". The latter is commonly abbreviated $P \to Q$ because it is equivalent to "P implies Q".

Principia Mathematica gave certain axioms for propositional logic, such as $(P \vee P) \to P$, and certain rules of inference such as the classical rule of *modus ponens*: from P and $P \to Q$, infer Q. Post proved that all valid formulas of propositional logic follow from the axioms by means of these rules, so *Principia* is *complete* as far as propositional logic is concerned.

Post also showed that propositional logic is consistent, by introducing the now familiar device of truth tables. Truth tables assign to each axiom the value "true," and each rule of inference preserves the value "true," so all theorems have the value "true" and hence are true in the intuitive sense. The same idea also shows that propositional logic is consistent in the formal sense. That is, it does not prove any proposition P together with its negation $\neg P$, since if one of these has the value "true" the other has the value "false." Together, the two results solve what Post called the *finiteness problem* for propositional logic: to give an algorithm that determines, for any given proposition, whether it is a theorem.

We now know that propositional logic is far easier than the full *Principia*. Indeed Post's results were already known to Bernays and Hilbert in 1918, though not published (see, for example, Zach [**16**]). However, what is interesting is that Post went straight ahead, attempting to analyze *arbitrary* rules of inference. He took a "rule of inference" to consist of a finite set of *premises*

$$g_{11}P_{i_{11}}g_{12}P_{i_{12}}\cdots g_{1m_1}P_{i_{1m_1}}g_{1(m_1+1)}$$

$$g_{21}P_{i_{21}}g_{22}P_{i_{22}}\cdots g_{2m_2}P_{i_{2m_2}}g_{2(m_2+1)}$$

$$\cdots$$

$$g_{k1}P_{i_{k1}}g_{k2}P_{i_{k2}}\cdots g_{km_k}P_{i_{km_k}}g_{k(m_k+1)},$$

which together produce a *conclusion*

$$g_1P_{i_1}g_2P_{i_2}\cdots g_mP_{i_m}g_{m+1}.$$

The g_{ij} are certain specific symbols or strings of symbols, such as the $\rightarrow$ symbol in modus ponens, and the P_{kl} are arbitrary strings (such as the P and Q in modus ponens). Each P_{kl} in the conclusion is in at least one of the premises. Such "rules" include all the rules of *Principia* and, Post thought, any other rules that draw conclusions from premises in a determinate way.

The problem of analyzing such "production systems" amounts to understanding all possible formal systems, a task of seemingly overwhelming proportions. However, Post initially made surprising progress. By the end of the 1920–21 academic year he had proved his *normal form theorem*, which says that the theorems of any production system can be produced by a *normal system* with a single axiom and rules of only the simple form

$$gP \quad \text{produces} \quad Pg'.$$

In other words, any string beginning with the specific string g may be replaced by the string in which g is removed and g' is attached at the other end.

Normal systems include an even simpler class of systems that Post called "tag" systems, in which each g' depends only on the initial letter of g and all g have the same length. One such system uses only the letters 0 and 1, and each g has length 3. If g begins with 0, then $g' = 00$, and if g begins with 1 then $g' = 1101$. The left-hand end of the string therefore advances by three places at each step, trying to "tag" the right-hand end which advances by two or four places. For example, here is what happens when the initial string is 1010:

1010
01101
0100
000
00
0

and then the string becomes empty. In all cases that Post tried, the result was either termination (as here) or periodicity, but he was unable to decide whether this was always the case. In fact, as far as I know the general behavior of this tag system is still not known. Post tried reducing the length of g and allowing more than two symbols, but it did not help.

> ... when this possibility was explored in the early summer of 1921, it rather led to an overwhelming confusion of classes of cases, with the solution of the corresponding problem depending more and more on problems in ordinary number theory. Since it had been our hope that the known difficulties of number theory would, as it were, be dissolved in the particularities of this more primitive form of mathematics, the solution of the general problem of "tag" appeared hopeless, and with it our entire program of the solution of finiteness problems. [**10**, p. 24]

After a few fruitless attempts to escape the difficulties with different normal forms, Post realized what the true situation must be: Theorems can indeed be produced by simple rules, but only because *any* computation can be reduced to simple steps. Predicting the outcome of simple rules, however, is no easier than deciding whether arbitrary sentences of mathematics are theorems. This fuller realization of the significance of the previous reductions led to a reversal of our entire program. [**10**, p. 44]

The reverse program was easier than the one he had set himself initially, which was essentially the following:

1. Describe all possible formal systems.
2. Simplify them.
3. Hence solve the deducibility problem for all of them.

Post's success in reducing complicated rules to simple ones convinced him that, for *any* system generating strings of symbols, there is a normal system that generates the same strings. But *it is possible to enumerate all normal systems*, since each consists of finitely many strings of symbols on a finite alphabet, and hence it is possible to enumerate all systems for generating theorems. This invites an application of the diagonal argument, described below. The outcome is that *for certain formal systems the deducibility problem is unsolvable.*

After this dramatic change of direction Post saw the true path as follows:

1. Describe all possible formal systems.
2. Diagonalize them.
3. Show that some of them have unsolvable deducibility problem.

And he also saw one step further—the incompleteness theorem—because:

4. No formal system obtains all the answers to an unsolvable problem.

Post's approach to incompleteness

We shall deal with Step 4 of Post's program first, because it is quite simple, and it dispels the myth that incompleteness is a difficult concept. Certainly, it rests on the concept of computability, but today we can define computability as "computable by a program in some standard programming language," and most readers will have a reasonable idea what this means.

Let us define an algorithmic problem, or simply *problem*, to be a computable list of questions:

$$P = \langle Q_1, Q_2, Q_3, \ldots \rangle$$

For example, the problem of recognizing primes is the list

$$\langle \text{“Is 1 prime?”}, \text{“Is 2 prime?”}, \text{“Is 3 prime?”}, \ldots \rangle$$

A problem is said to be *unsolvable* if the list of answers is not computable. The problem of recognizing primes is of course *solvable.*

Now suppose that an unsolvable $P = \langle Q_1, Q_2, Q_3, \ldots \rangle$ exists.

Then no consistent formal system F proves all correct sentences of the form

$$\text{“The answer to } Q_i \text{ is } A_i\text{.”},$$

since by systematically listing all the theorems of F we could compute a list of answers to problem P.

Thus any consistent formal system F is *incomplete* with respect to sentences of the form "The answer to Q_i is A_i": there are some true sentences of this form that F does not prove.

It is true that there are several matters arising from this argument. What is the significance of consistency? Are there unprovable sentences in mainstream mathematics? But for Post incompleteness was a simple consequence of the existence of unsolvable problems. He also saw unsolvable problems as a simple consequence of the diagonal argument (described in the next section).

The really *big* problem, in Post's view, was to show that all computation is reflected in normal systems. Without a precise definition of computation, the concept of unsolvable problem is meaningless. Gödel was lucky not to be aware of this very general approach to incompleteness. His approach was to analyze *Principia Mathematica* (and "related systems") and prove its incompleteness directly. He did not see incompleteness as a consequence of unsolvability, in fact did not *believe* that computability could be precisely defined until he read Turing's paper [**14**], where the concept of Turing machine was defined.

Thus Post's proof of incompleteness was delayed because he was trying to do so much: The task he set himself in 1921 was in effect to do most of what Gödel, Church, and Turing did among them in 1931–36. In 1936, Church published a definition of computability [**1**] and gave the first published example of an unsolvable problem. But "Church's thesis"—that here was a precise definition of computability—was not accepted until the equivalent Turing machine concept appeared later in 1936, along with Turing's very lucid arguments for it.

As mentioned above, Post arrived at a similar concept independently [**9**], so in fact he completed his program in 1936. By then, unfortunately, it was too late for him to get credit for anything except a small share of the computability concept.

The diagonal argument

The diagonal argument is a very flexible way of showing the incompleteness of infinite lists: lists of real numbers, lists of sets of natural numbers, and lists of functions of natural numbers. It was perhaps implicit in Cantor's 1874 proof of the uncountability of the real numbers, but it first became clear and explicit in his 1891 proof, which goes as follows.

Suppose that $x_1, x_2, x_3, \ldots$ is a list of real numbers. More formally, suppose that to each natural number n there corresponds a real number x_n, and imagine a tabulation of the decimal expansions of these numbers one above the other, say

$$\begin{aligned} x_1 &= 3.\underline{1}4159\ldots \\ x_2 &= 2.7\underline{1}828\ldots \\ x_3 &= 1.41\underline{4}21\ldots \\ x_4 &= 0.577\underline{2}1\ldots \\ x_5 &= 1.6180\underline{3}\ldots \\ &\vdots \end{aligned}$$

A number x *not* on the list can always be constructed by making x differ from each x_n in the nth decimal place. For example, one can take the nth decimal place of x to be 1 if the nth decimal place of x_n is not 1, and 2 if the nth decimal place of x_n is 1. With the list above, we get the number

$$x = 0.22111\ldots.$$

The method for producing this new number x is called "diagonal," because it involves only the diagonal digits in the tabulation of $x_1, x_2, x_3, \ldots$.

It is commonly thought that the diagonal method is *nonconstructive*, but in fact the diagonal number x is clearly computable from the tabulation of $x_1, x_2, x_3, \ldots$. Indeed, one needs to compute only one decimal place of x_1, two decimal places of x_2, three decimal places of x_3, and so on. Turing observed that this tells us something interesting about computable real numbers [**14**].

It is *not* the case that there are uncountably many computable reals, because there are only countably many Turing machines (or programs in a fixed programming language, as we would prefer to define the concept of computation today) and at most one computable number is defined by each machine. Indeed, a real number is defined only if the machine behaves in a special way. In Turing's formulation the machine must print the successive digits of the number on specified squares of the machine's tape and must not change any digit once it is printed.

It would therefore seem, by the diagonal argument, that we could compute a number x *different* from each of the computable numbers $x_1, x_2, x_3, \ldots$. What is the catch?

There is no problem computing a list of all Turing machines, or programs. All of them are sequences of letters in a fixed finite alphabet, so they can be enumerated in lexicographic order. Also, once each machine is written down we can run it to produce digits of the number it defines, if any. The catch is that *we cannot identify all the machines that define computable real numbers.* The problem of recognizing all such machines is *unsolvable* in the sense that no Turing machine can correctly answer all the questions

Does machine 1 define a computable real?
Does machine 2 define a computable real?
Does machine 3 define a computable real? ....

There cannot be a Turing machine that solves this problem, otherwise we could hook it up to a machine that diagonalizes all the computable numbers and hence compute a number that is not computable.

What prevents the identification of machines that define computable numbers? When one explores this question, other unsolvable problems come to light. For example, we could try to catch all machines that fail to define real numbers by attaching to each one a device that halts computation as soon as the machine makes a misstep, such as changing a previously printed digit. As Turing pointed out, this implies the unsolvability of the *halting problem*: to decide, for any machine and any input, whether the machine eventually halts (or performs any other specific act). This problem is a perpetual thorn in the side of computer programmers, because it means that there is no general way to decide whether programs do what they are claimed to do. Unsolvable problems also arise in logic and mathematics, because systems such as predicate logic and number theory are capable of simulating all Turing machines. This is how Church and Turing proved the unsolvability of the *Entscheidungsproblem*, the problem of deciding validity of formulas in predicate logic.

Post's application of the diagonal argument Post also used the diagonal argument, but in the form used by Cantor (1891) to prove that any set has more subsets than elements. Given any set X, suppose each member $x \in X$ is paired with a subset $S_x \subseteq X$. Then the *diagonal* subset $D \subseteq X$ defined by

$$x \in D \leftrightarrow x \notin S_x$$

is different from each S_x, with respect to the element x.

The computable version of the diagonal argument takes X to be the set $\mathbb{N}$ of natural numbers, and for each $n \in \mathbb{N}$, what Post called the nth *recursively enumerable* subset S_n of $\mathbb{N}$. A recursively enumerable (r.e.) set is one whose members may be computably listed, and there are various ways to pair Turing machines with r.e. sets. For example, S_n may be defined as the set of input numbers m for which the nth machine has a halting computation. There is no loss of generality in considering the elements of an r.e. set to be numbers, because any string of symbols (in a fixed alphabet) can be encoded by a number.

A typical r.e. set is the set of theorems of a formal system, which is why Post was interested in the concept. Each theorem T is put into a machine, which systematically applies all rules of inference to the axioms, halting if and only if T is produced. Another example, which gives the flavor of the concept in a setting more familiar to mathematicians, consists of the strings of digits between successive 9s in the decimal expansion of π. Since

$$\pi = 3.14159\underline{265358}9\underline{7}9\underline{323846264338327}9\underline{5028841}9\underline{716}9\underline{3}9937510\ldots,$$

the set in question is

$$S = \{265358, 7, 323846264338327, 5028841, 716, 3, \ldots\}.$$

It is clear that a list of members of S can be computed, since π is a computable number, but otherwise S is quite mysterious. We do not know how to decide membership for S, or even whether S is infinite. This is typical of r.e. sets, and useful to keep in mind when constructing r.e. sets that involve arbitrary computations.

The diagonal set D is not r.e., being different from the nth r.e. set S_n with respect to the number n; however, its complement $\overline{D}$ *is* r.e. This is because

$$n \in \overline{D} \leftrightarrow n \in S_n,$$

so any $n \in \overline{D}$ will eventually be found by running the nth machine on input n. Thus *$\overline{D}$ is an example of an r.e. set whose complement is not r.e.*. It follows that no machine can decide, for each n, whether $n \in \overline{D}$ (or equivalently, whether $n \in S_n$). If there were such a machine, we could list all the members of D by asking

$$\begin{gathered}\text{Is } 1 \in S_1?\\ \text{Is } 2 \in S_2?\\ \text{Is } 3 \in S_3?\\ \ldots\end{gathered}$$

and collecting the n for which the answer is no.

It also follows that *no consistent formal system can prove all theorems of the form* $n \notin S_n$, since this would yield a listing of D. This is a version of the incompleteness theorem, foreseen by Post in 1921, but first published by Gödel in 1931 [**4**].

Differences between Post and Gödel

As we have seen, Post's starting point was the concept of computation, which he believed could be formalized and made subject to the diagonal argument. Diagonalization yields problems that are *absolutely* unsolvable, in the sense that no computation can solve them. In turn, this leads to *relatively* undecidable propositions, for example, propositions of the form $n \notin S_n$. No consistent formal system F can prove all true propositions of this form, hence any such F must fail to prove some true proposition $n_0 \notin S_{n_0}$. But this proposition is only relatively undecidable, not absolutely, because F can be consistently extended by adding it as an axiom.

Gödel did not at first believe in absolutely unsolvable problems, because he did not believe that computation is a mathematical concept. Instead, he proved the existence of relatively undecidable propositions directly, by constructing a kind of diagonal argument inside *Principia Mathematica*. Also, he *arithmetized* the concept of proof there, so provability is expressed by a number-theoretic relation, and his undecidable proposition belongs to number theory. Admittedly, Gödel's proposition is not otherwise interesting to number theorists, but Gödel saw that it is interesting for another reason: *it expresses the consistency of* Principia Mathematica.

This remarkable fact emerges when one pinpoints the role, in the incompleteness proof, of the assumption that the formal system F is consistent, as we will soon explain. It seems that Gödel deserves full credit for this observation, which takes logic even higher than the level reached with the discovery of incompleteness.

Outsmarting a formal system We now reflect on Post's incompleteness proof for a formal system F, to find an explicit n_0 such that $n_0 \notin S_{n_0}$ is true but not provable by F.

It is necessary to assume that F is consistent, because an inconsistent formal system (with a modicum of ordinary logic) proves everything. In fact it is convenient to assume more, namely, that F proves only true propositions. Now consider the r.e. set of propositions of the form $n \notin S_n$ proved by F. The corresponding numbers n also form an r.e. set, with index n_0 say. That is,

$$S_{n_0} = \{n : F \text{ proves } n \notin S_n\}.$$

By definition of S_{n_0}, $n_0 \in S_{n_0}$ implies that F proves the proposition $n_0 \notin S_{n_0}$. But if so, $n_0 \notin S_{n_0}$ is true, and we have a contradiction. Thus the truth is that $n_0 \notin S_{n_0}$, *but F does not prove this fact.*

It seems that we know more than F, but how come? The "extra smarts" needed to do better than F lie in the ability to recognize that F is consistent (or, strictly speaking, that all theorems of F are true). In fact, what we have actually proved is the theorem

$$\text{Con}(F) \to n_0 \notin S_0,$$

where $\text{Con}(F)$ is a proposition that expresses the consistency of F. It follows that $\text{Con}(F)$ *is not provable in* F, otherwise the proposition $n_0 \notin S_{n_0}$ would also be provable (by modus ponens). But if we can "see" $\text{Con}(F)$, then we can "see" $n_0 \notin S_{n_0}$.

If F is a really vast system, like *Principia Mathematica* or a modern system of set theory, then it takes a lot of chutzpah to claim the ability to see $\text{Con}(F)$. But the incompleteness argument also applies to modest systems of number theory, which everybody believes to be consistent, *because we know an interpretation of the axioms*: $1, 2, 3, \ldots$ stand for the natural numbers, $+$ stands for addition, and so on. Thus the ability to see meaning in a formal system F actually confers an advantage: It allows us to see $\text{Con}(F)$, and hence to see propositions not provable by F.

Now recall how this whole story began. *Principia Mathematica* and other formal systems F were constructed in the belief that there was everything to gain (in rigor, precision, and clarity) and nothing to lose in treating deduction as computation with meaningless symbols. Gödel showed that this is not the case. Loss of meaning causes loss of theorems, such as $\text{Con}(F)$. It is surprising how little this is appreciated. More than 60 years ago Post wrote:

> It is to the writer's continuing amazement that ten years after Gödel's remarkable achievement current views on the nature of mathematics are thereby affected only to the point of seeing the need of many formal systems, instead of a universal one. Rather has it seemed to us to be inevitable that these developments will

result in a reversal of the entire axiomatic trend of the late 19th and early 20th centuries, with a return to meaning and truth. [**10**, p. 378]

Perhaps it is too much to expect a "reversal of the entire axiomatic trend," but a milder proposal seems long overdue. Post's words should be remembered every time we plead with our students not to manipulate symbols blindly, but to understand what they are doing.

Acknowledgments. The biographical information in this article is drawn mainly from Martin Davis's introduction to Post's collected works [**2**] and the web site of the American Philosophical Society, where most of Post's papers are held. I am also indebted to Martin Davis and John Dawson for supplying me with photocopies of correspondence between Post and Gödel.

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