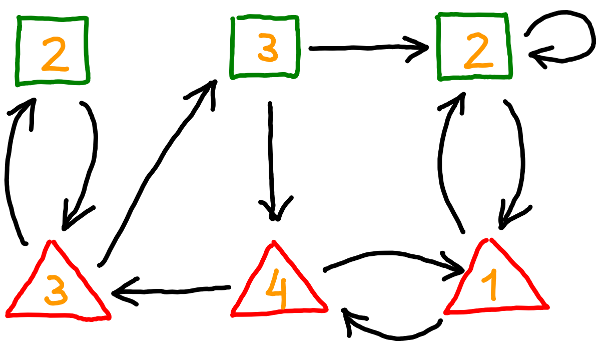
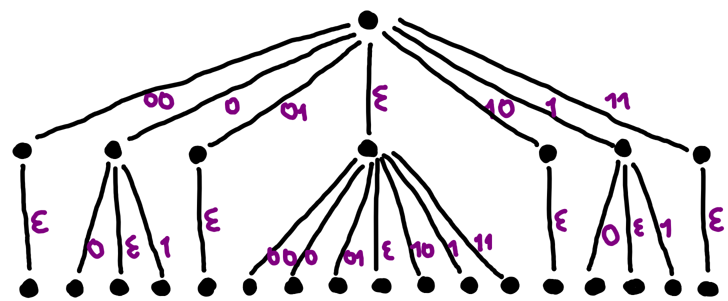
Parity Games and Universal Trees





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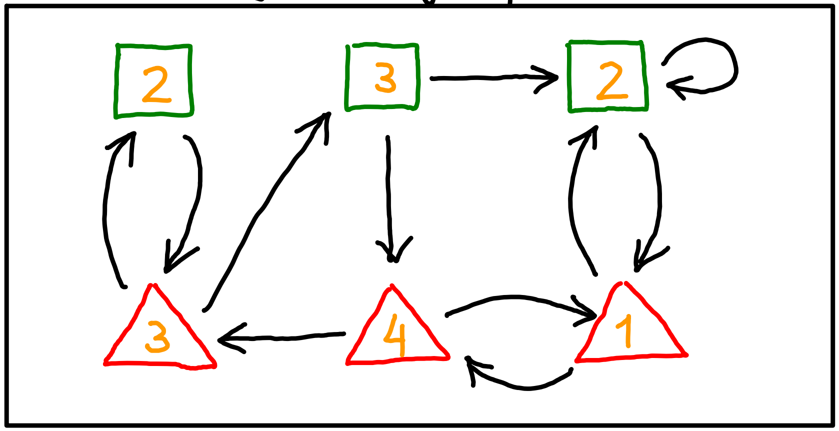
PARITY GAMES

$n = |V|$

A game graph



$$G = (V = V_{Even} \uplus V_{Odd}, E, \pi)$$

$$\pi : V \to \{1, 2, 3, 4, 5, \dots, d\}$$

PARITY GAMES

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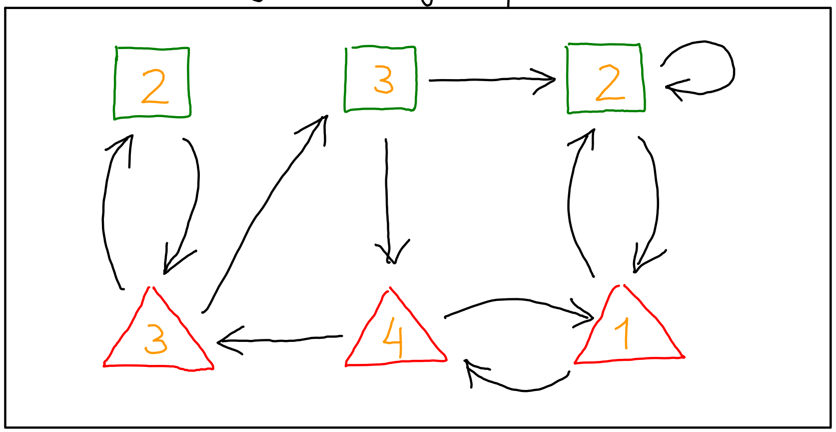

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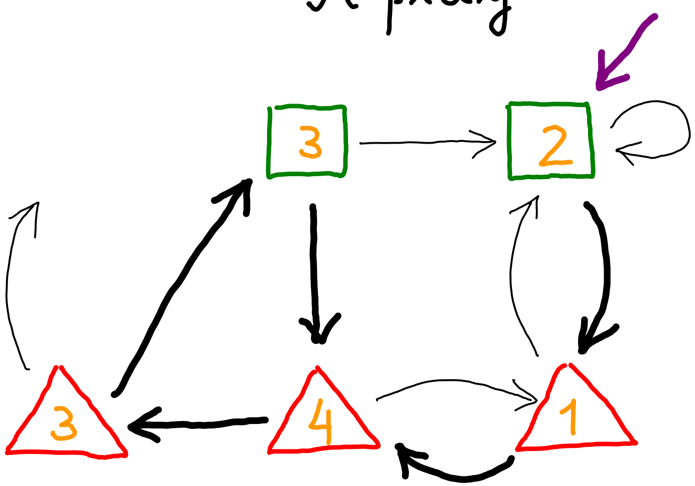
A play



PARITY GAMES

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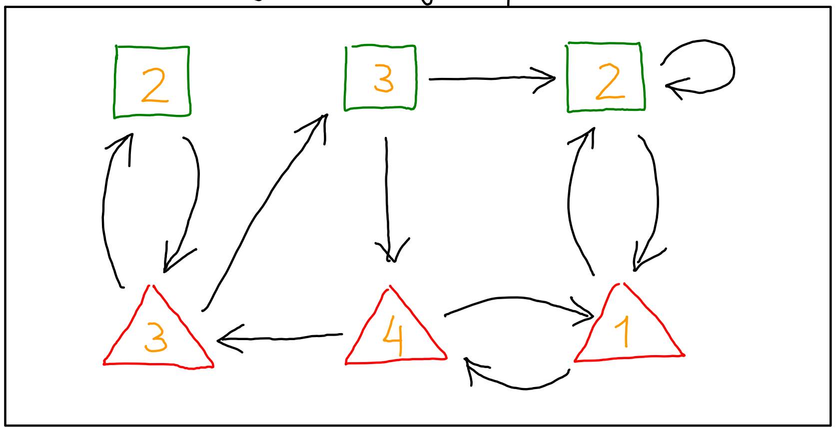

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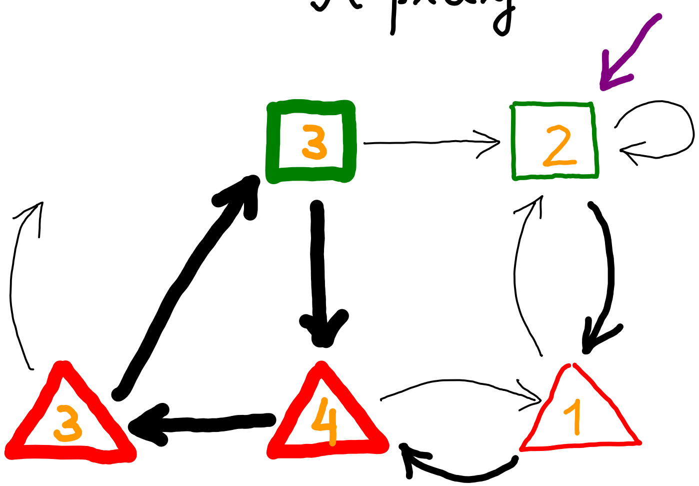
A play

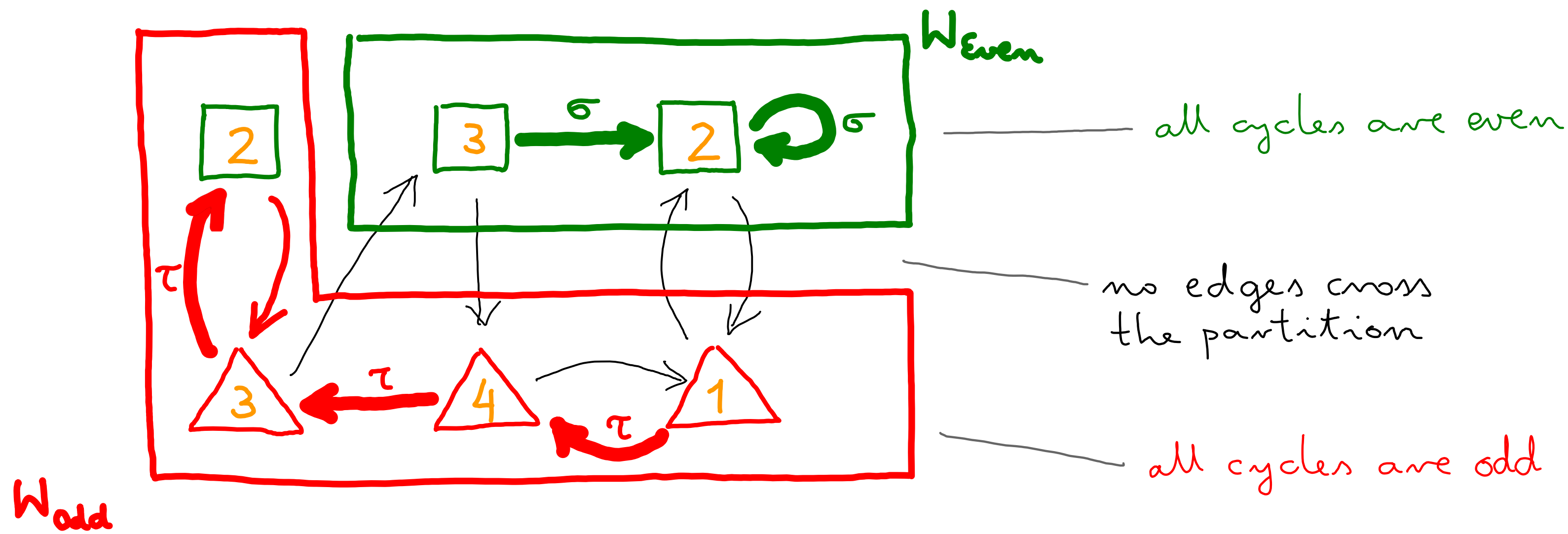


Even wins $\langle p_1, p_2, p_3, \ldots \rangle$
iff
max p_i on the cycle is even

SHORT WITNESSES FROM POSITIONAL DETERMINACY

THEOREM [Emerson, Jutla 1991; Mostowski 1991; re-proved since 1960's]

Parity games are positionally determined



COROLLARY [Emerson, Jutla, Sistla 1993]

Deciding the winner in parity games is $\boxed{\text{in } NP \cap co\text{-}NP}$

ALGORITHMS FOR SOLVING PARITY GAMES

- $\boxed{n^{d+O(1)}}$ [McNaughton 1993 ; Zielonka 1998]
- $\boxed{n^{\frac{d}{2}+O(1)}}$ [Browne, Clarke, Jha, Long, Marrero 1994; Seidl 1996; J. 2000]
- $\boxed{n^{d+O(1)}}$ strategy iteration [Vöge, J. 2000]

  $\boxed{2^{\Omega(n)}}$ [Friedmann 2009]

  - policy iteration for MDPs [Fearnley 2010]
  - randomized simplex [Hansen, Friedmann, Zwick 2011]
- $\boxed{n^{O(\sqrt{n})}}$ [Björklund, Sandberg, Vorobyov 2003; J., Paterson, Zwick 2006]
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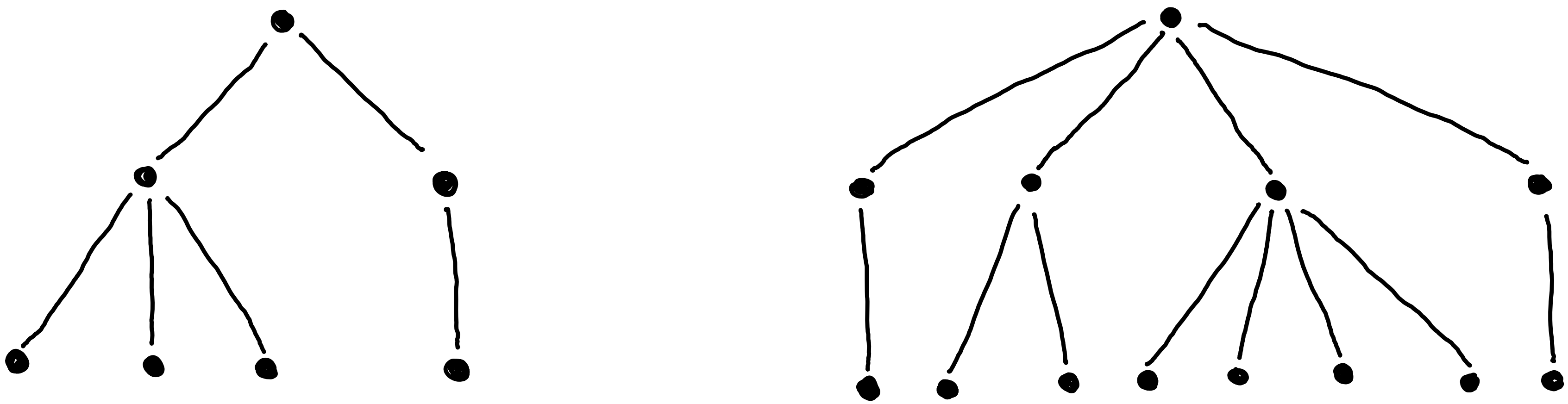
- $\boxed{n^{\lg d + O(1)}}$ [**Calude, Jain, Khoussainov, Li, Stephan 2017**
  J., Lazić 2017; Lehtinen 2018; **Parys 2019**]

ORDERED TREES



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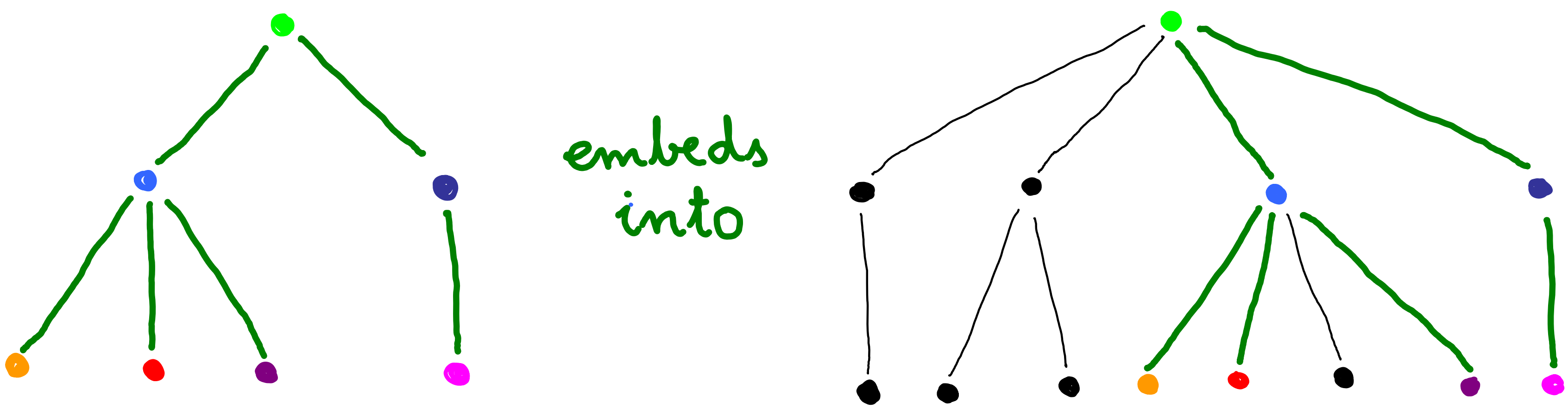
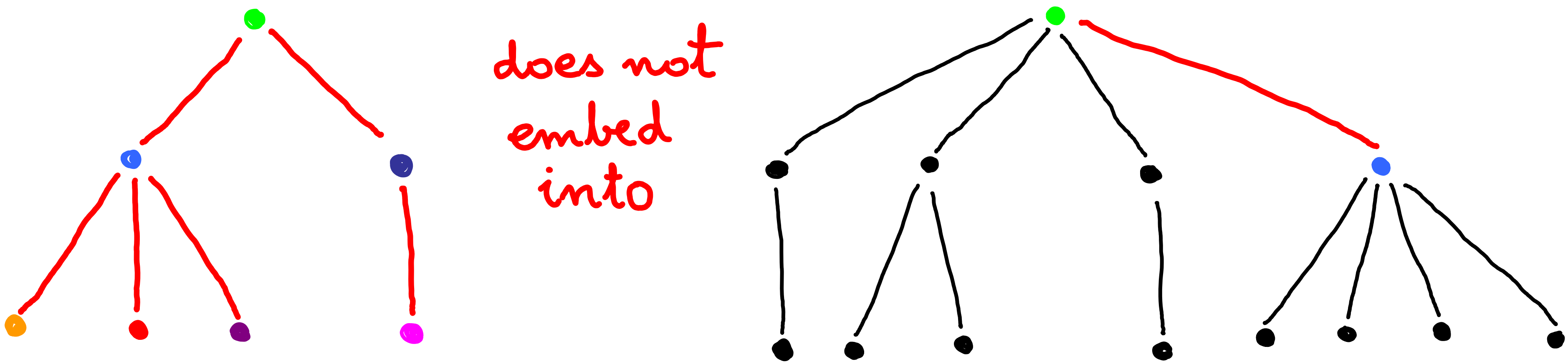
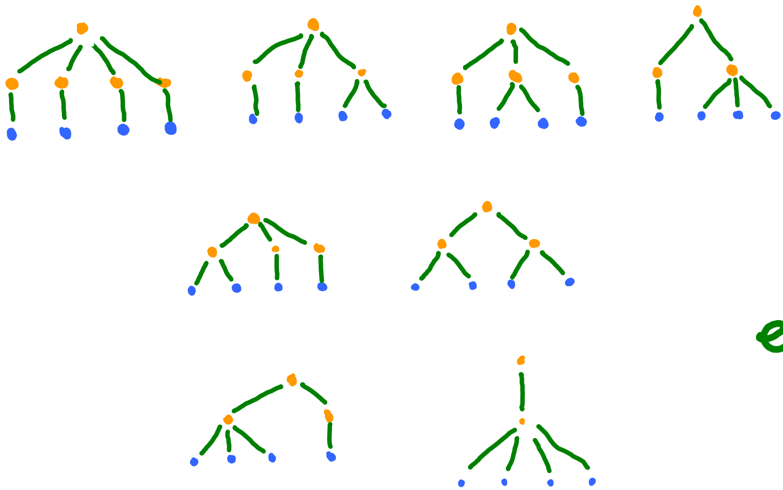


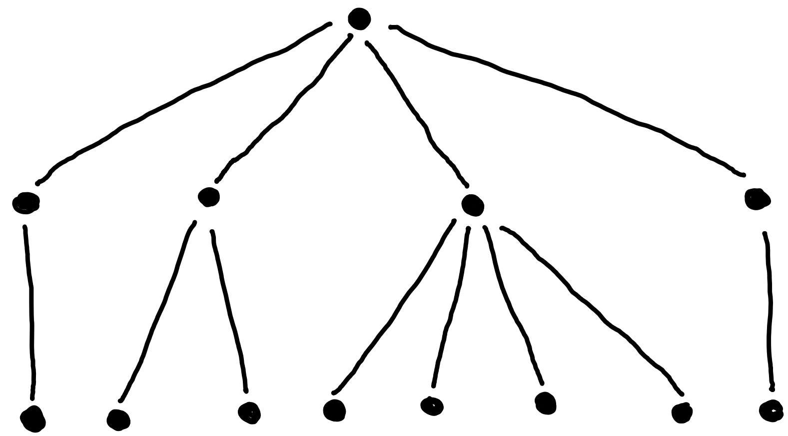
ORDERED TREES



UNIVERSAL TREES



all embed into

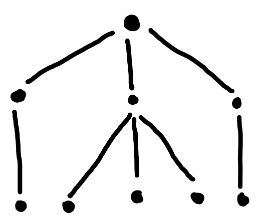


Leaves: 4
height: 2

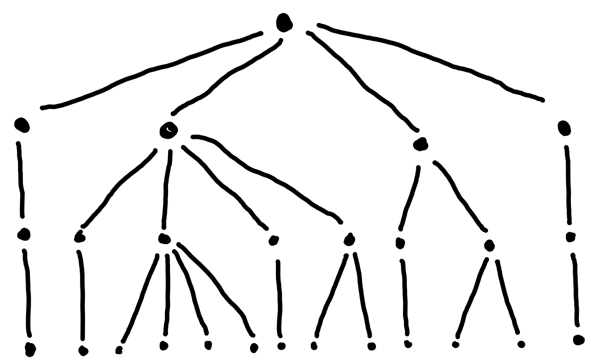
(4, 2)-universal tree

Universal trees

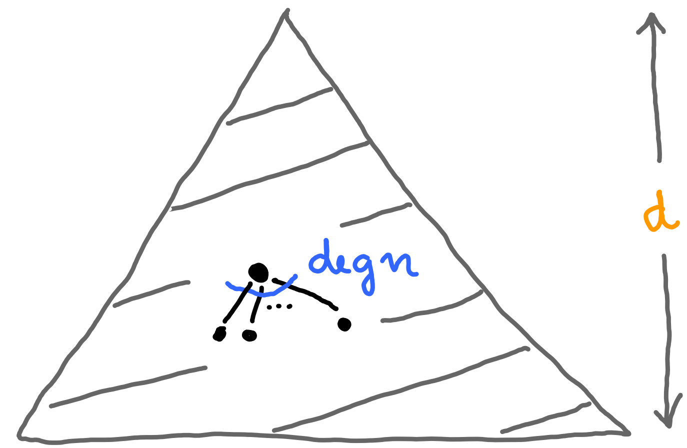
Definition T is an (n, d)-universal tree if every tree with n leaves and of height d embeds into T



$(3,2)$-universal



$(4,3)$-universal

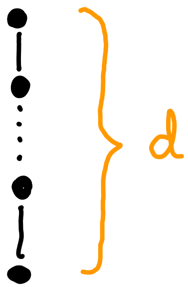


(n,d)-universal

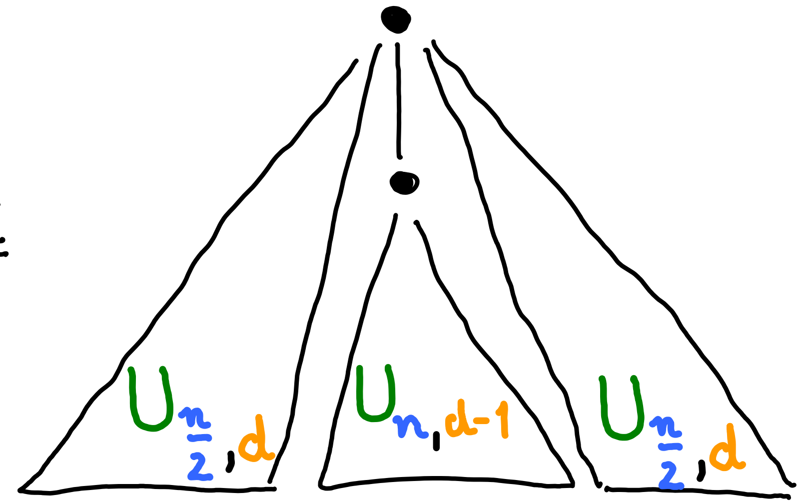
$\Theta(n^d)$ size

QUASI-POLYNOMIAL UNIVERSAL TREES

$U_{1,d} \stackrel{df}{=}$



$U_{n,d} \stackrel{df}{=}$



$U_{n,1} \stackrel{df}{=}$