PSI2 : Envelope Perfect Sampling of Non Monotone Systems

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***Abstract*—The famed perfect sampling method of Propp and Wilson [3] uses a backward coupling scheme to compute unbiased samples of the stationary distribution of Markov chains. It has been implemented in a software tool, called PSI2, that proved very efficient for monotone chains coming from queuing networks [4]. However, when the system includes at least one non-monotone event, the backward simulation scheme has to consider all possible states as starting points. This can be avoided by taking a new point of view that consists in bounding all possible trajectories of the Markov chain by envelopes. The new version of PSI2 presented here implements these latest improvements, including *envelope* techniques and *splitting*. Envelopes have been introduced by Bušić et al [1]. As soon as envelopes couple, then all trajectories must have coupled, so that an unbiased sample is obtained. As for splitting, it consists in generating all the trajectories of the Markov chains inside the envelopes to run a classical backward coupling technique from that point on.**

Combining these two techniques in PSI2 makes it a more efficient tool that covers a wider class of queuing networks than previously. This includes networks with non-monotone events such as negative customers, arrivals by batches, forks and joins as well as cox-distribution for services.

I. Envelope Perfect Sampling

Let $\{X_n\}_{n\in\mathbb{N}}$ be an irreducible and aperiodic discrete time Markov chain with a finite state space $\mathcal{X}$ and a transition matrix $P = (p_{i,j})$. Let π denote the steady state distribution of the chain: $\pi = \pi P$. The evolution of the Markov chain can always be described by a stochastic recurrence sequence $X_{n+1} = \Phi(X_n, e_{n+1})$, with $\{e_n\}_{n\in\mathbb{Z}}$ an independent and identically distributed sequence of events $e_n \in \mathcal{E},\ n \in \mathbb{N}$. The transition function $\Phi : \mathcal{X} \times \mathcal{E} \to \mathcal{X}$ verifies the property that $\mathbb{P}(\Phi(i,e) = j) = p_{i,j}$ for every pair of states $(i,j) \in \mathcal{X} \times \mathcal{X}$ and a random event $e \in \mathcal{E}$.

As proved in [3], there is $\ell \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \left|\Phi\left(\ldots\Phi\left(\Phi\left(\mathcal{X}, e_1\right), e_2\right), \ldots, e_n\right)\right| = \ell \text{ almost surely.}$$

The Markov chain couples if $\ell = 1$. In that case, this singleton is steady state distributed and its *coupling time* is the minimum value of n such that the set is reduced to the singleton.

A. Envelope Perfect Sampling Algorithm

When the state space $\mathcal{X}$ is equipped with a complete lattice order relation $\preceq$, we bound the trajectories of the Markov chain starting from all possible states by envelopes that can be used to build a new perfect sampler of a Markov chain, given in Algorithm 1.

Algorithm 1: Envelope Perfect Sampling Algorithm (EPSA)

```
Data: Φ , {e_{-n}}_{n∈ℕ}
Result: A state with stationary distribution
begin
    n = 1; M := Top; m := Bottom;
    repeat
        for i = n − 1 downto 0 do
            m := inf_{m⪯x⪯M} Φ(x, e);
            M := sup_{m⪯x⪯M} Φ(x, e);
        /* Splitting into S = {x ∈ X, m ⪯ x ⪯ M}
        trajectories is done here, if |M − m| < L */
        n := 2n;
    until M = m ;
    return M;
end
```

The trajectories of m, M as constructed in Algorithm EPSA are the envelopes of the chain. All states of the chain stay in $S = \{x \in \mathcal{X}, m \preceq x \preceq M\}$. Therefore, as soon as $m = M$ (after, say τ_e steps, called the coupling time of EPSA), the chain has coupled at state $m = M$ that is an unbiased sample of π.

The gain, comparing to classical perfect sampling algorithm, is that the complexity does not depend on the size of the state space. However, the coupling time of envelopes τ_e might be larger than the coupling time τ for the initial chain. Therefore, the efficiency of envelopes depends on the comparison of τ_e with $\tau|\mathcal{X}|$.

In the framework of queuing networks, this comparison is often in favor of envelopes [1]. For example, batch arrivals, negative customers, fork and join nodes, Cox service times all produce events that are non-monotone but for which envelopes are easy to compute and have a short coupling time.

B. Splitting

Even if the two envelopes do not couple or if their coupling time is too long, they still may get close. One way to take benefit from the envelopes in that case is to continue the simulation once the gap between the envelopes reaches some threshold L using a classical PSA, *i.e.* simulating the chain starting with all states between the upper and lower trajectories. This is called splitting in the following and can be added in EPSA at the point where the comment is inserted.

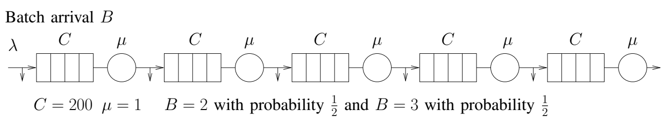
II. Implementation and Application Examples

The new version of Psi2 (version 4.4.6), `http://psi.gforge.inria.fr/`, handles envelopes and splitting in its kernel. Users define the computation of the envelopes of any event as a function in an external library. Then, the main loop of PSI2 will call that function over a couple of states (m, M) corresponding to the current lower and the upper envelope, anytime the corresponding event is drawn. Splitting is an option of envelopes, where the users specify the condition on the size of the interval $[m, M]$ to trigger the splitting of trajectories.

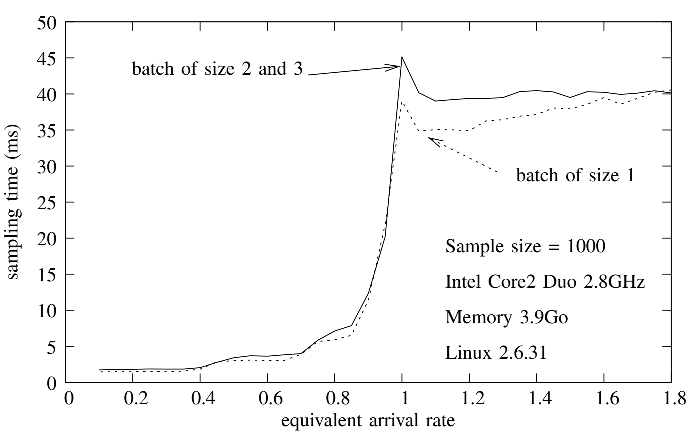
In the following subsections, we illustrate the efficiency of the new PSI2 by showing its behavior in two examples.

Batch arrivals

Consider a network of 5 $M/M/1$ queues in tandem with overflow and batch arrivals, as displayed below.



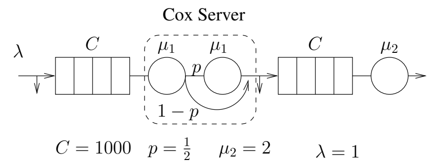
The whole batch is rejected if it cannot enter the queue entirely. This is a non monotone event. Indeed, if the number of packets in the first queue is 199 (resp. 198) and a batch arrival of size 2 occurs, then the number of packets becomes 199 (resp. 200), so that the order between the states is changed.



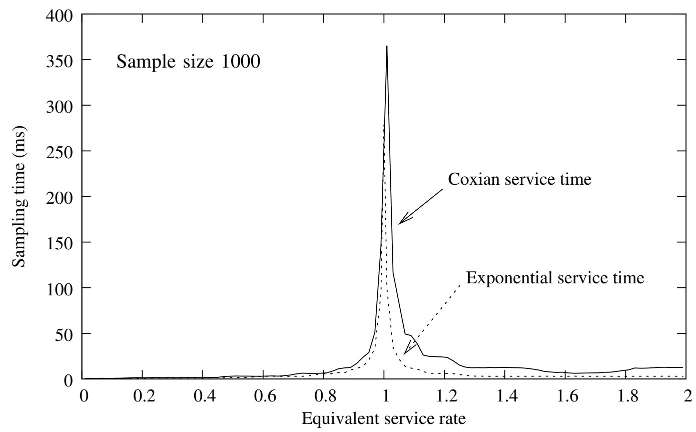
The time (in *ms*) to compute 1000 samples of the stationary distribution of the system using PSI2 is displayed in the figure above (solid line). It never gets over 50 ms while the state space size is $|\mathcal{X}| = 200^5 \geq 3 \cdot 10^{11}$. Furthermore, the execution time is almost the same as for 5 queues in tandem with the same load but individual arrivals that is monotone (dashed line). In that case, the envelope method extends the applicability of perfect sampling at almost no cost.

Tandem queues and Cox service

Here, we consider two queues in tandem where the first queue has a service with a Cox-2 distribution.



Again, the Cox service event is not monotone.



As shown in the figure above, once again, PSI2 performance is almost the same when the service in the first queue is Coxian (solid line) and when it is exponential with an equivalent rate. The noticeable pick when the load is one is because the coupling time behaves as the square of the capacity of queues in that case instead of being almost linear in all other cases [2].

III. Conclusions and Future Works

In the current version of PSI2, the envelopes for several non-monotone events for queues have been implemented. The current library contains generalized JSQ (index), negative customers, forks and joins, batch arrivals, cox-2 servers.

Future additions of available envelopes concern general phase-type service distributions.

Another improvement to get faster execution include selection of events taking into account the current envelope state and the tuning of the splitting threshold based on coupling time estimations.

References

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