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Research Article

The number of spanning trees in a superprism

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Abstract

Let the vertices of two disjoint and equal length cycles be denoted $u_0, u_1, \ldots, u_{n-1}$ in the first cycle and $v_0, v_1, \ldots, v_{n-1}$ in the second cycle for $n \geq 4$. The superprism $\breve{P}_n$ is defined as the graph obtained by adding to these disjoint cycles all edges of the form $u_i v_i$ and $u_i v_{i+2 \pmod n}$. In this paper, it is proved that the number of spanning trees in $\breve{P}_n$ is $n \cdot 2^{3n-2}$.

Keywords: spanning trees; antiprism graph; enumeration of trees; circulant graph; prism graph.

2020 Mathematics Subject Classification: 05C05, 05C30.

1. Introduction

In this paper, by a graph $G = (V, E)$, we mean an undirected graph without loops and parallel edges. Also, throughout this paper, the number of spanning trees in G, which represent the total number of distinct trees on all vertices of $V(G)$, is denoted by $t(G)$. Let the vertices of two disjoint and equal length cycles be denoted $u_0, u_1, \ldots, u_{n-1}$ in the first cycle and $v_0, v_1, \ldots, v_{n-1}$ in the second cycle for $n \geq 4$. The 4-regular, *superprism* $\breve{P}_n$ on $2n \geq 8$ vertices is defined as the graph obtained by adding to these disjoint cycles all edges of the form $u_i v_i$ and $u_i v_{i+2 \pmod n}$. We prove that the number of spanning trees in $\breve{P}_n$ is $n \cdot 2^{3n-2}$. Prism and antiprism are well-known graphs, which are closely related to superprism.

The prism P_n of order $2n$ is the cubic graph obtained from the cycles $(u_0, u_1, \ldots, u_{n-1})$ and $(v_0, v_1, \ldots, v_{n-1})$ by adding all edges of the form $u_i v_i$. The graph P_n can also be defined as a Cartesian product of the cycle C_n on n vertices and path K_2 on 2 vertices, denoted by $C_n \square K_2$ [3, 4]. The number of spanning trees in P_n that we established in [3] is

$$t(P_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2].$$

The antiprism $\bar{P}_n$ of order $2n$, for $n \geq 3$, is the quartic graph obtained from the cycles $(u_0, \ldots, u_{n-1})$ and $(v_0, \ldots, v_{n-1})$ by adding all edges of the form $u_i v_i$ and $u_i v_{i+1 \pmod n}$ [8, 11]. The following is known about $\bar{P}_n$:

Theorem 1.1 (see [11]). *Let $Ci_{2n}(1,2)$ be a circulant of order $2n$. Then, $\bar{P}_n \simeq Ci_{2n}(1,2)$.*

Circulant $Ci_n(1,2)$ is also called the square of a cycle [1]. The following is also known:

Theorem 1.2 (see [1]). *The number of spanning trees in the square of cycle for $n \geq 5$ is given by $\frac{n}{5}[(\frac{3+\sqrt{5}}{2})^n - (\frac{3-\sqrt{5}}{2})^n - 2(-1)^n]$.*

Based on Theorems 1.1 and 1.2, we obtain the number of spanning trees in $\bar{P}_n$ for $n \geq 3$, which is not explicitly published in literature, by substituting n with $2n$ in $t(Ci_{2n}(1,2))$. So, we have, $t(\bar{P}_n) = \frac{2n}{5}[(\frac{3+\sqrt{5}}{2})^{2n} - (\frac{3-\sqrt{5}}{2})^{2n} - 2]$.

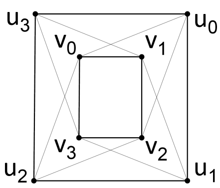

Figure 1.1: Smallest superprism, i.e., $\breve{P}_4$.

Note that the superprism is not planar, as opposed to either a prism or an antiprism. In Figure 1.1, we illustrate the smallest example of the superprism, i.e., $\breve{P}_4$. It is easy to verify that $\breve{P}_4$ is not planar.

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In general, the number of spanning trees can be determined for any graph based on the Kirchhoff Matrix Tree Theorem [9], as opposed to other parameters, e.g., the number of cycles, which cannot be determined as easily. The Kirchhoff characteristic matrix A_k of graph of order k with vertices $V(G) = \{v_1, v_2, \ldots, v_k\}$ is $k \times k$ symmetric matrix $[a_{i,j}]$, where (1) $a_{i,j} = -1$ if v_i, v_j are adjacent, (2) $a_{i,i}$ equals the degree of v_i, and (3) $v_{i,j} = 0$ in all other cases. The Kirchhoff matrix tree theorem states that for any graph G with at least two vertices, all the cofactors of A_k are equal, and they are equal to $t(G)$. Nevertheless, for the special family of graphs knowing the number of spanning trees based on an explicit formula turned out to be quite useful in many instances, because the Kirchhoff matrix tree theorem requires calculation of a determinant of the Kirchhoff characteristic matrix. This might become problematic for very large matrices. One of the first and simplest derived formulas for the number of spanning trees is due to Cayley [5]. It states that the complete graph K_n on n vertices has n^{n-2} number of spanning trees. In this paper, we derive a simple formula for the number of spanning trees of the superprism $\breve{P}_n$. This formula is almost as simple as Cayley's formula for $t(K_n)$ and it is much simpler than the formulas for prism and antiprism described above. Other explicit formulas for the special families of graphs can be found in the number of publications, e.g., [2,6,7,10,13–17]. In particular, there are many papers covering the number of spanning trees in the circulant graphs [1,4,6,8,10,16,17], which are related to our superprism through antiprism, as we indicated above.

In Section 2, a set of matrices is defined, and the relations between determinants of these matrices are derived. Based on these relations, in Section 3, a recurrence relationship for $t(\breve{P}_n)$ is derived, which proves the main result (Theorem 3.1).

2. Preliminary results

Let $t(G)$ denote the number of spanning trees in G. In order to derive the number of spanning trees $t(\breve{P}_n)$, we establish a recursion that is satisfied by the Kirchhoff cofactor of $\breve{P}_n$.

Based on the definition of $\breve{P}_n$, we first assign labels to the vertices of $\breve{P}_n$ as follows: (1) assign odd numbers $1, 3, \ldots, 2n-1$ to $u_0, u_1, \ldots, u_{n-1}$, and (2) assign even numbers $2, 4, \ldots, 2n$ to $v_0, v_1, \ldots, v_{n-1}$. We then form the Kirchhoff characteristic matrix A_{2n} based on these labels, and focus our attention on the principal $(2n-1) \times (2n-1)$ submatrix of A_{2n} obtained by canceling its last row and column corresponding to vertex v_{n-1} (e.g., vertex v_3 in Figure 1.1) labeled with $2n$. So, the number of spanning trees of $\breve{P}_n$ equals $t(\breve{P}_n) = \mathsf{det}(A_{2n-1})$, where $A_{2n-1} = [a_{i,j}]$ is defined as follows:

$$a_{i,j} = \begin{cases} 4 & \text{for } i = j, \\ -1 & \text{for } |i-j| = 2, \\ -1 & \text{if } |i-j| = 1 \text{ or } |i-j| = 5, \text{ and } (i+j+1) \equiv 0 \pmod 4, \\ -1 & \text{if } i = 1 \text{ and } j = 2n-1, \text{ or } i = 2n-1 \text{ and } j = 1, \\ -1 & \text{if } i = 2 \text{ and } j = 2n-3, \text{ or } i = 2n-3 \text{ and } j = 2, \\ -1 & \text{if } i = 4 \text{ and } j = 2n-1, \text{ or } i = 2n-1 \text{ and } j = 4, \\ 0 & \text{otherwise} \end{cases}$$

Our matrix A_{2n-1} is given as follows:

$$A_{2n-1} = \begin{bmatrix} 4 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & & 0 & 0 & -1 \\ -1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & & -1 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 & 0 & -1 & & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 & & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & 4 & 0 & -1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 & & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 & 4 & & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 4 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & & -1 & 0 & 4 \end{bmatrix}.$$

For convenience and clarity of the proofs, we define the following two matrices associated with A_{2n-1}. First, matrix B_{2n-1} is obtained from A_{2n-1} by subtracting the fourth row from the first row, followed by subtracting the fourth column from the first column. Let $\mathsf{det}(A_{2n-1}) = a_{2n-1}$ and $\mathsf{det}(B_{2n-1}) = b_{2n-1}$. Second, matrix C_{2n-1} is obtained from A_{2n-1} by replacing six unity elements $a_{1,2n-1}, a_{2,2n-3}, a_{4,2n-1}, a_{2n-1,1}, a_{2n-3,2}, a_{2n-1,4}$ by 0. Let $C_{2k-1} = C_{2n-1}$ for $n = k$, and let C_{2k-3} be obtained from C_{2k-1} by cancelling first two rows and first two columns of C_{2k-1} for $n \geq k \geq 4$. This recursion defines matrices $C_{2n-1}, C_{2n-3}, C_{2n-5}, \ldots, C_7, C_5$:

$$B_{2n-1} = \left[\begin{array}{cccccccccccc}
8 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\
0 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & & -1 & 0 & 0 \\
0 & 0 & 4 & -1 & -1 & 0 & 0 & -1 & & 0 & 0 & 0 \\
-4 & -1 & -1 & 4 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 & & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 4 & 0 & -1 & & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 & & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & -1 & 4 & & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & & 4 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 4 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & & -1 & 0 & 4
\end{array}\right],$$

where $a_{2n-1} = b_{2n-1}$,

$$C_{2k-1} = \left[\begin{array}{cccccccccccc}
4 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & & 0 & 0 & 0 \\
-1 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\
-1 & 0 & 4 & -1 & -1 & 0 & 0 & -1 & & 0 & 0 & 0 \\
0 & -1 & -1 & 4 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 4 & -1 & -1 & 0 & & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 & 4 & 0 & -1 & & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 4 & -1 & & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & -1 & 4 & & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 4 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & -1 & 0 & 4
\end{array}\right].$$

In addition, in order to derive the recurrence relationships in the supporting lemmas, presented in the next section, we also define auxiliary matrices as follows:

1. Matrix D_{2n-2} is obtained by canceling the first row and first column in A_{2n-1}.

$$D_{2n-2} = \left[\begin{array}{ccccccccccc}
4 & 0 & -1 & & & & & & -1 & 0 & 0 \\
0 & 4 & -1 & -1 & 0 & 0 & -1 & & & & 0 \\
-1 & -1 & 4 & 0 & -1 & & & & & & -1 \\
 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
 & 0 & -1 & \cdot & & & & & & & \\
 & 0 & & \cdot & & & & & & & \\
 & -1 & & \cdot & & & & & & & \\
 & & & \cdot & & & C_{2n-5} & & & & \\
 & & & \cdot & & & & & & & \\
 & & & \cdot & & & & & & & \\
-1 & & & \cdot & & & & & & & \\
0 & & & \cdot & & & & & & & \\
0 & 0 & -1 & \cdot & & & & & & &
\end{array}\right].$$

2. Matrix E_{2n-3} is obtained by canceling first and fourth rows, followed by canceling first and fourth columns in A_{2n-1}.

$$E_{2n-3} = \left[\begin{array}{cccccccccc}
4 & 0 & & & & & & -1 & 0 & 0 \\
0 & 4 & -1 & 0 & 0 & -1 & & & & \\
 & -1 & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\
 & 0 & \cdot & & & & & & & \\
 & 0 & \cdot & & & & & & & \\
 & -1 & \cdot & & & & & & & \\
 & & \cdot & & & C_{2n-5} & & & & \\
 & & \cdot & & & & & & & \\
 & & \cdot & & & & & & & \\
-1 & & \cdot & & & & & & & \\
0 & & \cdot & & & & & & & \\
0 & & \cdot & & & & & & &
\end{array}\right].$$

3. Matrix F_{2k-2} is obtained by canceling first row and first column in C_{2k-1}, where $k \leq n$.
4. Matrix G_{2k-2} is obtained by canceling last row and last column in C_{2k-1}, where $k \leq n$.
5. Matrix H_{2k-2} is obtained by appending first row and first column to C_{2k-3} as follows:

$$H_{2k-2} \;=\; \begin{bmatrix} 8 & 0 & -4 & & & & & \\ 0 & . & . & . & . & . & . & \\ -4 & . & & & & & & \\ & . & & C_{2k-3} & & & & \\ & . & & & & & & \\ & . & & & & & & \end{bmatrix}.$$

6. Matrix M_{2k-1} is obtained by appending first row and first column to G_{2k-2} as follows:

$$M_{2k-1} \;=\; \begin{bmatrix} 8 & 0 & -4 & & & & & \\ 0 & . & . & . & . & . & . & \\ -4 & . & & & & & & \\ & . & & G_{2k-2} & & & & \\ & . & & & & & & \\ & . & & & & & & \end{bmatrix}.$$

Lemma 2.1. *Let* $b_{2n-1}, c_{2k-1}, d_{2n-2}, e_{2n-3}, f_{2k-2}, g_{2k-2}, h_{2k-2}, m_{2k-3}$ *be respective determinants of matrices* B_{2n-1}, C_{2k-1}, D_{2n-2}, E_{2n-3}, F_{2k-2}, G_{2k-2}, H_{2k-2}, M_{2k-3} *for* $k \leq n$. *Then the following relations hold:* (i) $a_{2n-1} = b_{2n-1} = 8d_{2n-2} - 16e_{2n-3}$, (ii) $d_{2n-2} = 8c_{2n-3} - 8h_{2n-4} - 16g_{2n-4}$, (iii) $e_{2n-3} = 8h_{2n-4} - 16m_{2n-5}$, (iv) $h_{2k-2} = 8c_{2k-3} - 16h_{2k-4}$, (v) $m_{2k-1} = 8g_{2k-2} - 16m_{2k-3}$.

Proof. (i). By the definition of B_{2n-1}, $a_{2n-1} = b_{2n-1}$. Also, by expanding B_{2n-1} with respect to the first row, we obtain $b_{2n-1} = 8d_{2n-2} - 16e_{2n-3}$.
(ii). We subtract the last row from the first row, and then we subtract the last column from the first column in D_{2n-2}, which results in the following matrix D'_{2n-2}:

$$D_{2n-2} \to D'_{2n-2} \;=\; \begin{bmatrix} 8 & & & & & & & & -4 \\ & . & . & . & . & . & . & . & 0 \\ & . & & & & & & . & 0 \\ & . & & & & & & . & -1 \\ & . & & & G_{2n-4} & & & . & \\ & . & & & & & & . & \\ & . & & & & & & . & -1 \\ & . & . & . & . & . & . & . & 0 \\ -4 & & & & & & -1 & 0 & 4 \end{bmatrix}.$$

So, $d'_{2n-2} = d_{2n-2}$ and there are exactly two nonzero elements in the first row and first column of D'_{2n-2}. Expanding D'_{2n-2} with respect to the first row, we have

$$d_{2n-2} = 8\left(c_{2n-3} - \det \begin{bmatrix} 4 & -1 & 0 & 0 & -1 & & & & \\ -1 & . & . & . & . & . & . & . & . \\ 0 & . & & & & & & & \\ 0 & . & & & & & & & \\ -1 & . & & & & C_{2n-5} & & & \\ & . & & & & & & & \\ & . & & & & & & & \\ & . & & & & & & & \\ & . & & & & & & & \end{bmatrix} \right) - 16g_{2n-4}.$$

By subtracting the third row from the first row, followed by subtracting the third column from the first column, we obtain

$$d_{2n-2} = 8\left(c_{2n-3} - \det \begin{bmatrix} 8 & 0 & -4 & & & & & & \\ 0 & . & . & . & . & . & . & . & . \\ -4 & . & & & & & & & \\ & . & & & & & & & \\ & . & & & C_{2n-5} & & & & \\ & . & & & & & & & \\ & . & & & & & & & \\ & . & & & & & & & \\ & . & & & & & & & \end{bmatrix} \right) - 16g_{2n-4} = 8c_{2n-3} - 8h_{2n-4} - 16g_{2n-4}.$$

(iii). We subtract the last row from the first row, and then we subtract the last column from the first column in E_{2n-3}. Subsequently, we subtract the fourth row from the second row, and finally, we subtract the fourth column from the second column, which results in the following matrix E'_{2n-3}.

$$E_{2n-3} \to E'_{2n-3} = \begin{bmatrix} 8 & & & & & -4 \\ & 8 & 0 & -4 & & \\ & 0 & \cdot & \cdot & \cdots & \cdot \\ & -4 & \cdot & & & \\ & & \vdots & & C_{2n-5} & \\ -4 & & \cdot & & & \end{bmatrix}.$$

Clearly, $e'_{2n-3} = e_{2n-3}$ and there are exactly two nonzero elements in the first row and first column of E'_{2n-3}. Expanding E'_{2n-2} with respect to the first row results in the following:

$$e_{2n-3} = 8 \cdot \det \begin{bmatrix} 8 & 0 & -4 & & \\ 0 & \cdot & \cdot & \cdots & \cdot \\ -4 & \cdot & & & \\ & \vdots & & C_{2n-5} & \\ & \cdot & & & \end{bmatrix} - 16 \cdot \det \begin{bmatrix} 8 & 0 & -4 & & \\ 0 & \cdot & \cdot & \cdots & \cdot \\ -4 & \cdot & & & \\ & \vdots & & G_{2n-6} & \\ & \cdot & & & \end{bmatrix} = 8h_{2n-4} - 16m_{2n-5}.$$

(iv). By subtracting the fourth row from the second row, followed by subtracting the fourth column from the second column in H_{2k-2}, and followed by expanding H_{2k-2} with respect to the first row, we obtain the following:

$$h_{2k-2} = 8 \cdot c_{2k-3} - 16 \cdot \det \begin{bmatrix} 8 & 0 & -4 & & \\ 0 & \cdot & \cdot & \cdots & \cdot \\ -4 & \cdot & & & \\ & \vdots & & C_{2k-5} & \\ & \cdot & & & \end{bmatrix} = 8c_{2k-3} - 16h_{2k-4}.$$

(v). By subtracting the fourth row from the second row, followed by subtracting the fourth column from the second column in M_{2k-1}, and followed by expanding M_{2k-1} with respect to the first row, we obtain the following:

$$m_{2k-1} = 8 \cdot g_{2k-2} - 16 \cdot \det \begin{bmatrix} 8 & 0 & -4 & & \\ 0 & \cdot & \cdot & \cdots & \cdot \\ -4 & \cdot & & & \\ & \vdots & & G_{2k-4} & \\ & \cdot & & & \end{bmatrix} = 8g_{2k-2} - 16m_{2k-3}.$$

□

3. Main result

Before presenting our main result, we need additional lemmas that explore linear recurrence relations with constant coefficients. To this end, we use the following notation. For a sequence $\{f_i\}$ and the recurrence relation with constant coefficients $\alpha_0, \alpha_1, \ldots, \alpha_k$ given by $\alpha_k f_{i+k} + \alpha_{k-1} f_{i+k-1} + \cdots + \alpha_0 f_i = 0$, we say that the sequence $\{f_i\}$ satisfies the equation $\alpha_k x^k + \alpha_{k-1} x^{k-1} + \cdots + \alpha_0 x^0 = 0$, which is called the *characteristic equation* [12], where x is the shift operator satisfying $x \cdot f_{i-1} = f_i$ and $x^0 = 1$.

Lemma 3.1. *Let $\widehat{c}_k = c_{2k-1}$ and $k \geq 4$. The sequence $\{\widehat{c}_k\}$ satisfies the characteristic equation $(x-8)^2 = 0$.*

Proof. Let r_i, c_i be i-th row and i-th column in C_{2k-1} respectively for $k \geq 4$. We obtain matrix Q_{2k-1} from matrix C_{2k-1} by subtracting r_{2i} from r_{2i-3} followed by subtracting c_{2i} from c_{2i-3} obtaining new row r_{2i-3} and new column c_{2i-3}, for every integer i, where $2 \leq i \leq k-1$. Hence, $c_{2k-1} = \det(Q_{2k-1}) = q_{2k-1}$, where $Q_{2k-1} = [q_{i,j}]$ is defined as follows:

$$q_{i,j} = \begin{cases} 8 & \text{for } i = j,\ i \text{ odd, and } 2i \leq 2k-5, \\ 4 & \text{for } i = j, \text{ and either } i \text{ even or } 2i > 2k-5, \\ -1 & \text{for } |i-j| = 2 \text{ and } (i+j-2) \equiv 0 \pmod 4, \\ -1 & \text{if } i = 2k-3 \text{ and either } j = 2k-2 \text{ or } j = 2k-1, \\ -1 & \text{if } j = 2k-3 \text{ and either } i = 2k-2 \text{ or } i = 2k-1, \\ -4 & \text{for } |i-j| = 3 \text{ and } (i+j-1) \equiv 0 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

So, we have,

$$Q_{2k-1} = \begin{bmatrix} 8 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 4 & 0 & -1 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & -4 & 0 & 0 & & 0 & 0 & 0 \\ -4 & -1 & 0 & 4 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & -4 & & 0 & 0 & 0 \\ 0 & 0 & -4 & -1 & 0 & 4 & 0 & -1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & -1 & 0 & 4 & & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & 4 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & -1 & 0 & 4 \end{bmatrix}.$$

Let matrix R_{2k-2} be obtained by canceling the first row and first column in matrix Q_{2k-1}, and let $r_{2k-2} = \det(R_{2k-2})$. Expanding Q_{2k-1} with respect to the first row, we get

$$q_{2k-1} = 8 \cdot r_{2k-2} - 64 \cdot \det \begin{bmatrix} 8 & 0 & -4 & & \\ 0 & \cdot & \cdot & \cdots & \cdot \\ -4 & \cdot & & & \\ & \cdot & & Q_{2k-5} & \\ & \cdot & & & \\ & \cdot & & & \end{bmatrix} = 8r_{2k-2} - 64s_{2k-4}, \quad \text{where}$$

$$S_{2k-2} = \begin{bmatrix} 8 & 0 & -4 & & \\ 0 & \cdot & \cdot & \cdots & \cdot \\ -4 & \cdot & & & \\ & \cdot & & Q_{2k-3} & \\ & \cdot & & & \\ & \cdot & & & \end{bmatrix} \quad \text{and} \quad s_{2k-2} = \det(S_{2k-2}).$$

Expanding R_{2k-2} with respect to the first row, we get

$$r_{2k-2} = 4 \cdot q_{2k-3} - \det \begin{bmatrix} 8 & 0 & -4 & & \\ 0 & \cdot & \cdot & \cdots & \cdot \\ -4 & \cdot & & & \\ & \cdot & & Q_{2k-5} & \\ & \cdot & & & \\ & \cdot & & & \end{bmatrix} = 4q_{2k-3} - s_{2k-4}.$$

Expanding S_{2k-2} with respect to the first row, we get $s_{2k-2} = 8q_{2k-3} - 16s_{2k-4}$. Consequently, solving the following three relations $q_{2k-1} = 8r_{2k-2} - 64s_{2k-4}$, $r_{2k-2} = 4q_{2k-3} - s_{2k-4}$, $8q_{2k-1} = s_{2k} + 16s_{2k-2}$, we obtain $s_{2k} - 16s_{2k-2} + 64s_{2k-4} = 0$. This means that sequence $\{\widehat{s}_k\}$, where $\widehat{s}_k = s_{2k}$, satisfies the characteristic equation $(x-8)^2 = 0$. Furthermore, since $q_{2k-1} = c_{2k-1} = \widehat{c}_k$ then relation $8q_{2k-1} = s_{2k} + 16s_{2k-2}$ together with sequence $\{\widehat{s}_k\}$ satisfying $(x-8)^2 = 0$ imply that sequence $\{\widehat{c}_k\}$ also satisfies $(x-8)^2 = 0$. □

Lemma 3.2. *Let $\widehat{g}_k = g_{2k-2}$ and $k \geq 4$. The sequence $\{\widehat{g}_k\}$ satisfies the characteristic equation $(x-8)^2 = 0$.*

Proof. Let matrix Q'_{2k-2} be obtained from Q_{2k-1}, defined in Lemma 3.1, by deleting the last row and last column. Let $q'_{2k-2} = \det(Q'_{2k-2})$. Let matrix R'_{2k-3} be obtained by canceling the first row and first column in matrix Q'_{2k-2}, and let $r'_{2k-3} = \det(R'_{2k-3})$. Expanding Q'_{2k-2} with respect to the first row, we get

$$q'_{2k-2} = 8 \cdot r'_{2k-3} - 64 \cdot \det \begin{bmatrix} 8 & 0 & -4 & & \\ 0 & \cdot & \cdot & \cdots & \cdot \\ -4 & \cdot & & & \\ & \cdot & & Q'_{2k-6} & \\ & \cdot & & & \\ & \cdot & & & \end{bmatrix} = 8r'_{2k-2} - 64s'_{2k-5}, \text{ where}$$

$$S'_{2k-1} = \begin{bmatrix} 8 & 0 & -4 & & \\ 0 & \cdot & \cdot & \cdots & \cdot \\ -4 & \cdot & & & \\ & \cdot & & Q'_{2k-2} & \\ & \cdot & & & \\ & \cdot & & & \end{bmatrix} \quad \text{and} \quad s'_{2k-1} = \det(S'_{2k-1}).$$

Expanding R'_{2k-3} with respect to the first row, we get

$$r'_{2k-3} = 4 \cdot q'_{2k-4} - \det \begin{bmatrix} 8 & 0 & -4 & & & & \\ 0 & . & . & . & . & . & . \\ -4 & . & & & & & \\ & . & & Q'_{2k-6} & & & \\ & . & & & & & \\ & . & & & & & \end{bmatrix} = 4q'_{2k-4} - s'_{2k-5}.$$

Expanding S'_{2k-3} with respect to the first row, we get

$$s'_{2k-3} = 8q'_{2k-4} - 16s'_{2k-5}.$$

Consequently, solving the relations

$$q'_{2k-2} = 8r'_{2k-3} - 64s'_{2k-5},$$
$$r'_{2k-3} = 4q'_{2k-4} - s'_{2k-5},$$
$$8q'_{2k-2} = s'_{2k-1} + 16s'_{2k-3},$$

we obtain

$$s'_{2k-1} - 16s'_{2k-3} + 64s'_{2k-5} = 0.$$

So, our sequence $\{\widehat{s'}_k\}$, where $\widehat{s'}_k = s'_{2k-1}$, satisfies the characteristic equation $(x-8)^2 = 0$. Furthermore, since $q'_{2k-2} = g_{2k-2} = \widehat{g}_k$ then relation $8q'_{2k-2} = s'_{2k-1} + 16s'_{2k-3}$ together with sequence $\{\widehat{s'}_k\}$ satisfying $(x-8)^2 = 0$ imply that sequence $\{\widehat{g}_k\}$ also satisfies $(x-8)^2 = 0$. □

Lemma 3.3. *Let $\widehat{a}_n = a_{2n-1}$ and $n \geq 4$. The sequence $\{\widehat{a}_n\}$ satisfies the characteristic equation*

$$(x-8)^2 = 0.$$

Proof. By Lemma 2.1(v), $m_{2k-1} + 16m_{2k-3} = 8g_{2k-2}$, and by Lemma 3.2, sequence $\{\widehat{g}_k\}$ satisfies $P(x) = (x-8)^2 = 0$. This implies that sequence $\{\widehat{m}_i\}$, where $\widehat{m}_i = m_{2i-1}$, also satisfies $P(x) = 0$.

Since by Lemma 3.1 sequence $\{\widehat{c}_k\}$ satisfies $P(x) = (x-8)^2 = 0$, and by Lemma 2.1(iv), $h_{2k-2} + 16h_{2k-4} = 8c_{2k-3}$ then sequence $\{\widehat{h}_i\}$, where $\widehat{h}_i = h_{2i}$, also satisfies $P(x) = 0$.

Because sequences $\{\widehat{m}_i\}, \{\widehat{h}_i\}$ satisfy $P(x) = (x-8)^2 = 0$, and by Lemma 2.1(iii), $e_{2n-3} = 8h_{2n-4} - 16m_{2n-5}$ then sequence $\{\widehat{e}_i\}$, where $\widehat{e}_i = e_{2i-1}$, also satisfies $P(x) = 0$.

Because sequences $\{\widehat{c}_i\}, \{\widehat{h}_i\}, \{\widehat{g}_i\}$ satisfy $P(x) = (x-8)^2 = 0$, and by Lemma 2.1(ii), $d_{2n-2} = 8c_{2n-3} - 8h_{2n-4} - 16g_{2n-4}$ then sequence $\{\widehat{d}_i\}$, where $\widehat{d}_i = d_{2i}$, also satisfies $P(x) = 0$.

Because sequences $\{\widehat{d}_i\}, \{\widehat{e}_i\}$ satisfy $P(x) = (x-8)^2 = 0$, and by Lemma 2.1(i), $a_{2n-1} = 8d_{2n-2} - 16e_{2n-3}$ then sequence $\{\widehat{a}_i\}$, where $\widehat{a}_i = a_{2i-1}$, also satisfies $P(x) = 0$. □

We can now state and prove the (following) main result of this paper.

Theorem 3.1. *Let $n \geq 4$. The number of spanning trees in $\breve{P}_n$ is*

$$t(\breve{P}_n) = n \cdot 2^{3n-2}.$$

Proof. Let $g(n)$ be defined as

$$g(n) = n \cdot 2^{3n-2}.$$

By direct calculation, we verify that $g(n)$ satisfies the characteristic equation

$$(x-8)^2 = 0,$$

where x is the shift operator such that $g(n) = x \cdot g(n-1)$ and $x^0 = 1$. According to Lemma 3.3, $\widehat{a}_n = a_{2n-1}$ satisfies the same characteristic equation. Based on the Kirchhoff matrix tree theorem, by numerically evaluating the determinants a_{2n-1} for $n = 4, 5$, and 6, we obtain that $g(n) = \widehat{a}_n$ for $n = 4, 5, 6$. This implies that $g(n) = \widehat{a}_n$ for $n \geq 4$, which proves the theorem. □

We point out that our formula $t(\breve{P}_n) = n \cdot 2^{3n-2}$ is much simpler than the ones for prism and antiprism, both of which involve $\sqrt{3}$ and $\sqrt{5}$, respectively. This is quite surprising and counterintuitive because both prism and antiprism are planar graphs. On the other hand, based on Kuratowski's celebrated theorem, it's easy to verify that our $t(\breve{P}_4)$ in Figure 1.1 is not planar, because it contains induced $K_{3,3}$. In fact, we derived in this paper one of the simplest formulas in graph theory. It suggests that this superprism might be of interest in other respects as well.

Finally, let $\hat{P}_n$ be defined as $\breve{P}_n$ but for $n \geq 1$. That is, a loop is considered as a cycle on a single vertex, and a cycle on two vertices is considered as two parallel edges. Then we can state the following:

Corollary 3.1. *Let $n \geq 1$. The number of spanning trees in $\hat{P}_n$ is*

$$t(\hat{P}_n) = n \cdot 2^{3n-2}.$$

Proof. It is directly verified that $t(\hat{P}_1) = 2$, $t(\hat{P}_2) = 32$, and $t(\hat{P}_3) = 384$, which satisfy the formula. In addition, based on Theorem 3.1, $t(\hat{P}_n)$ is also satisfied for $n \geq 4$. □

Note that for $n = 1$ we have a multigraph with two loops, for $n = 2$ we have a multigraph with four pairs of parallel edges, and for $n = 3$ we have $\hat{P}_n \simeq \bar{P}_n$.

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