Optimization of the Directed Information

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Notation

Causal Conditioning pmf

$$
\begin{aligned}
P(y^n||x^n) &\triangleq \prod_{i=1}^{n} P(y_i|x^i, y^{i-1}) \\
P(y^n|x^n) &= \prod_{i=1}^{n} P(y_i|x^n, y^{i-1})
\end{aligned}
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Causal Conditioning entropy

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\begin{aligned}
H(Y^n||X^n) &\triangleq E[-\log P(Y^n||X^n)] \\
H(Y^n|X^n) &\triangleq E[-\log P(Y^n|X^n)]
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Directed Information

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\begin{aligned}
I(X^n \to Y^n) &\triangleq H(Y^n) - H(Y^n||X^n) \\
I(X^n; Y^n) &\triangleq H(Y^n) - H(Y^n|X^n)
\end{aligned}
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Notation

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$$P(y^n||x^n) \triangleq \prod_{i=1}^{n} P(y_i|x^i, y^{i-1})$$

$$P(y^n||x^{n-1}) \triangleq \prod_{i=1}^{n} P(y_i|x^{i-1}, y^{i-1})$$

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Directed Information

$$I(X^n \to Y^n) \triangleq H(Y^n) - H(Y^n||X^n)$$

$$I(X^{n-1} \to Y^n) \triangleq H(Y^n) - H(Y^n||X^{n-1})$$

Directed information and causal conditioning characterizes

1. rate reduction in losless compression due to causal side information at the decoder,
2. the gain in growth rate in horse-race gambling due to causal side information
3. channel capacity with feedback,
4. rate distortion with feedforward,
5. causal MMSE for additive Gaussian noise,
6. stock investment with causal side information,
7. measure of causal relevance between processes,
8. actions with causal constraint such as "to feed or not to feed back",

Directed information optimization

How to find

$$\max_{p(x^n||y^{n-1})} I(X^n \to Y^n).$$

Recall

$$\begin{aligned} I(X^n \to Y^n) &= \sum_{i=1}^{n} I(X^i; Y_i | Y^{i-1}) \\ &= H(Y^n) - H(Y^n || X^n) \\ &= \sum_{y^n, x^n} p(x^n, y^n) \log \frac{p(y^n || x^n)}{p(y^n)} \end{aligned}$$

$P(x^n, y^n)$ can be expressed by the chain-rule

$$p(x^n, y^n) = p(x^n || y^{n-1}) p(y^n || x^n)$$

Property of the optimization problem

$$\max_{p(x^n||y^{n-1})} I(X^n \to Y^n)$$

Good news

- $I(X^n \to Y^n)$ is convex in $p(x^n||y^{n-1})$ for a fixed $p(y^n||x^n)$.
- $p(x^n||y^{n-1})$ is a convex set.

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Bad news

- Not easy to describe $p(x^n||y^{n-1})$ using linear equations. Contrary to $p(x^n)$ where

$$\begin{aligned} p(x^n) &\geq 0 \ \forall x^n. \\ \sum_{x^n} p(x^n) &= 1. \end{aligned}$$

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- $I(X^n \to Y^n)$ non-convex in $p(x_1), ..., p(x_n|x^{n-1}, y^{n-1})$
- Cannot optimize each term in $\sum_i I(X^i; Y_i|Y^{i-1})$ or in $\sum_i I(X_i; Y_i^n|X^{i-1}, Y^{i-1})$, separately.

The Alternating maximization procedure

Lemma (double maximization)

$$\max_{p(x^n\|y^{n-1})} I(X^n \to Y^n) = \max_{p(x^n\|y^{n-1}),q(x^n|y^n)} I(X^n \to Y^n).$$

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Let $f(u_1, u_2)$, be a convex fun and we want to find

$$\max_{u_1\in\mathcal{A}_1,u_2\in\mathcal{A}_2} f(u_1, u_2).$$

The procedure is

$$u_1^{(k+1)} = \arg\max_{u_1\in\mathcal{A}_1} f(u_1^{(k)}, u_2^{(k)}),\ \ u_2^{(k+1)} = \arg\max_{u_2\in\mathcal{A}_2} f(u_1^{(k+1)}, u_2^{(k)}).$$

$$f^{(k)} = f(u_1^{(k)}, u_2^{(k)}).$$

Theorem (The Alternating maximization procedure)

$$\lim_{k\to\infty} f^{(k)} = \max_{u_1\in\mathcal{A}_1,u_2\in\mathcal{A}_2} f(u_1, u_2).$$

BA for directed information

Compute by the alternating maximization procedure

$$\max_{p(x^n\|y^{n-1})} \max_{q(x^n|y^n)} I(X^n \to Y^n).$$

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1st Step

Lemma ($\max_{q(x^n|y^n)} I(X^n \to Y^n)$)

For fixed $p(x^n\|y^{n-1})$, $q^*(x^n|y^n)$ *that achieves* $\max_{q(x^n|y^n)} I(X^n \to Y^n)$, *is*

$$q^*(x^n|y^n) = \frac{p(x^n\|y^{n-1})p(y^n\|x^n)}{\sum_{x^n} p(x^n\|y^{n-1})p(y^n\|x^n)}.$$

2nd Step

Lemma ($\max_{p(x^n\|y^{n-1})} I(X^n \to Y^n)$)

For fixed $q(x^n|y^n)$, $p^*(x^n\|y^{n-1})$ *that achieves* $\max_{p(x^n\|y^{n-1})} I(X^n \to Y^n)$, *is:*
Starting from $i = n$, *compute* $p(x_i|x^{i-1}, y^{i-1})$

$$p_i = p^*(x_i|x^{i-1}, y^{i-1}) = \frac{p'(x^i, y^{i-1})}{\sum_{x_i} p'(x^i, y^{i-1})},$$

where

$$p'(x^i, y^{i-1}) = \prod_{x_{i+1}^n, y_i^n} \left[\frac{q(x^n|y^n)}{\prod_{j=i+1}^n p_j} \right]^{\prod_{j=i}^n p(y_j|x^j, y^{j-1}) \prod_{j=i+1}^n p_j},$$

and do so ***backwards*** *until* $i = 1$.

Main ideas of 2nd Step

- Exchange $p(x^n \| y^{n-1})$ by the set $\{p_i\}_{i=1}^n$ where $p_i = p(x_i | x^{i-1}, y^{i-1})$

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- $I(X^n \to Y^n)$ is concave in each p_i.
- For fixed $q(x^n|y^n)$, p_i^* that achieves $\max_{p_i} I(X^n \to Y^n)$, depends only on

$$q(x^n|y^n), p_{i+1}, p_{i+2}, \ldots, p_n$$

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- Hence we can find

$$\max_{p_1} \ldots \left(\max_{p_{n-1}} \left(\max_{p_n} I(X^n \to Y^n) \right) \right)$$

despite being nonconvex.

BA for directed information

- Using Step 1 and 2 we can compute

$$I_L = \sum_{y^n, x^n} p(y^n \| x^n) r(x^n \| y^{n-1}) \log \frac{q(x^n | y^n)}{p(x^n \| y^{n-1})}.$$

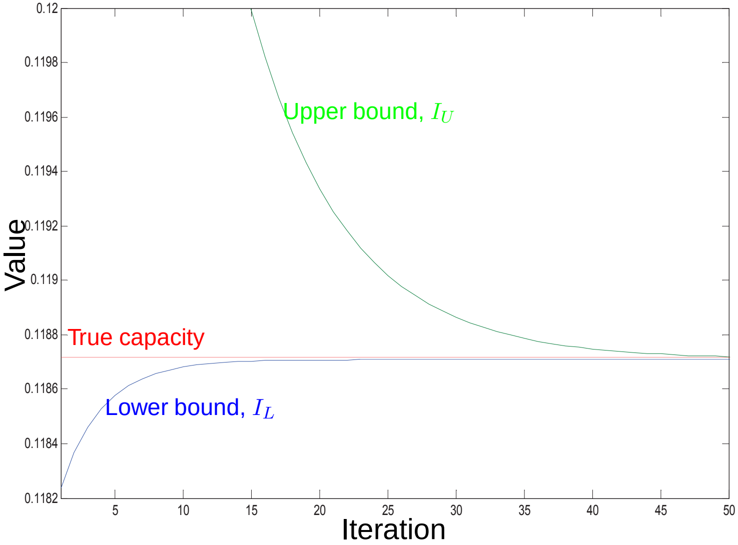
  which converges from below to $\max_{p(x^n \| y^{n-1})} I(X^n \to Y^n)$
- We also have an upper bound

$$I_U = \max_{x_1} \sum_{y_1} \max_{x_2} \cdots \sum_{y_{n-1}} \max_{x_n} \sum_{y_n} p(y^n \| x^n) \log \frac{p(y^n \| x^n)}{\sum_{x'^n} p(y^n \| x'^n) p(x'^n \| y^{n-1})}$$

- The algorithm terminate when

$$|I_U - I_L| \leq \epsilon$$

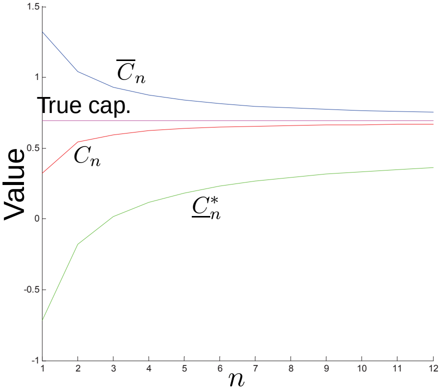
maximizing the directed information for BSC(0.3)



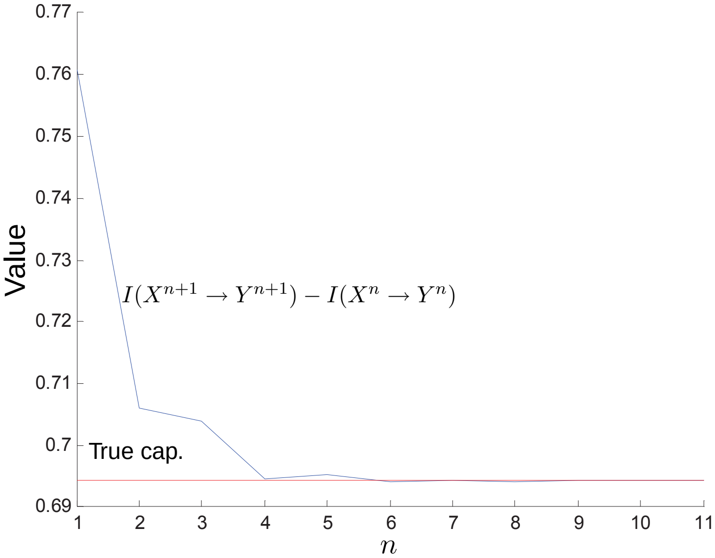
Bounds on capacity of any FSC

$$\overline{C}_n = \max_{s_0} \max_{p(x^n \| y^{n-1})} \frac{1}{n} I(X^n \to Y^n | s_0) + \frac{1}{n},$$

$$\underline{C}_n = \max_{p(x^n \| y^{n-1})} \min_{s_0} \frac{1}{n} I(X^n \to Y^n | s_0) - \frac{1}{n}.$$

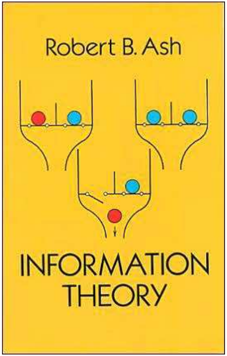


Directed information rate



Infinite-letter case

For two cases we have analytical solution using dynamic programming for unifilar channels.
<u>First case:</u> Trapdoor channel.



(a) Ash book

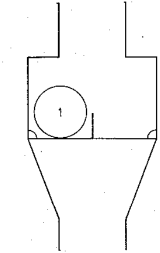


Fig. 7.1 A simple two-state channel.

(b) D. Blackwell

$C_{fb} = \log \phi$ Golden Ratio: $\phi = \frac{\sqrt{5}+1}{2}$

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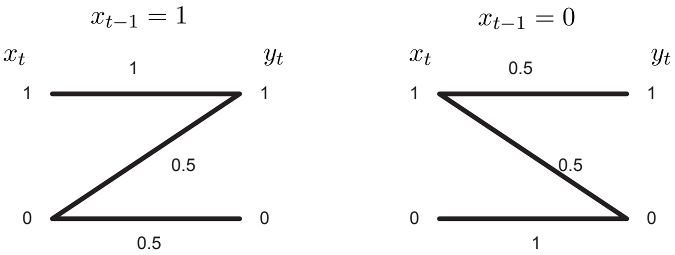
- The Ising channel graphical model:



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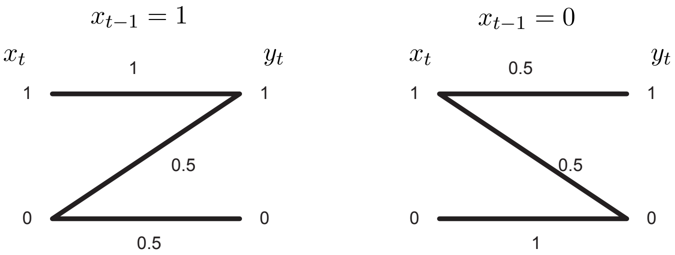

Q: How can one achieve $R = \frac{1}{2}$?

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- The DP leads to a simple capacity achieving coding scheme.

Channel notation and DP formulation

Notation	Meaning
t	Time $(\in \mathbb{N})$
x_t	Channel Input at time t $(\in \mathcal{X})$
$s_t(= x_{t-1})$	Channel State at time t $(\in \mathcal{S})$
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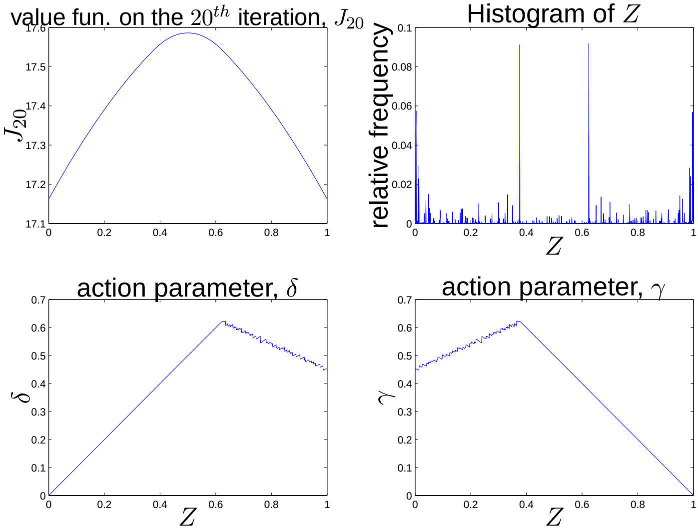
Ising channel	DP
$p(s_t = 0 \vert y^t)$, prob. of the channel state to be 0 given the output	z_t, the DP state
y_t, the channel output	w_t, the DP disturbance
$p(x_t \vert s_{t-1})$, channel input prob. given the channel state at time $t-1$	u_t, the DP action
$p(s_t = 0 \vert y^t)$ as a function of $p(s_{t-1} = 0 \vert y^{t-1})$ and input dist.	$z_t = F(z_{t-1}, u_{t-1}, w_{t-1})$, states evolving
$I(X_t, S_{t-1}; Y_t \vert y^{t-1})$	$g(z_{t-1}, u_t)$, reward function

DP numerical evaluation



DP and its relation to the coding scheme

	$z_t = p_0$	$z_t = p_1$	$z_t = p_2$	$z_t = p_3$
$y_t = 0$	$z_{t+1} = p_3$	$z_{t+1} = p_3$	$z_{t+1} = p_3$	$z_{t+1} = p_2$
$y_t = 1$	$z_{t+1} = p_1$	$z_{t+1} = p_0$	$z_{t+1} = p_0$	$z_{t+1} = p_0$
$p(x_t = 1 \vert x_{t-1} = 1)$	a	1	1	irrelevant
$p(x_t = 0 \vert x_{t-1} = 0)$	irrelevant	1	1	a

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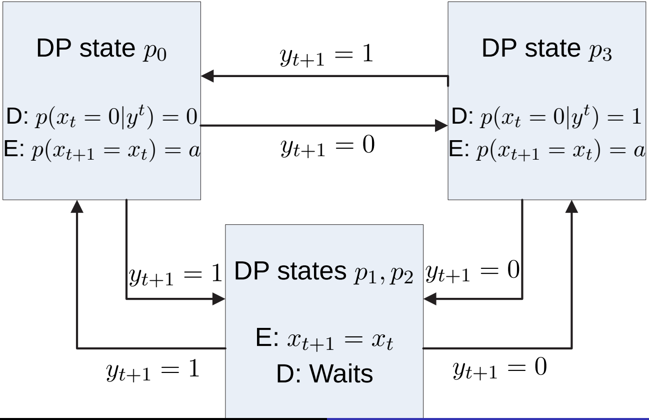
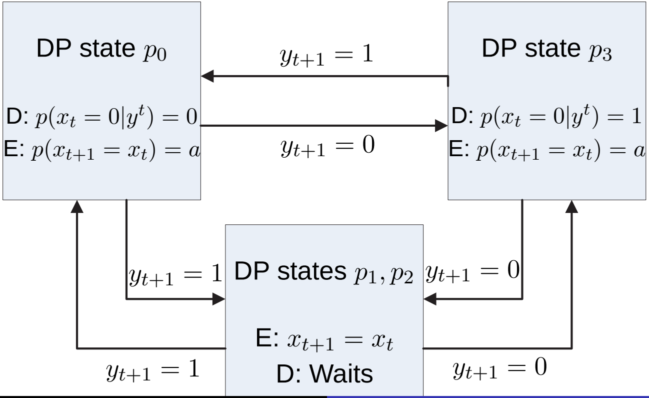


DP and its relation to the coding scheme

- Alternate between 0 and 1 with prob. $1 - a$.



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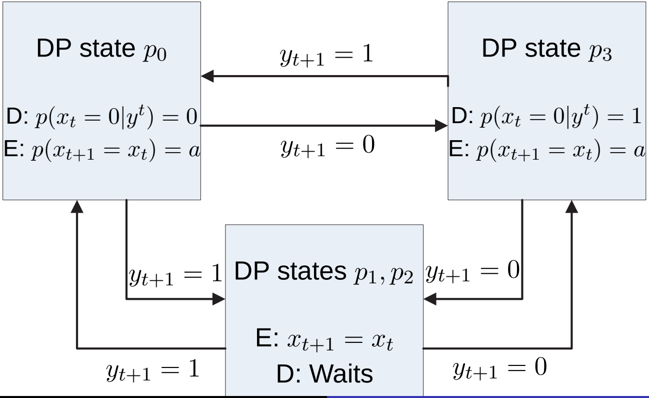
- If the output $y_{t+1} \neq s_t$, then decode $x_{t+1} = y_{t+1}$



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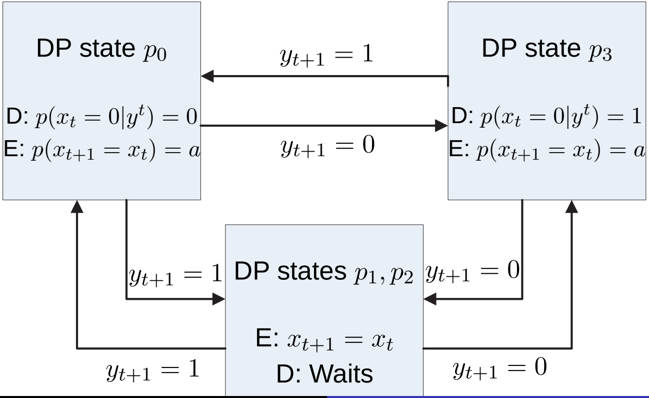

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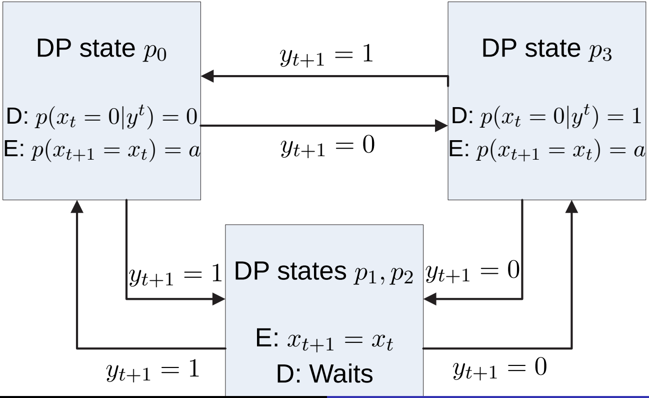
$$C = \frac{H(1-a)}{a \cdot 2 + (1-a) \cdot (2 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2})} = \frac{H(a)}{\frac{3}{2} + \frac{a}{2}}$$



Summary

- Convexity can be exploited to calculate

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  using alternating maximization procedure.
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Thank you very much!