ON RATS SEQUENCES IN GENERAL BASES

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Abstract

Conway's RATS sequences are generated by repeating the following process: begin with a positive integer, reverse the order of the digits, add the two integers together, then sort the sum's digits in increasing order from left to right. We consider this process for integers written in other bases besides 10 and study the long-term behavior of such sequences. In particular, we examine and show the existence of periodic and quasiperiodic (a special type of divergence) sequences for certain choices of base.

1. Introduction

Conway's RATS [1] game has very simple set of rules: begin with a positive integer n, **R**everse the digits of n, **A**dd the two integers together, and **T**hen **S**ort the digits of the sum in increasing order from left to right (discarding any zeros). Repeating this process generates an infinite sequence, but the game is said to end if any term in the generated sequence is ever repeated.

Example 1.1. The sequence generated by the RATS process beginning with $n = 888$ is given by:

$$888 \mapsto 1677 \mapsto 3489 \mapsto 12333 \mapsto 44556 \mapsto 111 \mapsto 222 \mapsto 444 \mapsto 888 \mapsto \cdots. \tag{1.1}$$

This particular example shows that starting with $n = 888$, the RATS game ends because the sequence is periodic.

Not all sequences generated by the RATS process are periodic, or even eventually periodic. Conway himself noted that the sequence generated by the RATS process starting with $n = 1$ diverges. Furthermore, he conjectured that this sequence is somewhat unique when it comes to the RATS game.

Conjecture (Conway, [1]). Every RATS sequence is either eventually periodic or eventually part of the sequence $1 \mapsto 2 \mapsto 4 \mapsto \cdots \mapsto 12333334444 \mapsto 55666667777 \mapsto 123333334444 \mapsto 556666667777 \mapsto \cdots$.

One can easily see that the RATS game can be extended to other integer bases besides 10. In this article, we will expand on some results of Cooper and Kennedy [2] and Cooper and Shattuck [7] on the behavior of RATS sequences for general bases. (Note that we always assume a nonunary base.) In particular, we will answer an open question posed by Cooper and Shattuck on the existence of divergent RATS sequences for certain bases.

2. Notation and Terminology

We begin with some definitions that will be used regularly throughout the following sections. Note that, unless otherwise specified, all variables used represent nonnegative integers.

Definition 2.1. For any positive integer n written in base b, let $\overline{n} := \overline{n_b}$ be the digit formed by reversing the digits of n, and let $n' := n'_b$ be the digit formed by ordering the nonzero digits of n in increasing order from left to right. Define the function $R_b \colon \mathbb{N} \to \mathbb{N}$ by $R_b(n) = (n + \overline{n})'$. To denote iterates of R_b, we adopt superscript notation. For any nonnegative integer t, let $R_b^t(n) = R_b(R_b^{t-1}(n))$ if $t \geq 1$ and $R_b^0(n) = n$.

Example 2.2.

$$\begin{aligned} R_3^2(1102) &= R_3((1102 + \overline{1102})') \\ &= R_3((1102 + 2011)') \\ &= R_3((10120)') \\ &= R_3(112) \\ &= (112 + 211)' \\ &= (1100)' \\ &= 11 \end{aligned}$$

Definition 2.3. We call $\{R_b^i(n)\}_{i=0}^{\infty}$ the **RATS sequence generated by n in base b.**

Definition 2.4. If $R_b^p(n) = n$ for some $p > 0$, we say that n and the RATS sequence generated by n in base b are **periodic**. If p is the least integer with this property, we say that the **period** is p. If $R_b^t(n)$ is periodic for some $t \geq 0$, we say that n and the RATS sequence generated by n in base b are **ultimately periodic**[1].

Note that periodic sequences are also ultimately periodic, but not vice versa. This helps to simplify some statements, avoiding the need to say that a result applies to periodic *and* ultimately periodic sequences.

[1]In [6], periodic sequences are referred to as "cycles" and ultimately periodic sequences are "tributaries" to a cycle.

Example 2.5. The sequence (1.1) shows that 888 is periodic with period 8.

For convenience, we will adopt the exponential notation used by Guy [5] when necessary. In exponential notation, the base denotes the digit and the exponent denotes the number of times that the digit appears; e.g., 118110005 will be denoted as $1^2 81^2 0^3 5$. For bases larger than 10, we place parentheses around a digit larger than 9. For example, in base 22, $1(11)^3(14)$ is the five digit number with digits 1, 11, 11, 11, 14.

Definition 2.6. If $n = 1^{m_1} \cdots (k-1)^{m_{k-1}} k^{m_k} (k+1)^{m_{k+1}} \cdots (b-1)^{m_{b-1}}$ and $R_b^{qt}(n) = 1^{m_1} \cdots (k-1)^{m_{k-1}} k^{m_k+t} (k+1)^{m_{k+1}} \cdots (b-1)^{m_{b-1}}$, with $m_i \geq 0$ for $1 \leq i \leq b-1$, for some $q > 0$ and all $t \geq 0$, we say that n and the RATS sequence generated by n in base b are **quasiperiodic**. If q is the least integer with this property, we say that the **quasiperiod** is q. The digit $k > 0$ which increases in count after q iterations is called the **growing digit**. If $R_b^t(n)$ is quasiperiodic for some $t \geq 0$, we say that n and the RATS sequence generated by n in base b are **ultimately quasiperiodic**[2].

Example 2.7. From Conway's conjecture we see that $123^5 4^4$ and $5^2 6^5 7^4$ are quasiperiodic elements in base 10 with quasiperiod 2 and growing digits 3 and 6, respectively.

Definition 2.8. Let $\mathcal{P}_b$, called the **period set in base** b, be the set of all p for which there are periodic elements with period p in base b.

From the first example (1.1), the sequence $888 \mapsto 1677 \mapsto \cdots$ shows that 888 is periodic with period 8, so $8 \in \mathcal{P}_{10}$. In fact, Cooper and Kennedy were able to completely characterize the period set $\mathcal{P}_{10}$ (see [2] and [3]).

Definition 2.9. Let $\mathcal{Q}_b$, called the **quasiperiod set in base** b, be the set of all q for which there are quasiperiodic elements with quasiperiod q in base b.

Conway's conjecture gives that $2 \in \mathcal{Q}_{10}$. Unlike the case for $\mathcal{P}_{10}$, we do not have a complete description of $\mathcal{Q}_{10}$. However, if Conway's conjecture is true, then it follows that $\mathcal{Q}_{10} = \{2\}$. In its full strength, this remains open, but one can show that $\mathcal{Q}_{10}$ can only contain even numbers.

One way to make further progress towards Conway's conjecture would be to show that, indeed, $\mathcal{Q}_{10} = \{2\}$. This is, however, not enough to prove the conjecture, since the possibility exists that there are elements that are neither ultimately periodic nor ultimately quasiperiodic, or that there are multiple disjoint sequences with quasiperiod 2. The difficulty in showing that certain quasiperiods are not in $\mathcal{Q}_{10}$ is that they require dealing with a tremendous number of cases. Even showing that there are no quasiperiodic RATS sequences, with quasiperiod 2, that do not intersect with Conway's sequence seems difficult.

[2] In [7], quasiperiodic sequences are called "divergent" and the quasiperiod is called the "length."

In the subsequent sections, we will study the sets $\mathcal{P}_b$ and $\mathcal{Q}_b$ for certain choices of b. In Section 3, we will show that there is an infinite family of bases b such that $|\mathcal{P}_b| = \infty$. In Section 4, we show how to construct a base b such that $\mathcal{Q}_b$ contains any desired finite set of positive integers. Lastly, in Section 5, we investigate the behavior of the growing digit of a quasiperiodic integer.

3. Existence of Periodic Sequences

From the work of Gentges [4] and Cooper and Shattuck [7], it follows that there are bases b such that $|\mathcal{P}_b| = \infty$, namely $b = 3$ and $b = 10$. In this section, we show that there is an infinite family of bases for which the same result is true.

By a sum written in the form

$$\begin{array}{ccccc} & 2^2 & 2^3 & 3^3 & 4^2 \\ + & 4^2 & 3^3 & 2^3 & 2^2 \\ \hline \end{array}$$

we mean the following sum:

$$\begin{array}{r} 2222233344 \\ +\ 4433322222 \\ \hline \end{array}$$

This will help visualize what the "reverse-add" portion of the RATS process produces. For example, if the above sums were in base 10, they would give the sum $2^5 3^3 4^2 + \overline{2^5 3^3 4^2} = 5^2 6^6 5^2$.

Theorem 3.1. *For $m > 1$ and $b = 3 \cdot 2^m - 2$, we have $|\mathcal{P}_b| = \infty$. More precisely, we have:*

(i) If m is odd, then $\mathcal{P}_b$ contains all sufficiently large even integers.

(ii) If m is even, then $\mathcal{P}_b$ contains all sufficiently large integers.

Proof. We will first show that, for any even $k > 0$, $1^{2^k(2^m-1)}(2^m-1)^{2^k-1}$ is periodic with period $m+1+k$. This proves that when m is odd, $\mathcal{P}_b$ contains all sufficiently large even numbers, and when m is even, $\mathcal{P}_b$ contains all sufficiently large odd numbers. In particular, $|\mathcal{P}_b| = \infty$.

We begin by iterating the RATS process $m+1$ times with the staring element $1^{2^k(2^m-1)}(2^m-1)^{2^k-1}$. Since $2^k(2^m-1) = 2^k - 1 + 2^k(2^m-2) + 1$, the reverse-add portion of the RATS process produces

$$\begin{array}{cccc} & 1^{2^k-1} & 1^{2^k(2^m-2)+1} & (2^m-1)^{2^k-1} \\ + & (2^m-1)^{2^k-1} & 1^{2^k(2^m-2)+1} & 1^{2^k-1} \\ \hline & (2^m)^{2^k-1} & 2^{2^k(2^m-2)+1} & (2^m)^{2^k-1} \end{array}$$

(Note that no carries occur since the base $b = 3 \cdot 2^m - 2$ satisfies $b > 2^m$.) After the sorting step, we obtain

$$R_b(1^{2^k(2^m-1)}(2^m-1)^{2^k-1}) = 2^{2^k(2^m-2)+1}(2^m)^{2(2^k-1)}.$$

Since $2^k(2^m-2)+1 = 2(2^k-1)+2^k(2^m-4)+3$, we get

$$\begin{array}{cccc} & 2^{2(2^k-1)} & 2^{2^k(2^m-4)+3} & (2^m)^{2(2^k-1)} \\ + & (2^m)^{2(2^k-1)} & 2^{2^k(2^m-4)+3} & 2^{2(2^k-1)} \\ \hline & (2^m+2)^{2(2^k-1)} & 4^{2^k(2^m-4)+3} & (2^m+2)^{2(2^k-1)} \end{array}$$

Therefore,

$$R_b(2^{2^k(2^m-2)+1}(2^m)^{2(2^k-1)}) = 4^{2^k(2^m-4)+3}(2^m+2)^{4(2^k-1)}.$$

Again, since $2^k(2^m-4)+3 = 4(2^k-1)+2^k(2^m-8)+7$, we get

$$\begin{array}{cccc} & 4^{4(2^k-1)} & 4^{2^k(2^m-8)+7} & (2^m+2)^{4(2^k-1)} \\ + & (2^m+2)^{4(2^k-1)} & 4^{2^k(2^m-8)+7} & 4^{4(2^k-1)} \\ \hline & (2^m+6)^{4(2^k-1)} & 8^{2^k(2^m-8)+7} & (2^m+6)^{4(2^k-1)} \end{array}$$

Therefore,

$$R_b(4^{2^k(2^m-4)+3}(2^m+2)^{4(2^k-1)}) = 8^{2^k(2^m-8)+7}(2^m+6)^{8(2^k-1)}.$$

By continuing this process, we see that if $0 < j \leq m$, then the jth term in the RATS sequence generated by $1^{2^k(2^m-1)}(2^m-1)^{2^k-1}$ is given by

$$(2^j)^{2^k(2^m-2^j)+2^j-1}(2^m+2^{j+1}-2)^{2^j(2^k-1)}. \tag{3.1}$$

In particular, by (3.1), the mth term in the RATS sequence is $(2^m)^{2^m-1}(2^{m+1}-2)^{2^m(2^k-1)}$.

The mth term is the first instance in which the larger digit has the higher exponent. So now the reverse-add portion of the RATS process has the following form:

$$\begin{array}{cccc} & (2^m)^{2^m-1} & (2^{m+1}-2)^{2^m(2^k-2)+1} & (2^{m+1}-2)^{2^m-1} \\ + & (2^{m+1}-2)^{2^m-1} & (2^{m+1}-2)^{2^m(2^k-2)+1} & (2^m)^{2^m-1} \\ \hline & 1^{2^m} & (2^m-1)^{2^m(2^k-2)+1} & 1^{2^m-2}\,0 \end{array}$$

(Note that carries occur at every step in the sum.) Therefore,

$$R_b((2^m)^{2^m-1}(2^{m+1}-2)^{2^m(2^k-1)}) = 1^{2^{m+1}-2}(2^m-1)^{2^m(2^k-2)+1}.$$

Again, continuing this process in a similar fashion, since $2^m(2^k-2)+1 = 2^{m+1}-2+2^m(2^k-4)+3$, we get

$$\begin{array}{rccc} & 1^{2^{m+1}-2} & (2^m-1)^{2^m(2^k-4)+3} & (2^m-1)^{2^{m+1}-2} \\ + & (2^m-1)^{2^{m+1}-2} & (2^m-1)^{2^m(2^k-4)+3} & 1^{2^{m+1}-2} \\ \hline & (2^m)^{2^{m+1}-2} & (2^{m+1}-2)^{2^m(2^k-4)+3} & (2^m)^{2^{m+1}-2} \end{array}.$$

Therefore,

$$R_b(1^{2^{m+1}-2}(2^m-1)^{2^m(2^k-2)+1}) = (2^m)^{2^{m+2}-4}(2^{m+1}-2)^{2^m(2^k-4)+3}.$$

Continuing this process, we see that if $m+1 < j \leq k$, then the jth term in the RATS sequence generated by $1^{2^k(2^m-1)}(2^m-1)^{2^k-1}$ is given by

$$(A)^{2^j(2^m-1)}(B)^{2^m(2^k-2^j)+2^j-1}, \tag{3.2}$$

where $A = 1$ and $B = 2^m - 1$ if j is even, and $A = 2^m$ and $B = 2^{m+1}-2$ if j is odd. Taking $j = k$ in (3.2), we get that, since k is even, the $(m+1+k)$th term in the RATS sequence is $1^{2^k(2^m-1)}(2^m-1)^{2^k-1}$, our starting element. Thus this sequence is periodic with period $m+1+k$, as claimed. This proves that $|\mathcal{P}_b| = \infty$ and part (i) of the theorem.

To obtain part (ii), it is enough to show that for all even $k > 0$,

$$1^{2^k(2^{m+1}-2)(2^{m+1}+1)}(2^m-1)^{2^k(2^{m+1}-1)}$$

is periodic with period $2(m+1)+k$. When m is even, this will give that $\mathcal{P}_b$ contains all sufficiently large even numbers. This can be done using the same method shown above. We omit the details. □

Notice that since $10 = 3 \cdot 2^2 - 2$, the theorem applies to base $b = 10$ with $m = 2$, and gives that $\mathcal{P}_{10}$ contains all sufficiently large integers. In this case, as a slightly stronger result, Cooper and Kennedy [2], [3] showed that the period set is as large as possible, that is, $\mathcal{P}_{10} = \{p : p \geq 2\}$. At this time it is not known if there are infinitely many bases such that their period sets contain all periods $p \geq p_0$ for some fixed $p_0 > 1$.

Another point of interest is that base 3 is not in the family described by Theorem 3.1, yet it too has the property that $|\mathcal{P}_3| = \infty$. Clearly we have not identified all such bases for which the period set is infinite. In fact, the family $\{3 \cdot 2^m - 2 : m > 1\}$, covered by Theorem 3.1, has asymptotic density zero. A reasonable follow-up to Theorem 3.1 would be to examine the possibility of the existence of a family of bases, with nonzero asymptotic density, having infinite period sets.

4. Existence of Quasiperiodic Sequences

The sequence from Conway's original RATS conjecture shows that $\mathcal{Q}_{10}$, the quasiperiod set in base 10, contains the element $q = 2$. In particular, the set $\mathcal{Q}_{10}$ is not empty. In this section we will construct infinite families of bases b for which $\mathcal{Q}_b \neq \emptyset$. We begin with a theorem of Cooper, Shattuck, and Gentges.

Theorem 4.1 (Cooper and Shattuck, [7], Gentges, [4]). *For any base b satisfying $b \equiv 1 \pmod{18}$ or $b \equiv 10 \pmod{18}$, with $b \geq 10$, we have $2 \in \mathcal{Q}_b$, i.e., there exists a quasiperiodic RATS sequence of period 2.*

Sketch of Proof. In the case $b = 18k + 1$ for $k > 0$, the element

$$123^3 4^4 5^{12} 6^{16} 7^{48} 8^{64} \cdots (6k)^m (6k+1)^{(23 \cdot 64^k - 32)/36}$$

is quasiperiodic with quasiperiod 2 if m is sufficiently large.

In the case $b = 18k + 10$ for $k \geq 0$, the element

$$123^3 4^4 5^{12} 6^{16} 7^{48} 8^{64} \cdots (6k+1)^{(3 \cdot 64^k)/4} (6k+2)^{64^k} (6k+3)^m (6k+4)^{(44 \cdot 64^k - 8)/9}$$

is quasiperiodic with quasiperiod 2 if m is sufficiently large. $\square$

Theorem 4.1 shows that bases containing at least one quasiperiodic sequence make up a positive proportion of all integers. The following corollary makes this statement explicit.

Corollary 4.2. *Let*

$$\delta = \liminf_{N \to \infty} \frac{\#\{b \leq N : \mathcal{Q}_b \neq \emptyset\}}{N}. \tag{4.1}$$

Then $\delta \geq 1/9$.

Along the lines of Theorem 4.1, Cooper and Shattuck [7] were also able to conclude that for bases $b = (2^q - 1)^2 + 1$, with q a prime or a base-2 Fermat pseudoprime, we have $q \in \mathcal{Q}_b$. In the next theorem, we expand on their result in two directions. First, we will remove the (pseudo)primality constraint on the quasiperiod q and second, we will exhibit an infinite family of bases b for which q is a quasiperiod.

Theorem 4.3. *Let $q > 2$. If $b \equiv 1 \pmod{(2^q - 1)^2}$ with $b > 1$, then $q \in \mathcal{Q}_b$.*

Proof. Suppose that $b = (2^q - 1)^2 k + 1$, where $k \geq 1$. Let $v = (2^q - 1)k$. We will construct a quasiperiodic integer in base b with quasiperiod q and growing digit v.

Consider an integer of the form

$$1^{m_1} 2^{m_2} \ldots (v-1)^{m_{v-1}} \boldsymbol{v}^{M + m_v} (v+1)^{\sum_{i=1}^{v} m_i}, \tag{4.2}$$

where $M > 0$, $m_i \geq 0$ for $i = 1, \ldots, v$, and where the digit in boldface denotes the proposed growing digit. We will derive a system of equations and conditions that the exponents m_i must satisfy in order for the integer (4.2) to have quasiperiod q. We then will show that this system has a solution provided M is large enough.

We see that the reverse-add portion of the RATS process produces

$$\begin{array}{cccccccc} & 1^{m_1} & 2^{m_2} & \ldots & \boldsymbol{v}^M & (v+1)^{m_v} & \ldots & (v+1)^{m_1} \\ + & (v+1)^{m_1} & (v+1)^{m_2} & \ldots & \boldsymbol{v}^M & v^{m_v} & \ldots & 1^{m_1} \\ \hline & (v+2)^{m_1} & (v+3)^{m_2} & \ldots & (\boldsymbol{2v})^M & (2v+1)^{m_v} & \ldots & (v+2)^{m_1} \end{array}.$$

(Note that no carries occur in the sum and every digit of the sum is nonzero.) Sorting the result gives the second term in the RATS sequence,

$$(v+2)^{2m_1}(v+3)^{2m_2}\ldots(2v-1)^{2m_{v-2}}(\boldsymbol{2v})^{M+2m_{v-1}}(2v+1)^{2m_v}. \tag{4.3}$$

In order for the growing digit $\boldsymbol{2v}$ of (4.3) to (approximately) line up against itself in the next sum, we need $2\sum_{i=1}^{v-2} m_i \approx 2m_v$. Let M_1 be such that

$$2m_v = M_1 + 2\sum_{i=1}^{v-2} m_i \tag{4.4}$$

and assume $M_1 \geq 0$ for now. We repeat the above process and get

$$\begin{array}{cccccccc} & (v+2)^{2m_1} & \ldots & (\boldsymbol{2v})^{M+2m_{v-1}-M_1} & (2v+1)^{M_1} & \ldots & (2v+1)^{2m_1} \\ + & (2v+1)^{2m_1} & \ldots & (\boldsymbol{2v})^{M+2m_{v-1}-M_1} & (2v)^{M_1} & \ldots & (v+2)^{2m_1} \\ \hline & (3v+3)^{2m_1} & \ldots & (\boldsymbol{4v})^{M+2m_{v-1}-M_1} & (4v+1)^{M_1} & \ldots & (3v+3)^{2m_1} \end{array}.$$

Sorting the result gives the third term in the RATS sequence,

$$(3v+3)^{4m_1}(3v+4)^{4m_2}\ldots(4v-1)^{4m_{v-3}}(\boldsymbol{4v})^{M+2m_{v-1}+4m_{v-2}-M_1}(4v+1)^{2M_1}. \tag{4.5}$$

In order for the growing digit $\boldsymbol{4v}$ of (4.5) to line up against itself in the next sum, we need $4\sum_{i=1}^{v-3} m_i \approx 2M_1$. Let M_2 be such that $2M_1 = M_2 + 4\sum_{i=1}^{v-3} m_i$ and assume $M_2 \geq 0$ for now.

Repeating this process a total of $q-1$ times gives that the qth term in the RATS sequences is

$$\begin{aligned} &((2^{q-1}-1)v+q)^{2^{q-1}m_1}((2^{q-1}-1)v+q+1)^{2^{q-1}m_2}\ldots \\ &\ldots(2^{q-1}v-1)^{2^{q-1}m_{q-1}}(\boldsymbol{2^{q-1}v})^{M+\sum_{i=1}^{q-1}2^i m_{v-i}-\sum_{i=1}^{q-2}M_i}(2^{q-1}v+1)^{2M_{q-2}}, \end{aligned} \tag{4.6}$$

where the M_j's are defined to be such that

$$2M_j = M_{j+1} + 2^{j+1}\sum_{i=1}^{v-(j+2)} m_j \tag{4.7}$$

for $j = 1, \ldots, q-2$ and are assumed to be nonnegative.

We repeat the reverse-add portion of the RATS process one more time and get

$$\begin{array}{cccccc} & ((2^{q-1}-1)v+q)^{2^{q-1}m_1} & \ldots & (\boldsymbol{2^{q-1}v})^{M'} & \ldots & (2^{q-1}v+1)^{2^{q-1}m_1} \\ + & (2^{q-1}v+1)^{2^{q-1}m_1} & \ldots & (\boldsymbol{2^{q-1}v})^{M'} & \ldots & ((2^{q-1}-1)v+q)^{2^{q-1}m_1} \\ \hline & 1\,(q+1)^{2^{q-1}m_1} & \ldots & \boldsymbol{v}^{M''} & \ldots & (q+1)^{2^{q-1}m_1-1}\,q \end{array}$$

where $M' = M + \sum_{i=1}^{q-1}(2^i m_{v-i} - M_i)$ and $M'' = M + \sum_{i=1}^{q-1}(2^i m_{v-i} - M_i)$. (This is the first instance where carries occur; in fact, they occur at every position in the sum, and the result has all nonzero digits.) Sorting the result gives the $(q+1)$th term in the RATS sequence,

$$1q(q+1)^{2^q m_1 - 1}(q+2)^{2^q m_2} \ldots \boldsymbol{v}^{M+2m_{v-1}+\cdots+2^q m_{v-q}-(M_1+\cdots+M_{q-1})}(v+1)^{2M_{q-1}}. \tag{4.8}$$

We seek to choose the m_i's such that the $(q+1)$th term is identical to the first term of the RATS sequence, except that the exponent of v in (4.8) should be one larger than the exponent of v in (4.2), the starting element. Comparing exponents in (4.2) and (4.8) for all digits except v leads to the following set of equations:

$$\begin{aligned} m_1 &= 1, \\ m_2 &= 0, \\ m_3 &= 0, \\ &\vdots \\ m_{q-1} &= 0, \\ m_q &= 1, \\ m_{q+1} &= 2^q m_1 - 1 = 2^q - 1, \\ m_{q+2} &= 2^q m_2 = 0, \\ m_{q+3} &= 2^q m_3 = 0, \\ &\vdots \\ m_{2q-1} &= 2^q m_{q-1} = 0, \\ m_{2q} &= 2^q m_q = 2^q, \\ m_i &= 2^q m_{i-q} \text{ for } 2q < i \leq v-1, \end{aligned} \tag{4.9}$$

and

$$2M_{q-1} = \sum_{i=1}^{v} m_i. \tag{4.10}$$

Note that these conditions automatically imply that the exponent of v in (4.8) must be one larger than that in (4.2). This is due to the fact that the total number of digits is preserved by the RATS process in every step leading from (4.2) to (4.8) except for the final step, where carries result in one additional digit. We see that (4.9) immediately gives nonnegative integer values of m_i for $1 \le i \le v-1$. What remains is to show that m_v and the quantities M_j defined by (4.7) and (4.4) are nonnegative integers. Furthermore, we need to ensure that the exponents of the growing digit in (4.2)–(4.6) and (4.8) are all nonnegative, in order for these expressions to make sense.

For m_v, combining (4.10), (4.4) and (4.7) for $j = 1, \ldots, q-2$, we get that

$$\begin{aligned}
2m_v &= M_1 + 2\sum_{i=1}^{v-2} m_i,\\
4m_v &= 2M_1 + 4\sum_{i=1}^{v-2} m_i,\\
4m_v &= M_2 + 4\sum_{i=1}^{v-3} m_i + 4\sum_{i=1}^{v-2} m_i,\\
8m_v &= 2M_2 + 8\sum_{i=1}^{v-3} m_i + 8\sum_{i=1}^{v-2} m_i,\\
&\vdots\\
2^q m_v &= 2M_{q-1} + 2^q\sum_{i=1}^{v-q} m_i + \cdots + 2^q\sum_{i=1}^{v-2} m_i,\\
2^q m_v &= \sum_{i=1}^{v} m_i + 2^q\sum_{i=1}^{v-q} m_i + \cdots + 2^q\sum_{i=1}^{v-2} m_i.
\end{aligned}$$

In particular, we get that $m_v = \frac{1}{2^q-1}L(q,v)$, where

$$L(q,v) = \left[2^q\left(\sum_{i=1}^{v-q} m_i + \cdots + \sum_{i=1}^{v-2} m_i\right) + \sum_{i=1}^{v-1} m_i\right]. \tag{4.11}$$

Assume that m_v is an integer. (We will establish below that this is indeed the case.) It immediately follows from (4.4) that M_1 is also an integer and that

$$\begin{aligned}
M_1 &= 2m_v - 2\sum_{i=1}^{v-2} m_i\\
&= \frac{2}{2^q-1}\left[2^q\left(\sum_{i=1}^{v-q} m_i + \cdots + \sum_{i=1}^{v-2} m_i\right) + \sum_{i=1}^{v-1} m_i\right] - 2\sum_{i=1}^{v-2} m_i\\
&= \left(\frac{2^{q+1}}{2^q-1} - 2\right)\sum_{i=1}^{v-2} m_i + \frac{2}{2^q-1}\left[2^q\left(\sum_{i=1}^{v-q} m_i + \cdots + \sum_{i=1}^{v-3} m_i\right) + \sum_{i=1}^{v-1} m_i\right].
\end{aligned}$$

It follows that $M_1 \geq 0$, since $\frac{2^{q+1}}{2^q-1} - 2 \geq 0$ and $m_i \geq 0$ for all $1 \leq i \leq v-1$.

Similarly, using (4.7) with $j = 1$, since M_1 is an integer, it follows that M_2 is also a nonnegative integer. Continuing in this way, we see that the numbers M_j defined by (4.7) and (4.4) are indeed nonnegative integers. Taking $M > 0$ sufficiently large, which we are free to choose, we get that every growing digit exponent in (4.2)–(4.6) and (4.8) is a nonnegative integer. Therefore, (4.2) is a quasiperiodic element with quasiperiod q.

To complete the proof, we will show that $m_v = \frac{1}{2^q-1}L(q,v)$ is an integer, or equivalently, that

$$L(q,v) \equiv 0 \pmod{2^q - 1}. \tag{4.12}$$

The remainder of the proof is broken down into cases, depending on the residue of v modulo q. We will only provide the proof for the case $v \equiv 0 \pmod q$, as the other cases are quite similar.

Suppose that $v \equiv 0 \pmod q$. Then $v = sq$ for some $s \geq 1$. By (4.11),

$$\begin{aligned} L(q,v) &= \left[2^q\left(\sum_{i=1}^{v-q} m_i + \cdots + \sum_{i=1}^{v-2} m_i\right) + \sum_{i=1}^{v-1} m_i\right] \qquad (4.13)\\ &= \left[2^q\left(\sum_{i=1}^{(s-1)q} m_i + \cdots + \sum_{i=1}^{sq-2} m_i\right) + \sum_{i=1}^{sq-1} m_i\right]. \end{aligned}$$

Using (4.9) and (4.10), we see that

$$\begin{aligned} \sum_{i=1}^{sq-1} m_i &= 1 + 1 + (2^q - 1) + 2^q + 2^q(2^q-1) + 2^{2q} + \cdots + 2^{(s-2)q} + 2^{(s-2)q}(2^q - 1)\\ &= 1 + 2^q + 2^{2q} + \cdots + 2^{(s-1)q}\\ &= \sum_{j=0}^{s-1} 2^{qj}. \end{aligned}$$

For the remaining sums in the definition of $L(q,v)$, notice that $m_i = 0$ for $v-q+1 < i < v-2$, so the sums inside the parenthesis of (4.13) are identical, with the exception of the first sum, $\sum_{i=1}^{(s-1)q} m_i$, which is missing the term $2^{(s-2)q}(2^q-1)$. Thus, we get that

$$\begin{aligned} L(q,v) &= 2^q\left((q-1)\sum_{j=0}^{s-1} 2^{qj} - 2^{(s-2)q}(2^q-1)\right) + \sum_{j=0}^{s-1} 2^{qj}\\ &= (2^q(q-1)+1)\sum_{j=0}^{s-1} 2^{qj} - 2^{(s-1)q}(2^q-1). \end{aligned}$$

Reducing modulo $2^q - 1$, it follows that

$$\begin{aligned}
&\equiv (1(q-1)+1)\sum_{j=0}^{s-1} 1 \pmod{2^q-1} \\
&\equiv sq \pmod{2^q-1} \\
&\equiv v \pmod{2^q-1} \\
&\equiv (2^q-1)k \pmod{2^q-1} \\
&\equiv 0 \pmod{2^q-1}.
\end{aligned}$$

Therefore, (4.12) holds, as desired.

Similar computations for the remaining cases exhaust all possibilities for the residue of v modulo q, which gives that m_v is always a nonnegative integer. □

Theorem 4.3 shows that the asymptotic density of bases with nonempty quasiperiod sets is higher than the lower bound given by Corollary 4.2.

Corollary 4.4. *With δ as in* (4.1)*, we have* $\delta \geq \frac{1}{9} + \frac{1}{49}\left(1 - \frac{1}{9}\right) \approx 0.129$.

Proof. By Theorem 4.1, any base $b \equiv 1, 10 \pmod{18}$ is such that $2 \in \mathcal{Q}_b$. By Theorem 4.3 for $q = 3$, any base $b \equiv 1 \pmod{49}$ is such that $3 \in \mathcal{Q}_b$. The set of bases that are either congruent to $1, 10 \pmod{18}$ or $1 \pmod{49}$ has asymptotic density $(1/9) + (1/49)(1 - 1/9)$ by the Chinese remainder theorem. It follows immediately that $\delta \geq (1/9) + (1/49)(1 - 1/9)$. □

Theorems 4.1 and 4.3 also imply that there exist bases b such that $\mathcal{Q}_b$ contains any prescribed finite set of distinct positive integers > 1.

Corollary 4.5. *For any set $A = \{q_1, q_2, \ldots, q_k\}$ of distinct positive integers > 1, there exists a base b such that $A \subseteq \mathcal{Q}_b$.*

Proof. Let $b = 18(2^{q_1} - 1)^2(2^{q_2} - 1)^2 \cdots (2^{q_k} - 1)^2 + 1$. Then $b \equiv 1 \pmod{18}$ and $b \equiv 1 \pmod{(2^{q_i} - 1)^2}$ for each i. The first condition implies that $2 \in \mathcal{Q}_b$ by Theorem 4.1 and the second condition implies that $q_i \in \mathcal{Q}_b$ for any $q_i > 2$, by Theorem 4.3. □

Corollary 4.5 shows that there are bases with arbitrarily large quasiperiod sets. However, it is still unknown if there are bases with distinct quasiperiodic RATS sequences with the same quasiperiod. It is conjectured that this is never the case.

Conjecture (Cooper and Shattuck, [7]). No base has two or more distinct quasiperiodic RATS sequences with the same quasiperiod.

The definition for quasiperiodic sequences can sometimes seem rather narrow. However, all computational evidence compiled so far has yielded that the behavior of RATS sequences always fit the descriptions of ultimately periodic or ultimately quasiperiodic sequences. We conjecture that this is always the case.

Conjecture. In any base, every RATS sequence is ultimately periodic or ultimately quasiperiodic.

5. Behavior of Growing Digits

In this section we introduce a finite map, f_b, that describes the behavior of growing digits in a quasiperiodic sequence in base b. We establish properties of this map and use these results to prove nonexistence results for quasiperiodic sequences in certain bases.

We begin with Conway's sequence

$$\cdots \mapsto 123^m4^4 \mapsto 5^26^m7^4 \mapsto 123^{m+1}4^4 \mapsto 5^26^{m+1}7^4 \mapsto \cdots . \tag{5.1}$$

as a motivating example.

By examining this sequence, it is clear that 3 is the growing digit associated with the quasiperiodic element 123^m4^4. When performing the reverse-add portion of the RATS process, we see that the digit 3 primarily lines up against itself in the sum. This is the reason that $6 = 3 + 3$ is the growing digit of the next term in the sequence, $5^26^m7^2$. Performing the reverse-add portion of the RATS process, again, we see that the digit 6 primarily lines up against itself in the sum. Due to a propagating carry, the most popular digit of the next iterate is again 3, since $3 \equiv 6 + 6 + 1 \pmod{10}$.

In general, if k is the growing digit of some quasiperiodic element in base b, and $k < b/2$, then $2k$ is the growing digit of the next term in the RATS sequence. If $k \geq b/2$, then a propagating carry makes $2k+1-b$ the growing digit of the next term in the RATS sequence. Motivated by this idea, we make the following definition.

Definition 5.1. Given a base b, let

$$f_b(k) = \begin{cases} 2k & \text{if } 0 < k < b/2, \\ 2k + 1 - b & \text{if } b/2 \leq k < b. \end{cases}$$

For a nonnegative integer t, let $f_b^t(k) = f_b(f_b^{t-1}(k))$ if $t \geq 1$ and $f_b^0(k) = f_b(k)$.

Proposition 5.2. *Suppose that b is a base with a quasiperiodic element n. If k is the growing digit of n, then $f_b(k)$ is the growing digit of $R_b(n)$. In particular, if n is quasiperiodic with quasiperiod q, then $f_b^q(k) = k$.*

Proof. The discussion above Definition 5.1 shows that given an element n, if the digit $k \neq (b-1)/2$ appears in n with a vastly higher count than all other digits combined, then $f_b(k)$ gives the digit that will be the most popular in $R_b(n)$. Applying this reasoning to the growing digit of a quasiperiodic element, we obtain the desired result.

The case where $k = (b-1)/2$ is the most popular digit if n requires some additional care. In this case, the predominant digit of $R_b(n)$ could be either $b-1$ or 0, depending on the existence of carries to the right of the block of digits k. However, since the RATS process discards zero digits, this would result in $R_b(n)$ having fewer digits than n (for n large enough), a contradiction to the definition of a quasiperiodic element. Thus, if n is a quasiperiodic element with growing digit $k = (b-1)/2$, then the growing digit of $R_b(n)$ must be $2k = b-1 = f_b(k)$.

From the above work, it follows that $f_b^q(k) = k$ by repeating the above reasoning q times. □

Example 5.3. From (5.1), the growing digit of 123^m4^4 is 3 and $f_{10}^2(3) = f_{10}(6) = 3$.

Definition 5.4. Given a base b and a digit $0 < k < b$, we say that k is a **fixed point** of f_b if there is a $t > 0$ such that $f_b^t(k) = k$. If t is the least integer with this property, we say that the **order** of k is t.

Corollary 5.5. *The quasiperiod of a quasiperiodic element is a multiple of the order of the corresponding growing digit with respect to the map f_b.*

Proof. Suppose that k is the growing digit of a quasiperiodic element with quasiperiod q. By Proposition 5.2, $f_b^q(k) = k$; that is, k is a fixed point of f_b. Suppose that the order of k is t. By the Euclidean algorithm, there are nonnegative integers s and r such that $q = st + r$ with $0 \leq r < t$. Then $k = f_b^q(k) = f_b^{st+r}(k) = f_b^r(f_b^{st}(k)) = f_b^r(k)$. If $r > 0$, then we have a contradiction to the assumed order of k. Hence $r = 0$ and $t|q$. Therefore, the quasiperiod of a quasiperiodic element is a multiple of the order of the corresponding digit. □

The above results show that the growing digit of a quasiperiodic element is a fixed point of the corresponding map f_b. However, the converse is not true in general. The element $b-1$ is always a fixed point (of order 1), but is never a growing digit of a quasiperiodic element, as the following results show.

Lemma 5.6. *For any base b, k is a fixed point of order 1 if and only if $k = b-1$.*

Proof. ($\Rightarrow$) If k is a fixed point of order 1, then either $k = f_b(k) = 2k$ or $k = f_b(k) = 2k + 1 - b$. The first case gives that $k = 0$, which is a contradiction, since $k > 0$, and the second case gives that $k = b-1$.

($\Leftarrow$) A simple calculation shows that, since $b > 1$, $b - 1 \geq b/2$, so $f_b(b-1) = 2(b-1) + 1 - b = b - 1$. □

Lemma 5.7. *For any base b, if a quasiperiodic element has growing digit k, then $k \neq b-1$.*

We leave the proof of Lemma 5.7 to the reader.

Theorem 5.8. *For any base b, we have $1 \notin \mathcal{Q}_b$. In other words, in any base, there are no quasiperiodic elements with quasiperiod 1.*

Proof. Some simple computations show that all RATS sequences in base 2 are periodic, giving that $\mathcal{Q}_2 = \emptyset$. So the assertion holds for $b = 2$.

Suppose $b > 2$ and that there is a quasiperiodic element n with quasiperiod 1 and growing digit k. By Corollary 5.5, k must be a fixed point of f_b of order 1. By Lemma 5.6, k must be $b-1$, a contradiction to Lemma 5.7. □

Theorem 5.8 serves as an analogue of the fact that $1 \notin \mathcal{P}_b$ (see [3, Theorem 1] for the case $b = 10$) for any base $b > 2$. The result is slightly different in that base 2 does not play a special role. Continuing in this same vein, we investigate the existence (or nonexistence) of fixed points of f_b of a given order for general bases.

Lemma 5.9. *For any base $b > 2$, there is a fixed point of f_b of order 2 if and only if $b \equiv 1 \pmod 3$.*

Proof. ($\Rightarrow$). Suppose that k is a fixed point of f_b of order 2. We consider four cases.

Case 1. $0 < k, f_b(k) < b/2$**.** Then $k = f_b^2(k) = 4k$, which implies that $k = 0$, a contradiction, since $k > 0$. So this case cannot happen.

Case 2. $0 < k < b/2 \leq f_b(k) < b$**.** Then $k = f_b^2(k) = 2(2k)+1-b = 4k+1-b$. This equation is equivalent to $3k = b-1$, which implies that $b \equiv 1 \pmod 3$.

Case 3. $0 < f_b(k) < b/2 \leq k < b$**.** Then $k = f_b^2(k) = 2(2k+1-b) = 4k+2-2b$. This equation is equivalent to $3k = 2(b-1)$, which implies that $b \equiv 1 \pmod 3$.

Case 4. $b/2 \leq k, f_b(k) < b$**.** Then $k = f_b^2(k) = 2(2k+1-b)+1-b = 4k+3-3b$. This equation is equivalent to $k = b-1$, a contradiction, since, by Lemma 5.7, $k < b-1$. So this case cannot happen.

Having exhausted all possibilities, we see that we must have that $b \equiv 1 \pmod 3$.

($\Leftarrow$) Suppose that $b \equiv 1 \pmod 3$. Let $k = (b-1)/3$. Then, since $0 < k < b/2$, $f_b(k) = 2k = 2(b-1)/3$. For $b \geq 4$ we have $2(b-1)/3 > b/2$, so $f_b^2(k) = 4(b-1)/3+1-b = (b-1)/3 = k$. Therefore, k is a fixed point of f_b of order 2. □

Corollary 5.10. *If $2 \in \mathcal{Q}_b$, for some base $b > 2$, then $b \equiv 1 \pmod 3$.*

Proof. The result follows from Lemma 5.9 and Proposition 5.2. □

Corollary 5.10 shows that the density of bases containing a quasiperiodic sequence with quasiperiod 2 is at most 1/3. On the other hand, Theorem 4.1 shows that quasiperiodic sequences with quasiperiod 2 exist in any base b satisfying $b \equiv 1 \pmod{18}$ or $b \equiv 10 \pmod{18}$. Hence this density is at least 1/9.

The following lemma shows that, for any given order t, there is a b such that f_b has a fixed point of order t. It should not be surprising that this is the case, as Theorems 4.1 and 4.3 show that quasiperiodic sequences can be constructed for any desired quasiperiod. What is interesting about the lemma is that the bases in the statement do not match the form of those in Theorems 4.1 and 4.3.

Lemma 5.11. *If $b = 2^t$, then 1 is a fixed point of order t.*

Proof. The lemma follows quickly from noticing that, for $b = 2^t$, we have $f_b^s(1) = 2^s$ for $1 \leq s < t$, and $f_b^t(1) = 2^t + 1 - b = 1$. □

Results like Corollary 5.10 and Lemma 5.11 can be used to narrow down bases that allow the existence of quasiperiodic RATS sequences of a specified quasiperiod. So if we are interested in finding quasiperiodic RATS sequences, by Proposition 5.2, it seems natural to search for bases b that have many fixed points of f_b.

Definition 5.12. For a base b, let $S_b = \{k : 0 < k < b, f_b^t(k) = k \text{ for some } t\}$. In other words, S_b is the set of all fixed points of f_b.

The following result gives an exact description of bases b for which f_b has only one fixed point. Notice that by Lemma 5.6, $b - 1$ is always a fixed point of f_b, so we always have $|S_b| \geq 1$.

Theorem 5.13. *For any base $b > 2$, we have $|S_b| = 1$ if and only if $b = 2^t + 1$ for some positive integer t.*

Proof. ($\Leftarrow$) Assume that $b = 2^t + 1$ for some positive integer t. If $t = 1$, then $b = 2^1 + 1 = 3$, and direct verification shows that in this case, $k = 2$ is the only fixed point of f_b. Hence, $|S_b| = 1$. Suppose now that $t \geq 2$. Let $\{1, 2, \ldots, b-1\} = \bigcup_{i=0}^{\infty} A_i$, where $A_i = \{k : 0 < k < b, 2^i \mid\mid k\}$. In other words, we partition the set $\{1, 2, \ldots, b-1\}$ according to the highest power of 2 dividing each digit. Notice that all but finitely many A_i's are empty; in particular, since $b - 1 = 2^t$, we have $\{1, 2, \ldots, b-1\} = \bigcup_{i=0}^{t} A_i$.

The set A_0 represents all odd numbers in $\{1, \ldots, 2^t\}$. Let $m \in A_0$. If $m < b/2$, then $f_b(m) = 2m$. If $m \geq b/2$, then $f_b(m) = 2m - b + 1 = 2m - 2^t$. Since $2 \mid f_b(m)$ and $4 \nmid f_b(m)$ in either case, we get that $f_b(A_0) \subseteq A_1$. Now let $m' \in A_1$. Then $m' = 2m''$ with $2 \nmid m''$. In particular, $m'' \in A_0$. Since $m'' = m'/2 < b/2$, $f_b(m'') = 2m'' = m'$. So we get that $A_1 \subseteq f_b(A_0)$. Therefore, we have $f_b(A_0) = A_1$.

Continuing in this fashion, we get that $f_b(A_i) = A_{i+1}$ for $0 \leq i < t$. Notice that $A_t = \{2^t\}$, so for any starting digit k, there is a t' such that $f_b^{t'}(k) = 2^t$. Since, by

Lemma 5.6, $b-1=2^t$ is a fixed point of order one, there cannot be any other fixed points. This is equivalent to the statement that $|S_b|=1$.

($\Rightarrow$) For the converse direction, we first note that if m is such that $0<m<b$, then, by Definition 5.1,

$$f_b^{-1}(m) = \begin{cases} \left\{\frac{m}{2}, \frac{m}{2}+\frac{b-1}{2}\right\} & \text{if } m \text{ is even and } b \text{ is odd,} \\ \emptyset & \text{if } m \text{ is odd and } b \text{ is odd,} \\ \left\{\frac{m}{2}\right\} & \text{if } m \text{ is even and } b \text{ is even,} \\ \left\{\frac{m}{2}+\frac{b-1}{2}\right\} & \text{if } m \text{ is odd and } b \text{ is even.} \end{cases} \tag{5.2}$$

We will use this fact extensively in this portion of the proof.

Assume now that $|S_b|=1$. By Lemma 5.6, we know that $b-1$ is a fixed point of f_b, so $S_b=\{b-1\}$. By the pigeonhole principle, for every $0<k<b-1$ there is a $t>0$ such that $f_b^t(k)$ is a fixed point of f_b. By assumption, $b-1$ is the only fixed point, so $f_b^t(k)=b-1$. In particular, $|f_b^{-1}(b-1)|>1$, which, by (5.2), implies $b-1$ is even and

$$f_b^{-1}(b-1) = \left\{\frac{c(b-1)}{2} : 1 \le c \le 2\right\},$$

where c runs through integers in the given range. Thus b must be of the form $b-1=2^{i_0}b_0$ with $i_0>0$ and $2 \nmid b_0$. If $i_0>1$, then using (5.2) again, we get

$$f_b^{-2}(b-1) = \left\{\frac{c(b-1)}{4} : 1 \le c \le 4\right\}.$$

Continuing in this way, we see that, for $1 \le i \le i_0$,

$$f_b^{-i}(b-1) = \left\{\frac{c(b-1)}{2^i} : 1 \le c \le 2^i\right\}.$$

In particular, for $1 \le i \le i_0$,

$$|f_b^{-i}(b-1)| = 2^i. \tag{5.3}$$

Notice that for $i<i_0$, the elements in $f_b^{-i}(b-1)$ are all even, but in $f_b^{-i_0}(b-1)$, half of the elements are odd. This implies, by (5.2), that

$$f_b^{-(i_0+1)}(b-1) = f_b^{-i_0}(b-1)$$

and hence $f_b^{-(i_0+t)}(b-1) = f_b^{-i_0}(b-1)$ for all $t>0$. Hence $\{0<k<b : k \text{ odd}\} \subseteq f_b^{-i_0}(b-1)$. Since, by (5.3), $f_b^{-i_0}(b-1)$ contains 2^{i_0} elements, and, by the above observation, half of these elements are odd, this implies that $\#\{0<k<b : k \text{ odd}\} \le 2^{i_0-1}$ and hence $(b-1)/2 = 2^{i_0-1}b_0 \le 2^{i_0-1}$. Thus $b_0=1$ and $b=2^{i_0}b_0+1=2^{i_0}+1$, so b is of the desired form.

In the case $i_0=1$, a similar argument shows that $b=2^1+1=3$, which is also of the desired form. $\square$

Corollary 5.14. *For any base $b = 2^t + 1$, where $t \geq 0$, we have $\mathcal{Q}_b = \emptyset$. In other words, there are no quasiperiodic RATS sequences in base $b = 2^t + 1$.*

Proof. Suppose that there is a quasiperiodic element n in base $b = 2^t + 1$. By Proposition 5.2, the growing digit of n must be fixed point of f_b. Theorem 5.13 shows that if $b = 2^t + 1$, then the only fixed point of f_b is $b - 1$. It follows that $b - 1$ must be the growing digit of n, a contradiction to Lemma 5.7. Therefore, there are no quasiperiodic RATS sequences in base b. $\square$

Corollary 5.14 shows that there is an infinite family of bases for which no quasiperiodic RATS sequences exist. Unfortunately, this is not enough to conclude that all sequences in such bases are ultimately periodic. For example, McMullen's conjecture [5], that all RATS sequences are ultimately periodic for bases $4 \leq b < 10$, is still open in the cases $b = 5 = 2^2 + 1$ and $b = 9 = 2^3 + 1$.

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