IRIA
laboria

laboratoire de recherche
en informatique
et automatique

ON THE UNIFORM HALTING PROBLEM FOR TERM REWRITING SYSTEMS

Gérard HUET
Dallas LANKFORD

Rapport de Recherche N° 283
Mars 1978

Institut de Recherche
d'Informatique
et d'Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105 78150 - Le Chesnay
France
Tél.: 954 90 20

ON THE UNIFORM HALTING PROBLEM FOR TERM REWRITING SYSTEMS

Gérard Huet*, Dallas Lankford**

Résumé :

On montre que le problème de l'arrêt uniforme des systèmes de réécriture de termes est indécidable de degré 0'', même en se restreignant aux symboles de fonction monadiques. On montre que par contre ce problème est décidable pour les termes sans variables.

Abstract :

We show that the uniform halting problem for term rewriting systems is undecidable of degree 0'', even when terms are restricted to monadic function symbols. We also show that the uniform halting problem for ground term rewriting systems is decidable.

* IRIA/LABORIA

** USC - ISI

1. Introduction

Let T be the set of terms of a first-order logic. A term rewriting system over T is a finite set :

$$R = \{\langle\gamma_i,\delta_i\rangle \mid 1\le i\le N\} \text{ with } \forall i\le N \; V(\delta_i)\subset V(\gamma_i)$$

where $V(t)$ denotes the set of variables appearing in term t.

We define, in the same way as in [2], the relation $\underset{R}{\rightarrow}$ over T as the smallest relation containing R and closed by :

a) for every substitution σ $\quad t \underset{R}{\rightarrow} t' \Longrightarrow \sigma(t) \underset{R}{\rightarrow} \sigma(t')$

b) $t_i \underset{R}{\rightarrow} t'_i \Longrightarrow Ft_1\cdots t_n \underset{R}{\rightarrow} Ft_1\cdots t_{i-1}t'_i t_{i+1}\cdots t_n$

for any function symbol F of arity n.

In the following we abbreviate $\underset{R}{\rightarrow}$ by $\rightarrow$.

A term t is said to be *immortal* in R iff there exists an infinite $\rightarrow$ sequence starting from t :

$$t = t_0 \rightarrow t_1 \rightarrow\cdots t_n \rightarrow\cdots .$$

If no term t is immortal in R, we say that R is *noetherian*. The uniform halting problem for R is to determine whether or not R is noetherian.

In order to do that, we shall show how to map this problem into the uniform halting problem for Turing machines.

2. The construction of R_M

Let M be any Turing machine, with

input alphabet $\Sigma = \{s_0,s_1,\cdots,s_n\}$ and

state alphabet $Q = \{q_0,q_1,\cdots,q_p\}$.

We assume $s_0=\star$, the blank symbol.

We consider the set T_M of terms built from the following function symbols :

$$F_1 = \{\overleftarrow{S}_0,\cdots,\overleftarrow{S}_n,\overrightarrow{S}_0,\cdots,\overrightarrow{S}_n,Q_0,\cdots,Q_p,L\}$$

and $F_0 = \{R\}$. All symbols in F_1 have arity 1, R has arity 0. We shall write the terms in T_M as strings over $F_1^{\star}(F_0\cup\{x\})$.

Let I be an instantaneous description of machine M :

$$I = \langle w_1,q_i,w_2\rangle$$

where $w_1\in\Sigma^{\star}$ is the contents of the tape to the left of the head, $w_2\in\Sigma^{\star}$ is the contents of the tape to the right of the head, q_i is the current state (we assume, as usual, the tape blank everywhere except in a finite portion). We code I by the term :

$$t_I = L\ \vec{w}_1\ Q_i\ \overleftarrow{w}_2\ R \in T_M.$$

<u>Example</u> : With $I = \langle s_1 s_2, q_2, s_1 s_4 \rangle$,

corresponding to the configuration :

...	*	*	s_1	s_2	s_1	s_4	*	*	...

$\uparrow q_2$

we get : $$t_I = L\ \vec{S}_1\ \vec{S}_2\ Q_2\ \overleftarrow{S}_1\ \overleftarrow{S}_4\ R.$$

Let us now show how to code the program of M with a term rewriting system R_M over T_M.

For any right-moving instruction of M :

"in state q_i reading s_j, write s_k and go right in state q_ℓ"

we put in R_M the rule $\langle Q_i\ \overleftarrow{S}_j\ x,\ \vec{S}_k\ Q_\ell\ x \rangle$.

If j=0 we add also the rule :

$$\langle Q_i R,\ \vec{S}_k\ Q_\ell R \rangle ,$$

corresponding to reading a new portion of tape to the right.

For any left-moving instruction of M :

"in state q_i reading s_j, write s_k and go left in state q_ℓ"

we put in R_M the rules :

$$\langle \vec{S}_m\ Q_i\ \overleftarrow{S}_j\ x,\ Q_\ell\ \overleftarrow{S}_m\ \overleftarrow{S}_k\ x \rangle \text{ for all } m,\ 0 \le m \le n,$$

and $$\langle L\ Q_i\ \overleftarrow{S}_j\ x,\ L\ Q_\ell\ \overleftarrow{S}_0\ \overleftarrow{S}_k\ x \rangle.$$

If j=0 we add also the rules :

$$\langle \vec{S}_m\ Q_i\ R,\ Q_\ell\ \overleftarrow{S}_m\ \overleftarrow{S}_k\ R \rangle \text{ for all } m,\ 0 \le m \le n,$$

and $$\langle L\ Q_i\ R,\ L\ Q_\ell\ \overleftarrow{S}_0\ \overleftarrow{S}_k\ R \rangle.$$

The construction of R_M is such that

$$I \underset{M}{\rightarrow} I' \Leftrightarrow t_I \underset{R_M}{\longrightarrow} t_I'.$$

From this follows immediately :

<u>Lemma 1</u> : If the instantaneous description I of M is immortal, then the term t_I is immortal in R_M.

(We recall that an instantaneous description of a Turing machine M is immortal iff starting with it M will never halt).

Corollary

The problem of determining, given $\mathcal{R}$ and t, whether t is immortal in $\mathcal{R}$, is undecidable of degree 0'.

Proof :

The halting problem for M on input x reduces to determining whether t_I is immortal in $\mathcal{R}_M$, where I is the initial instantaneous description of M with tape x. □

Note that the converse of lemma 1 holds also. However, this is too weak to give an answer to the uniform halting problem, since there are terms in $\mathcal{T}_M$ which do not code an instantaneous description. The next section will show how to get a stronger converse.

3. Equivalence of the uniform halting problems of M and $\mathcal{R}_M$

Let us introduce the notation :

$$\vec{\mathcal{S}} = \{\vec{S}_0, \cdots, \vec{S}_n\}$$

$$\overleftarrow{\mathcal{S}} = \{\overleftarrow{S}_0, \cdots, \overleftarrow{S}_n\}$$

$$\mathcal{Q} = \{Q_0, \cdots, Q_p\} .$$

Lemma 2 : If a term t is immortal in $\mathcal{R}_M$, M possesses some immortal instantaneous description.

Proof :

Any word t in $\mathcal{T}_M$ may be written as :

$$t = u_1 \, v_1 \, u_2 \, v_2 \cdots v_q \, u_{q+1} \, R \qquad q \geq 0$$

with $\quad u_i \in (\vec{\mathcal{S}} \cup \overleftarrow{\mathcal{S}} \cup \{L\})^* \qquad i \leq q+1$

and $\quad v_i \in \overleftarrow{\mathcal{S}}^* \, \mathcal{Q} \, \vec{\mathcal{S}}^* \qquad i \leq q.$

We assume furthermore that the v_i's are maximal, i.e. u_i does not end with an $\vec{S}_k$ $(1 \leq i \leq q)$, and u_i does not begin with an $\overleftarrow{S}_k$ $(2 \leq i \leq q+1)$.

<u>Claim</u> : If $t \xrightarrow[R_M]{} t'$ then $q>0$ and there exists a j, $1 \le j \le q$ such that

$\circ$ $t' = u_1 v_1 \cdots u_j\ v'_j u_{j+1} \cdots v_q u_{q+1}\ R$,

$\circ$ $Lv_j R \xrightarrow[R_M]{} Lv'_j R$

$\circ$ v'_j is of the form above.

This claim is easy to establish, by cases on the rule used to derive t into t'.

Now, if t is immortal, there must exist a $j \le q$ whose v_j is rewritten infinitely often, and therefore such that $Lv_j R$ is immortal in R_M. $Lv_j R$ is the code of an instantaneous description of M which is immortal by lemma 1. □

Combining lemma 1 and lemma 2, we get that R_M is noetherian iff M has no immortal instantaneous description. This second problem being undecidable of degree $0''$, using the results in Herman [1], we get :

<u>Theorem 1</u>

The uniform halting problem for term rewriting systems is undecidable of degree 0", even for terms restricted to monadic function symbols.

<u>Remark</u> :

In [3] it is claimed, without proof, that the uniform halting problem is undecidable, even when $|R| \le 3$.

We shall show in the next section that the uniform halting problem is decidable for ground term rewriting systems.

4. Ground term rewriting systems

We shall now restrict ourselves to the case where T consists only of *ground terms*, i.e. with no variables. A ground term rewriting system is a finite set :

$$R = \{<\gamma_1,\delta_i>|1\leq i\leq N\} \quad \forall i\leq N \quad V(\gamma_i) = V(\delta_i) = \emptyset.$$

Lemma 3 :

If t is immortal in R, there exists $i\leq N$ such that δ_i is immortal in R.

Proof : By induction on t.

Let $t = t_0 \to t_1 \to t_2 \to \cdots$ be an infinite $\to$-sequence, and let $u_i \in N_+^*$ be the occurrence of t_i which is reduced at step i.

If $\exists k\geq 0$ such that $u_k = \Lambda$ (the null, or top occurrence) then $t_{k+1} = \delta_k$ is immortal.

Otherwise, the sequence is internal, and let F be the common leading function symbol of all t_i's, p its arity : $t_i = Ft_i^1\cdots t_i^p \ \forall i\geq 0$. For every j, $1\leq j\leq p$, let $I_j = \{i|u_i \text{ begins with } j\}$. At least one of the I_j's is infinite, let us say I_k : $I_k = \{k_1,k_2,\cdots\}$ with $\forall i\geq 0 \quad 0\leq k_i\leq k_{i+1}$.

Then the subsequence

$$t_0^k = t_{k_1}^k \to t_{k_2}^k \to \cdots \quad \text{is an infinite} \to\text{-sequence, and the result}$$

follows by induction hypothesis. □

Lemma 4 :

If $\exists i\leq N$ such that δ_i is immortal in R, then $\exists j\leq N$ such that $\exists t \ \ \exists u\in O(t) \ \gamma_j \overset{*}{\to} t \ \& \ t/u = \gamma_j$.

Proof : By induction on $N = |R|$.

- If N=0 this is trivially true.
- Otherwise, assume δ_i is immortal in R :

$$\delta_i = t_0 \to t_1 \to t_2 \to \cdots$$

If the i-th rule is used in this reduction, then we can take j=i.

Otherwise, δ_i is immortal in $R' = R - \{<\gamma_i,\delta_i>\}$, and, using lemma 3, there exists $<\gamma_k,\delta_k>\in R'$ such that δ_k is immortal in R'. By induction hypothesis, there exists $<\gamma_j,\delta_j>\in R'\subset R$ answering the condition. □

Theorem 2 :

The uniform halting problem for ground term rewriting systems is decidable.

Proof :

Let R be a ground term rewriting system. Either R is noetherian, in which case all reductions from a given term terminate, or else using lemmas 3 and 4 there exists a rule $<\gamma,\delta>\in R$ such that δ reduces to a term containing γ as a subterm. We can therefore decide the uniform halting problem for R by enumerating, level by level, all the reductions of the right sides δ's of R, checking for the occurrence of the corresponding left side γ. Note that these reduction trees are finitely branching, R being finite. □

Bibliography

[1] G. Herman
Strong computability and variants of the uniform halting problem.
Z. Math. Logik Grundl. Math., 17 (1971), 115-131.

[2] G. Huet
Confluent reductions : abstract properties and applications to term rewriting systems.
Proceedings of 18th annual IEEE Symposium on Foundations of Computer Science, Oct. 77.

[3] R. Lipton & L. Snyder
On the halting of tree replacement systems.
Conference on theoretical computer science, University of Waterloo, July 1977.