Accepted Manuscript

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PII: S0304-4068(16)30029-5
DOI: http://dx.doi.org/10.1016/j.jmateco.2016.05.006
Reference: MATECO 2077

To appear in: *Journal of Mathematical Economics*

Received date: 23 July 2015
Revised date: 7 March 2016
Accepted date: 23 May 2016

Please cite this article as: Malamud, S., Rui, H., Whinston, A., Optimal reinsurance with multiple tranches. *Journal of Mathematical Economics* (2016), http://dx.doi.org/10.1016/j.jmateco.2016.05.006

Optimal Reinsurance with Multiple Tranches [*]

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Abstract

Motivated by common practices in the reinsurance industry and in insurance markets such as Lloyd's, we study the general problem of optimal insurance contracts design in the presence of multiple insurance providers. We show that the optimal risk allocation rule is characterized by a hierarchical structure of risk sharing where all agents take on risks only above the endogenously determined thresholds, or agent-specific deductibles. Linear risk sharing between two adjacent thresholds is shown to be optimal when all agents have CARA utilities. Furthermore, we show that the optimal thresholds can be efficiently calculated through the fixed point of a contraction mapping.

Keywords: risk sharing, insurance, reinsurance, contract design

JEL Classifications: G32, G21, G24

[*]The research of S. Malamud was supported in part by NCCR FINRISK, Project A5. The research of Andrew Whinston was supported in part by NSF grant 1228990. We thank Peter DeMarzo, Damir Filipovic, Günter Franke, Erwan Morellec, Christine Parlour, Stathis Tompaidis and Robert Wilson for useful comments. We are also thankful for the comments from participants in the 10th World Congress of Econometric Society in Shanghai. Existing errors are our sole responsibility.

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1 Introduction

The modern theory of efficient risk sharing goes back to the fundamental paper by Borch (1962), who characterized efficient risk sharing among several agents (typically more than two) with heterogeneous preferences. Based on this research, Wilson (1968) further developed the theory of syndicates. Both Borch and Wilson based their analysis on an important assumption that a complete set of state-contingent contracts is available for risk allocation. In many real-life situations, however, insurers are willing to take only risks that do not exceed a certain level. This situation is particularly true for insurance contracts, for which the corresponding insurance reimbursements (coverage functions) are always assumed to be non-negative and lower than the total loss. As has been shown by Arrow (1971, 1973) and Raviv (1979), such a feature of insurance policy may significantly alter the structure of optimal risk allocation. Namely, the efficient risk-sharing rule *between two agents* (i.e., the insured and the single insurerance provider) is generally characterized by the presence of a deductible. The goal of this paper is to extend Raviv's (1979) seminal characterization of optimal insurance design to the case of multiple insurers.

For insurance against loss that can potentially be very large, multiple insurance providers are typically involved to achieve more efficient risk sharing.[1] A well-known example is the so-called *subscription model* at Lloyd's, the world's leading insurance market providing specialist insurance services to businesses.[2] At Lloyd's, almost any

[1]By distributing large risk across many entities, insurance companies, large and small, can offer coverage limits to meet their policyholders' needs. This is very important for a more competitive insurance market.

[2]In his speech on the future of the insurance industry, Lord Levene, the former chairman of Lloyd's, said that "The first point which I want to make about the future of insurance is that the subscription model is not just alive and well — it is thriving. Lloyd's made record profits in 2009. Throughout the financial crisis, it maintained A+ ratings. Over three hundred years, it has never failed to pay a valid claim." Source: http://www.lloyds.com/Lloyds/Press-Centre/Speeches/2011/03/The-Future-of-the-Insurance-Industry.

single risk is insured by multiple insurers. As is stated on its website, "much of Lloyd's business works by subscription, where more than one syndicate takes a share of the same risk."[3] This is also a well-established practice among insurers generally.[4] Another example of allocating risk among multiple insurance providers is when an insurance company purchases insurance from multiple reinsurers.[5] Reinsurance is an indispensable and significant part of the insurance industry and "many reinsurance placements are not placed with a single reinsurer but are shared between a number of reinsurers."[6]

Despite its practical importance, there has been limited amount of research on optimal risk sharing in the presence of more than two insurance providers and the practical constraint that the insurance reimbursement is nonnegative and cannot exceed the size of the loss. The industry practice, which typically involves both proportional and excess of loss contracts with multiple agents offering insurance coverage, seems to be ad-hoc and lacks a strong theoretical basis.[7] This paper fills the gap in the literature by studying the optimal design of insurance contracts with multiple agents offering insurance coverage that satisfies the practical constraints. We also take into account of the intertemporal nature of insurance, which is a realistic aspect given that there is always a (sometimes significant) delay between the insurance premium payment and the arrival of an insurance event. We endogenize this by introducing intertemporal utility maximization for all agents. The framework in this paper applies both to the insurance scenario and the rein-

[3]Source: http://www.lloyds.com/lloyds/about-us/what-is-lloyds.

[4]See, for example, page 167 of Thoyts (2010) for more discussion.

[5]According to Reinsurance Association of America, "reinsurance is a transaction in which one insurance company indemnifies, for a premium, another insurance company against all or part of the loss that it may sustain under its policy or policies of insurance."

[6]Source:http://en.wikipedia.org/wiki/Reinsurance.

[7]Reinsurance policies can be categorized according to whether they are proportional or non-proportional with excess of loss contract being the prime example of the later. An insurance company often purchases several insurance policies of different types from multiple reinsuers and combine these policies to form multiple layers of insurance protection. Chapter 7 of Thoyts (2010) explains with detailed examples how different types of reinsurance policies work.

surance scenario. For ease of illustration, we call the agent seeking insurance coverage, whether a client of an insurance company or an insurance company itself, the insured, and the agents offering insurance coverage, whether insurance companies or reinsurance companies, the insurers.

Our first result implies that the practical constraints on insurance contracts, together with insurers' heterogeneity, naturally give rise to *optimal claims splitting through a tranche structure,* with different tranches characterized as the regions for which these constraints are binding for different groups of insurers. The total uncertain loss is divided into several tranches, whose boundaries are the insurer-specific deductibles. Different insurers provide partial coverage for losses inside multiple tranches. This prioritized tranche-sharing structure with multiple deductibles is very intriguing. It arises because of insurers' risk aversion and the heterogeneity of their marginal valuations. The insured optimally insures the first tranche above the minimal deductible with the insurer requiring the lowest marginal premium. Because this insurer is risk averse, the marginal premium increases with the level of losses. Just as the level of losses reaches the next deductible level, the first insurer's marginal premium reaches that of the second-highest ranked insurer, and it becomes optimal for the insured to buy co-insurance of the subsequent tranche from this second-highest ranked insurer. Continuing the process gradually, as the level of losses increases, insurers with higher marginal premia start participating in the trade, until the whole range of loss is exhausted.

To efficiently compute the optimal deductible levels and the co-insurance scheme within each tranche, one needs to compute the endogenously determined minimal marginal rate of intertemporal substitution (MMRIS) of each agent. Our second result is that the insurers' MMRIS can be calculated through the fixed point of an explicitly constructed contraction mapping. This result is crucial, both for the computation of optimal indemnities and for studying the dependence of deductibles on microeconomic characteristics.

In particular, we use this result to compute numerical examples of the optimal insurance contracts.

The rest of this paper is organized as follows: In Section 2, we review the relevant literature. In Section 3, we formulate the optimal insurance design problem and characterize optimal indemnities for a finite number of insurers. In Section 4, we show how the optimal contracts can be computed using the fixed point of a contraction mapping and provide several important comparative statics results. In Section 5, we conclude the paper and point out some future research directions. All proofs are in the Appendix.

2 Related Literature

This paper extends the classical results of Borch (1962) and Wilson (1968) and can therefore be applied to a large variety of economic problems such as Walrasian equilibrium allocations in complete markets under constraints. In particular, since we allow for heterogeneous discount factors, our results are related to those of Gollier and Zeckhauser (2005), who studied the effect of such a heterogeneity on efficient intertemporal allocations.[8] We show that the practical constraints on insurance contracts together with heterogeneity in discount factors may lead to the failure of classical aggregation results.

In the literature on optimal insurance design, the study most closely related to ours is that of Raviv (1979). He considered the same optimal insurance problem as ours, but with a single insurer and provided necessary and sufficient conditions for the optimality of a deductible. Thus, our results on the optimal insurance design can be viewed as an extension of Raviv (1979) to the case of multiple insurers. In addition, in contrast to Arrow (1971) and Raviv (1979), we also study the intertemporal aspect of optimal insurance design. This allows us to express the optimal allocation in terms of the marginal

[8]See also a recent paper by Kazumori and Wilson (2009) that studied general efficient intertemporal allocations and extended Wilson (1968) to a dynamic setting.

rates of intertemporal substitution and to link them to various agents' characteristics.

Numerous papers have studied the optimality of deductibles in optimal insurance design in various settings, extending the original model of Raviv (1979). See, for example, Doherty and Schlesinger (1983), Huberman, Mayers and Smith (1983), Blazenko (1985), Gollier (1987), Gollier (1996), Gollier and Schlesinger (1995,1996), Gollier (2004), and Dana and Scarsini (2007). Eeckhoudt, Gollier, and Schlesinger (1991) studied the dependence of the optimal deductible on the distribution of losses. Researchers in all of these studies assumed that there is a *single insurer*. The only class of models with multiple insurers that has been extensively studied in the insurance literature corresponds to risk sharing among insurers through a secondary complete capital market, which is not always available in many actual situations. See Aase (2014) for an overview and Citanna and Siconolfi (2015) for more recent development.

Cohen and Einav (2007) and Cutler, Finkelstein and McGarry (2008) found empirical support for the importance of preferences heterogeneity in insurance design and its impact on the optimal deductible choice.

3 The Model

The model's participants consist of an insurance buyer (the insured) and a set of N insurance sellers (the insurers). The insurance buyer faces the risk of a random loss, described by a nonnegative bounded random variable X with the largest potential loss $\operatorname{esssup} X = \bar{X}$. In addition, the insurance buyer is endowed with other (not explicitly modeled) assets, generating a non-stochastic cash flow $(w_0\,,\, w_1)$.[9] The insurance buyer

[9] The assumption of non-stochastic cash flows can be relaxed as long as w_{1i} is independent of X. Indeed, in this case we can redefine the utilities $\tilde{u}_i(c) = E[u_i(w_{1i} + c)]$, and then rewrite the problem in their terms. The assumption of independence does make sense for many real world settings where insurance is acquired against specific risks (e.g., a local natural disaster). However, if insurers' income is correlated with X, the structure of the risk sharing may change completely. We leave it as an interesting topic for future research. Our techniques can also be directly extended to allow for hedging and raising cash using a bond market.

is an intertemporal expected utility maximizer, with von Neumann-Morgenstern utility U and a discount factor δ.

To (partially) insure against potential random loss X, the insured designs a basket $F_i(X)\,,\, i = 1, \cdots, N$ of insurance contracts (also known as indemnity schedules, or, coverage functions), contingent on the realization of the loss X. Because we are interested in the risk sharing problem, we assume that there is no asymmetric information and therefore the true probability distribution of X is known to all market participants.[10] A basket of coverage functions is called admissible if, for all i, $F_i(X) \geq 0$ for all values of X and

$$F \;=\; \sum_{i=1}^{N} F_i \;\leq\; X.$$

That is, we assume that insurance reimbursement is always nonnegative and the total reimbursements cannot exceed the size of the loss. Given an insurance contracts design $\{F_i\}_{i=1}^{N}$, the insured retains exposure to the residual loss $X \,-\, F$.

Insurance can be bought from N insurers. Insurer i is endowed with a non-stochastic income flow $(w_{0i}\,,\, w_{1i})$. Each insurer is an intertemporal expected utility maximizer, with a von Neumann-Morgenstern utility u_i and a discount factor δ_i. All utility functions are assumed to be twice continuously differentiable, increasing, and concave on their domain of definition.

We assume that the insured can choose any basket satisfying the aforementioned admissibility conditions. The price, paid by the insured to insurer i (i.e., the insurance premium for the coverage function F_i) is denoted by $P_i = P_i(F_i)$. Both insurance provision and insurance design are potentially costly due to administrative expenses, underwriting cost, broker commission, and so on. These costs are a deadweight loss to both insured and insurers and we assume the cost is proportional to the insurance premium

[10]The probability distribution of X is exogenously given and all market participants agree on it. Note that we actually do not need to require that market participants know the true distribution of X, but rather that they have the same beliefs about it.

P_i.[11] Without loss of generality, we assume only the insured incurs cost and denote α the proportion.[12]

As is common in the literature on optimal insurance design (see, e.g., Raviv (1979)), we assume that insurer i is willing to provide insurance coverage for $F_i(X)$ if and only if the premium P_i satisfies the insurer's participation constraint

$$u_i(c_{0i}) \; + \; \delta_i \, E[u_i(c_{1i})] \; \geq \; L_i, \tag{1}$$

where

$$c_{0i} \; = \; w_{0i} + P_i \; , \; c_{1i} \; = \; w_{1i} - F_i(X) \tag{2}$$

is the insurer's consumption after entering the contract and

$$L_i \; = \; u_i(w_{0i}) \; + \; \delta_i \, u_i(w_{1i})$$

is the insurer's reservation utility.[13] Given the contracts $(P_i \, , \, F_i) \, , \, i \, = \, 1 \, , \, \cdots \, , \, N$, the

[11]Arrow (1971, p.204) writes: "It is very striking to observe that among health insurance policies of insurance companies in 1958, expenses of one sort or another constitute 51.6 percent of total premium income for individual policies, and only 9.5 percent for group policies." This supports the assumption that the cost is proportional to the premium size, and suggests that a proportional between 0.1 and 0.5 may be a reasonable, depending on the precise circumstances. Insurance broker commission is a fixed percentage of the premium quoted by an insurer. This also gives direct evidence of our way of modeling insurance cost. Source: http://www.willis.com/documents/publications/General_Publications/How_We_Get_Paid.pdf

[12]Indeed, if we assume both the insured and the insurers incur proportional cost, with proportion τ and θ respectively. Then, insurer i is only getting a fraction of $(1-\theta)P_i$, whereas the insured is actually paying $(1+\tau)P_i$. Therefore, this model is equivalent to one in which insurance provision costs are zero (i.e., $\theta = 0$), whereas insurance coverage costs are give by $\alpha \equiv \frac{1+\tau}{1-\theta} - 1$.

[13]The assumption that the reservation utility coincides with the utility before entering the contract is made for technical purposes, to avoid discontinuities in the price P_i.

insured's consumption is given by:

$$c_0 \;=\; w_0 \,-\, (1 \,+\, \alpha) \sum_{i=1}^{N} P_i, \qquad c_1 \;=\; w_1 \,-\, X \,+\, F(X)\,. \tag{3}$$

The problem of the insured is thus to design an admissible basket (F_i) so as to maximize his expected utility,

$$U(c_0) \;+\; \delta\, E\big[\,U(c_1)\,\big],$$

under the budget constraints (3) and participation constraints (1).

Clearly, the insured will always optimally choose the premium to bind participation constraints (1) for the insurers, and therefore the insurance premium satisfies

$$P_i(F_i) \;=\; -w_{0i} \;+\; v_i(L_i \,-\, \delta_i\, E[u_i(w_{1i} - F_i(X))])\,, \tag{4}$$

where v_i is the inverse of the insurer's utility: $v_i(u_i(x)) \;=\; x$.

Here, it should be pointed out that the preference parameters $(\delta_i\,,\, u_i)$ should not be interpreted directly as the "true" preferences of the insurers. Rather, it is a simple (and necessarily stylized) way of incorporating intertemporal substitution attitudes and risk aversion into insurance pricing. For example, if insurer i is risk neutral, we get $P_i(F_i) \;=\; \delta_i E[F_i]$. This result is the classical actuarial fair value premium rule (see, e.g., Borch (1962)), and the difference $\ell_i \equiv \delta_i - 1$ is commonly referred to as the fixed percentage loading. In particular, it is always assumed that $\ell_i > 0$, that is $\delta_i > 1$. Thus, in our setting, we will allow for discount factors $\delta_i > 1$ and assume that δ_i incorporates both time discounting and fixed percentage loadings.

Before we characterize the optimal contract structure, it is important to understand the intuition why purchasing insurance from multiple agents might be desirable. Let us first examine the case in which all insurers are risk neutral. In that case insurer i

is willing to accept the premium $P_i(F_i) = \delta_i\, E[F_i(X)]$ for an indemnity F_i. Therefore, diversifying between different insurers is never optimal for the insured. The insurer with the smallest discount factor $\delta_{\min}$ will always be the one to provide the cheapest insurance, and the insured will always buy insurance against the total indemnity $F = \sum_i F_i$ from this insurer because the premium is linear.[14] However, when insurers are risk averse, the situation is completely different because the marginal premium that an insurer requires for providing insurance against an additional unit of X is monotone increasing with the level of X. We illustrate this using a conceptual example with two insurers, 1 and 2, as in Figure 1. After the deductible is reached, insurer 1 provides coverage first because her insurance premium is lower than the insurance premium of insurer 2. This is clearly demonstrated from the right panel which shows the marginal rate of intertemporal substitution (MRIS) of each agent. However, as the level of X becomes sufficiently high, insurer 1's period-1 consumption decreases, pushing up her MRIS which eventually becomes higher than that of insurer 2. It thus becomes desirable for the insured to buy (partial) insurance against the high-level portion of X from insurer 2. With F_i optimally designed, the MRIS of those agents who absorb loss are equalized at any loss level.

Since the optimal allocation for the insurance design problem is constrained efficient, it can be solved in two steps: (1) solve the constrained social planner problem with fixed Pareto weights assigned to all market participants (both insurers and the insured); (2) find the endogenous Pareto weights from the insurers' participation constraints. In order to formulate the social planner's problem, define $F_{N+1} = X - \sum_i F_i(X)$ and let $u_{N+1}(c) = U(c)$, $c_{0(N+1)} = c_0$, $\delta_{N+1} = \delta$, and $w_{1(N+1)} = w_1$. Then, the constrained

[14]Indeed, $\sum_i P_i(F_i) \geq \delta_{\min} \sum_i E[F_i]$.

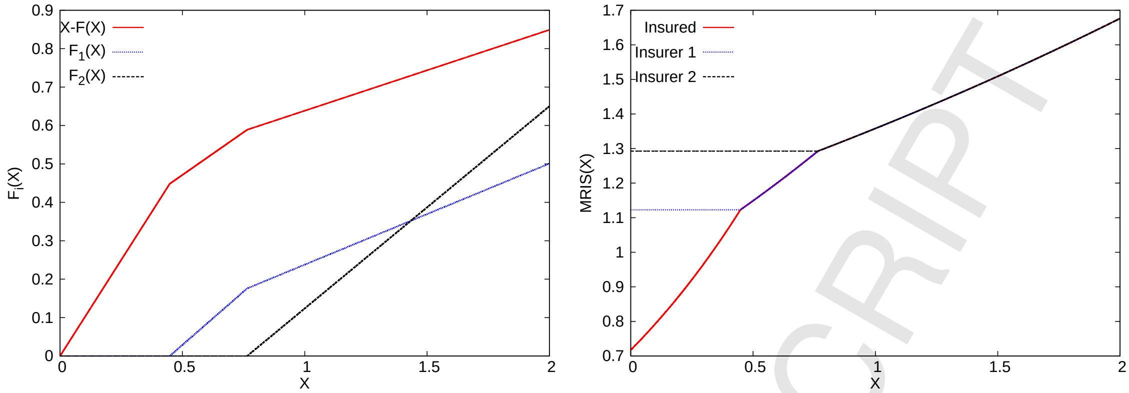

Figure 1: The left panel plots the coverage functions (i.e., $F_1(X)$ and $F_2(X)$) and retained loss (i.e., $X - F_1(X) - F_2(X)$) against the loss (X). The right panel plots the MRIS of each agent against the loss.

optimal allocation solves the social planner's problem

$$\max \left\{ \sum_{i=1}^{N+1} \mu_i \left(u_i(c_{0i}) + \delta_i \, E[u_i(w_{1i} - F_i(X))] \right) \right\} \tag{5}$$

under budget constraints (2) and (3), and the solution, which depends on μ_i — the weight that the social planner assigns to agent i, will be evaluated using the insurers' participation constraints. The Pareto weights are given by

$$\mu_i \;=\; \frac{1}{u_i'(c_{0i})}\,,\; i = 1, \cdots, N, \qquad \text{and } \mu_{N+1} \;=\; \frac{1}{(1+\alpha)U'(c_0)}\,.$$

As we will show below, the optimal allocation will be fully determined by the *minimal marginal rates of intertemporal substitution* (MMRIS)[15] [16]

$$Y_i \;=\; \frac{\delta_i \, u_i'(w_{1i})}{u_i'(c_{0i})}\,,\; i = 1, \cdots, N,$$

[15]It is minimal because $c_{1i} = w_{1i} - F_i(X) \leq w_{1i}$ and therefore $\frac{\delta_i \, u_i'(c_{1i})}{u_i'(c_{0i})} \geq \frac{\delta_i \, u_i'(w_{1i})}{u_i'(c_{0i})}$.

[16]Note that fixing Y_i is equivalent to fixing the insurance premia $P_i(F_i)$ because $c_{0i} = w_{0i} + P_i(F_i)$ and $c_0 = w_0 - (1+\alpha) \sum_i P_i(F_i)$.

of the insurers and the insured's MMRIS

$$Y \;=\; \frac{\delta\, U'(w_1)}{(1+\alpha)\, U'(c_0)}\,.$$

Definition 1 *For an insurer i, we denote by* $\text{rank}(i)$ *the number that insurer i will have when all insurers are reordered so that smaller* $\text{rank}(i)$ *implies larger Y_i. Furthermore, we denote by J the number of insurers for which Y_i is larger than Y.*

Note that the MMRIS of insurer i depends on c_{i0}, which is an endogenous object determined by the optimal choice of P_{i0} by the insured. Consequently, the rank that insurer i gets assigned is also endogenous and depends on all other parameters of the model.

Denote $q_i(x) = (u_i')^{-1}(x)$ as the inverse function of insurer i's marginal utility function and $Q(x) = (U')^{-1}(x)$ as the inverse function of insured's marginal utility function. We define insurer-specific deductibles $Z_1, \cdots, Z_N$ below.

Definition 2 *For each $i = 1, \cdots, N$, let:*

$$a_i \;\equiv\; (\delta\mu_{N+1})^{-1}\delta_i\mu_i \;=\; \frac{\delta^{-1}\,(1+\alpha)\,U'(c_0)}{\delta_i^{-1}\,u_i'(c_{0i})}\,. \tag{6}$$

Fix $k \in \{0, 1, \cdots, N, N+1\}$.

- *For $k = 0$ we define $Z_0 = \bar{X}$.*
- *For $1 \le k \le J$, let $K = \text{rank}^{-1}(k)$ be the insurer whose rank is equal to k and*

$$\tilde{Z}_k \;=\; w_1 - Q(a_K\, u_K'(w_{1K})) \;+ \sum_{i\,:\,\text{rank}(i)\ge k+1} \left(w_{1i} - q_i\left(a_i^{-1}\, a_K\, u_K'(w_{1K})\right)\right) \tag{7}$$

and

$$Z_k \;=\; \min\{\bar{X}\,,\, \tilde{Z}_k\}.$$

- *For $k = J+1$ we define:*

$$\tilde{Z}_{J+1} \;=\; \sum_{i\;:\;\mathrm{rank}(i)\geq J+1} \left(w_{1i} \;-\; q_i\left(U'(w_1)\,a_i^{-1}\right)\right), \tag{8}$$

and

$$Z_{J+1} \;=\; \min\{\bar{X}\,,\,\tilde{Z}_{J+1}\}.$$

- *For $J+2 \leq k \leq N$, let $K \;=\; \mathrm{rank}^{-1}(k-1)$ and*

$$Z_k \;=\; \sum_{i\;:\;\mathrm{rank}(i)\geq k} \left(w_{1i} - q_i\left(a_i^{-1}\,a_K\,u_K'(w_{1K})\right)\right)\,, \tag{9}$$

and

$$Z_k \;=\; \min\{\bar{X}\,,\,\tilde{Z}_k\}.$$

- *For $k = N+1$, we define $Z_{N+1} = 0$.*

We denote $\mathrm{Tranche}_j \equiv\; \mathrm{Tranche}(Z_{j+1}, Z_j)$ where $\mathrm{Tranche}(a,b)$ is defined as

$$\mathrm{Tranche}(a,b) \;=\; \begin{cases} 0 & ,\; x < a \\ x-a & ,\; x \in (a,b) \\ b-a & ,\; x > b \end{cases},$$

The definitions above is summarized below:

$$0 = \overbrace{Z_{N+1} \quad \leq \quad Z_N}^{\mathrm{Tranche}_N = \mathrm{Tranche}(Z_{N+1},Z_N)} \leq \cdots \leq \underbrace{Z_{J+1} \quad \leq \quad Z_J}_{\mathrm{Tranche}_J} \leq \cdots \leq \overbrace{Z_1 \quad \leq \quad Z_0}^{\mathrm{Tranche}_0} = \bar{X}.$$

The following theorem is an extension of the seminal Raviv (1979) characterization of the optimal insurance design for the case of multiple insurers.

Theorem 1 *There always exists a unique optimal allocation* $\{F_i\}_{i=1}^N$. *It is non-zero (i.e.,* $F(X) \not\equiv 0$*) if and only if:*

$$\frac{\delta\, U'(w_1 - \bar{X})}{(1+\alpha)\, U'(w_0)} \;>\; \min_i \frac{\delta_i\, u_i'(w_{1i})}{u_i'(w_{0i})}. \tag{10}$$

If (10) holds, then the following is true:

(1) Optimal indemnities F_i *and the uninsured part* $X - F(X)$ *are continuous and (weakly) monotone increasing in* X*;*

(2) If $Y_i > Y$*, then insurer* i *only participates in* Tranche_j *if* $j \le \text{rank}(i) - 1$*;*

(3) If $Y_i \le Y$*, then insurer* i *only participates in* Tranche_j *if* $j \le \text{rank}(i)$*;*

(4) The insured buys full insurance (i.e., $F(X) = X$*) against the part of* X *below* Z_{J+1} *and retains a partial exposure to* X *(i.e.,* $F(X) < X$*) for* $X > Z_{J+1}$*.*

(5) For each Tranche_j*, there exists a function* $\xi_j(X)$ *such that:*

$$\frac{\delta_i\, u_i'(c_{1i})}{u_i'(c_{0i})} \;=\; \xi_j(X) \tag{11}$$

for each insurer i *participating in* Tranche_j*. Furthermore,*

(5-a) If $j \ge J+1$ *(full insurance region), then:*

$$\xi_j(X) \;<\; \frac{\delta\, U'(c_1)}{(1+\alpha)\, U'(c_0)} \quad \textit{and}$$

(5-b) If $j < J+1$*, then:*

$$\xi_j(X) \;=\; \frac{\delta\, U'(c_1)}{(1+\alpha)\, U'(c_0)}. \tag{12}$$

First, we note that Equation (11) and Equation (12) uniquely determine the allocation. Indeed, substituting $c_{1i} = w_{1i} - F_i(X)$ into (11) gives $F_i \;=\; w_{1i} - q_i(\xi_j(X)\, u_i'(c_{0i})\, \delta_i^{-1})$.

Then, for $j \leq J+1$, the function ξ_j is uniquely determined by the constraint $\sum_i F_i(X) = X$, and for $j \geq J$, the function ξ_j is uniquely determined by (12) and the insured's budget constraint $c_1 = w_0 - X + \sum_i F_i(X)$.

Since both the insured's consumption $c_1 = w_1 - X + F(X)$ and the insurers' consumption $c_{1i} = w_{1i} - F_i(X)$ are (weakly) decreasing in X, the MRIS of the insured and the MRIS of the insurers' consumption are (weakly) increasing in X.

If the MMRIS of all insurers are larger than that of the insured, then there exists a *strictly positive deductible* Z_N and the tranche Tranche_N is not insured at all. After reaching the deductible, co-insurance becomes desirable. If there is at least one insurer i whose MMRIS is smaller than that of the insured, then the insured buys full insurance coverage against low levels of X (i.e., $x \in [0, Z_{J+1}]$) from the insurer with rank N and gradually from insurers with ranks above J. As the level of X hits Z_{J+1}, co-insurance becomes desirable and the insured starts to absorb loss.

After co-insurance is triggered, whenever X hits the next deductible level Z_i, it becomes desirable to purchase insurance coverage from the insurer with rank i.

We conclude this section with a result characterizing the nature of risk sharing within each tranche. Recall that

$$R_i(x) = -\frac{u_i'(x)}{u_i''(x)}$$

is the absolute risk tolerance of agent i. Wilson (1968) showed that the slopes of the sharing rules in a Pareto-efficient allocation can be characterized in terms of agents' absolute risk tolerances. The following result is an extension of Wilson's characterization for the constrained Pareto-efficient allocation in our model.

Corollary 1

$$\frac{d}{dx}F_i(x) \;=\; \begin{cases} 0, & x \le \; Z_{\mathrm{rank}(i)} \\ \frac{R_i(c_i)}{\sum_{j\,:\,\mathrm{rank}(j)\,\ge\,k+1} R_j(c_j)}, & x \in (Z_{k+1}, Z_k), 0 \le k \le \mathrm{rank}(i) - 1 \end{cases} .$$

The intuition behind the formula for the slope is the same as in Wilson (1968): The fraction of the aggregate risk agent i ends up taking is proportional to agent i's risk tolerance. The set of agents with whom agent i is sharing risks, however, depends on the level of X and changes from tranche to tranche.

4 Fixed-Point Algorithm

By Theorem 1, the optimal allocation is uniquely determined as soon as we know the rank of every insurer, as well as the thresholds Z_k. By Definitions 1 and 2, both the ranks and the thresholds are uniquely determined by the N-tuple of numbers (a_i).[17] Given the N-tuple (a_i), we denote $b_i \;=\; a_i^{-1}$ as their reciprocals and denote by $\mathbf{b} \;=\; (b_i)$ the vector of these reciprocals. We denote by $(Z_i(\mathbf{b})\,,\, i = 0, \cdots, N+1)$ the corresponding thresholds and by $(F_i(\mathbf{b})\,,\, i = 1, \cdots, N)$ the corresponding allocation. By definition (see (6)), the optimal allocation must satisfy:

$$b_i \;=\; \frac{\delta_i^{-1}\, u_i'(c_{0i})}{(1+\alpha)\,\delta^{-1}\, U'(c_0)} \;=\; \frac{\delta_i^{-1}\, u_i'\,(w_{0i} + P_i(F_i(\mathbf{b})))}{(1+\alpha)\,\delta^{-1}\, U'\left(w_0 \;-\; (1+\alpha)\;\sum_j\; P_j(F_j(\mathbf{b}))\right)} \tag{13}$$

for all $i = 1, \cdots, N$. This is a highly non-linear system of equations for vector $\mathbf{b}$. It is by no means clear how to solve it analytically or even numerically, nor is it clear how the solution would depend on the microeconomic characteristics of the model.

In this section we prove that this N-tuple is the unique fixed point of a contraction

[17] Note that $Y_i = \frac{\delta a_i u_i'(w_{1i})}{(1+\alpha)U'(c_0)} = a_i Y \frac{u_i'(w_{1i})}{U'(w_1)}$

mapping defined on an explicitly given compact set and can therefore be easily calculated by successive iterations.

We use the common notation b_{-i} to denote the vector of all coordinates of $\mathbf{b}$ except for b_i. Let

$$P_i^{\max} \;=\; -w_{0i} \,+\, v_i\,(L_i - \delta_i\, E[u_i(w_{1i} - X)])$$

be the premium that the insurer i is asking for providing full insurance against X,[18]

$$C_{\max} \;=\; \big((1+\alpha)\,\delta^{-1}\,U'\,(w_0)\big)^{-1}\;,\; C_{\min} \;=\; \left((1+\alpha)\,\delta^{-1}\,U'\left(w_0 \,-\, (1+\alpha)\sum_i P_i^{\max}\right)\right)^{-1}$$

and

$$\beta_i^{\min} \;=\; \log\big(C_{\min}\,\delta_i^{-1}\,u_i'\,(w_{0i} + P_i^{\max}))\big)\;,\; \beta_i^{\max} \;=\; \log\big(C_{\max}\,\delta_i^{-1}\,u_i'(w_{0i})\big)\;.$$

We denote $\Omega \;=\; \times_i\,[\beta_i^{\min}\,,\,\beta_i^{\max}]$ and let $\|x\|_{l_\infty} \;=\; \max_i\,|x_i|$ be the l_∞-norm of a finite sequence, equal to the maximal absolute value of its elements. The following lemma is the main technical result of this section.

Lemma 1 (contraction lemma) *For each $i = 1, \cdots, N$, there exists a unique, piecewise continuously differentiable function $H_i \;=\; H_i(C, b_{-i})$ solving*

$$H_i(C, b_{-i}) \;=\; \delta_i^{-1}\,C\,u_i'\,\big(w_{0i} \,+\, P_i\big(F_i\big(X, \big(H_i(C, b_{-i}), b_{-i}\big)\big)\big)\,\big)\,. \tag{14}$$

For any $C > 0$, the mapping G_C defined via $(G_C)_i(\mathbf{d}) \;=\; \log H_i(C, e^{d_{-i}})$ maps the compact set Ω into itself and is a strict contraction with respect to $\|\cdot\|_{l_\infty}$. Consequently,

[18] For simplicity, we always assume that the price $P_i^{\max}$ is well defined for any insurer i. This assumption is only necessary when dealing with utilities that are either defined on a half-line or are bounded from above. It can be relaxed at the cost of more technicalities, and we omit it for the reader's convenience.

there exists a unique fixed point $\mathbf{d}^*(C) \in \Omega$ *of this map, solving* $\mathbf{d}^*(C) = G_C(\mathbf{d}^*(C))$. *For any* $\mathbf{d}_0 \in \Omega$, *we have* $\mathbf{d}^*(C) = \lim_{n\to\infty}(G_C)^n(\mathbf{d}_0)$.

The result of Lemma 1 is quite surprising because it holds under absolutely no restrictions on model parameters. In particular, we do not need to impose any smallness conditions typically used in economic applications of the contraction mapping theorem.

With the fixed point mapping $\mathbf{d}^*(C)$ defined, we are ready to state the main result of this section.

Theorem 2 *Let* $\mathbf{b}^*(C^*) = e^{\mathbf{d}^*(C^*)}$ *where* $C^* \in (C_{\min}, C_{\max})$ *uniquely solves*

$$C = \left((1+\alpha)\delta^{-1}U'\Big(w_0 - (1+\alpha)\sum_i \Big(q_i\big(e^{d_i^*(C)}\delta_i C^{-1}\big) - w_{0i}\Big)\Big)\right)^{-1}. \tag{15}$$

The optimal allocation is given by $(F_i)(\mathbf{b}^*(C^*))$.

Theorem 2 provides a directly implementable algorithm for calculating the optimal allocation: Vector $\mathbf{d}(C)$ can be calculated by successive iterations using Lemma 1, and then C^* can be found using any standard numerical procedure for solving (15). Figure 4 provides the flow chart for the implementation of the fixed-point algorithm.

The characterization of the optimal allocation provided by Theorem 2 is perfectly suited for studying comparative statics. We need the following lemma.

Lemma 2 (comparative statics lemma) *If the right-hand sides of* (14) *and* (15) *are monotone increasing in some parameter, then so are* C^* *and* $\mathbf{d}^*(C^*)$.

By (7)-(9) and Theorem 1, all deductibles and other characteristics of the optimal indemnities can be expressed in terms of $a_i = e^{-d_i}$, thus we can use Lemma 2 to study the dependence of the optimal allocation on various model parameters.

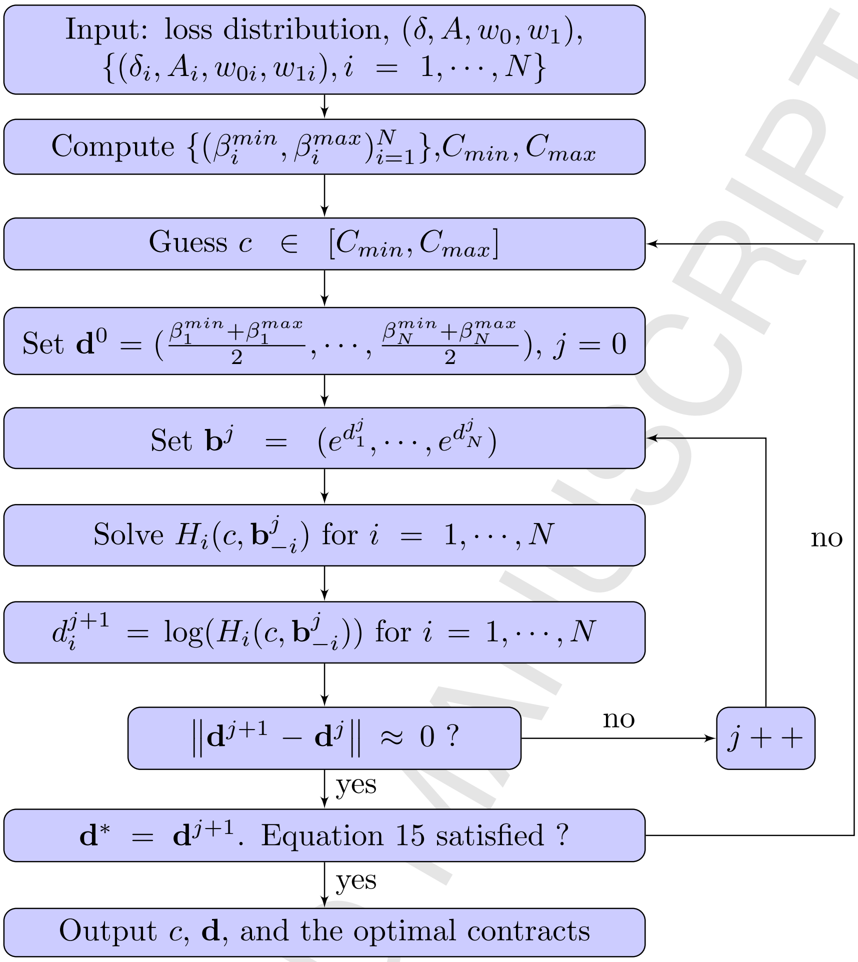

Figure 2: Flow chart of the fixed point algorithm

Define

$$Z_{\text{full coverage}} \equiv \max\{x : F(x) = x\}, \quad Z_{\text{deductible}} \equiv \max\{x : F(x) = 0\}$$

By Theorem 1, $Z_{\text{full coverage}}$ is positive if and only if $\min_i Y_i < Y$, in which case $Z_{\text{full coverage}} = Z_{J+1}$, and $Z_{\text{deductible}}$ is positive if and only if $\min_i Y_i > Y$, in which case $Z_{\text{deductible}} = Z_N$.

We refer to the insurance tranches that are fully insured as senior tranches and let

$$\#\{\text{senior}\} \;=\; \#\{\mathrm{i} : \mathrm{Y_i} < \mathrm{Y}\}$$

be the number of insurers participating in the tranches that are fully insured. By construction, #{senior} is also the number of senior insurance tranches. Finally, for each insurer i we define:

$$\text{index}(i) \;=\; \begin{cases} 1\,, & if \ \text{rank}(i) > J \\ 0\,, & if \ \text{rank}(i) \leq J \end{cases}.$$

That is, an insurer's index is one if the insurer participates in the tranches that are fully insured and zero otherwise.

Definition 3 *We say that a change in the parameters of the model leads to more insurance coverage if it leads to an increase (in the weak sense) in* #{senior}*,* $Z_{\text{full coverage}}$*, and* index(i) *for each* i*, and to a decrease (in the weak sense) in* $Z_{\text{deductible}}$*.*

That is, more insurance implies that a larger part of X is fully insured and more insurers participate in the fully insured senior tranches.

The next result describes the effect of a first-order stochastic dominant (FOSD) shift in the distribution of X, as well as the effect of changes in the insured's initial wealth and discount factor on the optimal allocation.

Corollary 2 *A decrease in the distribution of* X *in the FOSD sense, an increase in* w_0*, or an increase in* δ *lead to more insurance. In particular, there exists a threshold value for* δ *such that* $Z_{\text{deductible}}$ *is positive if and only if* δ *is below this threshold,*[19] *and similarly for* w_0*.*

[19]Here, we allow δ to vary and keep the rest of the parameters fixed.

The fact that larger initial endowment and larger discount factor lead to more insurance is very intuitive: Indeed, if δ is large, future shocks are more important for the insured, forcing the insured to acquire more insurance. Similarly, a larger initial capital w_0 leads to a smaller marginal loss from buying insurance at time zero, again making it optimal to buy more insurance. By contrast, the fact that *larger losses*[20] *lead to less insurance* is quite surprising. The reason is that, with larger risk, insurance gets more expensive on average. Therefore, it is optimal for the insured to increase exposure to low levels of X (achieved by raising the deductible level) and simultaneously increase insurance coverage for higher levels on X, reducing the probability of large losses.

In general, the ranks that the insured assigns to the insurers may depend in a non-trivial way on insurers' preferences and endowments. It turns out, however, that when all insurers have exponential utility functions, ranks can be characterized explicitly.

Corollary 3 *Suppose that all agents have exponential utility functions. Then, the ranks of the insurers follow the order of their pre-trade MRIS. That is* $\mathrm{rank}(i) > \mathrm{rank}(j)$ *if and only if*

$$\frac{\delta_i\, u_i'(w_{1i})}{u_i'(w_{0i})} \;<\; \frac{\delta_j\, u_j'(w_{1j})}{u_j'(w_{0j})}.$$

Suppose for simplicity that the insurers' endowments satisfy $w_{0i} = w_{1i}$. In this case, Corollary 3 implies that the rank of an insurer is determined solely by his discount factor δ_i and is independent of his risk aversion A_i. The reason is that, when an insurer's risk aversion is constant, the risk premium *per unit of risk* that the insurer is charging is independent of the level of X. For any insurer i, the insured optimally chooses the fraction of the total coverage $F(X)$ that insurer i covers to be proportional to his risk tolerance A_i, thereby equalizing marginal risk premia across the insurers. Therefore, only

[20] In the sense of first order stochastic dominance.

discount factors δ_i matter for ranking. In particular, if several insurers have identical pre-trade MRIS, the tranches in which they participate will be the same, and their coverage functions will differ only by constant multiples. This situation leads to the following interesting aggregation result. Denote by A_I^{-1} the sum of insurers' risk tolerances:

$$A_I^{-1} = \sum_{j=1}^{N} A_j^{-1}. \tag{16}$$

The following is true:

Corollary 4 *Suppose that* $w_{0i} = w_{1i}$[21] *for all* i*, and* $\delta_i = \delta_I$ *is independent of* i*. Then, the risk-sharing is linear:*

$$F_i(x) = \frac{A_i^{-1}}{A_I^{-1}} F(x)$$

and $F(x)$ *coincides with the optimal indemnity schedule that the insured would choose with a single representative insurer with risk aversion* A_I*.*

The result of Corollary 4 has a natural interpretation in the framework of the theory of syndicates developed by Wilson (1968). Namely, Corollary 4 implies that insurers with identical pre-trade MRIS effectively form a syndicate with the group (syndicate) utility given by that of the representative insurer. It is interesting to note that, without the practical constraints on insurance contracts, the syndicate result and the CARA linear risk sharing rule of Corollary 4 always hold,[22] independent of the MRIS of the agents. However, this aggregation result does not generally hold.

[21] Due to translation invariance of CARA preferences, optimal allocation depends only on the differences $w_{1i} - w_{0i}$ and $w_1 - w_0$ of endowments at times zero and one.

[22] In fact, the risk-sharing rule will generally be affine linear: indemnities may also differ by additive constants).

5 Conclusions

In the present study, we extended Raviv's (1979) seminal characterization of the optimal insurance design to the case of an insured facing multiple insurers with heterogeneous risk attitudes, discount factors, and endowments, and without asymmetric information. We showed that optimal indemnities can be characterized by insurer-specific deductibles and a hierarchical structure: The insured optimally assigns ranks to insurers depending on their MMRIS, and based on these ranks, the insured determines the optimal deductible level for each insurer. The insured then either fully insures all risks below an endogenous threshold with several (≥ 1) insurers having the highest ranks, or chooses a strictly positive minimal deductible. Afterwards, the insured gradually insures subsequent tranches with insurers of lower ranks, so that every subsequent tranche is co-insured by multiple insurers in a Pareto-efficient way.

Our model could also be viewed as surplus extraction through price discrimination when the insured has complete information. It would be interesting to extend our model to the case in which the insured has incomplete information about insurer types, as in the model of Cremer and McLean (1985). It would also be interesting to extend our model to a dynamic, multi-period setting and allow for asymmetric information.

Appendix

A Kuhn-Tucker First-Order Conditions for Multiple Insurers

By strict concavity, an allocation is optimal if and only if it satisfies the first-order Kuhn-Tucker conditions. They are:

$$\mu_{N+1}\delta U'\big(w_1 - X + F(X)\big) - \mu_i \delta_i u_i'(w_{1i} - F_i) = 0 \tag{17}$$

if the constraints $F_i \geq 0$ and $\sum_j F_j \leq X$ are not binding, and

$$\mu_{N+1}\delta U'\big(w_1 - X + F(X)\big) - \mu_i \delta_i u_i'(w_{1i} - F_i) < 0 \tag{18}$$

if the constraint $F_i \geq 0$ is binding but the constraint $\sum_j F_j \leq X$ is not binding.

Finally, if the constraint $\sum_j F_j \leq X$ is binding, there will be a Lagrange multiplier $\nu(X)$ for this constraint, and the first-order condition will be

$$\mu_{N+1}\delta U'\big(w_1 - X + F(X)\big) - \mu_i \delta_i u_i'(w_{1i} - F_i) = \nu(X) > 0 \tag{19}$$

if the constraint $F_i \geq 0$ is not binding, and

$$\mu_{N+1}\delta U'\big(w_1 - X + F(X)\big) - \mu_i \delta_i u_i'(w_{1i} - F_i) < \nu(X) \tag{20}$$

if the constraint $F_i \geq 0$ is binding.

Using (a_i) defined in (6), we can rewrite the conditions as the following.

$$a_i\, u_i'(w_{1i} \;-\; F_i(X)) \;=\; U'(w_1 - X + F(X)) \tag{21}$$

when none of the constraints is binding, and

$$a_i\, u_i'(w_{1i} \;-\; F_i(X)) \;>\; U'(w_1 - X + F(X)) \tag{22}$$

when the constraint $F_i \geq 0$ is binding but the constraint $\sum_j F_j \leq X$ is not.

When the constraint $\sum_j F_j(X) \leq X$ is binding, we have $\sum_j F_j(X) = X$. If we set:

$$\lambda(X) \;\equiv\; -\delta^{-1}\,\nu(X) \,+\, U'(w_1),$$

then, (19) and (20) take the form

$$a_i\, u_i'(w_{1i} \;-\; F_i(X)) \;=\; \lambda(X) \;<\; U'(w_1) \tag{23}$$

when $F_i \;\geq\; 0$ is not binding and

$$a_i\, u_i'(w_{1i} \;-\; F_i(X)) \;>\; \lambda(X) \tag{24}$$

when it is binding.

For ease of illustration, we from now on reorder the insurers in the increasing order of their rank. In other words, insurer i means from now on the insurer whose rank is equal to i.

By the uniqueness of optimal allocation, it suffices to show that the allocation, described in Theorem 1, indeed satisfies the first-order conditions (21) through (24). This is done in subsequent lemmas.

Lemma 3 *Let* $k \;\geq\; J+1$. *Then, for all* $X \in [Z_{k+1}, Z_k]\,(=\,\text{Tranche}_k)$, *the constraint* $\sum_j \,F_j(X) \;\leq\; X$ *is binding and the constraint* $F_j(x) \;\geq\; 0$ *is binding for all* $j \;<\; k$. *The* *optimal allocation for* $X \in \text{Tranche}_k$ *is uniquely determined via*

$$F_j(X) \;=\; \begin{cases} w_{1j} - q_j(\lambda_k(X)\, a_j^{-1})\,, & j \,\geq\, k \\ 0\;, & j \,<\, k \end{cases}. \tag{25}$$

Here, $\lambda_k(X)$ *is the unique solution to*

$$X \;=\; \sum_{j\,\geq\,k} \left(w_{1j} - q_j(\lambda_k(X)\, a_j^{-1})\right). \tag{26}$$

The slope of $F_j(X)$, $j \,\geq\, k$ *satisfies*

$$\frac{d}{dx} F_j(X) \;=\; \frac{R_j(c_{1j})}{\sum_{i\,\geq\,k} R_i(c_{1i})}.$$

Proof. By construction, the conjectured optimal allocation satisfies

$$\sum_j F_j(X) \;=\; X$$

for all $X \;\leq\; Z_{J+1}$. Thus, we need to verify that (23) and (24) hold in this case. Here, the connection between $\xi_k(X)$ from Theorem 1 and $\lambda_k(X)$ is given by:

$$\xi_k(X) \;=\; \frac{\lambda_k(X)\,\delta}{(1+\alpha)\,U'(c_0)}.$$

By (25) and (26), $F_i(X)$ satisfies

$$a_i\, u_i'(w_{1i} \;-\; F_i(X)) \;=\; \lambda_k(X) \qquad \text{and} \qquad \sum_i F_i(X) \;=\; X,$$

and it remains to check that equation (26) has a solution $\lambda_k(X)$ such that

$$\lambda_k(X) \,\le\, U'(w_1) \tag{27}$$

(constraint $\sum_j F_j \le X$ is binding) and

$$F_j \;=\; w_{1j} \;-\; q_j(\lambda_k(X)\, a_j^{-1}) \;\ge\; 0 \text{ for all } j \,\ge\, k \tag{28}$$

(constraint $F_j \ge 0$ is not binding for $j \ge k$) and (24) holds, that is,

$$a_j\, u_j'(w_{1j}) \;\ge\; \lambda_k(X) \tag{29}$$

for all $j < k$ (constraint $F_j \ge 0$ is binding for $j < k$). First, let $k > J + 1$. Recall now that

$$\tilde{Z}_{k+1} \;=\; \sum_{i=k+1}^{N} (q_i(a_i^{-1}\, a_k\, u_k'(w_{1\,k})) \;-\; w_{1i}) \;=\; \sum_{i=k}^{N} (q_i(a_i^{-1}\, a_k\, u_k'(w_{1\,k})) - w_{1i}),$$

and therefore $X \in [\tilde{Z}_{k+1}, \tilde{Z}_k]$ if and only if

$$\sum_{i=k}^{N} (w_{1i} - q_i(a_i^{-1}\, a_k\, u_k'(w_{1\,k}))\,) \;\le\; X \;\le\; \sum_{i=k}^{N} (w_{1i} - q_i(a_i^{-1}\, a_{k-1}\, u_{k-1}'(w_{1\,k-1})))\,.$$

Recalling that

$$X \;=\; \sum_{i\,=\,k}^{N} \left(w_{1i} \;-\; q_i\left(a_i^{-1}\, \lambda_k(X)\,\right)\,\right), \tag{30}$$

we get that

$$\lambda_k(X) \;\in\; [a_k\, u_k'(w_{1\,k})\,,\, a_{k-1}\, u_{k-1}'(w_{1\,k-1})]. \tag{31}$$

If $k = J+1$, the same argument implies that

$$\lambda_{J+1}(X) \;\in\; [a_{J+1}\, u'_{J+1}(w_{1\,J+1})\,,\, U'(w_1)]\,. \tag{32}$$

Recall that the insurers are ordered in such a way that the sequence

$$Y_i \;=\; \frac{\delta_i\, u'_i(w_{1i})}{u'_i(c_{0i})} \;=\; \frac{a_i\, u'_i(w_{1i})\,\delta}{U'(c_0)}$$

is monotone decreasing in i, and the inequality

$$Y_i \;<\; Y$$

only holds true if $i \geq J+1$. Consequently,

$$a_N\, u'_N(w_{1N}) \;\leq\; \cdots \;\leq\; a_{J+1}\, u'_{J+1}(w_{1\,J+1}) \;\leq\; U'(w_1) \;\leq\; a_J\, u'_J(w_{1J}) \;\leq\; \cdots \;\leq\; a_1\, u'_1(w_{11})\,. \tag{33}$$

Inequalities (31), (32), and (33) immediately yield (27) and (29). Finally, for $j \geq k$,

$$a_j\, u'_j(w_{1j}) \;\leq\; a_k\, u'_k(w_{1k}) \;\Leftrightarrow\; a_j^{-1} \;\geq\; u'_j(w_{1j})\,(a_k\, u'_k(w_{1k}))^{-1}$$

and, using that $\lambda_k(X) \;\geq\; a_k\, u'_k(w_{1k})$, we get

$$F_j \;=\; w_{1j} \;-\; q_j(\lambda_k(X)\, a_j^{-1}) \;\geq\; w_{1j} \;-\; q_j(a_k\, u'_k(w_{1k})\, u'_j(w_{1j})\,(a_k\, u'_k(w_{1k}))^{-1}) \;=\; 0,$$

and (28) follows.

It remains to prove the identity for the derivative. Differentiating (25), we get

$$F'_j(X) \;=\; -(u''_j(c_{1j}))^{-1}\, a_j^{-1}\, \lambda'_k(X),$$

and, differentiating (26), we get

$$\lambda_k'(X) \;=\; -\frac{1}{\sum_{i\,\geq\,k}\, q_i'\, a_i^{-1}}\,.$$

Differentiating $u_i'(q_i'(z)) \;=\; z$ at $z = a_i^{-1}\lambda_k(X)$, we get

$$(u_i''(c_{1i}))^{-1} \;=\; q_i'(a_i^{-1}\,\lambda_k(X))\,.$$

Thus,

$$\begin{aligned} F_j'(X) \;&=\; \frac{(u_j''(c_{1j}))^{-1}\,a_j^{-1}}{\sum_{i\geq k} q_i'\,a_i^{-1}} \;=\; \frac{(u_j''(c_{1j}))^{-1}\,a_j^{-1}}{\sum_{i\geq k}(u_i''(c_{1i}))^{-1}\,a_i^{-1}} \\ &=\; \frac{(u_j''(c_{1j}))^{-1}\,\lambda_k(X)\,a_j^{-1}}{\sum_{i\geq k}(u_i''(c_{1i}))^{-1}\,\lambda_k(X)\,a_i^{-1}} \;=\; \frac{(u_j''(c_{1j}))^{-1}\,u_j'(c_{1j})}{\sum_{i\geq k}(u_i''(c_{1i}))^{-1}\,u_i'(c_{1i})}\,, \end{aligned} \tag{34}$$

which is what had to be proved. ■

It remains to cover the case when the constraint $\sum_i F_i(X) \;\leq\; X$ is not binding. This is done in the following lemma.

Lemma 4 *Let $k \;\leq\; J$. Then, for all $X \;\in\; [Z_{k+1}, Z_k]\,(=\; \mathrm{Tranche}_k)$, the constraint $\sum_j F_j(X) \;\leq\; X$ is not binding, and the constraint $F_j(x) \;\geq\; 0$ is binding for all $j \;\leq\; k$. The optimal allocation for $X \in \mathrm{Tranche}_k$ is uniquely determined via*

$$F_j(X) \;=\; \begin{cases} w_{1j} - q_j(U'(w_1 - X + F(X))\,a_j^{-1})\;, & j \;>\; k \\ 0\;, & j \;\leq\; k\,. \end{cases} \tag{35}$$

Here, $F(X)$ is the unique solution to:

$$F(X) \;-\; \sum_{j\,\geq\,k+1}\big(w_{1j} - q_j(U'(w_1 - X + F(X))\,a_j^{-1})\big) \;=\; 0. \tag{36}$$

The slope of $F_j(X),\ j \geq k+1$ *satisfies*

$$\frac{d}{dx}F_j(X) \;=\; \frac{R_j(c_{1j})}{R(c_1) \,+\, \sum_{i>k} R_i(c_{1i})}\,.$$

Proof. We need to show that the allocation (35) and (36) satisfy the Kuhn-Tucker conditions:

$$a_i\, u_i'(w_{1i} \,-\, F_i) \;=\; U'(w_1 - X + F(X)),$$

with $F_i \geq 0$ for all $i > k$ and

$$a_i\, u_i'(w_{1i}) \;\; - U'(w_1 - X + F(X)) \;>\; 0$$

for all $i \;\leq\; k$.

For simplicity let $k \;<\; J$. By assumption, $X \in [\tilde{Z}_{k+1}\,,\, \tilde{Z}_k]$; that is,

$$\begin{aligned} & w_1 - Q(a_k\, u_k'(w_{1k})) + \sum_{i\,:\,\geq k+1} \left(w_{1i} - q_i\left(a_i^{-1}\, a_k\, u_k'(w_{1k})\right)\right) \\ & > \;X\; > \;w_1 - Q(a_{k+1}\, u_{k+1}'(w_{1\,k+1})) \;+ \sum_{i\,:\,\geq k+1} \left(w_{1i} -\, q_i\left(a_i^{-1}\, a_{k+1}\, u_{k+1}'(w_{1k})\right)\right). \end{aligned} \tag{37}$$

We show that the unique solution F to (36) satisfies

$$X \,-\, (w_1 - Q(a_k\, u_k'(w_{1\,k}))) \;\leq\; F \;\leq\; X \,-\, (w_1 - Q(a_{k+1}\, u_{k+1}'(w_{1\,k+1}))). \tag{38}$$

Indeed,

$$
\begin{aligned}
X &- (w_1 - Q(a_{k+1}\, u'_{k+1}(w_{1\,k+1})) \\
&\quad - \sum_{j\,\geq\,k+1} \big(w_{1j} - q_j(U'(w_1 - X + (X - w_1 + Q(a_{k+1}\, u'_{k+1}(w_{1\,k+1})))) \, a_j^{-1})\big) \\
&= X - Z_{k+1} \geq 0, \quad (39)
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
X &- (w_1 - Q(a_k\, u'_k(w_{1\,k}))) \\
&\quad - \sum_{j\,\geq\,k+1} \big(w_{1j} - q_j(U'(w_1 - X + (X - w_1 + Q(a_k\, u'_k(w_{1\,k})))) \, a_j^{-1})\big) \\
&= X - Z_k \leq 0. \quad (40)
\end{aligned}
$$

Consequently, by continuity and monotonicity, the right-hand side of (36) crosses zero at a single point F, satisfying (38). Hence, for $j \geq k+1$, by (33), we get:

$$
\begin{aligned}
F_j(X) &= w_{1j} - q_j(U'(w_1 - X + F(X))\, a_j^{-1}) \\
&\geq w_{1j} - q_j(a_{k+1}\, u'_{k+1}(w_{1\,k+1})\, a_j^{-1}) \geq w_{1j} - q_j(a_j\, u'_j(w_{1\,j})\, a_j^{-1}) = 0. \quad (41)
\end{aligned}
$$

It remains to be shown that the constraint $F_j(X) \geq 0$ is binding for $j \leq k$. By (38) and (33),

$$
a_j\, u'_j(w_{1j}) - U'(w_1 - X + F(X)) \geq a_j\, u'_j(w_{1j}) - a_k\, u'_k(w_{1k}) \geq 0,
$$

and the claim follows.

■

To complete the proof of Theorem 1, we only need to show that there is no trade if and only if (10) is violated. That is, the allocation $F_i = 0, i = 1, \cdots, N$ satisfies the first-order Kuhn-Tucker conditions if and only if (10) does not hold. Since in this case Y_i and Y coincide with the pre-trade MRIS, we need to show that

$$\frac{\delta_i\, u_i'(w_{1i})}{u_i'(w_{0i})} \;\geq\; \frac{\delta\, U'(w_1 \,-\, X)}{(1+\alpha)U'(w_0)}$$

for all $i \,=\, 1\,, \cdots, N$ and all $X \in [0, \bar{X}]$. Since $U'(c)$ is monotone decreasing in c, this holds if and only if

$$\min_i \frac{\delta_i\, u_i'(w_{1i})}{u_i'(w_{0i})} \;\geq\; \frac{\delta\, U'(w_1 \,-\, \bar{X})}{(1+\alpha)U'(w_0)},$$

and the claim follows.

B Contraction Mapping

For each $i \,=\, 1\,, \cdots, N$, let: $\Omega^{-i} \;\equiv\; \times_{j\neq i}\, [\beta_i^{\min}\,,\, \beta_i^{\max}]\,.$

To prove the contraction lemma, we need a few more technical results.

Lemma 5 *For any X inside a tranche, F_i is a piecewise C^1-function of $\mathbf{b}$. For all $j \neq i$, F_i satisfies*

$$\frac{\partial F_i}{\partial b_i} \;\geq\; 0, \quad \frac{\partial F_i}{\partial b_j} \;\leq 0, \quad b_i\, \frac{\partial F_i}{\partial b_i} \;\geq\; -\sum_{j\neq i} b_j\, \frac{\partial F_i}{\partial b_j}\,.$$

Proof. Suppose first that we are in the regime $F(X) < X$. Then, by (35),

$$F_i(X) \;=\; w_{1i} - q_i(b_i\, U'(w_1 - X + F(X)))$$

and

$$F(X) \;=\; F(\mathbf{b}\,,\, x)$$

solves

$$F(X) \;-\; \sum_j (w_{1j} - q_j(b_j\, U'(w_1 - X + F(X)))) \;=\; 0\,.$$

Here, the summation is only over those insurers j that participate in the tranche. Thus,

$$\frac{\partial F}{\partial b_j} \;=\; -\frac{q_j'(b_j\,U'(c_1))\,U'(c_1)}{1\,+\,\sum_k q_k'(b_k\,U'(c_1))\,b_k\,U''(c_1)};$$

and, hence, for $j \neq i$,

$$\frac{\partial F_i}{\partial b_j} \;=\; q_i'(b_i\,U'(c_1))\,b_i\,U''(c_1)\,\frac{q_j'(b_j\,U'(c_1))\,U'(c_1)}{1\,+\,\sum_k q_k'(b_k\,U'(c_1))\,b_k\,U''(c_1)} \;<\; 0$$

if insurer j participates in the tranche, and the derivative is zero otherwise. Consequently,

$$\sum_{j\neq i} b_j\,\frac{\partial F_i}{\partial b_j} \;=\; (q_i'(b_i\,U'(c_1)))\,b_i\,U'(c_1)\,\frac{\sum_{k\neq i} q_k'(b_k\,U'(c_1))\,b_k\;U''(c_1)}{1\,+\,\sum_k q_k'(b_k\,U'(c_1))\,b_k\,U''(c_1)}$$

and

$$\begin{aligned}
\frac{\partial F_i}{\partial b_i} \;&=\; -q_i'(b_i\,U'(c_1))U'(c_1)\;+\;q_i'(b_i\,U'(c_1))\,b_i\,U''(c_1)\,\frac{q_i'(b_i\,U'(c_1))\,U'(c_1)}{1\,+\,\sum_k q_k'(b_k\,U'(c_1))\,b_k\,U''(c_1)}\\
&=\; -q_i'(b_i\,U'(c_1))U'(c_1)\left(1\;-\;\frac{q_i'(b_i\,U'(c_1))\,b_i\,U''(c_1)}{1\,+\,\sum_k q_k'(b_k\,U'(c_1))\,b_k\,U''(c_1)}\right)\\
&=\; -q_i'(b_i\,U'(c_1))U'(c_1)\,\frac{1\,+\,\sum_{k\neq i} q_k'(b_k\,U'(c_1))\,b_k\,U''(c_1)}{1\,+\,\sum_k q_k'(b_k\,U'(c_1))\,b_k\,U''(c_1)}\,.
\end{aligned}$$

Therefore,

$$b_i\,\frac{\partial F_i}{\partial b_i} \;>\; -\sum_{j\neq i} b_j\,\frac{\partial F_i}{\partial b_j}\,.$$

Suppose now that the constraint $F(X) \leq X$ is binding, so that $F(X) = X$. Then, by (25),

$$F_i(X) \;=\; w_{1i} - q_i(\lambda(X)\,b_i),$$

with $\lambda(X)$ solving

$$X \;-\; \sum_i (w_{1i} - q_i(\lambda(X)\,b_i)) \;=\; 0$$

where the summation is only over insurers i participating in the tranche. Differentiating, we get

$$\frac{\partial \lambda(X)}{\partial b_j} \;=\; \frac{-q_j'(b_j\,\lambda(X))\,\lambda(X)}{\sum_k\, q_k'(b_k\,\lambda(X))\,b_k}$$

and, hence,

$$\frac{\partial F_i}{\partial b_j} \;=\; b_i\,q_i'(\lambda(X)\,b_i)\,\frac{q_j'(b_j\,\lambda(X))\,\lambda(X)}{\sum_k\, q_k'(b_k\,\lambda(X))\,b_k} \;<\; 0$$

if the insurer $j \neq i$ participates in the tranche and the derivative is zero otherwise. Similarly,

$$\begin{aligned}\frac{\partial F_i}{\partial b_i} \;&=\; -q_i'(\lambda(X)\,b_i)\,\lambda(X) \;+\; b_i\,q_i'(\lambda(X)\,b_i)\,\frac{q_i'(b_i\,\lambda(X))\,\lambda(X)}{\sum_k\, q_k'(b_k\,\lambda(X))\,b_k}\\ &=\; -q_i'(\lambda(X)\,b_i)\,\lambda(X)\,\frac{\sum_{k\neq i}\, q_k'(b_k\,\lambda(X))\,b_k}{\sum_k\, q_k'(b_k\,\lambda(X))\,b_k} \;>\; 0\,. \qquad (42)\end{aligned}$$

if $F_i(X) \neq 0$ (that is, if insurer i participates in the tranche), and is zero otherwise. A direct calculation implies that

$$-b_i\,\frac{\partial F_i}{\partial b_i} \;=\; \sum_{j\neq i} b_j\,\frac{\partial F_i}{\partial b_j}\,.$$

■

Lemma 6 *$H_i(C, \mathbf{b})$ is monotone increasing in $C \in [C_{\min}\,,\,C_{\max}]$ and $b_{-i} \in e^{\Omega^{-i}}$ and takes values in $[e^{\beta_i^{\min}}\,,\,e^{\beta_i^{\max}}]$.*[23] *Furthermore, there exists an $\eta < 1$ such that*

$$\sum_{j\neq i} b_j\,\frac{\partial H_i}{\partial b_j} \;\le\; \eta\,H_i$$

for all $b_{-i} \in \Omega^{-i}$ except for points in a finite union of hyperplanes, for which the derivatives do not exist.

[23]Note that $H_i(C, b_{-i})/C$ is decreasing in C because H_i is increasing in C.

Proof of Lemma 6. Consider the function:

$$\psi_i(y\,,\,b_{-i}\,,\,C) \;\equiv\; \delta_i^{-1}\,C\,u_i'\,(v_i\,(L_i - \delta_i E[u_i(w_{1i} - F_i(X,(y,b_{-i})))])) \;.$$

Then, the defining equation for H_i can be rewritten as

$$H_i \;=\; \psi_i(H_i\,,\,b_{-i}\,,\,C)\,.$$

To complete the proof of the first part of the lemma, it remains to be shown that (1) ψ_i is monotone decreasing in y; (2) for each fixed $C \in [C_{\min}\,,\,C_{\max}]$ and each fixed b_{-i}, it maps the whole $\mathbb{R}$ into the compact interval $[e^{\beta_i^{\min}}\,,\,e^{\beta_i^{\max}}]$; and (3) it is monotone increasing in $b_{-i}\,,\,C$ and is piecewise C^1 with respect to all variables.

By definition, the form of the function F_i depends on the relative ranking of insurers, which, in turn, is determined by the ordering of the numbers $b_i/u_i'(w_{1i})$ (see (33)). For each permutation π of $\{1,\cdots,N\}$, define the corresponding "sector": the subset of $\mathbb{R}_+^n$ such that, for all $\mathbf{b}$ in this sector, the sequence $b_{\pi(i)}/u'_{\pi(i)}(w_{1\pi(i)})$ is monotone increasing in i. The borders of these sectors belong to hyperplanes for which $b_i u_i'(w_{1i}) = b_j u_j'(w_{1j})$ for some $i \neq j$.

Clearly, since the function ψ_i is continuous, it suffices to prove the required result for each fixed sector.[24]

As above, by abuse of notation, we reorder the insurers for each fixed sector so that (33) holds, and thus insurer i will mean the insurer whose rank is equal to i.

First, the fact that the image of the function ψ_i is inside the interval $[e^{\beta_i^{\min}}\,,\,e^{\beta_i^{\max}}]$

[24] Here, one should in general take additional care of the situation when H_i hits the boundaries of the sectors for an open set of parameters. Clearly, this cannot happen for generic values of parameters (discount factors and endowments), and we therefore ignore it. The proof can be easily modified to cover this non-generic situation.

follows directly from the definition and the inequality:

$$0 \ \leq \ F_i(X) \ \leq \ X.$$

Note that the function $F_i(x)$ is continuous and is a smooth function of all b_i as long as $\mathbf{b}$ varies inside a fixed sector. Therefore,

$$\begin{aligned}
&\frac{\partial}{\partial b_k} E[u_i(w_{1i} - F_i(\mathbf{b}, X))] \\
&= \frac{\partial}{\partial b_k} \sum_j \int_{Z_{j+1}}^{Z_j} u_i(w_{1i} - F_i(\mathbf{b}, x)) p(x) dx \\
&= -\sum_j \int_{Z_{j+1}}^{Z_j} u_i'(w_{1i} - F_i(\mathbf{b}, x)) \left(\frac{\partial}{\partial b_k} F_i(\mathbf{b}, x) \right) p(x) dx \\
&= -E\left[u_i'(w_{1i} - F_i(\mathbf{b}, X)) \left(\frac{\partial}{\partial b_k} F_i(\mathbf{b}, X) \right) \right] .
\end{aligned} \tag{43}$$

The derivatives of $Z_i(b_j)$ do not appear on the right-hand side of (43) because the boundary terms cancel.

Denote

$$\tilde{c}_{i0} \ = \ v_i \left(L_i - \delta_i E[u_i(w_{1i} + F_i(X, (H_i(C, b_{-i}), b_{-i})))] \right).$$

Taking partial derivative with respect to b_j on

$$H_i(C, b_{-i}) = \delta_i^{-1} \, C \, u_i' \left(v_i \left(L_i - \delta_i E[u_i(w_{1i} - F_i(X, (y, b_{-i})))] \right) \right) \, .$$

yields

$$\begin{aligned}
\frac{\partial H_i}{\partial b_j} &= \delta_i^{-1} C u_i''(\tilde{c}_{0i}) \frac{1}{u_i'(\tilde{c}_{0i})} \cdot (-\delta_i) E\left[u_i'(w_{1i} - F_i)\big(- \frac{\partial F_i}{\partial b_j} - \frac{\partial F_i}{\partial b_i} \frac{\partial H_i}{\partial b_j} \big) \right] \\
&= C A_i(\tilde{c}_{0i}) E\left[u_i'(w_{1i} - F_i)\big(- \frac{\partial F_i}{\partial b_j} - \frac{\partial F_i}{\partial b_i} \frac{\partial H_i}{\partial b_j} \big) \right]
\end{aligned}$$

where we have used the identity $v_i'(x) = (u_i'(v(x)))^{-1}$ and $A_i(x) = -u_i''(x)/u_i'(x)$. Hence,

$$\frac{\partial H_i}{\partial b_j} = \frac{C\,A_i(\tilde{c}_{i0})\,E\left[u_i'(w_{1i} - F_i(X))\left(-\frac{\partial}{\partial b_j}F_i(X)\right)\right]}{1 + C\,A_i(\tilde{c}_{i0})\,E\left[u_i'(w_{1i} - F_i(X))\left(\frac{\partial}{\partial b_i}F_i(X)\right)\right]}$$

Lemma 5 implies that

$$\begin{aligned}\sum_{j\neq i} b_j \frac{\partial H_i}{\partial b_j} &= \frac{C\,A_i(\tilde{c}_{i0})\,E\left[u_i'(w_{1i} - F_i(X))\sum_j b_j\left(-\frac{\partial}{\partial b_j}F_i(X)\right)\right]}{1 + C\,A_i(\tilde{c}_{i0})\,E\left[u_i'(w_{1i} - F_i(X))\left(\frac{\partial}{\partial b_i}F_i(X)\right)\right]} \\ &\leq \frac{C\,A_i(\tilde{c}_{i0})\,E\left[u_i'(w_{1i} - F_i(X))\,b_i\left(\frac{\partial}{\partial b_i}F_i(X)\right)\right]}{1 + C\,A_i(\tilde{c}_{i0})\,E\left[u_i'(w_{1i} - F_i(X))\left(\frac{\partial}{\partial b_i}F_i(X)\right)\right]} \leq \eta\, b_i = \eta\, H_i\end{aligned} \tag{44}$$

where we have defined

$$\eta = \max_{e^{\Omega}} \frac{C\,A_i(\tilde{c}_{i0})\,E\left[u_i'(w_{1i} - F_i(X))\left(\frac{\partial}{\partial b_i}F_i(X)\right)\right]}{1 + C\,A_i(\tilde{c}_{i0})\,E\left[u_i'(w_{1i} - F_i(X))\left(\frac{\partial}{\partial b_i}F_i(X)\right)\right]}.$$

It follows from the proof of Lemma 5 that the derivative $\frac{\partial}{\partial b_i}F_i(X)$ stays uniformly bounded when $\mathbf{b}$ varies on the compact subset e^{Ω} and therefore $\eta < 1$. The proof is complete. ■

Lemma 7 *Consider a map $G = (G_i) : \Omega \to \Omega$ with coordinate maps $G_i(b_1, \cdots, b_N)$, such that the following is true:*

- *The map G is continuous;*
- *There exists a finite set $\mathcal{S}$ of smooth hyper-surfaces such that G is C^1 on $\Omega \setminus \mathcal{S}$; and*

- *There exists a constant $\eta < 1$ such that*

$$\sum_j \left| \frac{\partial G_i}{\partial d_j} \right| \;\leq\; \eta$$

for all i and all $\mathbf{d} = (d_j) \in \Omega \setminus \mathcal{S}$.

Then, the map G is a contraction in the l_∞ norm $\|\mathbf{d}\|_{l_\infty} \;=\; \max_i |d_i|$, so that

$$\|G(\mathbf{d}^1) - G(\mathbf{d}^2)\|_{l_\infty} \;\leq\; \eta\, \|\mathbf{d}^1 - \mathbf{d}^2\|_{l_\infty}.$$

In particular, G has a unique fixed point $\mathbf{d}_*$ that satisfies

$$\mathbf{d}^* \;=\; \lim_{n\to\infty} G^n(\mathbf{d}^0)$$

for any $\mathbf{d}^0 \in \Omega$.

Proof of Lemma 7. With continuity, we may assume that the two points $\mathbf{d}^1$ and $\mathbf{d}^2$ are in a generic position, so that the segment,

$$\mathbf{d}(t) \;=\; \mathbf{d}^1 + t\,(\mathbf{d}^2 - \mathbf{d}^1)\;,\; t \in [0,1]$$

connecting $\mathbf{d}^1$ and $\mathbf{d}^2$, intersects the hyperplanes from $\mathcal{S}$ for a finite set

$$t_1 \;<\; t_2 \;<\; \cdots \;<\; t_{m+1}.$$

Then,

$$\begin{aligned} |G_i(\mathbf{d}^1) - G_i(\mathbf{d}^2)| \;&=\; \left| \sum_{k=1}^{m} \int_{t_k}^{t_{k+1}} \sum_j \frac{\partial G_i}{\partial d_j}(\mathbf{d}(t))\,(d_j^2 - d_j^1)\,dt \right| \\ &\leq\; \max_j |d_j^2 - d_j^1|\,\eta \;=\; \eta\,\|\mathbf{d}^1 - \mathbf{d}^2\|_{l_\infty}. \end{aligned} \tag{45}$$

The last claim follows from the contraction mapping theorem (see Lucas and Stockey (1989), Theorem 3.2 on p. 50). ■

Proof of Lemma 1. Let $b_j = e^{d_j}$. By Lemma 6,

$$\sum_j \frac{\partial (G_C)_i}{\partial d_j} = \sum_{j \neq i} (H_i)^{-1} \frac{\partial H_i}{\partial b_j} b_j \leq \eta,$$

and the claim follows from Lemma 7. ■

Proof of Theorem 2. By the definition of H_i and $\mathbf{d}^*$, we have

$$q_i\big(e^{d_i^*(C)}\delta_i C^{-1}\big) - w_{0i} = q_i\big(H_i(C, e^{d_{-i}^*})\delta_i C^{-1}\big) - w_{0i} = P_i\big(F_i\big(X, \big(H_i(C, b_{-i}^*), b_{-i}^*\big)\big)\big). \quad (46)$$

Using the definition of C_{min} and C_{max}, and the fact that

$$0 \leq P_i\big(F_i\big(X, \big(H_i(C, b_{-i}^*), b_{-i}^*\big)\big)\big) \leq P_i^{max},$$

it follows from continuity that there exists a solution to equation 15 which we copy below for convenience:

$$C = \left((1+\alpha)\delta^{-1} U'\left(w_0 - (1+\alpha)\sum_i \left(q_i\big(e^{d_i^*(C)}\delta_i C^{-1}\big) - w_{0i}\right)\right)\right)^{-1}.$$

The uniqueness of C^* follows from monotonicity of $H_i(C, b_{-i})/C$.

To show $\{F_i(\mathbf{b}^*(C^*))\}_{i=1}^{i=N}$ is optimal, we only need to show $\mathbf{b}^*$ satisfies equation 13, that is,

$$b_i^* = \frac{\delta_i^{-1}\, u_i'\,(w_{0i} + P_i(F_i(\mathbf{b}^*)))}{(1+\alpha)\,\delta^{-1}\, U'\Big(w_0 \;-\; (1+\alpha)\, \sum_j\, P_j(F_j(\mathbf{b}^*))\Big)}.$$

Because C^* solves equation 15, by equation 46, the above condition is equivalent to

$$b_i^* = \delta_i^{-1} C^* u_i'\,(w_{0i} + P_i(F_i(\mathbf{b}^*)))\,,$$

which is true by the definition of the function $H_i(C, b_{-i})$. ■

Proof of Lemma 2. Pick a parameter ζ and suppose that

$$G_C(\mathbf{d}, \zeta_1) \,\geq\, G_C(\mathbf{d}, \zeta_2)$$

for all $\mathbf{d}$ and, for any fixed $\mathbf{d} = (d_i)$, the expression

$$\left((1+\alpha)\, \delta^{-1}\, U' \left(w_0 \,-\, (1+\alpha) \sum_i \left(q_i(e^{d_i}\, \delta_i\, C^{-1}) - w_{0i} \right) \right) \right)^{-1}$$

is larger for ζ_1 than for ζ_2. Pick a point $\mathbf{d}_0 \in \Omega$. Then, since G_C is monotone increasing in $\mathbf{d}$, we get:

$$\begin{aligned} G_C^2(\mathbf{d}_0, \zeta_1) \;&=\; G_C(G_C(\mathbf{d}_0, \zeta_1), \zeta_1) \;\geq\; G_C(G_C(\mathbf{d}_0, \zeta_2), \zeta_1) \\ &\geq\; G_C(G_C(\mathbf{d}_0, \zeta_2), \zeta_2) \;=\; G_C^2(\mathbf{d}_0, \zeta_2). \end{aligned} \quad (47)$$

Repeating the same argument, we get:

$$G_C^n(\mathbf{d}_0, \zeta_1) \;\geq\; G_C^n(\mathbf{d}_0, \zeta_2)$$

for any $n \in \mathbb{N}$. Sending $n \to \infty$ and using Lemma 1 and Lemma 7, we get:

$$\mathbf{d}^*(C, \zeta_1) \;\geq\; \mathbf{d}^*(C, \zeta_2)$$

for any C. This immediately yields that $C^*(\zeta_1) \,\geq\, C^*(\zeta_2)$, and therefore

$$\mathbf{d}^*(C^*(\zeta_1)\,,\, \zeta_1) \;\geq\; \mathbf{d}^*(C^*(\zeta_2)\,,\, \zeta_1) \;\geq\; \mathbf{d}^*(C^*(\zeta_2)\,,\, \zeta_2)$$

and the claim follows. ■

Lemma 8 *Suppose that an increase in a parameter* ζ *leads to a increase in the optimal* $\mathbf{d}^*$. *Then, this also leads to more insurance.*

Proof of Lemma 8. An increase in $d_i\,,\, i\,=\,1\,,\,\cdots\,,\,N$ is equivalent to a decrease in all coordinates of $\mathbf{a} = (a_i) \;=\; (e^{-d_i})$. Consequently, the number of the coordinates of $\mathbf{a}$ for which $a_i\, u_i'(w_{1i}) \,<\, U'(w_1)$ increases. This is precisely $\#\{\text{senior}\}$. Similarly, by definition, $Z_{\text{full coverage}} = Z_{J+1}$ is monotone decreasing in all a_i (see (8)), and $Z_{\text{deductible}}$ is monotone increasing in all coordinates of $\mathbf{a}$. Finally, the participation index is equal to 1 if $a_i\, u_i'(w_{1i}) \,<\, U'(w_1)$ and therefore stays equal to 1 if a_i decreases. ■

Proof of Corollary 2. By the definition of FOSD dominance, an increase in the distribution of X in the FOSD sense leads to a decrease of

$$E[u_i(w_{1i} \,-\, F_i(\mathbf{b}, X))]\,,$$

for all $i = 1\,,\,\cdots\,,\,N$ and, consequently, to a decrease in the right-hand side of (14) for any fixed $\mathbf{a}$. Therefore, the solution H_i to (14) also decreases in response to this change in the distribution of X. By Lemma 2, this leads to decrease of all coordinates of vector $\mathbf{b}$. The claims follow now from Lemma 8.

Similarly, an increase in w_0 and a increase in δ lead to an increase in the right-hand side of (15). This leads to an increase in C, and therefore, by Lemma 2, all coordinates of vector $\mathbf{b}$ increase. ■

Proof of Corollary 3. A direct calculation shows that, under the CARA assumption, the vector $\mathbf{b} \,=\, (b_i)$ solves

$$b_i \;=\; \delta_i^{-1}\, C\, \left(e^{-A_i\, w_{0i}} \,+\, e^{-\delta_i - A_i w_{1i}}\, E[1 - e^{A_i\, F_i(X)}]\right)\;,\; i \,=\, 1\,,\,\cdots\,,\,N. \tag{48}$$

Suppose that

$$\frac{\delta_i\, e^{-A_i\, w_{1i}}}{e^{-A_i\, w_{0i}}} \;>\; \frac{\delta_j\, e^{-A_j\, w_{1j}}}{e^{-A_j\, w_{0j}}} \tag{49}$$

for some insurers i and j, but $\text{rank}(i) < \text{rank}(j)$. By definition, this means that

$$b_i\, e^{A_i\, w_{1i}} \;\leq\; b_j\, e^{A_j\, w_{1j}}. \tag{50}$$

We now claim that the inequality $\text{rank}(i) < \text{rank}(j)$ implies

$$A_i\, F_i \;\leq\; A_j\, F_j. \tag{51}$$

Indeed, for all tranches in which insurer i participates, the slopes of $A_i\, F_i$ and $A_j\, F_j$ coincide by Corollary 1. Since j has a higher rank, $A_i\, F_i(X) = 0$ for all X for which $A_j\, F_j(X) = 0$. The claim (51) follows now by continuity of F_i and F_j. Consequently,

$$E[1 - e^{A_i\, F_i(X)}] \;\geq\; E[1 - e^{A_j\, F_j(X)}],$$

and therefore (48) and (49) together yield

$$\begin{aligned} b_i\, e^{A_i\, w_{1i}} \;&=\; \delta_i^{-1}\, C\, e^{A_i\,(w_{1i} - w_{0i})} \;+\; C\, E[1 - e^{A_i\, F_i(X)}] \\ &\geq\; \delta_j^{-1}\, C\, e^{A_j\,(w_{1j} - w_{0j})} \;+\; C\, E[1 - e^{A_j\, F_j(X)}] \;=\; b_j\, e^{A_j\, w_{1j}}, \end{aligned} \tag{52}$$

which contradicts (50). The proof is complete. ■

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