Equilibrium Driven by Discounted Dividend Volatility [*]

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August 26, 2009

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Abstract

We derive representations for the stock price drift and volatility in the equilibrium of agents with arbitrary, heterogeneous utility functions and with the aggregate dividend following an arbitrary Markov diffusion. We introduce a new, intrinsic characteristic of the aggregate dividend process that we call the "rate of discounting volatility" and show that, in equilibrium, the size of market price of risk is determined by the market price of discounted dividend volatility (DDV), discounted at that rate, and multiplied by the aggregate risk aversion. The stock price volatility is equal to the market price of DDV plus a volatility risk premium. In particular, stock price volatility is larger than the dividend volatility if the aggregate risk aversion is decreasing, dividend volatility is countercylical and the rate of discounting volatility is procyclical. We also obtain a representation for the optimal portfolios. Under the above cyclicality conditions, we show that the non-myopic (hedging) component of an agent's portfolio is positive (negative) if the product of agent's prudence and risk tolerance is below (above) two, and the sign is reversed for the reversed cyclicality conditions.

Keywords: equilibrium, heterogeneous agents, volatility, optimal portfolios

JEL Classification. D53, G11, G12

1 Introduction

What is the equilibrium prediction for the market price of risk and volatility of the risky asset in a complete market economy populated by heterogeneous agents? And what is the hedging behavior of those agents in equilibrium? Those are the two main questions we answer in this paper.

The basic intuition comes from Merton's continuous-time analog of CAPM. That is, with a CRRA representative agent with risk aversion γ and the dividend following a geometric Brownian motion with volatility σ, equilibrium behavior resembles that of the standard CAPM: the market price of risk is given by $\gamma\sigma$, stock price volatility equals σ, and the optimal portfolio is myopic and instantaneously mean-variance efficient. We extend these results and allow for

- an arbitrary dividend process;

- heterogeneous agents with arbitrary utility functions.

We show that

- the market price of risk is obtained as the expected value, under the risk-neutral probability, of the aggregate (relative) risk aversion multiplied by dividend volatility discounted at the rate we call the *"rate of discounting volatility"*. This rate depends only on the structure of the dividends process;

- the rate of discounting volatility can be interpreted as the speed of mean-reversion of the log-dividend process[1] if it is positive, and rate of growth if it is negative;

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[1]Possibly after a deterministic transformation.

- the stock price volatility can be decomposed into *excess* component and *fundamental* component. The fundamental component is given by the market price of dividend volatility, discounted at the above mentioned rate;

- excess volatility is given by a volatility risk premium, whose sign is determined by the co-movement of the dividend with aggregate risk aversion and discounted dividend volatility;

- the non-myopic (hedging) component of an agent's portfolio is given by a portfolio risk premium, whose sign is determined by the co-movement of agent's wealth and risk tolerance with aggregate risk aversion and discounted dividend volatility.

The most important general message from the above results is that the volatility of the dividends by itself is not enough to determine equilibrium properties. For example, bounds on the product of risk aversion and dividend volatility may substantially under- (over-) estimate the true equilibrium risk premium if the economy if growing (mean-reverting).

In the case of lognormal dividend, dividend volatility has been commonly interpreted as the fundamental component of the stock price volatility (see, e.g., Bhamra and Uppal (2009)). As follows from above, when dividend is not a geometric Brownian motion, this may lead to under- or over-estimation depending on the sign of the discount rate.

The above mentioned representations allow us to make predictions about equilibrium behavior. Suppose that the product of aggregate risk aversion and dividend volatility is countercyclical,[2] dividend volatility is countercyclical and the rate of discounting volatility is pro-cyclical. Then,

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[2]For example, aggregate risk aversion is always countercyclical in a heterogeneous economy with CRRA agents. See, Benninga and Mayshar (2000).

(a) the market price of risk is counter-cyclical;

(b) excess volatility is positive;

(c) optimal portfolios are monotone decreasing in risk aversion;

(d) the non-myopic (hedging) component of an agent's portfolio is positive (negative) if the product of the agent's prudence and risk tolerance is below (above) two.

All signs are reversed for the reversed cyclicality conditions.

The fact that countercyclical risk aversion leads to countercyclical market price of risk is very intuitive and agrees with analogous results in the literature on habit formation (see, Campbell and Cochrane (1999)). However, our result implies that market price of risk may be countercyclical even if risk aversion is procyclical, but countercyclicality of dividend volatility is sufficiently strong.

Excess volatility is a well known stylized fact. See, e.g., Shiller (1981), LeRoy and Porter (1981), Mankiw, Romer, and Shapiro (1985, 1991) and West (1988). Therefore, the fact that in our setup counter-cyclical risk aversion always leads to excess stock volatility gives credence to the model. The intuition behind it is as follows. In good states with high expected future dividends, aggregate risk aversion is low and so the agent is willing to hold the stock even it the return is low (i.e., the price is high). This makes the price go up very high in good states and, by the same arguments, go down fast in bad states, and therefore drives price volatility up.

The result about monotonicity of optimal portfolios is also important. Almost all papers on heterogeneous equilibria use this monotonicity property as the basis for economic intuition. See, e.g., Dumas (1989), Wang (1996), Basak and Cuoco (1998), Basak (2005), Bhamra and Uppal (2009). However,

to the best of our knowledge, no proof of this property has ever been given even in an economy with only two agents.

The relation between hedging portfolios and prudence is intriguing. It is known that prudence is responsible for the precautionary savings effect (see, Kimball (1991)). The above result shows that the sign of the hedging portfolio is determined by the relative strength of two motives: precautionary saving and risk aversion. When risk aversion is small relative to prudence, the precautionary saving (risk aversion) motive dominates and hedging portfolio becomes negative (positive).

We conclude the introduction with a discussion of related literature. Most of the work which extends standard CAPM and CCAPM to heterogeneous risk preferences is done in very special models with two CRRA agents only. Dumas (1989) considers a production economy of such a type, performing a numerical analysis. Similarly, Wang (1996) studies the term structure of interest rates in an economy populated by two CRRA agents, maximizing time-additive utility from intermediate consumption and the aggregate dividend following a geometric Brownian motion. Bhamra and Uppal (2009) consider the same economy, and derive conditions under which excess volatility is positive.[3] Basak and Cuoco (1998) study equilibria with two agents and limited stock market participation.

Cvitanić and Malamud (2009a, 2009b) study asymptotic equilibrum dynamics with an arbitrary number of CRRA agents maximizing utility from terminal consumption, as the horizon becomes large.

Only few papers study general properties of equilibria with non-CRRA preferences and/or a general dividend process. Bick (1990) and Wang (1993)

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[3]The main message of Bhamra and Uppal (2009) is that allowing the agents to trade in an additional derivative security, making the market complete, may increase the market volatility. Because of completeness, their equilibrium is an Arrow-Debreu equilibrium of Wang (1996).

allow for an arbitrary dividend process and describe the set of viable price processes. That is, all price processes that can be attained by varying the utility of the representative agent. Berrada, Hugonnier and Rindisbacher (2007) provide necessary and sufficient conditions for zero equilibrium trading volume in a general continuous-time model with heterogeneous agents, multiple goods, and multiple securities.

There are models in which heterogeneity comes from beliefs and asymmetric information, as in Basak (2005), Jouini and Napp (2009) and Biais, Bossaerts and Spatt (2009). Some papers study static, one period economies with heterogeneous preferences. See, e.g., Benninga and Mayshar (2000), Gollier (2001), Hara, Huang and Kuzmics (2007). Methodologically, our paper is close to Detemple and Zapatero (1991), Detemple, Garcia and Rindisbacher (2003) and Bharma and Uppal (2009), as we use Malliavin calculus to derive the main results.

Our paper is also related to a recent work of Mele (2007). He studies monotonicity/concavity properties of equilibrium stock price and their relation to equilibrium volatility dynamics. He introduces a new object, the risk adjusted discount rate, and rewrites the equilibrium stock price as the present value of risk adjusted future dividends, discounted at this rate. Even though the idea of such a representation is similar to the one of this paper, there is no direct connection between the risk adjusted discount rate and the rate of discounting volatility. The former depends on the endogenous equilibrium market price of risk, whereas the latter is an intrinsic property of the exogenously given dividend process. Furthermore, we obtain representations for the instantaneous moments of the price, and not of the price itself.

The paper is organized as follows. In Section 2, we describe the setup

and notation. In Section 3 we introduce the rate of discounting volatility and derive representations for market price of risk, volatility, as well as drift and volatility of the market price of risk, and study their behavior. Section 4 is devoted to equilibrium optimal portfolios. Section 5 concludes.

2 Setup and Notation

2.1 The Model

We consider a standard setting similar to that of Wang (1996). The economy has a finite horizon and evolves in continuous time. Uncertainty is described by a one-dimensional, standard Brownian motion B_t, $t \in [0, T]$ on a complete probability space $(\Omega, \mathcal{F}_T, P)$, where $\mathcal{F}$ is the augmented filtration generated by B_t. There is a single share of a risky asset in the economy, the stock, which pays a terminal dividend D_T such that

$$D_t^{-1} \, dD_t \;=\; \mu^D(D_t) \, dt \;+\; \sigma^D(D_t) \, dB_t.$$

This diffusion process lives on $(0, +\infty)$.

We assume that $\sigma^D(D_t) > 0$, and that μ^D and σ^D are such that a unique strong solution exists. Moreover, we assume $\mu^D \in C^1(\mathbb{R}_+)$, $\sigma^D \in C^2(\mathbb{R}_+)$. [4]

We also assume that a zero coupon bond with instantaneous constant risk-free rate r is available in zero net supply.[5]

There are K competitive agents behaving rationally, and agent k is initially endowed with $\psi_k > 0$ shares of stock, an the total supply of the stock is normalized to one,

$$\sum_{k=1}^{K} \psi_k \;=\; 1.$$

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[4]In general, whenever we use a derivative of a function, we implicitly assume it exists.
[5]The assumption of constant r is introduced only for simplicity of exposition.

Agent k chooses portfolio strategy π_{kt}, the portfolio weight at time t in the risky asset, as to maximize the expected utility

$$E\left[u_k(W_{kT})\right]$$

of its final wealth W_{kT}, where the wealth W_{kt} of agent k evolves as

$$dW_{kt} \;=\; W_{kt}(rdt \;+\; \pi_{kt}\left(S_t^{-1}dS_t - rdt\right)).$$

Here, S_t is the stock price at time t. The instantaneous drift and volatility of the stock price S_t are denoted by μ_t^S and σ_t^S respectively,

$$S_t^{-1}\,dS_t \;=\; \mu_t^S dt \;+\; \sigma_t^S dB_t.$$

We assume that u_k is $C^2(\mathbb{R}_+)$, and for the results involving prudence, we assume u_k is $C^3(\mathbb{R}_+)$.

2.2 Equilibrium

Definition 2.1. *We say that the market is in equilibrium if the agents behave optimally and both the risky asset market and the risk-free market clear.*

It is well known that the above financial market is complete, if the volatility process σ_t^S of the stock price is almost everywhere strictly positive. When the market is complete, there exists a unique stochastic discount factor (SDF) $M = M_T$ such that the stock price is given by

$$S_t \;=\; e^{r(t-T)}\,\frac{E_t[M_T\,D_T]}{E_t[M_T]}.$$

Equivalently, $\frac{M_T}{E[M_T]}$ is the density of the equivalent martingale measure Q and

$$S_t \;=\; e^{r(t-T)}\,E_t^Q[\,D_T\,].$$

Because of the market completeness, any equilibrium allocation is Pareto-efficient and can be characterized as an Arrow-Debreu equilibrium. See, e.g. Duffie and Huang (1986), Wang (1996).[6]

Introduce the inverse of the marginal utility

$$I_k(x) := (u_k')^{-1}(x) \tag{1}$$

It is well known that in this complete market setting the optimal terminal wealth is of the form

$$W_{kT} = I_k(y_k M_T)$$

where y_k is determined via the *budget constraint*[7]

$$E[I_k(y_k M_T) M_T] = W_{k0} = \psi_k S_0 = \psi_k E[M_T D_T].$$

We formalize this in

Proposition 2.1. *Equilibrium SDF M and parameters y_k solve the equations*

$$\sum_{k=1}^{K} I_k(y_k M_T) = D_T \tag{2}$$

$$E[I_k(y_k M_T) M_T] = \psi_k E[M_T D_T], \quad i = 1, \dots, K. \tag{3}$$

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[6]Because the endowments are co-linear (all agents hold shares of the same single stock), it can be shown that, under some conditions on agents' utilities, the equilibrium is in fact unique up to a multiplicative factor, and unique if we fix the risk-free rate. See, e.g., Dana (1995), Dana (2001). If the endowment is neither bounded away from zero nor from infinity, some additional care is needed to verify the existence of equilibrium. See, e.g., Dana (2001) and Malamud (2008). We implicitly assume throughout the paper that an equilibrium exists.

[7]We assume a unique such y_k exists.

3 The Rate of Discounting Volatility and the Equilibrium Stock Price

Since the market is complete, it is well known that the prices in our heterogeneous economy coincide with those in an artificial economy, populated by a single, representative agent with a utility function U, and the equilibrium stochastic discount factor equals the marginal utility of the representative agent, evaluated at the aggregate endowment,

$$M_T = U'(D_T). \tag{4}$$

That is, the function $U'(x)$ satisfies the equation

$$\sum_k I_k(y_k U'(x)) = x. \tag{5}$$

Let

$$\gamma^U(x) = -\frac{x U''(x)}{U'(x)}$$

be the relative risk aversion of the representative agent.

Definition 3.1. *Introduce the function*

$$c(x) = c^D(x) \stackrel{def}{=} -x (\mu^D)'(x) + x (\sigma^D)'(x) \sigma^D(x)^{-1} \mu^D(x)$$
$$+ (\sigma^D)'(x) \sigma^D(x) x + 0.5 (\sigma^D)''(x) x^2 \sigma^D(x) \tag{6}$$

We call the process $c(D_t)$ the rate of discounting volatility.[8]

The name "rate of discounting volatility" is justified by the following

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[8]The quantity $c(x)$ appears also in the paper Detemple, Garcia and Rindisbacher (2003) in a partial equilibrium setting, as an auxiliary process that helps improve the computational efficiency, without having a direct economic interpretation.

Theorem 3.1. *The equilibrium market price of risk*

$$\lambda_t \;=\; \frac{\mu_t^S - r}{\sigma_t^S}$$

is given by

$$\lambda_t \;=\; E_t^Q \left[\gamma^U(D_T)\, \sigma^D(D_T)\, e^{-\int_t^T c(D_s)\, ds} \right].$$

Theorem 3.1 shows that the dividend volatility is priced at a discount, with the rate of discounting being equal to $c(D_s)$. The following result is a direct consequence of Theorem 3.1:

Corollary 3.1. *Under the equilibrium risk neutral measure, the drift of the equilibrium market price of risk is always equal to $c(D_t)$.*

The result of Corollary 3.1 is somewhat surprising. It means that, even with arbitrary dividend process, the drift (under risk-neutral measure) of the equilibrium market price of risk is *independent of the representative agent's utility* and is determined solely by $c(x)$, an intrinsic characteristic of the dividend process. Thus, it is important to understand the nature of the function $c(x)$. We first note that the following is true.

Lemma 3.1. *The rate of discounting volatility is invariant under transformations: if $D_t \;=\; g(\tilde{D}_t)$ for a one-to-one function $g \in C^3(\mathbb{R}_+)$, then*

$$c^D(g(x)) \;=\; c^{\tilde{D}}(x).$$

This invariance property has important consequences. It means that structural properties of $c(x)$ (such as, e.g., monotonicity in x) are determined solely by the dynamical properties of the diffusion process D_t. In particular, since any diffusion process can be reduced to a constant volatility process by a functional transformation, we get that the following is true, as a consequence of Lemma 3.1.

Corollary 3.2. *The rate of volatility discounting $c(D_t)$ is constant if there exists a one-to-one function $g \in C^3(\mathbb{R}_+)$ such that $D_t = g(A_t)$ where*

$$dA_t = (a - bA_t)\,dt + \sigma^A dB_t.$$

In that case, $c = b$. Conversely, $c = b$ implies $D_t = F^{-1}(A_t)$ for some values of a and σ^A, where $F(x) = \int_{x_0}^{x} \frac{1}{y\,\sigma^D(y)}\,dy.$[9]

Thus, the rate c of discounting volatility can also be interpreted as the speed of mean-reversion or mean growth rate of the log-dividend process $\log D_t$, possibly after a transformation. If $c = b > 0$, there is mean reversion and the volatility is priced at discount. On the other hand, if $b < 0$, there is growth in the dividend process and the volatility is priced at premium, appreciated at growth rate $|b|$.

We will now need the following

Definition 3.2. *We denote by*

$$\sigma^D(t, T) \overset{def}{=} e^{-\int_t^T c(D_s)\,ds}\,\sigma^D(D_T)$$

the disounted volatility. The market price V_t of discounted volatility (MPDV) is given by

$$V_t = E_t^Q\left[\sigma^D(t, T)\right].$$

Let

$$\gamma_k^{\inf}\,,\ \gamma_k^{\sup}$$

denote the infinum and supremum of the relative risk aversion of agent k, and $\gamma_{\inf}^U$, $\gamma_{\sup}^U$ be the infinum and supremum of the relative risk aversion of the representative agent. It is known (see, Wilson (1968) and Hara, Huang and

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[9]Since, by assumption $\sigma^D(y)$ is positive and C^2, $F(x)$ is strictly increasing and therefore, by the implicit function theorem, $F^{-1} \in C^3$ and is also strictly increasing.

Kuzmics (2007)) that the representative agent's risk aversion is a weighted average of individual risk aversions. In particular, the following is true

Lemma 3.2. *We have*

$$\min_k \gamma_k^{\inf} \;\leq\; \gamma_{\inf}^U \;\leq\; \gamma_{\sup}^U \;\leq\; \max_k \gamma_k^{\sup}.$$

The following result is a direct consequence of Lemma 3.2 and Theorem 3.1.

Proposition 3.1. *The equilibrium market price of risk satisfies*

$$V_t \min_k \gamma_k^{\inf} \;\leq\; \lambda_t \;\leq\; V_t \max_k \gamma_k^{\sup}.$$

In particular,
$$\lambda_t \;=\; \gamma\, V_t$$

if $\gamma_k \;=\; \gamma^U$ *is constant and independent of* k.

That is, the size of the equilibrium market price of risk is determined by the size of relative risk aversion and MPDV. Thus, even if the volatility σ^D is constant, equilibrium market price of risk may happen to be larger or smaller than $\gamma\,\sigma^D$ if $c < 0$ or $c > 0$ respectively. This is intuitive, as $c < 0$ corresponds to a growing economy, and $c > 0$ to a mean-reverting economy.

The representation in Theorem 3.1 can also be used to get dynamic properties of the equilibrium market price of risk . We will need the following auxiliary[10]

Lemma 3.3. *Suppose that* f *and* g *are both increasing (decreasing). Then, the following is true*

    (1) the function
$$E\left[f(D_T)\, e^{\int_t^T g(D_s)\, ds} \,\big|\, D_t \;=\; x \right]$$

    is also monotone increasing (decreasing);

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[10]Item (1) of Lemma 3.3 is contained in Mele (2007).

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(2) we have

$$\text{Cov}_t \left(h(D_T),\, f(D_T)\, e^{\int_t^T g(D_s)\,ds} \right) \geq 0$$

if h has the same direction of monotonicity as f and g, and the inequality reverses if h has the opposite direction of monotonicity.

Since D_t is the only state variable driving the state of the economy, its fluctuations determine the business cycle of the economy. We make a formal

Definition 3.3. *Economic quantity $f_t = f(t, D_t)$ is pro-(counter-)cyclical if f is non-decreasing (non-increasing) in D_t for any fixed t.*

The following result is a direct consequence of Lemma 3.3

Corollary 3.3. *We have*

(1) *if $\gamma^U(x)\,\sigma^D(x)$ is decreasing and $c(D_t)$ is procyclical, then λ_t is countercyclical;*

(2) *if $\gamma^U(x)\,\sigma^D(x)$ is increasing and $c(D_t)$ is countercyclical, then λ_t procyclical.*

Note that there is a strong empirical evidence that the market price of risk λ_t is countercyclical (see, Fama and French (1989), Ferson and Harvey (1991)). Therefore, we would like to understand how plausible the assumptions of item (1) are.

Arrow (1965) argues that a utility of an individual investor should exhibit increasing relative risk aversion. However, the aggregate risk aversion γ^U of the representative agent may have a very different behavior. In fact, in the benchmark case, when the economy is populated by heterogeneous CRRA agents, γ^U is *decreasing.* See, Benninga and Mayshar (2000) and Cvitanić and Malamud (2009b). In fact, it is possible to show that γ^U is decreasing

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even if γ_k is increasing for some agents, but heterogeneity is sufficiently large. Furthermore, there is some empirical evidence that dividend volatility is counter-cyclical (see, e.g., French and Sichel (1993)). Under these conditions, our model can generate countercyclical market price of risk as long as the rate of discounting volatility is procyclical (e.g., when $c =$const; see Example 3.1). If rate c is strictly increasing, we can interpret this as a slowly growing economy, if c is negative, or a rapidly mean-reverting economy, if c is positive.

Note that it is possible generate counter-cyclical risk aversion (and, as a consequence, countercyclical risk premium) though state-dependent preferences such as, e.g., habit formation. See, Campbell and Cochrane (1999).

It turns out that monotonicity properties of $\sigma^D(x)$ and $c(x)$ are crucial for determining various static and dynamic properties of the equilibrium quantities. Let

$$\mathrm{Cov}_t^Q(X,Y) \;=\; E_t^Q[\,XY\,] \;-\; E_t^Q[\,X\,]\,E_t^Q[\,Y\,]$$

be the conditional covariance of random variables X and Y under the risk-neutral measure Q. We have the following is result.

Theorem 3.2. *The equilibrium stock price volatility is given by*

$$\sigma_t^S \;=\; V_t \;-\; \frac{1}{E_t^Q[D_T]}\,\mathrm{Cov}_t^Q\left(\left(\gamma^U(D_T) - 1\right)\sigma^D(t\,,T)\,,D_T\right). \qquad (7)$$

Furthermore, S_t is always procyclical and $\sigma_t^S > 0$ almost surely.

The market price V_t of discounted volatility is the *fundamental volatility* component, determined by solely by the size of the underlying dividend volatility. Theorem 3.2 implies that the equilibrium volatility is given by fundamental volatility plus a *volatility risk premium.* Even though, in con-

trast to standard CAPM (or, CCAPM) risk premium, the covariance in (7) is under the risk-neutral measure Q, the interpretation of the volatility risk premium is similar: the spread $\sigma_t^S - V_t$ between the stock volatility and the fundamental volatility is determined by the co-movement of aggregate risk aversion and discounted volatility with the dividend.

It is well known (see, Shiller (1981), LeRoy and Porter (1981), Mankiw, Romer, and Shapiro (1985, 1991) and West (1988)) that the stock volatility cannot be explained by the volatility of future dividends. Usually (see, e.g., Bhamra and Uppal (2009)), stock volatility is decomposed into the fundamental and excess volatility, with the latter being responsible for the large discrepancy between σ_t^S and σ_t^D. Expression (7) shows that the fundamental volatility V_t by itself may already substantially exceed dividend volatility if the economy is growing (that is, the rate of discounting volatility is negative).

Using Lemma 3.3, it is possible to determine the sign of excess volatility.

Corollary 3.4. *The following is true:*

(1) if the relative risk aversion γ^U is decreasing and $\gamma^U(x) \geq 1$, the volatility $\sigma^D(D_T)$ is countercyclical and the rate $c(D_s)$ of discounting volatility risk is procyclical, then

$$\sigma_t^S \ \geq \ V_t\,;$$

(2) if the relative risk aversion γ^U is increasing and and $\gamma^U(x) \geq 1$, the volatility is procyclical and the rate $c(D_s)$ of discounting volatility risk is counter-cyclical, then
$$\sigma_t^S \ \leq \ V_t\,.$$

The main message of Corollary 3.4 is that, to achieve a high stock price volatility, we need a negative, procyclical discount rate, that is, a slowly growing economy, as well as countercyclical risk aversion and dividend volatility.

Note that the conditions of item (1) are almost the same as those in Corollary 3.3, needed for the counter-cyclicality of the market price of risk . The intuition for this is as follows. By Theorem 3.2 stock price S_t is always procyclical. Thus, if the market price of risk is countercyclical, the stock is cheap in bad states (those with low D_t) and offers a high instantaneous return, that will force agents to buy more shares in those states and make the equilibrium price move faster and make it more volatile. The same argument applies in good states. On the contrary, if the market price of risk is procyclical, the high market price of risk offered by the the stock in good states (with high D_t) is offset by the high price S_t in those states. This drives down the trading volume and reduces the equilibrium volatility.

We would now like to have a closer look at the dynamics of the risk premium λ_t.

Proposition 3.2. *Assume that the representative agent's utility function U is $C^3(\mathbb{R}_+)$. We have*

$$\lambda_t^{-1} d\lambda_t = \left(c(D_t) + \sigma_t^\lambda \lambda_t \right) dt + \sigma_t^\lambda dB_t$$

with

$$\begin{aligned}
\lambda_t \sigma_t^\lambda &= - \operatorname{Var}_t^Q[\gamma^U(D_T) \sigma^D(t,T)] \\
&\quad + E_t^Q[e^{-\int_t^T 2c(D_s)ds} \left(\gamma^U \sigma^D(D_T) \right)'(D_T) D_T \sigma^D(D_T)] \\
&\quad - E_t^Q \left[\int_t^T e^{-\int_t^\theta 2c(D_r)dr} c'(D_\theta) \lambda_\theta D_\theta \sigma^D(D_\theta) d\theta \right] .
\end{aligned} \tag{8}$$

It is known that, in the benchmark model when D_t is a geometric Brownian motion and γ^U is constant, market price of risk is also constant and hence $\sigma_t^\lambda = 0$. The reason is that, in that case, c, σ^D and γ^U are all constant.

Identity (8) shows that stochastic volatility of the equilibrium market

price of risk is generated by the the structure of representative risk aversion and discounted volatility. To understand the effect of various components, we must separately consider the cases of pro- and counter-cyclical risk premia.

By Corollary 3.3, if γ^U, σ^D and $-c$ are all decreasing, market price of risk is countercyclical and, consequently, $\sigma_t^\lambda < 0$. In this case, all three components are negative and hence, strong cyclicality of $\gamma^U \sigma^D$ and c generates large volatility of the market price of risk.

The situation is different if γ^U, σ^D and $-c$ are increasing.. In that case, the variance term in (8) reduces the volatility $\sigma_t^\lambda > 0$. Thus, increasing cyclicality of $\gamma^U \sigma^D$ increases the second term, but also increases the variance of $\gamma^U \sigma^D$ and may therefore lead to a decrease in σ_t^λ. A similar effect takes place if we increase the cyclicality of c.

The term

$$E_t^Q \left[\int_t^T e^{-\int_t^\theta 2\, c(D_r)dr}\, c'(D_\theta)\, \lambda_\theta\, D_\theta\, \sigma^D(D_\theta)\, d\theta \right]$$

is the most complex. [11] Clearly, it is non-zero if and only if $c'(x)$ is non-zero. Thus, we can interpret it as the market price of discounted *changes in the rate of discounting volatility*.

We illustrate our results by the following

Example 3.1. Suppose we have

$$\frac{1}{D_t} dD_t \;=\; (a \,-\, b \log D_t)dt \;+\; \sigma\, dB_t.$$

In this case, the log-dividend process $A := \log D$ is a mean-reverting Gaussian process:

$$dA_t \;=\; (a - 0.5\sigma^2 - bA_t)dt \;+\; \sigma\, dB_t.$$

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[11]It coincides with the the price of an artificial security paying the dividend rate $c'(D_T)\, \lambda\, D_T\, \sigma^D(D_T)$, but discounted at the rate $2\, c(D_T)$.

In that case, $\sigma^D = \sigma$, $c = b$, and all the formulae substantially simplify and all the static and dynamic properties of equilibrium are determined solely by the properties of the aggregate risk aversion γ^U. In particular, we have the following

Proposition 3.3. *Suppose that γ^U is decreasing. Then,*

(1) *market price of risk is counter-cyclical and satisfies*

$$\min_k \gamma_k^{\inf} \ \leq \ \frac{\lambda_t}{e^{b(t-T)}\,\sigma} \ \leq \ \max_k \gamma_k^{\sup};$$

(2) *price volatility is larger than the discounted (or, appreciated, if $b < 0$) volatility,*

$$\sigma_t^S \ > \ e^{b\,(t-T)}\,\sigma^D.$$

All statements of Proposition 3.3 follow directly from the results above. The results of Proposition 3.3 are easily extended to the case when

$$g(\log D_t) \ = \ A_t$$

for some monotone increasing function g, because, by Lemma 3.1, the rate c of discounting volatility remains constant in this setting. By Ito's formula, $\sigma^D(D_t) = (g'(D_t))^{-1}\,\sigma$ and the results will depend on whether g is concave or convex.

4   Optimal Portfolios

We start with the following

Proposition 4.1. *The optimal portfolio π_{kt} is positive and is given by*

$$\sigma_t^S\,\pi_{kt} \ = \ \frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(t,\,T)\,W_{kT}\left(\gamma_k^{-1}(W_{kT}) - 1\right)\right]}{E_t^Q[W_{kT}]} \ + \ \lambda_t \qquad (9)$$

One important consequence of Proposition 4.1 is that there is no short-selling in equilibrium. The reason is that, because markets are complete, optimal wealths W_{kT} of all agents are increasing in D_T. Hence, there is no incentive for an agent to short the stock. In particular, this explains why market price of risk is always positive in equilibrium (see, Theorem 3.1). If the market price of risk were negative, it would be optimal for all agents to short the stock and markets would not clear. The same intuition implies that equity premium must be increasing in risk aversion: highly risk averse agents will only buy stock if it offers a sufficiently large equity premium.

In this section we will use Proposition 4.1 to derive various properties of equilibrium optimal portfolios. Denote by $\pi_{\log t}$ the optimal portfolio of the log investor. It is well known that

$$\pi_{\log t} \;=\; \frac{\lambda_t}{\sigma_t^S}.$$

An immediate consequence of Proposition 4.1 is

Proposition 4.2. *If $\gamma_k(x) \geq 1$ for all x, then*

$$\pi_{kt} \;\leq\; \pi_{\log t}$$

and the inequality reverses if, for all x, $\gamma_k(x) \leq 1$.

The result of Proposition 4.2 is intuitive. One would expect that more risk averse agents should buy less stock. This is indeed true in a static, one period optimization problem, as was shown by Arrow (1965). However, showing monotonicity of optimal portfolios with respect to risk aversion in a multi-period model is a highly non-trivial problem. We will need the following

Definition 4.1 (Ross (1981)). *Agent k is more risk averse than agent j in the sense of Ross if*

$$\inf_x \gamma_k(x) \;\geq\; \sup_x \gamma_j(x).$$

In this case we write $\gamma_k \geq_R \gamma_j$.

This definition was introduced by Ross (1981) in the context of a static, one period problem with two risky assets. Ross showed that the above mentioned monotonicity result of Arrow does not hold if we only require a weak, pointwise inequality in risk aversion. The reason is that optimal portfolio choice becomes a non-local problem and local properties of risk aversion are not sufficient for the analysis. A similar phenomenon arises in our dynamic, multi-period optimization: even though one of the assets is locally riskless, the amount of money invested into it changes over time and thus, effectively, we get a problem with two risky assets, and Definition 4.1 becomes the right concept to consider. In fact, we have the following monotonicity of optimal portfolios relative to risk aversion:

Proposition 4.3. *The following is true:*

(1) *suppose that the product $\gamma^U(x)\,\sigma^D(x)$ is decreasing and c is procyclical. Then, $\gamma_k \geq_R \gamma_j \geq 1$ implies*

$$\pi_{kt} \ \leq \ \pi_{jt}\,;$$

(2) *suppose that the product $\gamma^U(x)\,\sigma^D(x)$ is increasing and c is counter-cyclical. Then, $1 \geq \gamma_k \geq_R \gamma_j$ implies*

$$\pi_{kt} \ \leq \ \pi_{jt}\,.$$

Note however that we do not know whether the results of item (1) ((2)) hold for risk aversions below (above) one.

In the benchmark case when all agents in the economy have constant relative risk aversion, γ^U is decreasing (see, Benninga and Mayshar (2000)) and we arrive at

Corollary 4.1. *Suppose that the economy is populated by heterogeneous CRRA agents. Then,*

(1) if $\sigma^D(x)$ is countercyclical and c is procyclical, then optimal portfolio is decreasing in risk aversion for risk aversion above one;

(2) if $\sigma^D(x)$ is procyclical and c is countercyclical, then optimal portfolio is decreasing in risk aversion for risk aversion below one.

We will now study the structure of optimal portfolios in greater detail. Let

$$U_{kt}(x) = \sup_{\pi} E_t \left[\, u_k(W_{kT}) \,|\, W_{kt} = x \,\right]$$

be the value function of agent k[12] and

$$\gamma_{kt}(x) = -\frac{x\, U''_{kt}(x)}{U'_{kt}(x)}.$$

the effective relative risk aversion of agent k at time t. Also denote

$$\gamma_{kt} = \gamma_{kt}(W_{kt}).$$

It is known (see, Merton (1971)), that, when the market price of risk is constant, the optimal portfolio is myopic, instantaneously mean-variance efficient and is given by

$$\pi_{kt}^{\text{myopic}} \stackrel{def}{=} \frac{\lambda_t}{\gamma_{kt}\, \sigma_t^S}.$$

The following is true

Proposition 4.4. *In equilibrium,*

$$\gamma_{kt} = \frac{E_t^Q[W_{kT}]}{E_t^Q[\gamma_{kT}^{-1}\, W_{kT}]} \tag{10}$$

---

[12]Note that U_{kt} depends on D_t but we suppress this dependence.

and therefore myopic optimal portfolio is given by

$$\pi_{kt}^{\mathrm{myopic}} \;=\; \frac{\lambda_t}{\sigma_t^S} \, \frac{E_t^Q[\gamma_{kT}^{-1} W_{kT}]}{E_t^Q[W_{kT}]}.$$

Representation (10) shows that the effective relative risk tolerance γ_{kt}^{-1} (i.e., the risk tolerance of the value function) is given by the wealth-weighted market price of relative risk tolerance γ_{kT}^{-1}. That is, effectively, the attitude of agent k towards risk is much more affected by his risk tolerance in good states (where his wealth is large), than by that in bad states (with small wealth).

We will denote

$$\pi_{kt}^{\mathrm{hedging}} \;=\; \pi_{kt} \;-\; \pi_{kt}^{\mathrm{myopic}}$$

and refer to it as the hedging portfolio. This is the non-myopic component of the optimal portfolio that the agent uses to hedge against (or, take advantage of) future fluctuations in the stochastic opportunity set. Combining Propositions 4.1 and 4.4, we arrive at

Theorem 4.1. *We have*

$$\pi_{kt}^{\mathrm{hedging}} \;=\; - \, \frac{1}{\sigma_t^S \, E_t^Q[W_{kT}]} \, \mathrm{Cov}_t^Q\left(\gamma^U(D_T) \, \sigma^D(t\,,\,T)\,,\, W_{kT}\left(1 \,-\, \gamma_k^{-1}(W_{kT}) \right) \right). \tag{11}$$

There is a similarity between formula (11) and the expression (7) for the volatility risk premium . The reason is that $\pi_{kt}\sigma_t^S$ is simply the volatility of the wealth process W_{kt}. The role of MPDV is played here by the myopic component $\pi_{kt}^{\mathrm{myopic}}$, which is determined by the level of risk aversion and volatility. Similarly, in complete analogy with the volatility risk premium, $\pi_{kt}^{\mathrm{hedging}}$ is determined by the co-movement of the stochastic volatility and risk aversion.

The covariance representation for the hedging portfolio allows us to apply Lemma 3.3 and determine its sign. Let

$$P_k(x) \;=\; -\frac{x\,u'''(x)}{u''(x)} \;,\;\; r_k(x) \;=\; \gamma_k^{-1}(x)$$

be the relative prudence and relative risk tolerance of agent k.

Theorem 4.2. *The following is true*

(1) If $\gamma^U(x)\,\sigma^D(x)$ is decreasing and c is procyclical, then

$$
\begin{aligned}
\pi_{kt}^{\text{hedging}} &\geq 0 && if \quad \sup_x \left(P_k(x)\,r_k(x) \right) \leq 2 \\
\pi_{kt}^{\text{hedging}} &\leq 0 && if \quad \inf_x \left(P_k(x)\,r_k(x) \right) \geq 2
\end{aligned}
\tag{12}
$$

(2) If $\gamma^U(x)\,\sigma^D(x)$ is increasing and c is countercyclical, then

$$
\begin{aligned}
\pi_{kt}^{\text{hedging}} &\geq 0 && if \quad \inf_x \left(P_k(x)\,r_k(x) \right) \geq 2 \\
\pi_{kt}^{\text{hedging}} &\leq 0 && if \quad \sup_x \left(P_k(x)\,r_k(x) \right) \leq 2 \,.
\end{aligned}
\tag{13}
$$

The above result is somewhat unexpected at first glance. Since the optimal portfolio of a log investor is always myopic, one would expect that the sign of the hedging component only depends on whether risk aversion is above or below one. However, Theorem 4.2 shows that the hedging motives depend on the properties of *three* derivatives of the utility and, consequently, on the derivative of the relative risk aversion. The intuition behind this is as follows: when relative risk aversion is not constant, the agent anticipates future stochastic fluctuations in his risk aversion and uses the non-myopic part of the portfolio to hedge against these fluctuations.

This phenomenon is also related to precautionary savings. As Kimball (1990) showed in a static, one period model, the strength of the precautionary

savings motive for an agent anticipating stochastic fluctuations in his future income is determined by the relative prudence P_k. Here, P_k plays a similar role, determining the strength of savings/investment motive for an agent, anticipating future changes in the stochastic investment opportunity set.

Note also that under the conditions of item (1) in the above proposition, in a slowly growing economy, or in a rapidly mean-reverting economy, the hedging component of the portfolio is positive for investors whose risk aversion is large relative to prudence, and negative for investors whose risk aversion is small relative to prudence.

If we adopt Arrow (1965) hypothesis that $\gamma_k(x)$ is increasing, a direct calculation shows that this holds if and only if $P_k(x) \leq \gamma_k(x) + 1$ and therefore, if $\gamma_k(x) \geq 1$, $P_k\, r_k \leq 2$ and we arrive at

Corollary 4.2. *Suppose that $\gamma_k(x) \geq 1$ and is increasing. Then,*

(1) if $\gamma^U(x)\, \sigma^D(x)$ is decreasing and c is procyclical, then

$$\pi_{kt}^{\text{hedging}} \;\geq\; 0\,; \tag{14}$$

(2) if $\gamma^U(x)\, \sigma^D(x)$ is increasing and c is countercyclical, then

$$\pi_{kt}^{\text{hedging}} \;\leq\; 0\,. \tag{15}$$

For the benchmark, power utility case, $P_k\, r_k \;=\; 1 + \gamma_k^{-1}$ and the results take a simpler form

Corollary 4.3. *Suppose that $\gamma_k = $ const. The following is true*

(1) if $\gamma^U(x)\, \sigma^D(x)$ is decreasing and c is procyclical, then

$$\begin{aligned} \pi_{kt}^{\text{hedging}} &\geq 0 \qquad if\ \gamma_k \geq 1 \\ \pi_{kt}^{\text{hedging}} &\leq 0 \qquad if\ \gamma_k \leq 1\,; \end{aligned} \tag{16}$$

(2) if $\gamma^U(x)\,\sigma^D(x)$ is increasing and c is countercyclical, then

$$
\begin{aligned}
\pi_{kt}^{\text{hedging}} &\geq 0 \qquad if\ \gamma_k \leq 1 \\
\pi_{kt}^{\text{hedging}} &\leq 0 \qquad if\ \gamma_k \geq 1\,.
\end{aligned}
\tag{17}
$$

The intuition behind Corollary 4.3 is as follows. Under the conditions of item (1), Corollary 3.3 implies that the market price of risk is countercyclical. An agent with high risk aversion $\gamma_k > 1$ has a high marginal utility $u_k'(x) = x^{-\gamma_k}$ in bad states with low D_t and has a high valuation of any additional unit of consumption in those states. Since the market price of risk in bad states is high, stock is a highly attractive instrument for agent k to hedge against those states. On the other hand, an agent with low risk aversion $\gamma_k < 1$ is not "afraid" of the bad states and bets on the realization of good states with high D_t. However, since the market price of risk is low in those states, he sells some of his stock holdings to hedge against low market price of risk risk. The arguments reverse for item (2).

Example 4.1. We consider the setting of Example 3.1, when the log-dividend is a Gaussian mean-reverting process. We have, then, from the above results,

Proposition 4.5. *Suppose that γ^U is decreasing. Then,*

(1) optimal portfolios are increasing in risk aversion (in the sense of Ross) for risk aversion above one;

(2) the hedging portfolio of an agent k is positive (negative) if $P_k(x)\,r_k(x) \leq 2\ (\geq 2)$.

Again, the results can be extended to the case when $g(\log D_t)$ is Gaussian mean-reverting for some monotone increasing function g.

5 Conclusions

We consider equilibrium in a continuous time economy, populated by hetero-geneous agents maximizing expected utility from terminal wealth. We obtain representations of the equilibrium market price of risk , drift and volatility, as well as of the optimal portfolios, in terms of expected values under the risk-neutral measure. The equilibrium values depend on the aggregate relative risk aversion and dividend volatility, discounted by a specific discount factor, called the rate of discounted volatility. In special cases, this rate is equal either to the mean-reversion rate, or to the growth rate of the log-dividend process. Using the obtained representations, we derive results on the size of the risk premia and stock volatility, as well as on the size of optimal portfolios relative to the associated risk aversion. It would be of interest to extend these results to agents who consume throughout the period, and/or to agents who also differ in their beliefs regarding the dividend process. We leave such extensions for future research. In a different direction, it would be interesting to test empirically the following predictions from our model: in a slowly growing economy, or in a rapidly mean-reverting economy, the market price of risk is counter-cyclical, the risky asset volatility is high, and the hedging component of the portfolio is positive for investors whose risk aversion is large relative to prudence, and negative for investors whose risk aversion is small relative to prudence.

Appendix

A Proofs: Equilibrium Price Dynamics

Denote by $\mathcal{D}_t$ the Malliavin derivative operator. [13] The following is the main technical result of the paper.

Proposition A.1. *The drift and volatility of the stock price are given by*

$$
\begin{aligned}
\mu_t^S &= r + \sigma_t^S \frac{E_t[(\gamma^U(D_T)M_T\mathcal{D}_tD_T)/D_T]}{E_t[M_T]} \\
\sigma_t^S &= \frac{1}{E_t[M_T\,D_T]}E_t[(1-\gamma_U(D_T)M_T\mathcal{D}_tD_T)] - \frac{1}{E_t[M_T]}E_t[\mathcal{D}_tM_T]
\end{aligned}
\tag{18}
$$

and the optimal portfolio of agent k is given by

$$
\pi_{kt} = \frac{1}{\sigma_t^S}\frac{E_t\left[\mathcal{D}_tM\left(y_kMI_k'(y_k\,M_T)+I_k(y_k\,M_T)\right)\right]}{E_t[MI_k(y_k\,M_T)]} - \frac{1}{\sigma_t^S}\frac{E_t[\mathcal{D}_tM_T]}{E_t[M_T]}
$$

where

$$
\mathcal{D}_tD_T = D_t\sigma^D(D_t)e^{\delta_T-\delta_t}
\tag{19}
$$

with

$$
\begin{aligned}
\delta_t &= \int_0^t[D_s(\mu^D)'(D_s)-0.5(D_s(\sigma^D)'(D_s))^2-D_s(\sigma^D)'(D_s)\sigma^D(D_s)]ds \\
&\quad + \int_0^t D_s(\sigma^D)'(D_s)dB_s
\end{aligned}
\tag{20}
$$

and

$$
\mathcal{D}_tM_T = -\frac{1}{D}\gamma^U(D_T)M_T\mathcal{D}_tD_T\,.
$$

Proof of Proposition A.1. Recall that price S_t and the wealth of agent k satisfy

$$
\log S_t = r(t-T) + \log E_t[M_T\,D_T] - \log E_t[M_T]
$$

$$
\log W_{kt} = \log E_t\left[MI_k(y_k\,M_T)\right] - \log E_t[M_T]\,.
$$

We get the volatility σ_t^S as the Malliavin derivative $\mathcal{D}_t\log S_t$ and we get $\sigma_t^S\pi_{kt}$

---

[13]For an expedient introduction to Malliavin derivatives see Detemple, Garcia and Rindisbacher (2003).

as the Malliavin derivative $\mathcal{D}_t \log W_{kt}$. Thus, we have

$$\pi_{kt} = \frac{\mathcal{D}_t \log W_{kt}}{\mathcal{D}_t \log S_t} . \tag{21}$$

We will now calculate the Malliavin derivatives. For process D, it is well known that the Malliavin derivative

$$Y_t := \mathcal{D}_t D_u , \quad u \geq t$$

satisfies the linear SDE

$$dY_u = (D_u(\mu^D)'(D_u) + \mu^D(D_u))Y_u du + (D_u(\sigma^D)'(D_u) + \sigma^D(D_u))Y_u dB_u , \quad u \geq t$$

$$Y_t = D_t \sigma^D(D_t)$$

and (19) follows. Using this and (4), we can compute

$$\mathcal{D}_t M_T = U''(D_T)\mathcal{D}_t D . \tag{22}$$

Using the identity

$$\mathcal{D}_t E_t[X] = E_t[\mathcal{D}_t X]$$

we can compute

$$\begin{aligned}
&\mathcal{D}_t \log W_{\gamma t} \\
&= \frac{1}{E_t[M_T I_k(y_k M_T)]} E_t \left[y_k M_T I_k'(y_k M_T)\mathcal{D}_t M_T + I_k(y_k M_T)\mathcal{D}_t M_T \right] - \frac{E_t[M_T]}{E_t[\mathcal{D}_t M_T]}
\end{aligned} \tag{23}$$

and

$$\mathcal{D}_t \log S_t = \frac{1}{E_t[M_T D_T]} E_t[D\mathcal{D}_t M_T + M_T \mathcal{D}_t D] - \frac{1}{E_t[M_T]} E_t[\mathcal{D}_t M_T]. \tag{24}$$

It remains to show the expression for the drift. By the martingale property,

we can write,

$$\frac{dE_t[M_T\,D_T]}{E_t[M_T\,D_T]} \;=\; U_t dW_t \;,\; \frac{dE_t[M_T]}{E_t[M_T]} \;=\; V_t dW_t$$

where, by Clarke-Ocone formula and (22),

$$U_t \;=\; \frac{\mathcal{D}_t E_t[M_T\,D_T]}{E_t[M_T\,D_T]} \;=\; \frac{1}{E_t[M_T\,D_T]} E[M_T(1-\gamma_U(D_T))\mathcal{D}_t D_T]$$

and

$$V_t \;=\; \frac{\mathcal{D}_t E_t[M_T]}{E_t[M_T]} \;=\; -\,\frac{E_t[(\gamma^U(D_T)M_T\mathcal{D}_t D_T)/D_T]}{E_t[M_T]}\,.$$

Applying Ito's formula, we get

$$d\log S_t \;=\; rdt \;+\; d\log \frac{E_t[M_T\,D_T]}{E_t[M_T]} \;=\; \frac{1}{2}(2r \;+\; V_t^2 - U_t^2)dt \;+\; (U_t - V_t)dW_t.$$

Therefore,

$$\mu_t^S \;=\; r \;+\; \frac{1}{2}(V_t^2 - U_t^2 + (U_t - V_t)^2) \;=\; r \;+\; V_t\,(V_t - U_t)$$

and thus

$$\mu_t^S \;=\; r \;+\; \frac{E_t[(\gamma^U(D_T)M_T\mathcal{D}_t D_T)/D_T]}{E_t[M_T]} \times$$
$$\left(\frac{E_t[(\gamma^U(D_T)M_T\mathcal{D}_t D_T)/D_T]}{E_t[M_T]} + \frac{E[M_T(1-\gamma_U(D_T))\mathcal{D}_t D_T]}{E_t[M_T\,D_T]} \right) \quad (25)$$

Q.E.D.

The following result allows us to rewrite the Malliavin derivative $\mathcal{D}_t D$ without involving stochastic integrals. It has also been proved by Detemple, Garcia and Rindisbacher (2003) in a slightly different form, but we present a derivation here for the reader's convenience.

Lemma A.1. *We have*

$$\mathcal{D}_t D_T \;=\; D_T\,\sigma^D(D_T)\,e^{-\int_t^T c(D_s)\,ds}\,. \quad (26)$$

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Proof. By Ito's formula,

$$\log(D_T \, \sigma^D(D_T)) \; - \; \log(D_t \, \sigma^D(D_t)) \; = \; \int_t^T \left(\sigma^D(D_s) + D_s \, (\sigma^D)'(D_s)\right) dB_s$$

$$+ \int_t^T \left((\sigma^D(D_s) + D_s \, (\sigma^D)'(D_s)) \, \sigma^D(D_s)^{-1} \, \mu^D(D_s)\right) ds$$

$$+ \frac{1}{2} \int_t^T \left((2(\sigma^D)'(D_s) + D_s(\sigma^D)''(D_s)) \, \sigma^D(D_s)\right.$$

$$\left. - (\sigma^D(D_s) + (D_s \, (\sigma^D)'(D_s))^2)\right) ds \quad (27)$$

and the claim follows. Q.E.D.

Proof of Theorem 3.1. The proof follows directly by substituting (26) into (18). Q.E.D.

Proof of Lemma 3.1. By (26),

$$\mathcal{D}_t(f(D_T)) \; = \; f'(D_T) \, D_T \, \sigma^D(D_T) \, e^{-\int_t^T c^D(D_s)ds}$$

On the other hand, from

$$D_t \; = \; g(\tilde{D}_t),$$

and (26), applied to the process $\tilde{D}_t$, we get

$$\mathcal{D}_t(f(D_T)) \; = \; \mathcal{D}_t(f(g(\tilde{D}_T)))$$

$$= \; f'(D_T) \, g'(\tilde{D}_T) \, \mathcal{D}_t \tilde{D}_T \; = \; f'(D_T) \, g'(\tilde{D}_T) \, \tilde{D}_T \, \sigma^{\tilde{D}}(\tilde{D}_T) \, e^{-\int_t^T c^{\tilde{D}}(\tilde{D}_s) \, ds}. \quad (28)$$

By Ito's formula, comparing the diffusion terms, we get

$$D_T \, \sigma^D(D_T) \; = \; g'(\tilde{D}_T) \, \tilde{D}_T \, \sigma^{\tilde{D}}(\tilde{D}_T)$$

and therefore

$$c^D(D_s) \; = \; c^{\tilde{D}}(\tilde{D}_s),$$

which is what had to be proved. Q.E.D.

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Proof of Corollary 3.2. Pick any $x_0 > 0$. Let

$$F(x) = \int_{x_0}^x \frac{1}{y\,\sigma(y)}\,dy.$$

It is easily verified that the diffusion $F(D_t)$ has the dynamics of the form

$$d(F(D_t)) = \mu^F(D_t)\,dt + dB_t$$

and, therefore, $X_t = e^{F(D_t)}$ has the dynamics

$$X_t^{-1}\,dX_t = \mu^X(X_t)\,dt + dB_t$$

By Lemma 3.1, $c^D =$ const if and only if

$$c^X = -x\,(\mu^X)'(x) = \text{const}.$$

That is, $\mu^X(x) = \tilde{a} - b\log x$, and $A_t = \log X_t = F(D_t)$ has the dynamics

$$dA_t = (\tilde{a} - 0.5 - b\,Y_t)\,dt + dB_t,$$

and the claim follows. Q.E.D.

We will need the following known

Lemma A.2. *For any one-dimensonal diffusion, the function*

$$G(t,\,x) = E[g(D_T)\,|\,D_t = x]$$

is monotone increasing (decreasing) in x for all $t \in [0, T]$ if and only if so does $g(x)$.

Furthermore, if both $g(x)$ and $h(x)$ are increasing (or both decreasing), then

$$E_t[g(D_T)]\,E_t[h(D_T)] \leq E_t[g(D_T)\,h(D_T)].$$

If both g, h are strictly increasing (or both strictly decreasing), then the inequality is also strict unless D_T is constant almost surely. If one function is increasing and the other is decreasing, then the inequality reverses.

See, Herbst and Pitt (1991).

Lemma A.3. *Suppose that F and $G_1, \cdots, G_N$ are monotone increasing functions. Then, for any $N \in \mathbb{N}$ and any $\{t_1 \leq \cdots t_N\} \subset [t, T]$,*

$$E[F(X_T)\, G_1(X_{t_1})\, \cdots\, G_N(X_{t_N}) \,|\, X_t = x]$$

is monotone increasing in x and

$$E_t[F(X_T)\, G_1(X_{t_1})\, \cdots\, G_N(X_{t_N})] \;\geq\; E_t\,[F(X_T)]\; E_t\,[\,G_1(X_{t_1})\, \cdots\, G_N(X_{t_N})\,]\,.$$

Proof. The proof is by induction. For $N = 1$, we have

$$E_t[F(X_T)\, G_{t_1}(X_{t_1})] \;=\; E_t[\, E_{t_1}[F(X_T)]\, G_{t_1}(X_{t_1})].$$

By Lemma A.2, the function inside the expectation is increasing in X_{t_1} and another application of Lemma A.2 provides monotonicity of $E_t[F(X_T)\, G_{t_1}(X_{t_1})]$. Now, by Lemma A.2,

$$
\begin{aligned}
E_t[\, E_{t_1}[F(X_T)]\, G_1(X_{t_1})] \;&\geq\; E_t[\, E_{t_1}[F(X_T)]\,]\, E_t[G_{t_1}(X_{t_1})] \\
&=\; E_t[F(X_T)]]\, E_t[G_{t_1}(X_{t_1})] \quad (29)
\end{aligned}
$$

and we are done. Suppose now that the claim has been proved for N. Then,

$$
\begin{aligned}
E_t[F(X_T)\, G_1(X_{t_1})\, \cdots\, G_N(X_{t_N})] \\
=\; E_t[\, G_1(X_{t_1})\, E_{t_1}[\, F(X_T)\, G_2(X_{t_2})\, \cdots\, G_N(X_{t_N})]\,] \\
\geq\; E_t[E_{t_1}[\, F(X_T)\,]\, E_{t_1}[G_1(X_{t_1})\, G_2(X_{t_2})\, \cdots\, G_N(X_{t_N})]\,] \quad (30)
\end{aligned}
$$

and the claim follows from Lemma A.2 and the induction hypothesis. Q.E.D.

Proof of Lemma 3.3.. The claim follows from Lemma A.3, approximating the integral $\int_t^T$ by discrete integral sums. Q.E.D.

Proof of Theorem 3.2. The proof follows directly from Proposition A.1 and (26). Cyclicality of $S_t = S(t, D_t)$ follows from Lemma 3.3 and the identity

$S_t = e^{r(t-T)} E_t^Q[D_T]$. The fact that $\sigma_t^S > 0$ follows from

$$\sigma_t^S = \frac{\partial}{\partial D_t} S(t, D_t) \sigma^D(D_t).$$

Alternatively, by Clarke-Ocone formula, $S_t \sigma_t^S$ can be obtained as the conditional expectation (under Q) of the Malliavin derivative of D_T, which is positive. Q.E.D.

Proof of Proposition 3.2. We have

$$\lambda_t = e^{\int_0^t c(D_s)ds} \frac{E_t[-U''(D_T) \sigma^D(D_T) e^{-\int_0^T c(D_s)ds}]}{E_t[U'(D_T)]}.$$

By the Clarke-Ocone formula,

$$d E_t[U'(D_T)] = dE_t[M_T] = E_t[\mathcal{D}_t M_T] dB_t$$

and

$$d E_t[U''(D_T) \sigma^D(D_T) e^{-\int_0^T c(D_s)ds}]$$
$$= \left(E_t \left[U'''(D_T) \sigma^D(D_T) + U''(D_T) (\sigma^D)'(D_T) \mathcal{D}_t D_T e^{-\int_0^T c(D_s)ds} \right] \right.$$
$$\left. - E_t \left[\int_t^T c'(D_s) (\mathcal{D}_t D_s) ds U''(D_T) \sigma^D(D_T) e^{-\int_0^T c(D_s)ds} \right] \right) dB_t \quad (31)$$

The claim follows from Lemma A.1 and Ito's formula and the following identity

$$E_t^Q \left[\int_t^T e^{-\int_t^\theta 2c(D_r)dr} c'(D_\theta) \lambda_\theta D_\theta \sigma^D(D_\theta) d\theta \right]$$
$$= E_t^Q \left[\int_t^T e^{-\int_t^\theta 2c(D_r)dr} c'(D_\theta) E_\theta^Q[\gamma^U(D_T) \sigma^D(\theta, T)] D_\theta \sigma^D(D_\theta) d\theta \right] =$$
$$= E_t^Q \left[\gamma^U(D_T) \sigma^D(t, T) \int_t^T c'(D_\theta) D_\theta \sigma^D(t, \theta) d\theta \right]. \quad (32)$$

Q.E.D.

B Proofs: Optimal Portfolios

Proof of Proposition 4.1. The claim follows directly from Proposition A.1 and (26). Since, by Ito's formula,

$$\pi_{kt} = \frac{\partial}{\partial D_t} W_{kt}(D_t) \sigma^D(D_t),$$

we need to show that W_{kt} is monotone increasing in D_t. But,

$$W_{kt} = e^{-r(T-t)} E_t^Q[I_k(y_k M_T)]$$

and the claim follows from Lemma 3.3.   Q.E.D.

Proof of Proposition 4.3. By (9), we need to show that

$$\frac{E_t^Q \left[\gamma^U(D_T) \sigma^D(D_T) e^{-\int_t^T c(D_s)ds} W_{kT} \left(1 - \gamma_k^{-1}(W_{kT}) \right) \right]}{E_t^Q[W_{kT}]}$$

$$\geq \frac{E_t^Q \left[\gamma^U(D_T) \sigma^D(D_T) e^{-\int_t^T c(D_s)\,ds} W_{jT} \left(1 - \gamma_j^{-1}(W_{jT}) \right) \right]}{E_t^Q[W_{jT}]}. \quad (33)$$

We only prove case (1). Case (2) is analogous. Since, by assumption,

$$\inf \left(1 - \gamma_{kT}^{-1} \right) \geq \sup \left(1 - \gamma_{jT}^{-1} \right),$$

it suffices to show that

$$\frac{E_t^Q \left[\gamma^U(D_T) \sigma^D(D_T) e^{-\int_t^T c(D_s)ds} W_{kT} \right]}{E_t^Q[W_{kT}]} \geq \frac{E_t^Q \left[\gamma^U(D_T) \sigma^D(D_T) e^{-\int_t^T c(D_s)ds} W_{jT} \right]}{E_t^Q[W_{jT}]}.$$

Introduce a new probability measure

$$dQ^k = \frac{W_{kT}}{E^Q[W_{kT}]} dQ$$

and let

$$f(x) = \frac{I_j(y_j x)}{I_k(y_k x)}.$$

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and $z_i = I_i(\lambda_i x)$, $i \in \{j, k\}$. Then,

$$
\begin{aligned}
f'(x) &= \frac{I_j(y_j x)}{x \, I_k(y_k x)} \left(y_j \, x \, \frac{I'_j(y_j x)}{I_j(y_j x)} - y_k \, x \, \frac{I'_k(y_j x)}{I_k(y_k x)} \right) \\
&= \frac{I_j(y_j x)}{x \, I_k(y_k x)} \left(\frac{u'_j(z_1)}{z_1 \, u''_j(z_1)} - \frac{u'_k(z_2)}{z_2 \, u''_k(z_2)} \right) \\
&= \frac{z_1}{x \, z_2} \left(\gamma_k^{-1}(z_2) - \gamma_j^{-1}(z_1) \right) \leq 0, \quad (34)
\end{aligned}
$$

that is, f is decreasing. Therefore, $f(U'(D_T))$ is increasing and, by Lemma 3.3,

$$
\begin{aligned}
&\frac{E_t^Q \left[\gamma^U(D_T) \, \sigma^D(D_T) \, e^{-\int_t^T c(D_s) ds} \, W_{jT} \right]}{E_t^Q [W_{jT}]} \\
&\quad = \frac{E_t^{Q_k} \left[\gamma^U(D_T) \, \sigma^D(D_T) \, e^{-\int_t^T c(D_s) ds} \, f(U'(D_T)) \right]}{E_t^{Q_k} [f(U'(D_T))]} \\
&\leq E_t^{Q_k} \left[\gamma^U(D_T) \, \sigma^D(D_T) \, e^{-\int_t^T c(D_s) ds} \right] = \frac{E_t^Q \left[\gamma^U(D_T) \, \sigma^D(D_T) \, e^{-\int_t^T c(D_s) ds} \, W_{kT} \right]}{E_t^Q [W_{kT}]}
\end{aligned}
$$

$$(35)$$

Q.E.D.

Proof of Proposition 4.4. Let for simplicity $t = 0$. By definition, the value function is

$$
U_k(x) = E[u_k(I_k(M_T \, y_k))]
$$

where $y_k = y_k(x)$ solves

$$
x = E[M_T \, I_k(M_T \, y_k)].
$$

Differentiating this identity, we get

$$
y'_k = E[M_T^2 \, I'_k(M_T \, y_k)]^{-1}
$$

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and therefore

$$U_k'(x) \;=\; E\left[u_k'(I_k(M_T\,y_k))\,I_k'(M_T\,y_k)\,M_T\,\right]\lambda')k \;=\; E\left[\,M_T\,y_k\,I_k'(M_T\,y_k)\,M_T\,\right]y_k' \;=\; y_k.$$

Consequently,

$$U_k''(x) \;=\; y_k' \;=\; \frac{1}{E[M_T^2\,I_k'(M_Ty_k)]}$$

and

$$\gamma_{k0}(x) = = \;-\frac{x}{y_k\,E[M_T^2\,I_k'(M_Ty_k)]} \;=\; -\frac{E[M_T\,I_k(M_Ty_k)]}{E[M_T^2\,y_k\,I_k'(M_Ty_k)]}$$

Differentiating the identity

$$u_k'(I_k(x)) \;=\; x$$

we get

$$I_k'(x) \;=\; (u_k''(x))^{-1}$$

and therefore

$$y_k M_T\,I_k'(y_k M_T) \;=\; -\,\gamma_{kT}^{-1}W_{k\,T}.$$

The second claim follows from Theorem 3.1. Q.E.D.

Proof of Theorem 4.1. By Propositions 4.1 and 4.4,

$$
\begin{aligned}
\sigma_t^S\,\pi_{k\,t}^{\text{hedging}} \;&=\; \sigma_t^S\left(\pi_{k\,t}\;-\;\pi_{k\,t}^{\text{myopic}}\right)\\
&=\; \frac{E_t^Q\left[\gamma^U(D_T)\,\sigma^D(t\,,\,T)\,W_{k\,T}\,\left(\gamma_{kT}^{-1}-1\right)\right]}{E_t^Q[W_{kT}]} \;+\; E_t^Q[\gamma^U(D_T)\,\sigma^D(t\,,\,T)]\\
&\qquad\qquad -\; E_t^Q[\gamma^U(D_T)\,\sigma^D(t\,,\,T)]\frac{E_t^Q[\gamma_{kT}^{-1}\,W_{k\,T}]}{E_t^Q[W_{k\,T}]}\\
&=\; \frac{1}{E_t^Q[W_{k\,T}]}\Big(E_t^Q\left[\gamma^U(D_T)\,\sigma^D(t\,,\,T)\,W_{k\,T}\,\left(\gamma_{kT}^{-1}-1\right)\right]\\
&\qquad\qquad -\; E_t^Q[\gamma^U(D_T)\,\sigma^D(t\,,\,T)]\,E_t^Q[(\gamma_{kT}^{-1}-1)\,W_{k\,T}]\Big) \quad (36)
\end{aligned}
$$

which is what had to be proved. Q.E.D.

Proof of Theorem 4.2. The claim follows from Theorem 4.1 and Lemma 3.3

since

$$f(x) \;=\; x \,-\, r_k(x)$$

is increasing if and only if

$$f'(x) \;=\; 1 \,+\, \frac{(u_k'')^2 \,-\, u_k'(x)\,u_k'''(x)}{(u_k''(x))^2} \;=\; \frac{1}{(u_k''(x))^2}\,(2 \,-\, P_k(x)\,r_k(x)) \;\geq\; 0$$

and is decreasing otherwise. Q.E.D.

References

[1] Arrow, Kenneth J. (1965): *Aspects of the Theory of Risk-Bearing.* Helsinki: Yrjö Hahnsson Foundation.

[2] Basak, Suleyman (2005): "Asset Pricing with Heterogeneous Beliefs," *Journal of Banking and Finance,* **29**, 2849-2881.

[3] Basak, Suleyman and Domenico Cuoco (1998): "An Equilibrium Model with Restricted Stock Market Participation," *Review of Financial Studies,* **11**, 309-341.

[4] Bhamra, Harjoat S. and Raman Uppal (2009): "The Effect of Introducing a Non-Redundant Deerivative on the Volatility of Stock-Market Returns When Agents Differ in Risk Aversion", *forthcoming in Review of Financial Studies.*

[5] Benninga, Simon and Joram Mayshar (2000): "Heterogeneity and Option Pricing," *Review of Derivatives Research,* **4(1)**, 7-27.

[6] Berrada, Tony, Julien Hugonnier and Marcel Rindisbacher (2007): "Heterogenous Preferences and Equilibrium Trading Volume," *Journal of Financial Economics,* **83,** 719-750.

[7] Biais, Bruno, Bossaerts, Peter and Chester Spatt (2009): "Equilibrium Asset Pricing And Portfolio Choice Under Asymmetric Information," *forthcoming in The Review of Financial Studies.*

[8] Bick, Avi (1990): "On Viable Diffusion Price Processes of the Market Portfolio," *Journal of Finance,* **15,** 673-689.

[9] Campbell, John Y. and John Cochrane (1999): "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy,* **107,** 205-251.

[10] Cvitanić, Jaksa and Semyon Malamud (2009a): "Relative Extinction of Heterogeneous Agents," *working paper.*

[11] Cvitanić, Jaksa and Semyon Malamud (2009b): "Risk Aversion and Equilibrium Optimal Portfolios in Large Markets", *working paper.*

[12] Dana, Rose-Anne (1995): "An Extension of Milleron, Mitjushin and Polterovich's Result," *Journal of Mathematical Economics,* **24(3)**, 259-269.

[13] Dana, Rose-Anne (2001): "Risk Aversion and Uniqueness of Equilibrium. An Application to Financial Markets," *Review of Economic Design,* **6**, 155-173 (special issue in the honor of R. Radner, edited by Ischiishi.T. et Marshak T).

[14] Detemple, Jerome B., Garcia, Rene and Marcel Rindisbacher (2003): "A Monte Carlo Method for Optimal Portfolios," *The Journal of Finance,* **58**, 401-446.

[15] Detemple, Jerome B. and Fernando Zapatero (1991): "Asset Prices

in an Exchange Economy with Habit Formation," *Econometrica,* **59**, 1633-1657.

[16] Duffie, Darrell and Chi-Fu Huang (1986): "Implementing Arrow-Debreu Equilibria by Continuous Trading of Few Long-Lived Securities," *Econometrica,* **53(6)**, 1337-1356.

[17] Dumas, Bernard (1989): "Two-Person Dynamic Equilibrium in the Capital Market," *The Review of Financial Studies*, **2**, 157-188.

[18] Fama, Eugene F. and Kenneth R. French (1989): "Business Conditions and Expected Returns on Stock and Bonds," *Journal of Financial Economics,* **25**, 23-49.

[19] Ferson, William E. and Campbell R. Harvey (1991): "The variation of economic risk premiums," *Journal of Political Economy,* **99**, 385-415.

[20] French, Mark W. and Daniel E. Sichel (1993): "Cyclical Patterns in the Variance of Economic Activity," *Journal of Business and Economic Statistics,* **11(1)**, 113-119.

[21] Gollier, Christian (2001): "Wealth Inequality and Asset Pricing," *Review of Economic Studies,* **68**, 181-203.

[22] Herbst, Ira and Lauren Pitt (1991): "Diffusion Equation Techniques in Stochastic Monotonicity and Positive Correlations," *Probab. Th. Rel. Fields,* **87**, 275-312.

[23] Hara, Chiaki, Huang, James, and Christoph Kuzmics (2007): "Representative Consumers Risk Aversion and Efficient Risk-Sharing Rules," *Journal of Economic Theory,* **137**, 652-672.

[24] Jouini, Elyes and Clotilde Napp (2008): "Consensus Consumer and Intertemporal Asset Pricing with Heterogeneous Beliefs," *forthcoming in the Review of Financial Studies.*

[25] Kimball, Miles S. (1990): Precautionary Saving in the Small and in the Large, *Econometrica,* **58** 53-73.

[26] LeRoy, Stephen F. and Robert D. Porter (1981): "The Present-Value Relation: Tests Based on Implied Variance Bounds, *Econometrica,* **49**, 555-574.

[27] Malamud, Semyon (2008): "Long Run Forward Rates and Long Yields of Bonds and Options in Heterogeneous Equilibria," *Finance and Stochastics,* **12**, 245-264.

[28] Mankiw, Greory N., Romer, David and Matthew D. Shapiro (1985): An Unbiased Reexamination of Stock Market Volatility, *Journal of Finance,* **40**, 677-687.

[29] Mankiw, Gregory N., Romer, Daviw and Matthew D. Shapiro (1991): Stock Market Forecastability and Volatility: A Statistical Appraisal, *Review of Economic Studies,* **58**, 455-477.

[30] Mele, Antonio (2007): "Asymmetric Stock Market Volatility and the Cyclical Behavior of Expected Returns", *Journal of Financial Economics,* **86**, 446-478.

[31] Merton, Robert (1971): "Optimum Consumption and Portfolio Rules in a Continuous Time Model," *Journal of Economic Theory*, **3**, 373-413.

[32] Ross, Stephen A. (1981): "Some Stronger Measures of Risk Aversion in the Small and in the Large with Applications", *Econometrica,* **49(3)**, 621-639.

[33] Shiller, Robert J. (1981): "Do Stock Prices Move Too Much to be Justified by Subsequent Changes in Dividends?, *American Economic Review,* **71**, 421-436.

[34] Wang, Jiang (1996): "The Term Structure of Interest Rates In A Pure Exchange Economy With Heterogeneous Investors," *Journal of Financial Economics*, **41**, 75-110.

[35] Wang, Susheng (1993): "The Integrability Problem of Asset Prices," *Journal of Economic Theory,* **59(1),** 199-213.

[36] West, Kenneth D., (1988): "Dividend Innovations and Stock Price Volatility, *Econometrica*, **56**, 37-61.

[37] Wilson, Robert (1968): "The Theory of Syndicates," *Econometrica,* **36 (1)**, 119-132.