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Invariant measure for an infinite neural network.

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Abstract

We investigate the large-time limit behaviour of an infinite locally connected network. We show an exponential decay of the spatial covariances for the limit measure. We prove also the Gibbsian property for the limit measure.

1. Introduction

We study here an infinite-particle limit of a neural model [3], where it is assumed that an independent neuron can be described by some Markov process. This model takes into account the well-known facts from physiology: the spiking nature of the neuronal activity and the exponential decay of the membrane potential in the absence of afferent spikes. The interactions between the particles (neurons) in the model depend on time. Some results on the large-time behaviour of this model with excitatory connections one can find in [5]. The simulations carried out in [3], provide good evidence that this model explains a wide range of experimental data on existence of equilibrium and non-equilibrium phase transitions and metastable states. The theoretical proof in [3] of the existence of phase transitions in the considered model is based on the earlier result [2], which establishes the Gibbsian form for the invariant measure under certain conditions. Although there were given some arguments on the feasibility of these conditions, they are difficult to check.

The model described in [3] is non-Markovian. The notion of a process of inhibitions, which is a certain embedded Markov chain for this model, was developed in [6]. The inhibition of a neuron represents the duration of time before the first moment of firing of this neuron if no interaction takes place meanwhile. Note, that a process of inhibitions had been introduced and studied earlier in the literature as a simplified model for the neuronal activity (e.g., [1]).

Recently ergodicity of the infinite-particle limit of the process of inhibitions with local connections was proved in [7]. The provided proof is constructive, and is based on the method of cluster expansions [4]. Using cluster expansions, obtained in [7], we derive here some results on the invariant measure of the infinite-particle process of inhibitions.

The purpose of this paper is to describe the large-time limit measure for the infinite process of inhibitions. We show in particular, the exponential decay of the spatial covariances.

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Further we will establish the Gibbsian property of the invariant measure for the considered process. It is known (see, for example, [4]), that Gibbs random field is not unique for certain parameters. Hence, our results give new evidence of the existence of phase transitions in the considered neural model for some parameters. This together with previous results [2]-[3] gives us hope for the development of an adequate stochastic dynamical model for neuronal activity.

2. Model and results

Let us fix $K > 0$ and a finite set $\{l_k, k = 1, \dots, K\} \in Z^\nu \backslash \{0\}$ arbitrarily, and define for any $z \in Z^\nu$ its neighbourhood:

$$O(z) := \{z + l_k, k = 1, \dots, K\}.$$

We define on some probability space independent random positive variables $X_z(0)$, $z \in \mathbf{Z}^\nu$. The density of the distribution of $X_z(0)$ is

$$p_{0z}(u) = 2\alpha^{3/2} e^{2\alpha u} \frac{(y - a_z)}{\sigma\sqrt{\pi(e^{2\alpha u} - 1)^3}} \exp\left(-\frac{\alpha(y - a_z)^2}{\sigma^2(e^{2\alpha u} - 1)}\right)$$

for any $z \in \Lambda$ and $u \in \mathbf{R}_+$, where $\alpha > 0$, $\sigma > 0$, $y > 0$ and $a_z \leq y$ are the parameters of the model.

We define also on the same probability space mutually independent positive variables Y and $\theta(u)$, $u > 0$, with the following densities of the distributions, respectively:

$$p(v) = 2\alpha^{3/2} e^{2\alpha v} \frac{y}{\sigma\sqrt{\pi(e^{2\alpha v} - 1)^3}} \exp\left(-\frac{\alpha y^2}{\sigma^2(e^{2\alpha v} - 1)}\right),$$

and

$$p_{\theta(u)}(v) = \frac{2\alpha^{3/2} a}{\sigma\sqrt{\pi}} e^{-\alpha u} \frac{e^{2\alpha v}}{\sqrt{(e^{2\alpha v} - 1)^3}} \exp\left(-\frac{\alpha a^2 e^{-2\alpha u}}{\sigma^2(e^{2\alpha v} - 1)}\right),$$

$v > 0$, where $a > 0$ is a connection constant for our model (clearly, if $a = 0$ then $\theta(u) \equiv 0$).

Consider now Markov process $X(t) = (X_z(t), z \in \mathbf{Z}^\nu) \in R_+^{\mathbf{Z}^\nu}$, $t > 0$, with initial state $X(0) = (X_z(0), z \in \mathbf{Z}^\nu)$. The dynamics of the process $X(t)$ is the following. As long as all the components of $X(t)$ are strictly positive, they decrease from the initial configuration $X(0)$ linearly in time with rate one until the first time t_{z_1} that one of the components reaches zero for some $z_1 \in \mathbf{Z}^\nu$. At this moment t_{z_1} the trajectory $X_{z_1}(t_{z_1})$ jumps to a random value, which is an independent copy of Y, while every trajectory $X_j(t)$ with $j \in O(z_1)$ receives a positive increment $\theta(X_j(t_{z_1}))$ independent of the other processes. The rest trajectories $X_i(t)$ with $i \notin O(z_1)$, remain continuous at $t = t_{z_1}$. After moment t_{z_1} the dynamics are repeated.

The existence of the infinite-dimensional process $X(t), t \geq 0$, has been proved in [7]. As it is shown in [6], the z-th component of the process $X(t)$ represents the inhibition of the z-th neuron at time t in the original model [3].

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For any finite subset $A \subset \mathbf{Z}^\nu$ let us denote by $p(t, v_A)$, $v_A \in \mathbf{R}_+^A$, $t > 0$, a density of the distribution of the process $(X_z(t),\ z \in A)$, i.e.,

$$\mathbf{P}\{(X_z(t),\ z \in A) \in B\} = \int_B p(t, v_A) dv_A$$

for any $t > 0$ and any Borel set B in $\mathbf{R}_+^A$. (We use in this paper the following notations: $u_A = (u_z,\ z \in A)$ for any $A \subset \mathbf{Z}^\nu$.) In particular, we denote $p_0(t, v_A)$, $t > 0$, $v_A \in \mathbf{R}_+^A$, the density of an "unperturbed" process $X^0(t) = (X_z^0(t), z \in \mathbf{Z}^\nu)$ (i.e., with $a \equiv 0$). By the definition of $X^0(t)$ its coordinates $X_z^0(t)$ are independent renewal processes for different $z \in \mathbf{Z}^\nu$. More precisely, for any $z \in \mathbf{Z}^\nu$ process $X_z^0(t)$, $t \geq 0$, has left-continuous trajectories, which decrease with rate 1 until the first moment T of hitting zero, where a trajectory has a random gap:

$$\lim_{\epsilon \downarrow 0} X_z^0(T + \epsilon) =_d Y,$$

independent of the prehistory. It is not difficult to derive, that

$$p_0(t, v_A) = \prod_{z \in A} p_{0z}(t, v_z), \qquad (1)$$

with

$$p_{0z}(t, v) = p_{0z}(t + v) + \int_0^t p_{0z}(u) p(t - u + v) du +$$
$$+ \int_0^t p_{0z}(u) \left(\sum_{k=1}^{\infty} \int_0^{t-u} p_{S_k}(x) p(t - u + v - x)\, dx \right) du$$

for every $v \in \mathbf{R}_+$ and $t > 0$, where $S_k := \sum_{l=1}^k Y^l$, $k \geq 1$; Y^l, $l = 1, ..., k$, are independent copies of the variable Y, and p_{S_k} is the density of the distribution S_k. We put here $p(w) \equiv 0$ for $w \leq 0$. It is shown in [7], that for any finite t and any connection constant a a finite-particle density admits a cluster expansion, i.e., it can be written in the following form:

$$p(t, v_A) := p_0(t, v_A) + \sum_{k=1}^{\infty} \sum_{A-cluster\ z=(z_1,...,z_k)} G_z(t, k, v_A), \qquad (2)$$

where all the functions $G_z(t, k, v_A)$ in the right-hand side depend only on the finite-particle densities (1) of the "unperturbed" process $X^0(t)$, and the connection constant a. It is also proved in [7], that the series (2) converge uniformly in $t > 0$ and $v_A \in \mathbf{R}_+^A$. "An A-cluster" means here a certain finite set in $\mathbf{Z}^\nu$, connected with the set A in the topology, naturally induced by the connections between the neurons in our model. (Precise definitions one can find in [4] or [7].) Formula (2) allows one to derive the following result on the convergence of the finite-particle densities of the process $X(t)$ for some set of the connection constants.

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Theorem [7]. (a) For any $0 < q < 1$ there exist positive constants a_0, C, γ and h such that for all $\alpha \geq 1$,

$$0 < a \leq a_0 \alpha^{\frac{q}{2(1-q)}} \tag{3}$$

and for all t, $t' > 0$ and $u_A \in \mathbf{R}_+^A$

$$|p(t + t', u_A) - p(t, u_A)| \leq \alpha^{3/2} C^{|A|} e^{-\gamma \alpha t} e^{-h\alpha \sum_{x \in A} u_x}.$$

(b) For any $0 < \alpha < 1$ there exist positive constants a_1, C_1, γ_1 and h_1 such that for any

$$0 < a \leq a_1 \tag{4}$$

and for all t, $t' > 0$ and $u_A \in \mathbf{R}_+^A$

$$|p(t + t', u_A) - p(t, u_A)| \leq C_1^{|A|} e^{-\gamma_1 t} e^{-h_1 \sum_{x \in A} u_x}.$$

Making use of cluster expansions from [7], we obtain an exponential decay of the covariance functions

$$\mathbf{C}_t(x, x') := \mathbf{Cov}\{X_x(t), X_{x'}(t)\}, \ \ t > 0,$$

$x, x' \in \mathbf{Z}^\nu$, of the components of the process $X(t)$, uniformly in $t \in (0, \infty)$.

Theorem 1. For any parameters of the model such that (3) or (4) holds, there exist positive constants B and β such that

$$|\mathbf{C}_t(x, x')| \leq B e^{-\beta ||x - x'||}$$

for all $t > 0$ and $x, x' \in \mathbf{Z}^\nu$,
where $||x|| := \max_i |x_i|$ for any $x = (x_1, \ldots, x_\nu) \in \mathbf{Z}^\nu$.

Theorem [7] implies the existence and uniqueness of the following limit for the set of parameters satisfying the conditions of Theorem [7]:

$$\lim_{t \to \infty} p(t, u_A) := p(u_A) \tag{5}$$

for any finite $A \subset \mathbf{Z}^\nu$ and $u_A \in \mathbf{R}_+^A$. Thus, in this case the unique invariant measure for the process $X(t)$, $t \geq 0$, has the distribution in $\mathbf{R}^{Z^\nu}$, defined by its finite-particle densities (5). Next we will show the exponential fast decay of the dependence of the conditional densities on the values of the distant components.

Theorem 2. Let the parameters of the model satisfy the conditions (3) or (4). Then for any finite $A \subset B \subset \Lambda \in \mathbf{Z}^\nu$ there exist positive constants Γ and γ such that

$$|\, p(v_A \,|\, v_{B \setminus A}, u_{\Lambda \setminus B}) - p(v_A \,|\, v_{B \setminus A}, u'_{\Lambda \setminus B})| \leq \Gamma e^{-\gamma d(A, \Lambda \setminus B)}$$

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uniformly for any $v_B \in \mathbf{R}_+^B$ and $u_{\Lambda\setminus B}, u'_{\Lambda\setminus B} \in \mathbf{R}_+^{\Lambda\setminus B}$, where for any finite subsets A and C in $\mathbf{Z}^\nu$

$$d(A,C) := \min_{x\in A, y\in C} ||x-y||.$$

It follows from formula (2) that the conditional probabilities are strictly positive. Hence, applying results [4], we get from Theorem 2 the following corollary.

Corollary. The invariant measure for the process $X(t)$ is Gibbsian (for the parameters satisfying (3) or (4)).

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