CHARACTERIZATION OF SECOND-ORDER STRONG DIVISIBILITY SEQUENCES OF POLYNOMIALS

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1. INTRODUCTION

In [2], Kimberling posed the question of which recurrent sequences $\{t_n : n = 0, 1, 2, \dots\}$ have the property

$$\gcd(t_m, t_n) = t_{\gcd(m,n)} \quad \forall m, n \in \mathbb{N}. \tag{1}$$

A sequence with this property is called a strong divisibility sequence (SDS). For example, the Fibonacci polynomials, defined by the second-order linear recurrence $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$; $F_0(x) = 0$, $F_1(x) = 1$, is a SDS of polynomials (see [3]). This paper will present a characterization of all the second-order SDS of polynomials. The proofs are all elementary; the most advanced technique used is mathematical induction. I will not discuss the sequences that consist only of integers. For the characterization of second-order SDS of integers see [1].

2. THE SET S AND THE SUBSETS D, F, G, AND H

Let S be the set of second-order linear recurrent sequences of polynomials defined by

$$s_n(x) = p(x)s_{n-1}(x) + q(x)s_{n-2}(x); \quad s_0(x) = 0, \quad s_1(x) = 1$$

where $p(x), q(x) \in \mathbb{Z}[X]$. The subset of all of the SDS of S will be denoted by D.

Note we let $s_0(x) = 0$ because all terms of a strong divisibility sequence divide $s_0(x)$. We may also take $s_1(x) = 1$ without loss of generality because all the second-order strong divisibility sequences are obviously all the multiples of the sequences from D.

In pursuit of a description of D, consider the following subsets of S: F, F_1, G, and H; defined with the initial conditions 0 and 1.

$$\begin{aligned}
F &= \{(f_n) : f_n(x) = p(x)f_{n-1}(x) + q(x)f_{n-2}(x);\ f_0(x) = 0,\ f_1(x) = 1\}\\
F_1 &= F \text{ where } \gcd(p(x), q(x)) = 1\\
G &= \{(g_n) : g_n(x) = p(x)g_{n-1}(x);\ g_0(x) = 0,\ g_1(x) = 1\} \text{ (degenerate sequence)}\\
H &= \{(h_n) : h_n(x) = q(x)h_{n-2}(x);\ h_0(x) = 0,\ h_1(x) = 1\}.
\end{aligned}$$

There are some results which follow from defining G and H: $D \cap G = \emptyset$ and $D \cap H = \emptyset$. For the set G we can clearly see that $g_2(x) = p(x)$ and $g_3(x) = (p(x))^2$. So

$$\gcd(g_2(x), g_3(x)) = p(x) \neq 1 = g_{\gcd(2,3)}(x)$$

which contradicts (1). Similarly, consider $h_3(x)$ and $h_5(x)$ which equals $q(x)$ and $(q(x))^2$, respectively, to obtain a contradiction to (1).

3. THE SUBSETS F AND F_1

Clearly $F_1 \subset F$. We will see that $F_1 = D \cap F$ by showing $F_1 \cap D = F_1$ and $D \cap F \setminus F_1 = \emptyset$.

Theorem 1: *Let $f_n \in F$ then*

$$f_{n+k}(x) = f_{k+1}(x) f_n(x) + q(x) f_k(x) f_{n-1}(x).$$

Proof: We use strong induction on k. By definition of F,

$$f_{n+1}(x) = p(x) f_n(x) + q(x) f_{n-1}(x)$$

We see that $f_2(x) = p(x)$ and $f_1(x) = 1$ showing that the initial case is true. We assume $f_{n+k}(x) = f_{k+1}(x) f_n(x) + q(x) f_k(x) f_{n-1}(x)$ by the induction hypothesis. So it follows

$$\begin{aligned} f_{(n+k)+1}(x) &= p(x) f_{n+k}(x) + q(x) f_{(n+k)-1}(x) \\ &= p(x)[f_{k+1}(x) f_n(x) + q(x) f_k(x) f_{n-1}(x)] \\ &\qquad + q(x)[f_k(x) f_n(x) + q(x) f_{k-1}(x) f_{n-1}(x)] \\ &= f_{k+2}(x) f_n(x) + q(x) f_{k+1}(x) f_{n-1}(x). \quad \square \end{aligned}$$

Corollary 1: *Let $f_n \in F$ then*

$$m \mid n \text{ implies } f_m(x) \mid f_n(x).$$

Proof: Assume $m \mid n$ which implies $n = km$. To show $f_m(x) \mid f_{km}(x)$ we will use induction on k. $f_m(x) \mid f_{1 \cdot m}(x)$ clearly. Suppose $f_m(x) \mid f_{km}(x)$ by the induction hypothesis. So

$$f_m(x) \mid \alpha f_{km}(x) + \beta f_m(x) \quad \forall \alpha, \beta.$$

With Theorem 1, choose the appropriate α and β, $f_{m+1}(x)$ and $q(x) f_{km-1}(x)$ respectively, to yield $f_m(x) \mid f_{km+m}(x)$; moreover,

$$f_m(x) \mid f_{(k+1)m}(x). \quad \square$$

Theorem 2: *Let $f_n \in F_1$ then*

$$\gcd(f_n(x), f_{n+1}(x)) = 1.$$

Proof: We will use induction on n. $\gcd(f_1(x), f_2(x)) = 1$ since $f_1(x) = 1$. We know

$$\begin{aligned} f_{n+2}(x) &\equiv p(x) f_{n+1}(x) + q(x) f_n(x) \pmod{f_{n+1}(x)}, \text{ so} \\ f_{n+2}(x) &\equiv q(x) f_n(x) \pmod{f_{n+1}(x)}. \end{aligned}$$

Therefore $\gcd(f_{n+2}(x), f_{n+1}(x)) = \gcd(f_{n+1}(x), q(x) f_n(x))$. Notice that $\gcd(f_{n+1}(x), q(x)) = 1$ by the fact $f_n \in F_1$ giving us the property $\gcd(p(x), q(x)) = 1$. So with the induction hypothesis of $\gcd(f_{n+1}(x), f_n(x)) = 1$ and $\gcd(f_{n+1}(x), q(x)) = 1$, it follows that $\gcd(f_{n+1}(x), q(x) f_n(x)) = 1$ which yields $\gcd(f_{n+2}(x), f_{n+1}(x)) = 1$. $\square$

Corollary 2: *Let $f_n \in F_1$ then $m = qn + r$ implies*

$$\gcd(f_m(x), f_n(x)) = \gcd(f_n(x), f_r(x)).$$

Proof: Assume $m = qn + r$,

$$f_m(x) = f_{qn+r}(x) = f_{r+1}(x)f_{qn}(x) + q(x)f_r(x)f_{qn-1}(x),$$

by Theorem 1. Consider

$$\begin{aligned} f_m(x) &\equiv f_{r+1}(x)f_{qn}(x) + q(x)f_r(x)f_{qn-1}(x) \pmod{f_n(x)} \\ f_m(x) &\equiv q(x)f_r(x)f_{qn-1}(x) \pmod{f_n(x)}. \end{aligned}$$

Thus $\gcd(f_m(x), f_n(x)) = \gcd(f_n(x), q(x)f_r(x)f_{qn-1}(x))$. From Theorem 2 and Corollary 1 we see that $\gcd(f_{qn}(x), f_{qn-1}(x)) = 1$ and $\gcd(f_{qn}(x), f_n(x)) = f_n(x)$, respectively. So it follows $\gcd(f_n(x), f_{qn-1}(x)) = 1$ and since the $\gcd(f_n(x), q(x)) = 1$ we arrive at $\gcd(f_n(x), q(x)f_{qn-1}(x)) = 1$. Therefore,

$$\gcd(f_m(x), f_n(x)) = \gcd(f_n(x), q(x)f_r(x)f_{qn-1}(x)) = \gcd(f_n(x), f_r(x)). \quad \square$$

Theorem 3: *All sequences in F_1 are in D.*

Proof: Let $f_n \in F_1$ and consider the use of the Euclidean algorithm in conjunction with Corollary 2. We can see

$$\begin{array}{rcl} m = q_0 n + r_1 & n > r_1 \geq 0 \Rightarrow & \gcd(f_m(x), f_n(x)) = \gcd(f_n(x), f_{r_1}(x)) \\ n = q_1 r_1 + r_2 & r_1 > r_2 \geq 0 \Rightarrow & \gcd(f_n(x), f_{r_1}(x)) = \gcd(f_{r_1}(x), f_{r_2}(x)) \\ & \vdots & \\ r_{k-1} = q_k r_k + 0 & \Rightarrow & \gcd(f_{r_{k-1}}(x), f_{r_k}(x)) = \gcd(f_{r_k}(x), f_0(x)) = f_{r_k}. \end{array}$$

With $\gcd(m, n) = r_k$ and $\gcd(f_m(x), f_n(x)) = f_{r_k}(x)$, it follows

$$\gcd(f_n(x), f_m(x)) = f_{r_k}(x) = f_{\gcd(n,m)}(x)$$

for all sequences $f_n \in F_1$. $\square$

Theorem 4: $F \cap D = F_1$.

Proof: Let $f_n \in D \cap F \setminus F_1$ then $\gcd(p(x), q(x)) = d(x) \neq 1$, which implies $p(x) = d(x)P(x)$ and $q(x) = d(x)Q(x)$. Therefore,

$$f_n(x) = d(x)P(x)f_{n-1}(x) + d(x)Q(x)f_{n-2}(x).$$

Consider $f_2(x)$ and $f_3(x)$: $d(x)P(x)$ and $d(x)(P(x)d(x)P(x) + Q(x))$ respectively. This gives a contradiction to (1) since

$$\gcd(f_2(x), f_3(x)) = d(x) \neq 1 = f_1(x).$$

Thus $D \cap F \setminus F_1 = \emptyset$, and since $F_1 \subset F$ and $F_1 \cap D = F_1$, we can see $F \cap D = F_1$. $\square$

4. CONCLUSION

$D \cap G = \emptyset$, $D \cap H = \emptyset$, and $F \cap D = F_1$, show that D, all SDS of polynomials with the initial conditions $s_0(x) = 0$ and $s_1(x) = 1$, is the set of sequences F_1. Thus the set of multiples of F_1 is all the second-order strong divisibility sequences of polynomials, completing the characterization.

REFERENCES

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