COMPLEMENTARY FIBONACCI SEQUENCES

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(Submitted March 2006-Final Revision March 2007)

ABSTRACT

For given a_1, a_2 we determine the sequence (a_i) where (c_i) is the complement of (a_i) and (a_i) originates from (c_i) by the Fibonacci-like recurrence $a_i = c_{i-1} + c_{i-2}$. The sequences (a_i) turn out to be close to arithmetic progressions with difference 3.

1. INTRODUCTION

The complement of a sequence of positive integers is the strictly increasing sequence of all positive integers not being in the given sequence. Complements of sequences are discussed for example in [1–3]. Here we consider pairs of sequences (a_i) and (c_i) where (c_i) is the complement of (a_i) and (a_i) is determined by a Fibonacci-like recurrence from (c_i). That is, given a_1, a_2 with $a_1 \leq a_2$, the sequences (a_i) and (c_i) are determined by

$$\begin{aligned} a_i &= c_{i-1} + c_{i-2} \quad \text{for } i \geq 3, \\ c_1 &= \text{smallest number} \neq a_1, a_2, \\ c_i &= \text{smallest number} \neq a_1, a_2, \ldots, a_i, c_1, c_2, \ldots, c_{i-1} \quad \text{for } i \geq 2. \end{aligned} \tag{1}$$

Observe that (c_i) is the complement of (a_i) since $a_i > c_{i-1}$ and (c_i) is strictly increasing. The sequence (a_i) is strictly increasing at least for $i \geq 3$.

As an example we choose $a_1 = 2$, $a_2 = 5$ and obtain

$$\begin{aligned} (a_i) &= (2, 5, 4, 9, 13, 15, 18, 21, 23, 26, 30, 33, 36, 39, 42, 46, 49, \ldots) \\ (c_i) &= (1, 3, 6, 7, 8, 10, 11, 12, 14, 16, 17, 19, 20, 22, 24, 25, 27, \ldots) \end{aligned}$$

Here we collect properties of these complementary Fibonacci sequences.

2. RESULTS FOR $\mathbf{a_1 \equiv a_2 \equiv 0}$ (mod 3)

In this case the sequence (a_i) is an arithmetic progression with difference 3.

Theorem 1: For $a_1 \equiv a_2 \equiv 0 \pmod 3$ we have

$$a_i = 3i - 6 \qquad \text{for } i \geq 3 \quad \text{and}$$

$$c_i = \left\lfloor \frac{3i-1}{2} \right\rfloor \quad \text{for } i \geq 1.$$

Proof: It has to be checked that the asserted sequences fulfill the three equations in (1). We have

$$a_i = c_{i-1} + c_{i-2} = \left\lfloor \frac{3i-4}{2} \right\rfloor + \left\lfloor \frac{3i-7}{2} \right\rfloor = 3i - 6 \ \text{ for } i \geq 3.$$

Since $a_2 \geq a_1 \geq 3$ it follows $c_1 = 1$ as asserted. For the third equation of (1) we have $c_i \neq a_j$ for $j \geq 1$ since $c_i \not\equiv 0 \pmod 3$. The sequence (c_i) is monotonic increasing so that $c_i \neq c_j$ for $j < i$. For even i we have the smallest possible value $c_i = c_{i-1} + 1$. For odd i we have $c_i = c_{i-1} + 2$ since $c_{i-1} + 1$ is an a_j for $j \leq i$ as $c_{i-1} + 1 \equiv 0 \pmod 3$ and as $a_i = 3i - 6 > \frac{3i-5}{2} = c_{i-1}$ for $i \geq 3$. □

3. RESULTS FOR $\mathbf{a_1 = a_2 \not\equiv 0}$ (mod 3)

Here the differences $\Delta_i = a_{i+1} - a_i$, $i \geq 3$, of consecutive values of (a_i) are not always 3 as in the preceding case. There occur also differences 2 and 4 for indices with exponentially growing distances and the difference 5 occurs once.

Theorem 2: For $a_1 = a_2 = 3j + r \geq 5$, $r = 1$ or $r = 2$, we have $a_3 = 3$ and $\Delta_i = 3$, $i \geq 3$, except for the indices

$$\begin{aligned} i &= f_4(n, v, j, r) \\ &= (2j+1)4^n + 1 + (v-2)\left(\frac{(v+r-4)(v+r-3)4^n}{2} + \frac{4^n - 1}{3}\right) \end{aligned}$$

for $v = 1, 2, 3$ and $n = 0, 1, 2, \ldots$ where $\Delta_i = \begin{cases} 4 & \text{if } i \neq 2j+2, \\ 5 & \text{if } i = 2j+2 \end{cases}$

and

$$\begin{aligned} i &= f_2(n, v, j, r) \\ &= (4j+2)4^n + v - 1 + 2(v-2)\left(\frac{(v+r-4)(v+r-3)4^n}{2} + \frac{4^n - 1}{3}\right) \end{aligned}$$

for $v = 1, 2, 3$ and $n = 0, 1, 2, \ldots$ where $\Delta_i = 2$.

Proof: Since $a_1 = a_2 \geq 5$ it follows from (1) that $c_1 = 1$, $c_2 = 2$, and thus $a_3 = 3$. Then (1) and Theorem 1 imply $\Delta_i = 3$ for $3 \leq i < 2j + r$ since the sequence (c_i) starts as in Theorem 1 and since $\lfloor \frac{3i-1}{2} \rfloor = 3j + r = a_1 = a_2$, that is, $i = 2j + r$ determines the first c_i being different from the corresponding value $\lfloor\frac{3i-1}{2}\rfloor$ in Theorem 1. It follows that for $a_1 = a_2 \geq 5$ and $i = 2j + r - 1, \ldots, 2j + r + 1$ the values c_i, a_i, and Δ_i are as in Tables 1 and 2 for the cases $r = 1$ and 2, respectively.

i	c_i	a_i	Δ_i
$2j$	$3j-1$	$6j-6$	3
$2j+1$	$3j+2$	$6j-3$	4
$2j+2$	$3j+4$	$6j+1$	5

Table 1. The case $r = 1$.

i	c_i	a_i	Δ_i
$2j+1$	$3j+1$	$6j-3$	3
$2j+2$	$3j+4$	$6j$	5
$2j+3$	$3j+5$	$6j+5$	4

Table 2. The case $r = 2$.

Thus there exist exceptional differences $\Delta_i = 4$ or $\Delta_i = 5$ as asserted for $n = 0$ and $v = 1, 2, 3$. Note the double occurences of the index $i = 2j + 2$ corresponding to the difference $\Delta_i = 5$, that is,

$$\begin{aligned} 2j + 2 &= f_4(0, 2, j, 1) = f_4(0, 3, j, 1), \\ 2j + 2 &= f_4(0, 1, j, 2) = f_4(0, 2, j, 2). \end{aligned} \tag{2}$$

In the following we will see that the differences $\Delta_x = 3, 4$, and 5 in (a_i) determine $\Delta_x - 1$ consecutive numbers in (c_i) yielding $\Delta_x - 1$ consecutive differences Δ_i being 2 or 3. Differences $\Delta_i = 2$ result from differences $\Delta_x = 4$ and 5 only and yield a difference $\Delta_j = 4$ each. Thus these cases determine (Δ_i) completely.

For $\Delta_x = 3$ two differences 3 as in Table 3 are obtained using (1).

i	c_i	a_i	Δ_i
x		a_x	3
		$a_x + 3$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$a_x - 1$		
	$a_x + 1$		
	$a_x + 2$	$2a_x$	3
	$a_x + 4$	$2a_x + 3$	3
		$2a_x + 6$	

Table 3. Differences 3 determined by $\Delta_x = 3$.

Assuming $\Delta_x = 4$ and $a_x = 3x - d_x$ and using (1) we obtain further exceptional differences $\Delta_y = 2$ and $\Delta_z = 4$ as shown in Table 4. Note that $6x - 2d_x + 2$ and $6x - 2d_x + 6$ do not occur in (a_i) since (c_i) is strictly increasing, that is, $\Delta_i = c_i - c_{i-2} \geq 2$. Table 4 also implies that any other difference 2 or 4 in (a_i) between indices x and y causes a difference 4 or 2, respectively, between indices y and z. With $a_i = 3i - d_i$ we get

$$d_{i+1} = d_i + 3 - \Delta_i. \tag{3}$$

It follows that $d_z = d_x$ in Table 4. Then $a_z = 3z - d_z = 3z - d_x = 12x - 4d_x + 6$ determines $z = 4x - d_x + 2$. We obtain $y = 2x + 1 - (2d_x - d_y)/3$ from $a_y = 3y - d_y = 6x - 2d_x + 3$.

i	c_i	a_i	Δ_i
x		$3x-d_x$	4
$x+1$		$3x-d_x+4$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$3x-d_x-1$		
	$3x-d_x+1$		
	$3x-d_x+2$	$6x-2d_x$	3
$y=2x+1-(2d_x-d_y)/3$	$3x-d_x+3$	$6x-2d_x+3$	2
	$3x-d_x+5$	$6x-2d_x+5$	3
		$6x-2d_x+8$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$6x-2d_x+2$		
	$6x-2d_x+4$		
$z=4x-d_x+2$	$6x-2d_x+6$	$12x-4d_x+6$	4
		$12x-4d_x+10$	

Table 4. Differences 2, 3, and 4 determined by $\Delta_x=4$.

If $\Delta_x=5$ corresponding to Table 4 we obtain pairs of differences $\Delta_y=\Delta_{y+1}=2$ and $\Delta_z=\Delta_{z+1}=4$ in Table 5.

i	c_i	a_i	Δ_i
x		$3x-d_x$	5
$x+1$		$3x-d_x+5$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$3x-d_x-1$		
	$3x-d_x+1$		
	$3x-d_x+2$	$6x-2d_x$	3
$y=2x+1-(2d_x-d_y)/3$	$3x-d_x+3$	$6x-2d_x+3$	2
$y+1$	$3x-d_x+4$	$6x-2d_x+5$	2
	$3x-d_x+6$	$6x-2d_x+7$	3
		$6x-2d_x+10$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$6x-2d_x+2$		
	$6x-2d_x+4$		
$z=4x-d_x+2$	$6x-2d_x+6$	$12x-4d_x+6$	4
$z+1$	$6x-2d_x+8$	$12x-4d_x+10$	4
		$12x-4d_x+14$	

Table 5. Differences 2, 3, and 4 determined by $\Delta_x=5$.

By Tables 3, 4, and 5 together with Tables 1 and 2 as bases we conclude that the sequence of exceptional differences $\Delta_i\neq 3$ is as in the first rows of Tables 6 and 7 for $r=1$ and 2, respectively. It remains to check that the corresponding indices for $n>0$ and $v=1,2,3$

are $i = f_2(n,v,j,r)$ for $\Delta_i = 2$ and $i = f_4(n,v,j,r)$ for $\Delta_i = 4$ and $\Delta_i = 5$ as asserted in Theorem 2. Observe (2) for $\Delta_i = 5$.

Δ_i	4	5	2	2	2	4	4	4	2	2	2	4	4	4	...
n	0	0	0	0	0	1	1	1	1	1	1	2	2	2	...
v	1	2, 3	1	2	3	1	2	3	1	2	3	1	2	3	...
d_i	6	5	3	4	5	6	5	4	3	4	5	6	5	4	...

Table 6. Exceptional differences in the case $r = 1$.

Δ_i	5	4	2	2	2	4	4	4	2	2	2	4	4	4	...
n	0	0	0	0	0	1	1	1	1	1	1	2	2	2	...
v	1, 2	3	1	2	3	1	2	3	1	2	3	1	2	3	...
d_i	6	4	3	4	5	6	5	4	3	4	5	6	5	4	...

Table 7. Exceptional differences in the case $r = 2$.

For the last rows in Tables 6 and 7 we have from Tables 1 and 2 that $d_{2j+r} = 6$ for the first exceptional Δ_i. The following values change only for $\Delta_i \neq 3$ according to (3). Thus it holds

$$\begin{aligned} d_i &= 7 - v \ \text{ for } \Delta_i = 4 \quad \text{and} \\ d_i &= v + 2 \ \text{ for } \Delta_i = 2. \end{aligned} \tag{4}$$

For $\Delta_x = 4$ we obtain from $x = f_4(n,v,j,r)$ as in Theorem 2 with $d_x = 7 - v$ and $z = 4x - d_x + 2$ from Table 4 the induction step that

$$z = f_4(n+1,v,j,r) = 4f_4(n,v,j,r) + v - 5$$

as asserted in Theorem 2. Furthermore, with $d_y = v + 2$ and $y = 2x + 1 + (d_y - 2d_x)/3$ from Table 4 it follows

$$y = f_2(n,v,j,r) = 2f_4(n,v,j,r) + v - 3$$

as asserted.

For $\Delta_x = 5$ it remains to check

$$\begin{aligned} f_4(1, 4-r, j, r) &= f_4(1, 3-r, j, r) + 1 \quad \text{and} \\ f_2(0, 4-r, j, r) &= f_2(0, 3-r, j, r) + 1 \end{aligned}$$

since the indices y and z in Table 5 are the same as in Table 4. □

By Theorem 2 with (3) the elements of (a_i) can be expressed as follows.

Theorem 3: For $a_1 = a_2 = 3j + r \geq 5$, $r = 1, 2$, and with f_2 and f_4 from Theorem 2 we have

$$\begin{aligned} a_i &= 3i - 6 \quad \text{for } 3 \leq i \leq 2j + r = f_4(0,1,j,r) \quad \text{and} \\ a_i &= 3i - d_t \quad \text{for } s < i \leq t \end{aligned}$$

where $\Delta_s \neq 3$ and $\Delta_t \neq 3$ are two consecutive exceptional differences and

$$d_t = \begin{cases} 7 - v & \text{if } t = f_4(n, v, j, r), \\ 2 + v & \text{if } t = f_2(n, v, j, r). \end{cases}$$

If $a_1 = a_2 \geq 5$ we have found (Tables 6 and 7) that the sequence of exceptional differences consists of triples of $\Delta_i = 4$ and triples of $\Delta_i = 2$, alternatingly. If $a_1 = a_2 = 1, 2$, or 4 then the sequence of exceptional differences consists of alternating values $\Delta_i = 4$ and $\Delta_i = 2$.

Theorem 4: For $a_1 = a_2 = 1, 2$, and 4 the sequences (a_i), $i \geq 3$, are $(n \geq 0)$

$a_1 = a_2 = 1:$

$$\begin{aligned} &a_3 = 5,\ a_4 = 7,\ a_5 = 10, \\ &a_i = 3i - 4 \ \text{ for } f_4(n) = 4^{n+1} + 1 < i \leq f_2(n), \\ &a_i = 3i - 5 \ \text{ for } f_2(n) = 2 \cdot 4^{n+1} + 1 < i \leq f_4(n+1), \end{aligned}$$

$a_1 = a_2 = 2:$

$$\begin{aligned} &a_3 = 4, \\ &a_i = 3i - 4 \ \text{ for } f_4(n) = 2 \cdot 4^{n} + 1 < i \leq f_2(n), \\ &a_i = 3i - 5 \ \text{ for } f_2(n) = 4^{n+1} + 1 < i \leq f_4(n+1), \end{aligned}$$

$a_1 = a_2 = 4:$

$$\begin{aligned} &a_3 = 3,\ a_4 = 7, \\ &a_i = 3i - 4 \ \text{ for } f_4(n) = 3 \cdot 4^{n} + 1 < i \leq f_2(n), \\ &a_i = 3i - 5 \ \text{ for } f_2(n) = 6 \cdot 4^{n} + 1 < i \leq f_4(n+1). \end{aligned}$$

Proof: By Table 8 there are differences $\Delta_i = 4$ for $i = f_4(0) = 5$, 3, and 4 in the cases $a_1 = a_2 = 1$, 2, and 4, respectively. The asserted intervals for i follow inductively with $z = f_4(n+1) = 4f_4(n) - 3$ and $y = f_2(n) = 2f_4(n) - 1$ by Table 4 for $x = f_4(n)$ since $d_x = 5$ and $d_y = 4$ by (3). □

i	c_i	a_i	Δ_i	c_i	a_i	Δ_i	c_i	a_i	Δ_i
1	2	1		1	2		1	4	
2	3	1		3	2		2	4	
3	4	5	2	5	4	**4**	5	3	4
4	6	7	3	6	8	3	6	7	**4**
5	8	10	**4**	7	11	**2**	8	11	3
6	9	14	3		13		9	14	3
7	11	17	3				10	17	**2**
8	12	20	3					19	
9	13	23	**2**						
10		25							

Table 8. First exceptional differences for $a_1 = a_2 = 1, 2, 4$.

4. GENERAL CASES

At first we state the two simple subcases where (a_i) is as in Theorem 1 or 3.

Theorem 5: If $a_1 \equiv 0 \pmod 3$ then $a_i = a'_i$, $i \geq 3$, for the sequence (a'_i) with $a'_1 = a'_2 = a_2$ as in Theorem 1 or 3.

If $a_1 \not\equiv 0 \pmod 3$ and a_2 occurs in (a'_i) for $a'_1 = a'_2 = a_1$ (as in Theorem 3) then $a_i = a'_i$, $i \geq 3$.

Proof: In the first case a_1 occurs in (a'_i) due to Theorem 1 or 3 and $a_1 < a_2$. Therefore in both cases $(c'_i) = (c_i)$ by (1) and thus $a_i = a'_i$, $i \geq 3$. □

In the general case, starting with an exceptional difference $\Delta_i = 2$ or 4, repeated application of Table 4 generates an exponential sequence of indices belonging to differences 4 and 2 alternatingly and each being nearly twice the preceding index. Let $V(a_1, a_2)$ count the number of infinite exponential sequences of this kind for given a_1 and a_2.

Theorem 6: For all $a_1 \leq a_2$ we have $a_i = 3i - d_i$, $i \geq 3$, for $0 \leq d_i \leq 6$ and $V(a_1, a_2) = 0$, 1, 2, 3, 4, or 6 only.

Proof: In the cases of Theorems 1, 4, and 2 we have $V(a_1, a_2) = 0, 1$, and 3, respectively and $d_i = 3$, 4, 5, or 6 by Theorem 1 and Tables 6, 7, and 8. The cases of Theorem 5 reduce to the preceding cases.

It remains $a_1 \not\equiv 0 \pmod 3$ and a_2 does not occur in (a'_i) for $a'_1 = a'_2 = a_1$. These are the cases where both initial values a_1 and a_2 have an effect on (a_i) being treated in the following.

In addition to the occurence of $\Delta_x = 5$ as in Theorem 3 in the general case also $\Delta_x = 6$ may occur as exceptional difference. For $\Delta_x = 6$ which will occur in Tables 10 to 12 and 14 we obtain further exceptional differences $\Delta_y = \Delta_{y+1} = \Delta_{y+2} = 2$ in Table 9 corresponding to Tables 4 and 5.

i	c_i	a_i	Δ_i
x		$3x - d_x$	6
$x+1$		$3x - d_x + 6$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$3x - d_x - 1$		
	$3x - d_x + 1$		
	$3x - d_x + 2$	$6x - 2d_x$	3
$y = 2x + 1 - (2d_x - d_y)/3$	$3x - d_x + 3$	$6x - 2d_x + 3$	2
$y+1$	$3x - d_x + 4$	$6x - 2d_x + 5$	2
$y+2$	$3x - d_x + 5$	$6x - 2d_x + 7$	2
	$3x - d_x + 7$	$6x - 2d_x + 9$	3
		$6x - 2d_x + 12$	

Table 9. Exceptional differences 2 determined by $\Delta_x = 6$.

If $a_2 - a_1 \leq 4$ then Tables 10 to 13 prove that $V(a_1, a_2) = 6$ for the indicated values of a_1 and a_2. For $(a_1, a_2) = (1, 2)$, $(2, 3)$, $(4, 5)$, $(1, 3)$, $(4, 6)$, $(5, 7)$, $(1, 4)$, $(2, 5)$, $(4, 7)$, $(5, 8)$, $(7, 10)$, $(2, 6)$, $(4, 8)$, $(5, 9)$, and $(7, 11)$ we get the values $V(a_1, a_2) = 2$, 2, 2, 0, 4, 4, 2, 2, 1, 2, 4, 2, 2, 2, and 4, respectively. The values of d_i are in the asserted interval for the listed pairs (a_1, a_2) with small values of a_1. For the cases of Tables 10 to 13 we observe the first value $d_3 = 6$. By

(3) there are changes of d_i only after $\Delta_i \neq 3$. In all four cases at first d_i is decreased by 6 due to $\Delta_i = 4$, 5, or 6 before d_i is increased 6 times by 1 because of $\Delta_i = 2$. Thus d_i oscillates between 0 and 6.

c_i	a_i	Δ_i
	a_1	
	a_1+1	
	$\vdots$	
	a_1-4	
	a_1-1	
	a_1+2	
	$\geq a_1+3$	
	$\vdots$	
a_1-3		
a_1-2		
a_1+3	$2a_1-5$	6
a_1+4	$2a_1+1$	6
	$2a_1+7$	

Table 10. $a_2 = a_1 + 1$, $a_1 \equiv 1 \pmod 3$, $a_1 \geq 7$.

c_i	a_i	Δ_i
	a_1	
	a_1+2	
	$\vdots$	
	a_1-2	
	a_1+1	
	a_1+4	
	$\vdots$	
a_1-3		
a_1-1		
a_1+3	$2a_1-4$	6
a_1+5	$2a_1+2$	6
	$2a_1+8$	

Table 11. $a_2 = a_1 + 2$, $a_1 \equiv 2 \pmod 3$, $a_1 \geq 8$.

c_i	a_i	Δ_i
	a_1	
	a_1+3	
	$\vdots$	
	a_1-4	
	a_1-1	
	a_1+2	
	a_1+5	
	$\vdots$	
a_1-3		
a_1-2		
a_1+1	$2a_1-5$	4
a_1+4	$2a_1-1$	6
a_1+6	$2a_1+5$	5
	$2a_1+10$	

c_i	a_i	Δ_i
	a_1	
	a_1+3	
	$\vdots$	
	a_1-2	
	a_1+1	
	a_1+4	
	a_1+7	
	$\vdots$	
a_1-3		
a_1-1		
a_1+2	$2a_1-4$	5
a_1+5	$2a_1+1$	6
a_1+6	$2a_1+7$	4
	$2a_1+11$	

Table 12. $a_2 = a_1 + 3$, $a_1 \equiv 1, 2 \pmod 3$, $a_1 \geq 8$.

c_i	a_i	Δ_i
	a_1	
	a_1+4	
	$\vdots$	
	$\leq a_1-4$	
	a_1-1	
	a_1+2	
	a_1+5	
	$\geq a_1+8$	
	$\vdots$	
a_1-3		
a_1-2		
a_1+1	$2a_1-5$	4
a_1+3	$2a_1-1$	5
a_1+6	$2a_1+4$	5
a_1+7	$2a_1+9$	4
	$2a_1+13$	

Table 13. $a_2 = a_1 + 4$, $a_1 \equiv 1 \pmod 3$, $a_1 \geq 10$.

If $a_2 - a_1 \geq 5$, then that index i_0 where $c'_{i_0} = a_2$, that is, $c_{i_0} \neq c'_{i_0}$, we have $a'_i = a_i$ for $3 \leq i \leq i_0$ and a_{i_0+1} differs from a'_{i_0+1}. We distinguish the cases $\Delta_x = 2$, 3, 4, and 5 for that x with $a_x < a_2 < a_{x+1}$ where $a_2 = 3x - d_x + j$ for $1 \leq j \leq \Delta_x - 1$. In Tables 14 to 17 we present the essential values of c_i, a_i, and Δ_i beginning with $i = i_0 - 2$ and for $\Delta_x = 5$ and $j = 4$ with $i = i_0 - 3$. We use the abbreviations $a_i^* = a_i - (6x - 2d_x)$ and $c_i^* = c_i - (3x - d_x)$.

c_i^*	a_i^*	Δ_i
−2		
−1		
3	−3	5
4	2	5
	7	

$\Delta_{x-1} \geq 3$, $\Delta_{x+1} \geq 3$

c_i^*	a_i^*	Δ_i
−2		
−1		
3	−3	5
5	2	6
	8	

$\Delta_{x-1} \geq 3$, $\Delta_{x+1} = 2$

c_i^*	a_i^*	Δ_i
−3		
−1		
3	−4	6
4	2	5
	7	

$\Delta_{x-1} = 2$, $\Delta_{x+1} \geq 3$

Table 14. Effects of $a_2 = 3x - d_x + 1$ for $\Delta_x = 2$.

c_i^*	a_i^*	Δ_i
−2		
−1		
2	−3	4
4	1	5
	6	

$j = 1$, $\Delta_{x-1} \geq 3$

c_i^*	a_i^*	Δ_i
−3		
−1		
2	−4	5
4	1	5
	6	

$j = 1$, $\Delta_{x-1} = 2$

c_i^*	a_i^*	Δ_i
−1		
1		
4	0	5
5	5	4
	9	

$j = 2$, $\Delta_{x+1} \geq 3$

c_i^*	a_i^*	Δ_i
−1		
1		
4	0	5
6	5	5
	10	

$j = 2$, $\Delta_{x+1} = 2$

Table 15. Effects of $a_2 = 3x - d_x + j$ for $\Delta_x = 3$ and $j = 1, 2$.

c_i^*	a_i^*	Δ_i
−2		
−1		
2	−3	4
3	1	4
	5	

$j = 1$

c_i^*	a_i^*	Δ_i
−1		
1		
3	0	4
5	4	4
	8	

$j = 2$

c_i^*	a_i^*	Δ_i
1		
2		
5	3	4
6	7	4
	11	

$j = 3$

Table 16. Effects of $a_2 = 3x - d_x + j$ for $\Delta_x = 4$ and $j = 1, 2, 3$.

c_i^*	a_i^*	Δ_i
−2		
−1		
2	−3	4
3	1	4
4	5	2
	7	

$j = 1$

c_i^*	a_i^*	Δ_i
−1		
1		
3	0	4
4	4	3
	7	

$j = 2$

c_i^*	a_i^*	Δ_i
1		
2		
4	3	3
6	6	4
	10	

$j = 3$

c_i^*	a_i^*	Δ_i
1		
2		
3	3	2
6	5	4
7	9	4
	13	

$j = 4$

Table 17. Effects of $a_2 = 3x - d_x + j$ for $\Delta_x = 5$ and $j = 1, 2, 3, 4$.

Since $a_2 - a_1 \geq 5$, in Tables 14 to 17 there are no coincidences of a_1 with the essential values of c_i corresponding to c_i^*.

To determine the values of $V(a_1, a_2)$ in the general cases we first note that $V' = V(a_1, a_1)$ is 1 or 3 by Tables 6 to 8. In the cases of Table 14 we have a_2 within a gap $\Delta_x = 2$. In the leftmost case this results in 2 differences $\Delta = 5$ implying 2 pairs of $\Delta = 2$. Thus we obtain $V = V' - 1 + 4$. In the remaining two cases of Table 14 we have a_2 within one of 2 consecutive gaps $\Delta = 2$ resulting in differences 5 and 6 or 6 and 5 and implying 5 differences $\Delta = 2$. Thus we obtain $V = V' - 2 + 5$.

Correspondingly, in the first and third case of Table 15 we obtain $V = V' + 3$ and for the remaining cases $V = V' - 1 + 4$. In Table 16 we get $V = V' - 1 + 2$ in all three cases. Cases 1 and 4 of Table 17 yield $V = V' - 2 + 3$ and cases 2 and 3 yield $V = V' - 2 + 1$. Altogether, the value a_2 increases the number 1 or 3 of exceptional differences V' by 3, 1, or -1 to $V = 2$, 4, or 6 as asserted in Theorem 6. Values of $V(a_1, a_2)$ for small a_1, a_2 are presented in Table 18.

	1 2 3 4 5
$a_2 =$	1234567890123456789012345678901234567890123456789012345
$a_1 = 1$	1202141441222144144144141441441441441441441441222144144
2	121222144141441441441222144144144144144144144141441441
3	01303303303303303303303303303303303303303303303303303
4	1241222144144141441441441441441222144144144144144144
5	334224344436363663663636634443444366366366366344436
6	03303303303303303303303303303303303303303303303303
7	3634443422436636366366363636636636634443663663663
8	336634224344436636636363663663636636636636636634
9	03303303303303303303303303303303303303303303303
10	3636634443422436636636636366366363636636636636
11	336636634224344436636636636636363663663636636
12	03303303303303303303303303303303303303303303
13	3636636634443422436636636636636636366366363
14	336636636634224344436636636636636636636363
15	03303303303303303303303303303303303303303
16	3636636636634443422436636636636636636636
17	336636636636634224344436636636636636636
18	03303303303303303303303303303303303303
19	3636636636636634443422436636636636636
20	336636636636636634224344436636636636

Table 18. Values of $V(a_1, a_2)$ for small a_1, a_2.

For d_i we observe that Tables 6 to 8 imply $3 \leq d_i' \leq 6$ for (a_i') with $a_1' = a_2' = a_1$. In the cases of Table 14 and the second and fourth case of Table 15 there is one $\Delta = 2$ or a pair of consecutive differences $\Delta = 2$ and thus by (3) we have $d_i' \geq 4$ or $d_i' \geq 5$ before d_i is decreased by 4 or 5 due to $\Delta = 5$, 5 or $\Delta = 5$, 6, respectively. In the remaining cases of Tables 15 to 17 the decrease of d_i is at most 3. Since the corresponding exceptional differences $\Delta = 2$ always increase d_i by the same amount we have $0 \leq d_i \leq 6$ and Theorem 6 is proved. □

Now the sequences (a_i) are determined completely to be $a_i = 3i - d_i$, $i \geq 3$, where d_i oscillates within subintervals of $(0, 6)$ with exponentially growing step lengths. Most of the differences $\Delta_i = a_{i+1} - a_i$ are 3. There are $V(a_1, a_2)$ infinite sequences of exponentially growing indices with differences 4 and 2 alternatingly. Differences $\Delta_i = 5$ and 6 occur at most three times.

It may be future work to consider complementary sequences (a_i) being determined by other recurrences.

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AMS Classification Numbers: 11B37, 11B39, 11B25

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