ON SOME ARITHMETIC PROPERTIES OF A SEQUENCE RELATED TO THE QUOTIENT OF FIBONACCI NUMBERS

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Abstract. We examine the sequence $(T_n)_{n\geq 1}$ of numbers: 1, 11, 61, 451, 3001, 20801, 141961, ... given by $T_n = F_{5n}/(5F_n)$, where F_n is the Fibonacci number. Curious divisibility properties are obtained including related conditions resembling a strong divisibility sequence. In particular, we prove that all prime divisors of the numbers in this sequence end in one. Another result asserts that each integral power of a number in the sequence is a divisor of some other number in the sequence. Specifically, we prove that for any positive integers n and k, the term

$$T(nT(nT(\cdots nT(n)\cdots)))$$

with k occurrences of the number n is exactly divisible by T_n^k.

1. Introduction

The Fibonacci sequence $(F_n)_{n\geq 0}$ is defined by

$$F_0 = 0,\ F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2.$$

It is probably regarded as one of the most studied integer sequences of all time because of its rich and well-structured properties. The terms in the Fibonacci sequence are referred to as the Fibonacci numbers. Their most intriguing characters are based on the number-theoretic properties. For example, the Fibonacci sequence is a divisibility sequence in the sense that F_m divides F_n whenever m divides n for all nonnegative integers m and n. Some distinguished arithmetic properties of the Fibonacci sequence lie in the intricate structure of its subsequences as illustrated by a previous work of the authors of this note. In [6], we define a family of subsequences $(G_k(n))$ of the Fibonacci sequence as follows: for each nonnegative integer n,[1]

$$G_1(n) = F_n \quad \text{and} \quad G_k(n) = F(nG_{k-1}(n)) \quad \text{for } k \geq 2.$$

One of the most basic properties states that the number F_n^k exactly divides $G_k(n)$ for all positive integers k and n with $n > 3$. In this work, we find that this kind of dynamical property is shared by at least one more sequence. The terms of this sequence are denoted by T_n and defined as the quotient of Fibonacci numbers $F_{5n}/(5F_n)$ for each positive integer n. The first few terms of this sequence are

$$1, 11, 61, 451, 3001, 20801, 141961.$$

We notice immediately that each term of this sequence seems to end in one. This sequence appeared in [2] where the authors gave the Zeckendorf decomposition of each number in the sequence. We are to examine number-theoretic properties of these numbers, including the characters of their prime factorizations and related divisibility properties.

In the following discussion we recall the definition of exact divisibility as follows: a power of integer a^k is said to *exactly divide* an integer b, denoted $a^k \parallel b$, provided that $a^k \mid b$ and

[1] The notations a_n and $a(n)$ to denote the nth term of a sequence (a_n) are used interchangeably in this paper.

$a^{k+1} \nmid b$. We also recall the p-adic valuation $v_p(a)$ of a positive integer a to be the exponent of the prime p in the prime factorization of a.

2. Entry Point Function $Z(n)$

For a positive integer n, we define $Z(n)$ to be the so-called *entry point* of n in the Fibonacci sequence as the first positive index m such that $n \mid F_m$. For example, $Z(3) = 4$ since $F_4 = 3$ and $Z(11) = 10$ since $F_{10} = 55 \equiv 0 \pmod{11}$ and $F_j \not\equiv 0 \pmod{11}$ for all $1 \leq j \leq 9$. Arithmetic properties of $Z(n)$ are extensive and quite useful, including the generalization of the relations and the examination of a specific case such as when n is prime or a power of prime. Some of these results that are needed for this work are summarized in the following lemma. Its comprehensive investigation can be found in [7].

Lemma 2.1. *Let m and n be positive integers and p prime. Then the following statements hold.*

1. $n \mid F_m$ *if and only if* $Z(n) \mid m$.
2. $p \equiv \left(\frac{p}{5}\right) \pmod{Z(p)}$, *where* $\left(\frac{p}{5}\right)$ *is the Legendre symbol of p with respect to* 5.

3. Prime Divisors of T_n

We have observed earlier that the first few terms of the sequence (T_n) seem to end in one. The following theorem shows that this is indeed the case for all terms of the sequence.

Theorem 3.1. *Let n be a positive integer. Then*

1. $\gcd(F_n, T_n) = 1$, *and*
2. $T_n \equiv 1 \pmod{10}$.

Proof. This follows immediately from the relation

$$T_n = 5F_n^2(F_n^2 + (-1)^n) + 1. \tag{3.1}$$

This relation is a result of a more general one given in [5] which states that

$$F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{n(q+k)} \frac{2q+1}{q+k+1} 5^k \binom{q+k+1}{2k+1} F_n^{2k}, \quad n, q \geq 0.$$

By letting $q = 2$, we obtain the identity (3.1), as required. □

We may characterize the prime divisors of the terms of (T_n) based on divisibility properties of the entry point as follows.

Theorem 3.2. *Let p be prime. Then $p \mid T_n$ for some n if and only if $p \neq 5$ and $5 \mid Z(p)$.*

Proof. Let p be prime. Assume that $p \mid T_n$ for some n. By Theorem 3.1, we have $p \neq 5$ and $p \nmid F_n$. Now since $F_{5n} = 5F_nT_n$, we have $p \mid F_{5n}$. By Lemma 2.1, we have $Z(p) \mid 5n$ and $Z(p) \nmid n$ (from $p \nmid F_n$). Hence, $5 \mid Z(p)$.

For the converse, we assume that $p \neq 5$ and $5 \mid Z(p)$. Let $n = \frac{Z(p)}{5}$. By the definition of $Z(p)$, we have $p \mid F_{5n}$ and $p \nmid F_n$. Since $F_{5n} = 5F_nT_n$ and $p \neq 5$, it follows that $p \mid T_n$. □

Theorem 3.3. *Let p be prime. Then $p \mid T_n$ for some n if and only if $p \neq 5$, $Z(p) \mid 5n$, and $v_5(Z(p)) = v_5(n) + 1$.*

Proof. Let p be prime. Assume that $p \mid T_n$ for some n. Since $F_{5n} = 5F_nT_n$, we obtain $p \mid F_{5n}$. By Theorem 3.1, we also obtain $p \neq 5$ and $p \nmid F_n$. Hence, by Lemma 2.1, $Z(p) \mid 5n$ and $Z(p) \nmid n$. Now since $p \mid T_n$, it follows from Theorem 3.2 that $5^k \parallel Z(p)$ for some $k \in \mathbb{N}$. Write $Z(p) = 5^k m_1$ and $n = 5^\ell n_1$, where $5 \nmid m_1$ and $5 \nmid n_1$. Then $5^k m_1 \mid 5^{\ell+1} n_1$ and $5^k m_1 \nmid 5^\ell n_1$. This yields $k = \ell + 1$. Since $v_5(Z(p)) = k$ and $v_5(n) = \ell$, the result follows.

For the converse, assume that $p \neq 5$ $Z(p) \mid 5n$, $5^k \parallel Z(p)$, and $5^{k-1} \parallel n$ for some $k \in \mathbb{N}$. Since $Z(p) \mid 5n$, Lemma 2.1 implies that $p \mid F_{5n}$. Since $5^k \parallel Z(p)$ and $5^{k-1} \parallel n$, it follows that $Z(p) \nmid n$. Once again, Lemma 2.1 implies that $p \nmid F_n$. Now since $p \neq 5$ and $F_{5n} = 5F_nT_n$, we have $p \mid T_n$, as required. □

Theorem 3.4. *Let p be prime such that $p \mid T_n$ for some n. Then*

$$p \equiv 1 \pmod{10}.$$

Proof. Let p be prime such that $p \mid T_n$ for some n. By Theorem 3.1, we have $p \neq 2$. By Theorem 3.2, we have $5 \mid Z(p)$. This implies $Z(p) = 5n_1$ for some positive integer n_1. By Lemma 2.1, we obtain $p \equiv \pm 1 \pmod{5n_1}$. This implies $p \equiv \pm 1 \pmod 5$. By Lemma 2.1, $p \equiv 1 \pmod{5n_1}$. Since p is odd, this implies n_1 is even. Hence, $p = 10k + 1$ for some positive integer k and the theorem follows. □

Remark 3.5. *We note that the converse of Theorem 3.4 does not hold, i.e., not all primes p ending in one are divisors of some T_n. In fact, consider the prime $p = 211$. By direct computation, we have $Z(p) = 42$ and since $5 \nmid 42$, Theorem 3.2 implies that $211 \nmid T_n$ for all n.*

4. Almost Strong Divisibility Sequence

A sequence (a_n) of integers is said to be a *strong divisibility sequence* if $\gcd(a_m, a_n) = a_{\gcd(m,n)}$ for all m, n. The sequence $a_n = a^n - b^n$ where $\gcd(a, b) = 1$ is a nontrivial example of such sequence. Another well-known example includes the Fibonacci sequence. We show in this section that the sequence (T_n) also possesses in some sense the quality of being a strong divisibility sequence.

Lemma 4.1. *Let m and n be positive integers such that $m \mid n$ and $v_5(m) = v_5(n)$. Then $T_m \mid T_n$.*

Proof. Since $m \mid n$ and $v_5(m) = v_5(n)$, there exist positive integers r, s, and ℓ such that $m = \ell r$, $n = \ell s$, $r \mid s$, $5 \nmid r$, and $5 \nmid s$. Let p be a prime such that $p^k \parallel T_{\ell r}$. Then $p^k \mid F_{5\ell r}$ (by definition) and $p^k \nmid F_{\ell r}$ (by Theorem 3.1). By Lemma 2.1, $Z(p^k) \mid 5\ell r$ and $Z(p^k) \nmid \ell r$. Since $5 \nmid r$, the previous statement holds if and only if $5^i \parallel Z(p^k)$ and $5^{i-1} \parallel \ell$ for some $i \in \mathbb{N}$. Now since $5 \nmid s$ and $r \mid s$, we have $Z(p^k) \mid 5\ell s$ and $Z(p^k) \nmid \ell s$. Thus, by Lemma 2.1, $p^k \mid F_{5\ell s}$ and $p^k \nmid F_{\ell s}$. Now since, by Theorem 3.1, $p \neq 5$ and $\gcd(F_{\ell s}, T_{\ell s}) = 1$, it follows that $p^k \mid T_{\ell s}$. Hence, $v_p(T_m) \leq v_p(T_n)$. Since p was arbitrary, it follows that $T_m \mid T_n$ and the lemma follows. □

Lemma 4.2. $\gcd(T_m, T_n) \mid T_{\gcd(m,n)}$ *for all positive integers m and n.*

Proof. Let d be a common divisor of T_m and T_n. It suffices to prove that $d \mid T_{\gcd(m,n)}$. If $d = 1$, then the result is clear. Assume that $d > 1$. We have $d \mid T_m$ and $d \mid T_n$. By the definition of the sequence (T_n), we have $d \mid F_{5m}$ and $d \mid F_{5n}$. Consequently, $d \mid \gcd(F_{5m}, F_{5n})$. Since (F_n) is a strong divisibility sequence, this implies $d \mid F_{\gcd(5m,5n)}$. Now since $F_{\gcd(5m,5n)} = F_{5\gcd(m,n)} = 5F_{\gcd(m,n)}T_{\gcd(m,n)}$, it follows that $d \mid 5F_{\gcd(m,n)}T_{\gcd(m,n)}$. By Theorem 3.1, we have $\gcd(d, 5) = 1$ and $\gcd(d, F_m) = \gcd(d, F_n) = 1$, so that $\gcd(d, \gcd(F_m, F_n)) = 1$.

Again, since (F_n) is a strong divisibility sequence, it follows that $\gcd(d, F_{\gcd(m,n)}) = 1$. Hence, $d \mid T_{\gcd(m,n)}$ and the proof is complete. $\square$

We are now ready to prove the theorem that characterizes the sequence (T_n) as an *almost* strong divisibility sequence in the sense that $\gcd(T_m, T_n) = T_{\gcd(m,n)}$ if and only if $v_5(m) = v_5(n)$. The precise statement is as follows.

Theorem 4.3. *Let m and n be positive integers. Then*

$$\gcd(T_m, T_n) = \begin{cases} T_{\gcd(m,n)}, & \text{if } v_5(m) = v_5(n), \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Let m and n be positive integers. We consider two cases.
Case 1. $v_5(m) = v_5(n)$.
By Lemma 4.2, it suffices to show that $T_{\gcd(m,n)} \mid \gcd(T_m, T_n)$. We write $m = 5^k m_1$, $n = 5^k n_1$, and $d = \gcd(m, n) = 5^k d_1$, where $\gcd(m_1, 5) = \gcd(n_1, 5) = \gcd(d_1, 5) = 1$. Consequently, $d_1 \mid m_1$ and $d_1 \mid n_1$. By Lemma 4.1, we have $T_d \mid T_m$ and $T_d \mid T_n$, i.e., $T_{\gcd(m,n)} \mid \gcd(T_m, T_n)$, as required.
Case 2. $v_5(m) \neq v_5(n)$.
Assume to the contrary that there is a prime p such that $p \mid T_m$ and $p \mid T_n$. We write $m = 5^k m_1$ and $n = 5^\ell n_1$, where $\gcd(m_1, 5) = \gcd(n_1, 5) = 1$. By Theorem 3.2, there exists a positive integer i such that $5^i \parallel Z(p)$. Since $p \mid T_m$ and $p \mid T_n$, by Theorem 3.3, we have $i = k + 1$ and $i = \ell + 1$. Consequently, $k = \ell$, contradicting the fact that $v_5(m) \neq v_5(n)$. Hence, $\gcd(T_m, T_n) = 1$. $\square$

5. Dynamical Properties of (T_n)

In this section, we discuss some dynamical properties of (T_n) that are analogous to the work of Panraksa, Tangboonduangjit, and Wiboonton [6] of the Fibonacci numbers. For this purpose we define similar subsequences $(H_k(n))$ of the sequence (T_n) as follows: for each positive integer n, we let

$$H_1(n) = T_n \quad \text{and} \quad H_k(n) = T_{nH_{k-1}(n)} \quad \text{for } k \geq 2.$$

The first few terms of such sequence, therefore, are

$$T(n), \quad T(nT(n)), \quad T(nT(nT(n))), \quad T(nT(nT(nT(n)))).$$

We will show that each term of this sequence is exactly divisible by some power of T_n. We first prove some lemmas about the greatest common divisor of the Fibonacci numbers and some quotients of them.

Lemma 5.1. *Let n be a positive integer and p prime. Then*

$$\gcd\left(F_n, \frac{F_{pn}}{F_n}\right) = \begin{cases} p, & \text{if } p \mid F_n; \\ 1, & \text{otherwise.} \end{cases}$$

Proof. By the expansion formula of Fibonacci into the sum involving binomial coefficients and lower terms of Fibonacci numbers (see [3], for example), we obtain

$$F_{pn} \equiv \binom{p}{1} F_n F_{n-1}^{p-1} \equiv p F_n F_{n-1}^{p-1} \pmod{F_n^2}.$$

Thus, $\dfrac{F_{pn}}{F_n} \equiv p F_{n-1}^{p-1} \pmod{F_n}$ and the result follows. $\square$

Lemma 5.2. *Let k and n be positive integers such that $k \mid F_n$. Then*

$$\gcd\left(F_n, \frac{F_{kn}}{kF_n}\right) = 1.$$

Proof. Assume that $k \mid F_n$. Applying the same expansion formula of Fibonacci numbers as in the proof of Lemma 5.1, we have

$$F_{kn} \equiv \binom{k}{1} F_n F_{n-1}^{k-1} + \binom{k}{2} F_n^2 F_{n-1}^{k-2} \equiv kF_n F_{n-1}^{k-1} + \frac{k(k-1)}{2} F_n^2 F_{n-1}^{k-2} \pmod{F_n^3}.$$

Thus,

$$\frac{F_{kn}}{kF_n} \equiv F_{n-1}^{k-1} + \frac{(k-1)}{2} F_n F_{n-1}^{k-2} \pmod{\frac{F_n}{k} F_n}, \quad \text{so that} \quad \frac{F_{kn}}{kF_n} \equiv F_{n-1}^{k-1} \pmod{F_n}.$$

We therefore have $\gcd\left(\frac{F_{kn}}{kF_n}, F_n\right) = \gcd(F_{n-1}^{k-1}, F_n) = 1$, where we have used a well-known fact which states that $\gcd(F_{n-1}, F_n) = 1$ for all n.

□

Lemma 5.3. *Let k and n be positive integers. Then*

$$\gcd\left(F_{nH_k(n)}, T_n\right) = 1.$$

Proof. Let k and n be positive integers. Then, since (F_n) is a strong divisibility sequence, we have

$$\gcd\left(F_{nH_k(n)}, F_{5n}\right) = F_{\gcd(nH_k(n), 5n)} = F_{n \gcd(H_k(n), 5)} = F_n, \tag{5.1}$$

where the last equality follows from the fact that $5 \nmid H_k(n)$. Let

$$d = \gcd\left(F_{nH_k(n)}, T_n\right) = \gcd\left(F_{nH_k(n)}, \frac{F_{5n}}{5F_n}\right).$$

Then, by (5.1), we have $d \mid F_n$. Therefore, d is a common divisor of F_n and T_n. However, since $\gcd(F_n, T_n) = 1$ (by Theorem 3.1), it follows that $d = 1$. Hence, the proof is complete. □

The following lemma appeared as a step of the proof of a lemma in [6]. We repeat its proof here for the sake of completeness.

Lemma 5.4. *Let m, k, and ℓ be positive integers with $\ell \geq 3$, then*

$$\gcd(m^k, \ell) \mid m^{\ell-2}.$$

Proof. Let p be a prime divisor of m. Let $r = v_p(\gcd(m^k, \ell))$ and $s = v_p(m^{\ell-2})$. It suffices to prove that $r \leq s$. We have

$$m^k = p^{r+i} c_1 \quad \text{and} \quad \ell = p^{r+j} c_2,$$

where $i, j \geq 0$ and $\gcd(p, c_1) = \gcd(p, c_2) = 1$. We see that

$$s = \frac{r+i}{k}(p^{r+j} c_2 - 2).$$

Now since $\frac{r+i}{k} \geq 1$, it suffices to show that $r \leq p^{r+j} c_2 - 2$. Since $\ell \geq 3$, the statement is true when $r = 1$, and so we may assume that $r \geq 2$. Then

$$p^{r+j} c_2 - 2 \geq p^r - 2 \geq 2^r - 2 \geq r.$$

Hence, $r \leq s$, as desired. □

In the proof of the next two results, we identify the usual subscript notation of a term of sequence with its functional notation for the sake of readability.

Theorem 5.5. *Let k and n be positive integers. Then*

$$T_n^k \mid H_k(n).$$

Proof. Let n be a positive integer. We will prove the statement by induction on k. The case when $k = 1$ is clear. For the inductive step, we assume that $T^k(n) \mid H_k(n)$ for some $k \geq 1$. Then, by Lemma 4.1, $T(nT^k(n)) \mid H_{k+1}(n)$. Therefore, it suffices to prove that $T^{k+1}(n) \mid T(nT^k(n))$. The expansion formula of the Fibonacci numbers yields

$$\begin{aligned} F(5nT^k(n)) &= \sum_{j=1}^{T^k(n)} \binom{T^k(n)}{j} F^j(5n)F^{T^k(n)-j}(5n-1)F(j) \\ &= \sum_{j=1}^{T^k(n)} \frac{T^k(n)}{\gcd(T^k(n),j)} a_j F^j(5n)F^{T^k(n)-j}(5n-1)F(j), \quad a_j \in \mathbb{N} \\ &= \sum_{j=1}^{T^k(n)} T^k(n) a_j F(5n) b_j, \quad b_j \in \mathbb{N} \\ &= T^{k+1}(n)d, \quad d \in \mathbb{N}, \end{aligned}$$

where we have used a result by Hermite [4] in the second equality. The third equality follows from the definition of $T(n)$ and the fact that $\gcd(T^k(n), j)$ divides $T^{j-1}(n)$ for all j. Indeed, the case $j = 1$ is obvious; for the case $j = 2$, we have $\gcd(T^k(n), j) = 1$, since $T(n)$ is odd; for the case $j \geq 3$ we apply Lemma 5.4. The last equality follows again from the definition of $T(n)$. Therefore,

$$T(nT^k(n)) = \frac{F(5nT^k(n))}{5F(nT^k(n))} = \frac{T^{k+1}(n)d}{5F(nT^k(n))}.$$

Since $(F(n))$ is a divisibility sequence, we have $\gcd(F(nT^k(n)), F(5n)) = F(\gcd(nT^k(n), 5n)) = F(n\gcd(T^k(n), 5)) = F(n)$. Thus, $(F(nT^k(n)), T(n)) = 1$, so that $\frac{d}{5F(nT^k(n))}$ is an integer. This proves that $T^{k+1}(n) \mid T(nT^k(n))$, as desired. This establishes the inductive step and the proof by induction is complete. □

Theorem 5.6. *Let $n \geq 2$ and k be positive integers. Then*

$$T_n^k \parallel H_k(n).$$

Proof. Let $n \geq 2$ be a positive integer. We will prove the statement by induction on k. The case when $k = 1$ is obvious. For the inductive step, we assume that $T^k(n) \parallel H_k(n)$. We want to show that $T^{k+1}(n) \parallel H_{k+1}(n)$. We have

$$H_{k+1}(n) = \frac{F(5nH_k(n))}{5F(nH_k(n))}.$$

Let the numerator be denoted by P. By Lemma 5.3, it suffices to show that $T^{k+1}(n) \parallel P$. The expansion formula of the Fibonacci numbers, together with Theorem 5.5, yields, after taking modulo $T^{k+2}(n)$

$$P \equiv H_k(n)F(5n)F^{H_k(n)-1}(5n-1) + \frac{H_k(n)(H_k(n)-1)}{2} F^2(5n)F^{H_k(n)-2}(5n-1). \quad (5.2)$$

Since $H_k(n)$ is odd, as it is a term of $(T(n))$, we have $2 \mid H_k(n) - 1$ and therefore,

$$T^{k+2}(n) \mid H_k(n) \left(\frac{H_k(n) - 1}{2} \right) F^2(5n).$$

However, $T^{k+1}(n)$ exactly divides the first summand in (5.2). Therefore, $T^{k+1}(n) \parallel P$, as required. Hence, the proof by induction is complete. □

Theorem 5.7. *For each nonnegative integer i, let N_i be the set of all positive integers n such that $v_5(n) = i$ and let $\mathcal{T}_i$ be the set of all T_n with $n \in N_i$. Then the following statements hold.*

(1) *The collection $\{\mathcal{T}_i\}_{i \geq 0}$ partitions the image set of the sequence (T_n), i.e.,*

$$\{T_n \,:\, n \in \mathbb{N}\} = \bigcup_{i \geq 0} \mathcal{T}_i.$$

(2) *For each nonnegative integer i and positive integer k, we have*

$$H_k(N_i) \subset \mathcal{T}_i,$$

where $H_k(N_i) = \{H_k(n) \,:\, n \in N_i\}$.

Proof. The first statement follows directly from Theorem 4.3. To prove the second statement, we let i be a nonnegative integer and let $n \in N_i$. It suffices to prove that $H_k(n) \in \mathcal{T}_i$ for each $k \in \mathbb{N}$. For $k = 1$, the result is clear. Assume $k \geq 2$. For $n = 1$, the statement is obvious. Assume $n \geq 2$. Then, Theorem 5.5 implies that $\gcd(H_k(n), T_n) = T_n > 1$. Since $H_k(n) = T_{nH_{k-1}(n)}$, Theorem 4.3 implies that $v_5(n) = v_5(nH_{k-1}(n))$. By the definition, this yields $H_k(n) = T_{nH_{k-1}(n)} \in \mathcal{T}_i$. Hence, the statement follows. □

6. The Infinitude of Certain Primes

Upon considering some arithmetic properties of (T_n), one might be led to ask the question: Is the set of all prime divisors of the sequence (T_n) infinite? To answer this question, we need the following theorem by Carmichael [1, 9].

Theorem 6.1. *For a positive integer $n \neq 1, 2, 6, 12$, the Fibonacci number F_n has a prime divisor which does not divide any earlier Fibonacci number.*

The next theorem gives the analogue of this theorem for the sequence (T_n) and therefore provides the affirmative answer to the underlying question above.

Theorem 6.2. *For a positive integer $n \neq 1$, the term T_n has a prime divisor which does not divide any earlier term of (T_n).*

Proof. Let $n \neq 1$ be a positive integer. Then $5n \neq 1, 2, 6, 12$, so that Theorem 6.1 implies F_{5n} has a prime divisor which does not divide any F_k for all $k < 5n$. Since $T_n = F_{5n}/5F_n$, it follows that T_n has a prime divisor which does not divide any T_k for all $k < 5n$. In particular, since $n < 5n$, this implies T_n has a prime which does not divide any T_k for all $k < n$. □

In light of Theorem 3.2 which characterizes the prime divisors of the sequence (T_n), some interesting corollaries of this result follows.

Corollary 6.3. *Let $\mathcal{P}$ be the set of all primes p of the form $p = 10k + 1$ with $5 \mid Z(p)$. Then the set $\mathcal{P}$ is infinite.*

Proof. By Theorem 3.2, $\mathcal{P}$ is exactly the set of all prime divisors of the terms of (T_n). However, by Theorem 6.2, this set is known to be infinite (as T_n has a new prime divisor for each $n > 1$). Hence the conclusion follows. □

Corollary 6.4. *For each nonnegative integer i, let P_i be the set of all primes p such that $p \mid T_n$ with $v_5(n) = i$. Then the set P_i is infinite.*

Proof. Let i be a nonnegative integer. Let (m_j) be a subsequence of the sequence of all positive integers with the property that $\gcd(5, m_j) = 1$ for each j. Then P_i is the set of all prime divisors of the terms of (T_{n_j}) with $n_j = 5^i m_j$ for each $j \in \mathbb{N}$. Consequently, Theorem 6.2 implies that P_i is an infinite set, since, for each $j > 1$, the term T_{n_j} has a prime divisor that has never occurred before in prime factorization of T_{n_k} with $k < j$. □

Remark 6.5. *We have learned from Theorem 6.2 that the set $\mathcal{P}$ of all prime divisors of the terms of (T_n) is infinite. In fact, by Theorem 4.3, it is not difficult to see that the set $\mathcal{P}$ can be partitioned by the set of P_i's defined in Corollary 6.4, i.e.,*

$$\mathcal{P} = \bigcup_{i \geq 0} P_i,$$

where P_i's are nonempty and pairwise disjoint.

Acknowledgements

The first author is supported by Research Fund for DPST Graduate with First Placement 032/2555. He thanks Somkid Amornsamarnkul for his invaluable guidance. The second author is supported by MUIC Seed Grant Research Fund. We express our gratitude toward the anonymous referee for suggestions leading to improvements of this work.

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MSC2010: 11B39

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