ON PERIODS OF FIBONACCI SEQUENCES AND REAL QUADRATIC p-RATIONAL FIELDS

ZAKARIAE BOUAZZAOUI

Abstract. We relate the p-rationality of a real quadratic field K to properties of periods of a Fibonacci sequence associated to K.

1. Introduction

Let p be an odd prime number. A number field K is said to be p-rational if the Galois group of the maximal pro-p-extension of K which is unramified outside p is a free pro-p-group of rank r_2+1, where r_2 is the number of pairs of complex embeddings of K. The notion of p-rational number fields has been introduced by Movahhedi and Nguyen Quang Do [M-N], [Mo88], [Mo90], and was used for the construction of non-abelian extensions satisfying Leopoldt's conjecture. Greenberg [G] used complex abelian p-rational number fields for the construction of p-adic Galois representations with open images. In these notes, we focus on the p-rationality of real quadratic number fields. In [G, Corollary 4.1.5], Greenberg relates the p-rationality of the field $\mathbf{Q}(\sqrt{5})$ to properties of the classical Fibonacci numbers. In [B], the author gave a generalization of this result to any real quadratic field.

Let $d>0$ be a fundamental discriminant. Denote by ε_d and h_d the fundamental unit and the class number of the field $\mathbf{Q}(\sqrt{d})$ respectively, and let $N(.)$ be the absolute norm. We associate to the field $\mathbf{Q}(\sqrt{d})$ a Fibonacci sequence $F^{(\varepsilon_d+\overline{\varepsilon}_d,-N(\varepsilon_d))}=(F_n)_{n\geq 0}$ defined by $F_0=0$, $F_1=1$ and the recursion formula

$$F_{n+2}=(\varepsilon_d+\overline{\varepsilon}_d)F_{n+1}-N(\varepsilon_d)F_n,\ for\ n\geq 0.$$

Let $p\geq 5$ be a prime number such that $p\nmid(\varepsilon_d-\overline{\varepsilon_d})^2h_d$. Then,

$$\mathbf{Q}(\sqrt{d}) \text{ is } p\text{-rational if and only if } F_{p-(\frac{d}{p})}\equiv 0 \pmod{p^2}, \tag{1.1}$$

where $(\frac{d}{\cdot})$ denotes the Legendre symbol (see [B, Theorem 3.4]). The proof of this result uses mainly the equivalence (3.5) below, which relates the p-rationality of the quadratic field to congruence modulo p^2 of the p-adic regulator, hence to congruence modulo p^2 of powers of the fundamental unit, which is related to Fibonacci numbers.

For any positive integer m, the Fibonacci sequence $(F_n)_n$ modulo m is periodic. We define its period to be the smallest positive integer s for which $F_s\equiv 0 \pmod m$ and $F_{s+1}\equiv 1 \pmod m$. Periods of Fibonacci sequences have been studied for first by [Wall] for the classical Fibonacci sequence.

Using the equivalence (1.1), we prove a result which describes the p-rationality of real quadratic number fields in terms of a property of the periods of Fibonacci sequences.

Theorem 1.1. *Let $d>0$ be a fundamental discriminant. For every odd prime number p such that $p\nmid(\varepsilon_d-\overline{\varepsilon_d})^2h_d$, let $k(p)$ and $k(p^2)$ be the periods of the Fibonacci sequence associated to*

We thank the 19th International Fibonacci Conference referee for useful comments on an earlier version.

the field $\mathbf{Q}(\sqrt{d})$ *modulo* p *and* p^2 *respectively. The quadratic field* $\mathbf{Q}(\sqrt{d})$ *is* p*-rational precisely when* $k(p) \neq k(p^2)$.

The numerical examples we obtain support a conjecture of Gras asserting that a number field is p-rational for almost all primes p [Gr].

2. Real quadratic fields and Fibonacci numbers

The classical Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$ and the recurrence relation

$$F_{n+1} = F_n + F_{n-1}.$$

For every integer $n \geq 0$, the values of F_n are encoded in the powers of the matrix $T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, i.e., for any $n \geq 0$ we have

$$T^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

As a generalization we define, for any integers a and b, a Fibonacci sequence in the following way: $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = aF_n + bF_{n-1}$. The terms of the above sequence are generated by the integer powers of the matrix $U = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$, where for any $n \geq 0$ wee have

$$U^n = \begin{pmatrix} bF_{n-1} & F_n \\ bF_n & F_{n+1} \end{pmatrix}.$$

For a positive integer m, we define the period $k(m)$ of $(F_n)_n$ modulo m to be the smallest positive integer s for which

$$U^s \equiv I \pmod{m},$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

These periods have been studied at first for the classical Fibonacci sequence by D.D. Wall in [Wall]. For general Fibonacci sequences, see [R], [E-J], and [D-R].

Formulas are known for computing $k(m)$ based on the prime factorization of m, but if p is prime,there is no formula for $k(p)$. However, certain divisibility relations hold. Here we are interested in $k(p)$ and $k(p^2)$. The following theorem lists some of their properties for general Fibonacci sequences. If p is a prime number and b in an integer, we set $ord_p(b)$ for the order of b in the group $\mathbf{Z}/p\mathbf{Z}$.

Theorem 2.1. *Let* $(F_n)_n$ *be a Fibonacci sequence associated to the coprime integers* a, b *and let* U *be the corresponding matrix. Let* $\Delta := a^2 - 4b$ *and denote* $lcm(n, m)$ *for the least common multiple of* n *and* m.

1. $U^n \equiv I \pmod{m} \Leftrightarrow k(m) \mid n$.
2. *If* $m = p_1^{n_1} \cdots p_s^{n_s}$, *then* $k(m) = lcm(k(p_1^{n_1}), ..., k(p_s^{n_s}))$.
3. *If* $k(p) \neq k(p^2)$, *then* $k(p^2) = pk(p)$.
4. *if* Δ *is a(nonzero)quadratic residue modulo* p, *then* $k(p) \mid p - 1$.
5. *if* Δ *is a quadratic nonresidue modulo* p, *then* $k(p) \mid (p+1) \cdot ord_p(-b)$; *except if* $b \equiv -1 \pmod{p}$ *in which case* $k(p) \mid p + 1$.
6. *if* $\Delta \equiv 0 \pmod{p}$, *then* $k(p) = p \cdot ord_p(2^{-1}a)$.

Proof. All statements are proved in [R]. □

In [Wall], Wall raised the question whether the equality $k(p) = k(p^2)$ holds for the classical Fibonacci sequence and some prime number p? At this time, no such primes exist for classical Fibonacci sequence. The question for general Fibonacci sequences, as defined in these notes, has an affirmative answer for many examples (see section 5).

3. Real quadratic p-rational fields

In this section we give a characterization of the p-rationality of real quadratic fields in terms of values of the associated L-functions at odd negative integers. In fact, the p-rationality of totally real abelian number fields K is intimately related to special values of the associated zeta functions ζ_K. The relation is as follows. For any finite set Σ of primes of K, we denote by $G_\Sigma(K)$ the Galois group of the maximal pro-p-extension of K which is unramified outside Σ. Let S be the finite set of primes $S_p \cup S_\infty$, where S_∞ is the set of infinite primes of K and S_p is the primes above p in K. It is known that the group $G_{S_p}(K)$ is a free pro-p-group on $r_2 + 1$ generators if and only if the second Galois cohomology group $H^2(G_{S_p}(K), \mathbf{Z}/p\mathbf{Z})$ vanishes. This vanishing is related to special values of the zeta function ζ_K via the conjecture of Lichtenbaum. More precisely, let $\mathcal{G}_S$ be the Galois group of the maximal extension of K which is unramified outside S. The main conjecture of Iwasawa theory (now a theorem of Wiles [W90]) relates the order of the group $H^2(\mathcal{G}_S, \mathbf{Z}_p(i))$, for even integers i, to the p-adic valuation of $\zeta_K(1-i)$ by the p-adic equivalence:

$$w_i(K)\zeta_K(1-i) \sim_p |H^2(\mathcal{G}_S, \mathbf{Z}_p(i))|, \tag{3.1}$$

where for any integer i, $w_i(F)$ is the order of the group $H^0(G_F, \mathbf{Q}_p/\mathbf{Z}_p(i))$, and $\sim_p$ means having the same p-adic valuation, see e.g [Kol]. Moreover, the group $H^2(\mathcal{G}_S, \mathbf{Z}_p(i))$ vanishes if and only if $H^2(\mathcal{G}_S, \mathbf{Z}/p\mathbf{Z}(i))$ vanishes. Let μ_p be the group of p-th roots of unity. The periodicity of the groups $H^2(\mathcal{G}_S, \mathbf{Z}/p\mathbf{Z}(i))$ modulo $\delta = [K(\mu_p) : K]$ gives that

$$H^2(\mathcal{G}_S, \mathbf{Z}/p\mathbf{Z}(i)) \cong H^2(\mathcal{G}_S, \mathbf{Z}/p\mathbf{Z}(i + j\delta)),$$

for any integer j. In addition, since p is odd, the vanishing of the group $H^2(\mathcal{G}_S, \mathbf{Z}/p\mathbf{Z}(i))$ is equivalent to the vanishing of the group $H^2(G_{S_p}(K), \mathbf{Z}/p\mathbf{Z}(i))$. Number fields such that $H^2(G_{S_p}(K), \mathbf{Z}/p\mathbf{Z}(i)) = 0$ are called (p, i)-regular [A]. In particular, the field K is p-rational if and only if $w_{p-1}(K)\zeta_K(2-p) \sim_p 1$. This leads to the following characterization of the p-rationality of totaly real number fields.

Proposition 3.1. *Let p be an odd prime number which is unramified in an abelian totally real number field K. Then we have the equivalence*

$$K \textit{ is } p\textit{-rational} \Leftrightarrow L(2-p, \chi) \textit{ is a } p\textit{-adic unit}, \tag{3.2}$$

where χ is ranging over the set of irreducible characters of $\mathrm{Gal}(K/\mathbf{Q})$.

Proof. First, the zeta function ζ_K decomposes in the following way:

$$\zeta_K(2-p) = \zeta_{\mathbf{Q}}(2-p) \times \prod_{\chi \neq 1} L(2-p, \chi).$$

Second, it is known that $\zeta_{\mathbf{Q}}(2-p)$ is of p-adic valuation -1 and that $w_{p-1}(K)$ has p-adic valuation 1, giving that $w_{p-1}(K)\zeta_{\mathbf{Q}}(2-p) \sim_p 1$. Then from (3.1) we obtain the formula

$$\prod_{\chi \neq 1} L(2-p, \chi) \sim_p |H^2(\mathcal{G}_S, \mathbf{Z}_p(p-1))|.$$

Since, for every character $\chi \neq 1$, the value $L(2-p,\chi)$ is a p-integer [Wa, Corollary 5.13], we have $H^2(\mathcal{G}_S, \mathbf{Z}_p(p-1)) = 0$ if and only if for every $\chi \neq 1$, $L(2-p,\chi)$ is a p-adic unit. Furthermore, the vanishing of the group $H^2(\mathcal{G}_S, \mathbf{Z}_p(p-1))$ is equivalent to the vanishing of the group $H^2(\mathcal{G}_S, \mathbf{Z}/p\mathbf{Z}(p-1))$, which turns out to be equivalent to the vanishing of $H^2(G_{S_p}(K), \mathbf{Z}/p\mathbf{Z})$ (by the above mentioned periodicity statement). This last vanishing occurs exactly when the field K is p-rational. □

In the particular case of a real quadratic field $K = \mathbf{Q}(\sqrt{d})$, we have the decomposition

$$\zeta_K(2-p) = \zeta_{\mathbf{Q}}(2-p)L(2-p, (\tfrac{d}{\cdot})),$$

where $(\frac{d}{\cdot})$ is the quadratic character associated to the field $K = \mathbf{Q}(\sqrt{d})$.

Corollary 3.2. *For every odd prime number $p \nmid d$, we have the equivalence*

$$\mathbf{Q}(\sqrt{d}) \textit{ is } p\textit{-rational} \Leftrightarrow L(2-p, (\tfrac{d}{\cdot})) \not\equiv 0 \pmod{p}. \tag{3.3}$$

The properties of special values of p-adic L-functions tells us that the p-rationality is related to the class number and the p-adic regulator. More precisely, let K be a totally real number field of degree g. Under the Leopoldt conjecture, class field theory gives that $G_{S_p}(K)^{ab} \cong \mathbf{Z}_p^{r_2+1} \times \mathcal{T}_K$, where $\mathcal{T}_K$ is the $\mathbf{Z}_p$-torsion of $G_{S_p}(K)^{ab}$. Then the field K is p-rational precisely when $\mathcal{T}_K = 0$ [M-N, Théorème et Definition 1.2]. Moreover, the order of $\mathcal{T}_K$ satisfies

$$|\mathcal{T}_K| \sim_p w(K(\mu_p)) \prod_{v|p} (1 - N(v)^{-1}) \cdot \frac{R_p(K).h_K}{\sqrt{|d_k|}}, \tag{3.4}$$

([Coa, app]), where h_K is the class number, $R_p(K)$ is the p-adic regulator, $N(v)$ is the absolute norm of v, $w(K(\mu_p)) = |\mu(K(\mu_p))|$ the number of roots of unity of $K(\mu_p)$ and d_K is the discriminant of the number field K. Hence for every odd prime number p such that $(p, d_K h_K) = 1$, the field K fails to be p-rational if and only if $v_p(R_p(K)) > g - 1$.

Under the light of the above discussion, for a real quadratic field $\mathbf{Q}(\sqrt{d})$ we have the equivalence:

$$\mathbf{Q}(\sqrt{d}) \text{ is } p-\text{rational} \Leftrightarrow R_p(\mathbf{Q}(\sqrt{d})) \not\equiv 0 \pmod{p^2}. \tag{3.5}$$

Recall that $R_p(\mathbf{Q}(\sqrt{d})) = \log_p(\varepsilon_d)$, where ε_d is a fundamental unit of K and $\log_p$ is the p-adic logarithm.

4. A Wieferich phenomenon for p-rational fields

In 1909 Arthur Wieferich proved that if a prime number p satisfies

$$\frac{2^{p-1}-1}{p} \not\equiv 0 \pmod{p}, \tag{4.1}$$

then the first case of Fermat's last theorem is true at p. This was generalized by many authors, allowing 2 to be replaced by other primes. The only known primes satisfying (4.1) are 1093 (Meissner in 1913) and 3511 (Beeger in 1922), the next one, if any, must exceed 6.7×10^{15}.

Since Fermat's last theorem is proved in general, the motivation for going further disappeared, but the problem of understanding the distribution of primes for which (4.1) holds remains open, and seems to be interesting in itself. Here we mention two meaningful results about these primes, the first one is due to Heath-Brown [Heath], and the second is a result of Silverman [Si]. For this, we give the following general definition.

Definition 4.1. *For a nonzero integer a and a prime p not dividing a, we form the "Fermat quotient"*

$$q_p(a) := \frac{a^{p-1}-1}{p} \pmod{p}. \tag{4.2}$$

We say that p is a Wieferich prime of basis a if we have $q_p(a) \equiv 0 \pmod{p}$ and non-Wieferich otherwise.

The theorem of Heath-Brown establishes an equidistribution result for the Fermat quotients.

Theorem 4.2. [Heath, Theorem 2] *Let p be an odd prime number. The values $q_p(a)$ are uniformaly distributed modulo p for $1 \leq a < p$.*

The result of Silverman gives a lower bound for the number of non-Wieferich primes for a fixed basis. More precisely, we have:

Theorem 4.3. [Si, Theorem 2] *Let a be an integer such that $a \neq \pm 1$. If the abc-conjecture is true, then*

$$|\{p \leq X \ : \ q_p(a) \not\equiv 0 \pmod{p}\}| \gg_a \log{(X)}.$$

Silverman observed that the statement of this result also holds for commutative algebraic groups over $\mathbf{Q}$. In particular, for the group of points of an elliptic curve over $\mathbf{Q}$. For our purposes we consider groups of units of real quadratic fields and study the Wieferich primes for elements of infinite order in such groups and relate these primes to the primes for which the quadratic field is p-rational.

For this, let $d > 0$ be a fundamental discriminant and let ε be a unit of infinite order in the quadratic field $\mathbf{Q}(\sqrt{d})$.

Definition 4.4. *A rational odd prime p is said to be Wieferich of basis ε if the congruence*

$$\frac{\varepsilon^{p^r-1}-1}{p} \equiv 0 \pmod{p}$$

holds, where r is the residue degree of p in the quadratic field $\mathbf{Q}(\sqrt{d})$.
Otherwise, the prime number p is said to be non-Wieferich of basis ε.

The recent works of Boeckle et al. [B-G-K-K] also relate p-rationality to Wieferich phenomenon. C. Maire and M. Rougnant [M-R], assuming the *abc*-conjecture for number fields, obtain the analogue of Silverman bound.

For a quadratic field $K = \mathbf{Q}(\sqrt{d})$ we denote by ε_d and h_d the fundamental unit and the class number of K respectively.

Proposition 4.5. *Let p be an odd prime number such that $p \nmid dh_d$. Then the field $\mathbf{Q}(\sqrt{d})$ is p-rational if and only if p is a non-Wieferich prime of basis ε_d.*

Proof. To characterize the p-rationality of real quadratic fields in terms of non-Wieferich primes, we use the equality

$$\log_p((\varepsilon_d^{p^r-1}-1)+1) = (\varepsilon_d^{p^r-1}-1) - \frac{1}{2}(\varepsilon_d^{p^r-1}-1)^2 + ... \tag{4.3}$$

where $\log_p$ is the p-adic logarithm and as before r is the residue degree of p in the quadratic field $\mathbf{Q}(\sqrt{d})$. We denote $\mathcal{O}_d$ for the ring of integers of $\mathbf{Q}(\sqrt{d})$.

Note that if p is inert, then $(\mathcal{O}_d/p\mathcal{O}_d)^\times$ is a cyclic group of order p^2-1, but if p splits into two prime ideals, $(\mathcal{O}_d/p\mathcal{O}_d)^\times$ is a product of two cyclic groups of order $p-1$. Then, these facts and (4.3) show that

$$R_p(K) \equiv p\kappa \pmod{p^2},$$

where $R_p(K) = \log_p(\varepsilon_d)$ and κ is an element of $\mathcal{O}_d$. Hence, we have the equivalence

$$\frac{\varepsilon_d^{p^r-1}-1}{p} \not\equiv 0 \pmod{p} \quad \Leftrightarrow \quad R_p(K) \not\equiv 0 \pmod{p^2}. \tag{4.4}$$

Then combining this last equivalence with the equivalence (3.5) we obtain the desired result. □

This proposition leads us to relate the p-rationality of real quadratic fields to congruences of generalized Fibonacci numbers, hence to periods of Fibonacci numbers modulo p and p^2. This will be the subject of the next section.

5. Periods and p-rationality

In this section, we associate to a real quadratic field $\mathbf{Q}(\sqrt{d})$ a Fibonacci sequence $(F_n)_n$ defined as follows: $F_0 = 0$, $F_1 = 1$ and

$$F_{n+1} = (\varepsilon_d - \overline{\varepsilon_d})F_n - N_{\mathbf{Q}(\sqrt{d})}(\varepsilon)F_{n-1}.$$

Proof of Theorem 1.1. Using Theorem 3.4 of [B], it suffices to prove the following equivalence,

$$k(p) \neq k(p^2) \text{ if and only if } F_{p-(\frac{d}{p})} \not\equiv 0 \pmod{p^2}. \tag{5.1}$$

Suppose that $k(p) = k(p^2)$. Since $k(p) \mid p - (\frac{d}{p})$, we obtain $F_{p-(\frac{d}{p})} \equiv 0 \pmod{p^2}$. Then we have

$$F_{p-(\frac{d}{p})} \not\equiv 0 \pmod{p^2} \;\Rightarrow\; k(p) \neq k(p^2).$$

We suppose now that $k(p) \neq k(p^2)$, and assume to the contrary that $F_{p-(\frac{d}{p})} \equiv 0 \pmod{p^2}$. Then we have $k(p^2) \mid p - (\frac{d}{p})$, which contradicts the fact that $k(p^2) = pk(p)$. □

For the classical Fibonacci sequence, i.e. the sequence associated to the quadratic field $\mathbf{Q}(\sqrt{5})$, D.D. Wall asked in [Wall] the question whether they exist primes p such that $k(p) = k(p^2)$. Up to 10^{15}, such primes do not exist. It is likely to conjecture that the field $\mathbf{Q}(\sqrt{5})$ is p-rational for every prime number p. For other real quadratic fields, we have examples of primes p for which the equality $k(p) = k(p^2)$ holds. This is the case for $\mathbf{Q}(\sqrt{2})$ and the primes $p = 13$ and $p = 103$. The computations suggest that the set of primes for which the equality holds is small, i.e. it has density zero in the set of all primes. This is in accordance with a conjecture of Gras which asserts that a number field K is p-rational for almost all primes p, except possibly a set of primes of density zero.

We could compute for a given real quadratic field $K = \mathbf{Q}(\sqrt{d})$ ($d < 10^3$) the set of primes $p < 10^9$ for which K is not p-rational, hence $k(p) = k(p^2)$ for the corresponding Fibonacci sequence. We list here some examples. The computations are performed by pariGP program and are based on the computation of the fundamental units and the congruence modulo p of the associated Fermat quotient.

Discriminant	Primes
5	
8	13, 31, 1546463
12	103
13	241
17	
21	46179311
24	7, 523
28	
29	3, 11
33	29, 37, 6713797
37	7, 89,257, 631
40	191, 643, 134339, 25233137
41	29, 53, 7211
44	
53	5
56	6707879, 93140353
57	59, 28927, 1726079, 7480159
60	181, 1039, 2917, 2401457
61	

For higher dimensional totally real number fields, it would be of interest to obtain analogous characterizations of the p-rationality, for example for bi-quadratic and cubic totally real fields. In the case of imaginary quadratic fields, the methods used here don't apply, since the p-rationality of such fields is related only to the p-divisibility of their class numbers (cf. [G]). Another problem that could be considered is the following. Consider a set of local conditions, i.e. conditions on the ramification, the decomposition and the inertia of a finite set of primes. For a prime number p, can one find a real quadratic field with these local conditions and whose Fibonacci sequence satisfies $k(p) \neq k(p^2)$?

References

[A] J. Assim, Codescente en K-théorie étale et corps de nombres, *Manuscripta Math.* **86** (1995), 499–518.

[A-B] J. Assim and Z. Bouazzaoui, Half-integer weight modular forms and p-rational fields, *Funct. Approx. Comment. Math.* (to appear).

[B-G-K-K] G. Boeckle, D. Guiraud, S. Kalyanswamy, and C. Khar, Wieferich primes and a mod p Lepoldt conjecture. arXiv:1805.00131v1 (2018).

[B] Z. Bouazzaoui, Fibonacci numbers and real quadratic p-rational fields, *Period. Math. Hungar.* **81** (2020) 123–133.

[Coa] J. Coates, p-adic L-functions and Iwasawa's theory. In: *Algebraic number fields: L-functions and Galois properties* (Proc. Sympos., Univ. Durham, Durham, 1975), pp. 269–353. Academic Press, London, 1977.

[D-R] D. Kalman and R. Mena, The Fibonacci numbers: exposed, *Math. Mag.* **76** (2003), no. 3, 167–181.

[E-J] A. S. Elsenhans and J. Jahnel, The Fibonacci sequence modulo p^2 - An investigation by computer for $p < 10^{14}$. `arXiv:1006.0824` (2010).

[F-K] F. G. Dorais and D. Klyve, A Wieferich prime search up to 6.7×10^{15}, *J. Integer Seq.* **14** (2011), no. 9, Article 11.9.2, 14 pp.

[Gr] G. Gras, Les θ-régulateurs locaux d'un nombre algébrique: conjectures p-adiques. *Canad. J. Math.* **68** (2016), 571–624.

[G] R. Greenberg, Galois representations with open image. *Ann. Math. Qué.* **40** (2016), 83–119.

[G-M] H. Graves and M. Ram Murty, The abc conjecture and non-Wieferich primes in arithmetic progressions, *J. Number Theory* **133** (2013), no. 6, 1809–1813.

[Ha] K. Hatada, Mod 1 distribution of Fermat and Fibonacci quotients and values of zeta functions at $2-p$, *Comment. Math. Univ. St. Pauli* **36** (1987), 41–51.

[Heath] D. R. Heath-Brown, An estimate for Heilbronn's exponential sum. *Analytic number theory, Vol. 2* (Allerton Park, IL, 1995), 451–463, Progr. Math., 139, Birkhäuser Boston, Boston, MA, 1996.

[Kol] M. Kolster, K-theory and arithmetic. In: *Contemporary developments in algebraic K-theory*, 191–258, ICTP Lect. Notes, XV, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[M-R] C. Maire and M. Rougnant, A note on p-rational fields and the *abc*-conjecture, *Proc. Amer. Math. Soc.* **148** (2020), no. 8, 3263–3271.

[Mo88] A. Movahhedi, Sur les p-extensions des corps p-rationnels. Thèse, Paris VII (1988).

[Mo90] A. Movahhedi, Sur les p-extensions des corps p-rationnels, *Math. Nachr.* **149** (1990), 163–176.

[M-N] A. Movahhedi and Q. D. Nguyen, Sur l'arithmètique des corps de nombres p-rationnels. *Sém. de théorie des nombres de Paris*, 1987–1988, Bikhäuser, Vol. 81, 155–200 (1990).

[P] PARI Group. Bordeaux. PARI/GP. Version 2.10.0, (2017).

[R] M. Renault, The period, rank, and order of the (a, b)-Fibonacci sequence mod m, *Math. Mag.* **86** (2013), no. 5, 372–380.

[Si] J. H. Silverman, Wieferich's criterion and the abc-conjecture, *J. Number Theory* **30** (1988), no. 2, 226–237.

[Wall] D. .D Wall, Fibonacci series modulo m, *Amer. Math. Monthly* **67** (1960), 525–532.

[Wa] L. C. Washington, *Introduction to Cyclotomic Fields.* (2nd ed.), Springer-Verlag, 1997.

[W90] A. Wiles, The Iwasawa conjecture for totally real fields, *Ann. of Math. (2)* **131** (1990), no. 3, 493–540.

[Wi] H. C. Williams, Some formulas concerning the fundamental unit of a real quadratic field, *Discrete Math.* **92** (1991), no. 1–3, 431–440.

MSC2020: 11R80, 11B39, 19F27

Department of Mathematics, Moulay Ismail University, Meknes, Morocco
E-mail address: z.bouazzaoui@edu.umi.ac.ma