ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2020. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

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BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-835 Proposed by David M. Bloom, Brooklyn College of CUNY, Brooklyn, NY.
(Vol. 35.3, August 1997)

In a sequence of coin tosses, a *single* is a term (H or T) that is not the same as any adjacent term. For example, in the sequence HHTHHHTH, the singles are the terms in positions 3, 7, and 8. Let $S(n,r)$ be the number of sequences of n coin tosses that contain exactly r singles. If $n \geq 0$, and p is prime, find the value modulo p of $\frac{1}{2}\,S(n+p-1,p-1)$.

Editor's Note: This is another old problem whose solution has never appeared, so we would like to invite the readers to solve it.

B-1261 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Show that $\displaystyle\prod_{n=0}^{\infty} \frac{L_{3^n}^2+1}{L_{3^n}^2+3} = \prod_{n=0}^{\infty} \frac{5F_{3^n}^2-3}{5F_{3^n}^2-1}$, and determine the exact value of the limit.

B-1262 **Proposed by D. M. Bătineţu-Giurgiu, Mateo Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**

Compute $\lim_{n\to\infty}\left(\sqrt[3n+3]{(2n+1)!!\,F_{n+1}}-\sqrt[3n]{(2n-1)!!\,F_n}\right)\sqrt[3]{n^2}$.

B-1263 **Proposed by Stanley Rabinowitz, Milford, NH.**

Let P_n denote the nth Pell number. Find a recurrence relation for $X_n = F_n + P_n$.

B-1264 **Proposed by Pridon Davlianidze, Tbilisi, Republic of Georgia.**

Prove that

(A) $\displaystyle\prod_{n=2}^{\infty}\left(1+\frac{1}{F_{2n-1}^2}\right)=\frac{\alpha^2}{2}$

(B) $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{1}{F_{2n}^2}\right)=\frac{\alpha^2}{3}$

(C) $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{1}{F_{2n-1}^2}\right)\left(1+\frac{1}{F_{2n}^2}\right)=\frac{\alpha}{2}$

B-1265 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

For any integer $n \ge 1$, find a closed form expression for the sum $\displaystyle\sum_{k=1}^{n}\prod_{j=1}^{k}\left(L_{2^{j+1}}+L_{2^j}\right)$.

SOLUTIONS

An Oldie from the Vault

B-756 **Proposed by Stanley Rabinowitz, Chelmsford, MA.**
(Vol. 32.1, February 1994)

Find a formula expressing the Pell number P_n in terms of Fibonacci and/or Lucas numbers.

Solution 1 by G. C. Greubel, Newport News, VA.

Consider the generating functions of the Pell, Pell-Lucas, Fibonacci, and Lucas polynomials, which are given by

$$\sum_{n=0}^{\infty}P_n(x)t^n=\frac{t}{1-2xt-t^2},\qquad \sum_{n=0}^{\infty}Q_n(x)t^n=\frac{2(1-xt)}{1-2xt-t^2},$$
$$\sum_{n=0}^{\infty}F_n(x)t^n=\frac{t}{1-xt-t^2},\qquad \sum_{n=0}^{\infty}L_n(x)t^n=\frac{2-xt}{1-xt-t^2}.$$

Two relations will be given in this solution. For the first consider

$$\frac{t}{1-2xt-t^2}-\frac{t}{1-xt-t^2}=x\cdot\frac{t}{1-2xt-t^2}\cdot\frac{t}{1-xt-t^2},$$

which gives

$$\sum_{n=0}^{\infty}[P_n(x)-F_n(x)]\,t^n = x\left(\sum_{i=0}^{\infty}P_i(x)t^i\right)\left(\sum_{j=0}^{\infty}F_j(x)t^j\right).$$

After comparing coefficients of t^n, we obtain

$$P_n(x) = F_n(x) + x\sum_{s=0}^{n}P_{n-s}(x)\,F_s(x).$$

The second relation follows from

$$\frac{2-xt}{1-2xt-t^2} - \frac{2-xt}{1-xt-t^2} = x\cdot\frac{t}{1-2xt-t^2}\cdot\frac{2-xt}{1-xt-t^2},$$

which leads to

$$2P_{n+1}(x) - xP_n(x) = L_n(x) + x\sum_{s=0}^{n}P_{n-s}(x)\,L_s(x).$$

When $x=1$ these relations reduce to

$$P_n = F_n + \sum_{s=0}^{n}P_{n-s}F_s, \qquad \text{and} \qquad 2P_{n+1} - P_n = L_n + \sum_{s=0}^{n}P_{n-s}L_s.$$

Solution 2 by T. Goy, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

Define $D_0 = 1$, and let

$$D_n = \operatorname{perm}\begin{pmatrix} F_1 & 1 & 0 & \cdots & 0 & 0 \\ F_2 & F_1 & 1 & \cdots & 0 & 0 \\ F_3 & F_2 & F_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{n-1} & F_{n-2} & F_{n-3} & \cdots & F_1 & 1 \\ F_n & F_{n-1} & F_{n-2} & \cdots & F_2 & F_1 \end{pmatrix}.$$

It is easy to verify that $D_1 = P_1$ and $D_2 = P_2$, and $D_n = \sum_{i=1}^{n} F_i D_{n-i}$. It follows that

$$\begin{aligned} D_n &= F_1 D_{n-1} + \sum_{i=2}^{n}\big(F_{i-1}+F_{i-2}\big)D_{n-i} \\ &= D_{n-1} + \sum_{i=1}^{n-1}F_i D_{n-1-i} + \sum_{i=0}^{n-2}F_i D_{n-2-i} \\ &= 2D_{n-1} + D_{n-2}. \end{aligned}$$

Since D_n satisfies the Pell recurrence relation with $D_1 = P_1$ and $D_2 = P_2$, we determine that $D_n = P_n$ for all integers $n \geq 1$. Since

$$\operatorname{perm}\begin{pmatrix} a_1 & a_0 & \cdots & 0 & 0 \\ a_2 & a_1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{pmatrix} = \sum_{\substack{t_1,t_2,\ldots,t_n\geq 0 \\ t_1+2t_2+\cdots+nt_n=n}} a_0^{n-t_1-\cdots-t_n}\,\frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}\,a_1^{t_1}a_2^{t_2}\cdots a_n^{t_n},$$

we obtain

$$P_n = \sum_{\substack{t_1,t_2,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!} F_1^{t_1}F_2^{t_2}\cdots F_n^{t_n}.$$

Editor's Notes: Frontczak derived a similar result for the generalized Pell sequence with arbitrary initial values, thereby obtaining the same result in Solution 1. Using a different method, Edgar derived a formula in terms of compositions of n that can be expressed as

$$P_n = \sum_{k=1}^{n} \sum_{\substack{i_1,i_2,\ldots,i_k\geq 1\\ i_1+i_2+\cdots+i_k=n}} F_{i_1}F_{i_2}\cdots F_{i_k},$$

which is equivalent to Solution 2. Fedak and the proposer compared the Binet's formulas of P_n to those of F_n and L_n. Fedak found that

$$P_n = \frac{\sqrt{5}\,(\gamma^n+\delta^n)F_n + (\gamma^n-\delta^n)L_n}{4\sqrt{2}},$$

where $\gamma = \frac{1+\sqrt{2}}{\alpha}$ and $\delta = \frac{1-\sqrt{2}}{\beta}$, and the proposer proved that $P_n = \text{round}\left(\frac{r^n(L_n+F_n\sqrt{5})}{4\sqrt{2}}\right)$, where $r = \frac{2(1+\sqrt{2})}{1+\sqrt{5}}$.

Also solved by Tom Edgar, I. V. Fedak, Robert Frontczak, Raphael Schumacher, and the proposer.

<u>**A Not-So-Obvious Application of Cauchy-Schwarz Inequality**</u>

<u>**B-1241**</u> **Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**
(Vol. 57.1, February 2019)

For all positive integers n, prove that

$$\frac{F_{n+2}}{L_{n+2}} + \frac{F_{n+1}}{L_{n+1}} + \frac{F_n}{L_{n+1}+F_{n+2}} > 1.$$

Solution by Hideyuki Ohtsuka, Saitama, Japan.

By Cauchy-Schwarz inequality, for $a, b, c > 0$, we have

$$[a(a+2b)+b(b+2c)+c(c+2a)]\left(\frac{a}{a+2b}+\frac{b}{b+2c}+\frac{c}{c+2a}\right) \geq (a+b+c)^2.$$

From the above inequality and the identity

$$a(a+2b)+b(b+2c)+c(c+2a) = a^2+b^2+c^2+2ab+2bc+2ca = (a+b+c)^2,$$

we obtain

$$\frac{a}{a+2b}+\frac{b}{b+2c}+\frac{c}{c+2a} \geq 1,$$

where equality occurs if and only if $a = b = c$. Using the identity $L_m = F_{m+1}+F_{m-1}$, we find

$$\begin{aligned}\frac{F_{n+2}}{L_{n+2}} + \frac{F_{n+1}}{L_{n+1}} + \frac{F_n}{L_{n+1}+F_{n+2}} &= \frac{F_{n+2}}{F_{n+3}+F_{n+1}} + \frac{F_{n+1}}{F_{n+2}+F_n} + \frac{F_n}{F_n+2F_{n+2}}\\ &= \frac{F_{n+2}}{F_{n+2}+2F_{n+1}} + \frac{F_{n+1}}{F_{n+1}+2F_n} + \frac{F_n}{F_n+2F_{n+2}} > 1.\end{aligned}$$

Editor's Notes: Bataille and Metcalfe showed (independently) that the inequality is equivalent to $L_{3n+1} + 2(-1)^n L_n > 0$, and G. C. Gruebel reduced it to $F_{n+1}^3 - F_n^2 F_{n-1} > 0$. Both inequalities are easy to establish.

Also solved by Michel Bataille, Brian Beasley, Brian Bradie, Kenny B. Davenport, Dmitry Fleischman, G. C. Greubel, Wei-Kai Lai and John Risher (student) (jointly), Ehren Metcalfe, Ángel Plaza, and the proposer.

Generalizing a Curious Sum

B-1242 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 57.1, February 2019)

Let $r_1, r_2, \ldots, r_n$ be positive even integers. Prove that

$$\sum_{\epsilon_1,\cdots,\epsilon_n\in\{-1,1\}} F_{\epsilon_1 r_1+\cdots+\epsilon_n r_n} = 0, \qquad \text{and} \qquad \sum_{\epsilon_1,\cdots,\epsilon_n\in\{-1,1\}} L_{\epsilon_1 r_1+\cdots+\epsilon_n r_n} = 2\prod_{k=1}^{n} L_{r_k}.$$

Solution by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Let $\{G_n\}_{n\geq 0}$ be the generalized Fibonacci sequence defined by $G_n = G_{n-1}+G_{n-2}$ for $n \geq 2$, with initial values G_0 and G_1. We use induction on n to prove that

$$\sum_{\epsilon_1,\cdots,\epsilon_n\in\{-1,1\}} G_{\epsilon_1 r_1+\cdots+\epsilon_n r_n} = G_0\prod_{k=1}^{n} L_{r_k}.$$

In our proof, we will use the following fact about $\{G_n\}_{n\geq 0}$ (see [1] for instance):

$$G_{n+m} + (-1)^m G_{n-m} = G_n L_m.$$

For $n = 1$ and any even integer r_1, we have $G_{r_1} + G_{-r_1} = G_{0+r_1} + (-1)^{r_1} G_{0-r_1} = G_0 L_{r_1}$. Assume the identity is true for some fixed $n \geq 1$. Then, for any even integers $r_1, r_2, \ldots, r_{n+1}$,

$$\begin{aligned}
&\sum_{\epsilon_1,\cdots,\epsilon_n,\epsilon_{n+1}\in\{-1,1\}} G_{\epsilon_1 r_1+\cdots+\epsilon_{n+1} r_{n+1}} \\
&\quad= \sum_{\epsilon_1,\cdots,\epsilon_n\in\{-1,1\}} \left(G_{\epsilon_1 r_1+\cdots+\epsilon_n r_n + r_{n+1}} + G_{\epsilon_1 r_1+\cdots+\epsilon_n r_n - r_{n+1}}\right) \\
&\quad= \left(\sum_{\epsilon_1,\cdots,\epsilon_n\in\{-1,1\}} G_{\epsilon_1 r_1+\cdots+\epsilon_n r_n}\right) L_{r_{n+1}} = G_0 \prod_{k=1}^{n+1} L_{r_k}.
\end{aligned}$$

This completes the induction.

References

[1] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section*, Dover Press, New York, 2008.

Also solved by Michel Bataille, Brian Bradie, I. V. Fedak, Dmitry Fleischman, G. C. Greubel, Ángel Plaza, Raphael Schumacher, and the proposer.

In Need of a More Complicated Formula

B-1243 Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 57.1, February 2019)

For any positive integer k, the k-Fibonacci numbers are defined recursively by $F_{k,0} = 0$, $F_{k,1} = 1$, and $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for $n \geq 2$. Prove that

$$k\sum_{m=0}^{n} F_{k,m} = \left\lfloor \frac{(\sqrt{k^2+4}+k)^{n+1} - 2^{n+1}}{2^n\sqrt{k^2+4}\,(\sqrt{k^2+4}-k)} \right\rfloor.$$

Editor's Remark: The identity was an inequality in the proposer's original proposal. The section editor mistakenly entered it as an identity, thereby making it a much harder problem.

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

The given expression is not true in general. For example, with $k = 2$ and $n = 1$,

$$2\sum_{m=0}^{1} F_{2,m} = 2, \qquad \text{but} \qquad \left\lfloor \frac{(\sqrt{8}+2)^2 - 2^2}{2\sqrt{8}\,(\sqrt{8}-2)} \right\rfloor = 4.$$

To determine a correct expression, use the recurrence $kF_{k,m} = F_{k,m+1} - F_{k,m-1}$ to obtain

$$k\sum_{m=0}^{n} F_{k,m} = F_{k,n+1} + F_{k,n} - 1.$$

Now, let $\alpha_k = \frac{k+\sqrt{k^2+4}}{2}$ and $\beta_k = \frac{k-\sqrt{k^2+4}}{2}$, so that $F_{k,n} = \frac{\alpha_k^n - \beta_k^n}{\alpha_k - \beta_k}$. Observe that $\alpha_k > k \geq 1$, and $-1 < \beta_k < 0$ for all positive integers k. It follows that for all positive integers k and n,

$$\beta_k^{n+1} + \beta_k^n = \beta_k^n(1+\beta_k) < 1 + \beta_k < 1 < \alpha_k.$$

Moreover, for all integers $n \geq 1$,

$$|\beta_k^{n+1} + \beta_k^n| = |\beta_k^n(1+\beta_k)| < |\beta_k| = -\beta_k.$$

Hence, $\beta_k - \beta_k^{n+1} - \beta_k^n < 0$. Note that this inequality also holds when $n = 0$. Therefore,

$$\begin{aligned} F_{k,n+1} + F_{k,n} - 1 &= \frac{\alpha_k^{n+1} + \alpha_k^n - \alpha_k - \beta_k^{n+1} - \beta_k^n + \beta_k}{\alpha_k - \beta_k} \\ &< \frac{\alpha_k^{n+1} + \alpha_k^n - \alpha_k}{\alpha_k - \beta_k} < \frac{\alpha_k^{n+1} + \alpha_k^n - \beta_k^{n+1} - \beta_k^n}{\alpha_k - \beta_k} = F_{k,n+1} + F_{k,n}, \end{aligned}$$

which implies that

$$F_{k,n+1} + F_{k,n} - 1 = \left\lfloor \frac{\alpha_k^{n+1} + \alpha_k^n - \alpha_k}{\alpha_k - \beta_k} \right\rfloor.$$

Thus,

$$k\sum_{m=0}^{n} F_{k,m} = \left\lfloor \frac{(\sqrt{k^2+4}+k)^n + 2(\sqrt{k^2+4}+k)^{n-1} - 2^n}{2^{n-1}\sqrt{k^2+4}\,(\sqrt{k^2+4}-k)} \right\rfloor.$$

The invalidity of the identity was also noticed by I. V. Fedak and Dmitry Fleischman.

Making It Easier with Lagrange

B-1244 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany. (Vol. 57.1, February 2019)

Prove that following identities for the Fibonacci and Lucas numbers for $n \geq 2$:

a) $\displaystyle\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (F_k - F_j)^2 = nF_nF_{n+1} - (F_{n+2} - 1)^2$

b) $\displaystyle\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (L_k - L_j)^2 = n(L_nL_{n+1} - 2) - (L_{n+2} - 3)^2$

c) $\displaystyle\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (F_kL_j - F_jL_k)^2 = \begin{cases} F_nF_{n+1}(L_nL_{n+1} - 2) - (F_{n+1}L_n - 2)^2, & \text{if } n \text{ is even,} \\ F_nF_{n+1}(L_nL_{n+1} - 2) - F_{n+1}^2L_n^2, & \text{if } n \text{ is odd.} \end{cases}$

Solution by Hideyuki Ohtsua, Saitama, Japan.

Let G_n denote either F_n or L_n. We use the following identities [1, Identities 33, 35, and 44]:

$$\sum_{k=1}^{n} G_k = G_{n+2} - G_2, \qquad \sum_{k=1}^{n} G_{2k} = G_{2n+1} - G_1, \qquad \sum_{k=1}^{n} G_k^2 = G_nG_{n+1} - G_0G_1,$$

and the Lagrange identity

$$\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (a_kb_j - a_jb_k)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_kb_k\right)^2.$$

For a) and b), we have

$$\begin{aligned} \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (G_k \cdot 1 - G_j \cdot 1)^2 &= \left(\sum_{k=1}^{n} G_k^2\right)\left(\sum_{k=1}^{n} 1\right) - \left(\sum_{k=1}^{n} G_k\right)^2 \\ &= n(G_nG_{n+1} - G_0G_1) - (G_{n+2} - G_2)^2. \end{aligned}$$

Therefore, we obtain the desired identities.

For c), by Lagrange's identity and $F_kL_k = F_{2k}$, we have

$$\begin{aligned} \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (F_kL_j - F_jL_k)^2 &= \left(\sum_{k=1}^{n} F_k^2\right)\left(\sum_{k=1}^{n} L_k^2\right) - \left(\sum_{k=1}^{n} F_{2k}\right)^2 \\ &= F_nF_{n+1}(L_nL_{n+1} - 2) - (F_{2n+1} - 1)^2. \end{aligned}$$

Using the identity [1, Identity 15a] $F_aL_b = F_{a+b} + (-1)^bF_{a-b}$, we have $F_{n+1}L_n = F_{2n+1} + (-1)^nF_1$, that is, $F_{2n+1} = F_{n+1}L_n - (-1)^n$. Therefore, we obtain the desired identity.

We can improve the result in c). From

$$F_nL_{n+1} \cdot F_{n+1}L_n = \left[F_{2n+1} - (-1)^n\right]\left[F_{2n+1} + (-1)^n\right] = F_{2n+1}^2 - 1,$$
$$(F_{2n+1} - 1)^2 = F_{2n+1}^2 - 2F_{2n+1} + 1 = F_{2n+1}^2 - 2\left[F_nL_{n+1} + (-1)^n\right] + 1,$$

we obtain

$$F_nF_{n+1}(L_nL_{n+1} - 2) - (F_{2n+1} - 1)^2 = 2F_n(L_{n+1} - F_{n+1}) + 2(-1)^n - 2.$$

Since $L_{n+1} - F_{n+1} = F_{n+2} + F_n - F_{n+1} = 2F_n$, we deduce that

$$\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (F_k L_j - F_j L_k)^2 = 4F_n^2 + 2(-1)^n - 2.$$

References

[1] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section*, Dover Press, New York, 2008.

Also solved by Michel Bataille, Brian Bradie, Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, G. C. Greubel, Wei-Kai Lai, Ehren Metcalfe, and the proposer.

The Sum of Multiples of Cubes of Lucas Numbers

B-1245 Proposed by Kenny B. Davenport, Dallas, PA.
(Vol. 57.1, February 2019)

Show that, for any positive integer n,

$$\sum_{k=1}^{n} kL_k^3 = \frac{5\left[(2n+3)L_{3n-1} - L_{3n}\right] + 49}{4} - (n+2)L_{n-1}^3 - L_n^3.$$

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

We use induction to prove the identity. The identity clearly holds when $n = 1$. Let us assume that it is also true for some integer $n \geq 1$. Then for $n+1$ we should prove that

$$\sum_{k=1}^{n+1} kL_k^3 = \frac{5\left[(2n+5)L_{3n+2} - L_{3n+3}\right] + 49}{4} - (n+3)L_n^3 - L_{n+1}^3,$$

or, by induction hypothesis, that

$$\begin{aligned}(n+1)L_{n+1}^3 \quad &= \quad \frac{5\left[(2n+5)L_{3n+2} - L_{3n+3} - (2n+3)L_{3n-1} + L_{3n}\right]}{4} \\ &\quad - (n+3)L_n^3 - L_{n+1}^3 + (n+2)L_{n-1}^3 + L_n^3.\end{aligned}$$

In the last identity the first fraction is equal to $5(n+2)L_{3n}$, since $L_{3n+2} = 3L_{3n-1} + 2L_{3n-2}$, and $L_{3n} = L_{3n-1} + L_{3n-2}$. So, we have to prove that

$$(n+1)L_{n+1}^3 = 5(n+2)L_{3n} - (n+3)L_n^3 - L_{n+1}^3 + (n+2)L_{n-1}^3 + L_n^3,$$

or, equivalently, $L_{n+1}^3 + L_n^3 - L_{n-1}^3 = 5L_{3n}$. Since this is Identity 103 in [1, p. 92], the proof is complete.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York, 2001.

Also solved by Michel Bataille, Brian Bradie, I. V. Fedak, Dmitry Fleischman, G. C. Greubel, Wei-Kai Lai, Ehren Metcalfe, Raphael Schumacher, Jason L. Smith, David Terr, and the proposer.

We would like to belatedly acknowledge Michel Bataille for solving Problems B-1238, B-1239, and B-1240.