SUMS OF PARTITION SETS IN GENERALIZED PASCAL TRIANGLES I

V. E. HOGGATT, JR.
San Jose State University, San Jose, California 95192
and
G. L. ALEXANDERSON
University of Santa Clara, Santa Clara, California 95053

In the expansion

$$(1+x+x^2+\cdots+x^{k-1})^n = \sum_{i=0}^{(k-1)n} \left[{n \atop i} \right]_k x^i, \qquad k \geq 2, \quad n \geq 0,$$

clearly

$$\left[{n \atop 0} \right]_k = \left[{n \atop (k-1)n} \right]_k = 1$$

and

$$\left[{n \atop r} \right]_k = \sum_{j=0}^{k-1} \left[{n-1 \atop r-j} \right]_k, \left[{n \atop j} \right]_k = 0, \qquad j < 0, \quad j > (k-1)n.$$

For $k = 2$, these are the binomial coefficients and when dealing with these we shall use the usual notation:

$$\left[{n \atop r} \right]_2 = \binom{n}{r}.$$

The problem of calculating sums of the following type for $k = 2$ was first treated by Cournot [2] and Ramus [5] and Ramus' method is outlined in [4]:

$$S(n, k, q, r) = \sum_{j=0}^{N} \left[{n \atop r+jq} \right]_k ,$$

where

$$N = \left[\frac{(k-1)n-r}{q} \right] ,$$

[ ] denoting the greatest integer function. We wish here to investigate for certain fixed k and q the different values of these sums as r ranges from 0 to $q - 1$ and, further, the differences between the sums.

1. THE METHOD OF RAMUS

Let ω be a primitive q^{th} root of unity then

$$\omega = \cos \frac{2\pi}{q} + i \sin \frac{2\pi}{q} .$$

Then

$$(1+\omega^0)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n}$$

$$(1+\omega)^n = \binom{n}{0} + \binom{n}{1}\omega + \binom{n}{2}\omega^2 + \binom{n}{3}\omega^3 + \cdots + \binom{n}{n}\omega^n$$

$$(1+\omega^2)^n = \binom{n}{0} + \binom{n}{1}\omega^2 + \binom{n}{2}\omega^4 + \binom{n}{3}\omega^6 + \cdots + \binom{n}{n}\omega^{2n}$$

$$\cdots$$

$$(1+\omega^{q-1})^n = \binom{n}{0} + \binom{n}{1}\omega^{q-1} + \binom{n}{2}\omega^{2(q-1)} + \binom{n}{3}\omega^{3(q-1)} + \cdots + \binom{n}{n}\omega^{n(q-1)} .$$

Multiplying each successive row by $\omega, \omega^{-r}, \omega^{-2r}, \cdots, \omega^{-(q-1)r}$, $0 \leqslant r \leqslant q-1$, and adding the products we get

$$q\left[\binom{n}{r} + \binom{n}{r+q} + \binom{n}{r+2q} + \cdots\right] = \sum_{\ell=0}^{q-1} (1+\omega^{\ell})^n \omega^{-r\ell} = \sum_{\ell=0}^{q-1} (\omega^{\ell/2} + \omega^{-\ell/2})^n \omega^{-r\ell+\ell n/2}$$

$$= \sum_{\ell=0}^{q-1}\left(2\cos\frac{\ell\pi}{q}\right)^n \omega^{\frac{\ell(n-2r)}{2}} = \sum_{\ell=0}^{q-1}\left(2\cos\frac{\ell\pi}{q}\right)^n\left[\cos\frac{\ell(n-2r)2\pi}{2q} + i\sin\frac{\ell(n-2r)2\pi}{2q}\right] .$$

Since the left side is real, the coefficient of i on the right must be zero, hence

$$S(n,2,q,r) = \binom{n}{r} + \binom{n}{r+q} + \binom{n}{r+2q} + \cdots = \frac{1}{q}\sum_{\ell=0}^{q-1}\left(2\cos\frac{\ell\pi}{q}\right)^n \cos\frac{\ell(n-2r)\pi}{q} .$$

Applying the same technique to the expansion $(1+x+x^2+\cdots+x^{k-1})^n$ one finds that

$$S(n,k,q,r) = \left[{n \atop r}\right]_k + \left[{n \atop r+q}\right]_k + \left[{n \atop r+2q}\right]_k + \cdots = \frac{1}{q}\sum_{\ell=0}^{q-1}(1+\omega^{\ell}+\omega^{2\ell}+\cdots+\omega^{(k-1)\ell})^n\omega^{-r\ell}$$

$$= \begin{cases} \dfrac{1}{q}\displaystyle\sum_{\ell=0}^{q-1}\left[2\sum_{j=1}^{\frac{k-1}{2}}\cos\frac{\ell(k-2j+1)\pi}{q} + 1\right]^n \cos\frac{\ell(nk-n-2r)\pi}{q} & \text{for } k \text{ odd} \\ \dfrac{1}{q}\displaystyle\sum_{\ell=0}^{q-1}\left[2\sum_{j=1}^{k/2}\cos\frac{\ell(k-2j+1)\pi}{q}\right]^n \cos\frac{\ell(nk-n-2r)\pi}{q} & \text{for } k \text{ even} \end{cases}$$

2. THE CASES $k = 2,\ q = 3, 4$

This case is treated in [4] and more recently in [6]. From the formulas above one easily shows that

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots = \frac{1}{3}\left[2^n + \left(2\cos\frac{\pi}{3}\right)^n\cos\frac{n\pi}{3} + \left(2\cos\frac{2\pi}{3}\right)^n\cos\frac{2n\pi}{3}\right]$$

$$= \frac{1}{3}\left[2^n + \cos\frac{n\pi}{3} + (-1)^n\cos\frac{2n\pi}{3}\right] = \frac{1}{3}\left[2^n + 2\cos\frac{n\pi}{3}\right]$$

$$\binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \cdots = \frac{1}{3}\left[2^n + 2\cos\frac{(n-2)\pi}{3}\right]$$

$$\binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \cdots = \frac{1}{3}\left[2^n + 2\cos\frac{(n-4)\pi}{3}\right] .$$

By examining the table for $\cos (n\pi)/3$ one sees that the three differences

$$\frac{2}{3}\left[\cos\frac{n\pi}{3} - \cos\frac{(n-2)\pi}{3}\right], \quad \frac{2}{3}\left[\cos\frac{(n-2)\pi}{3} - \cos\frac{(n-4)\pi}{3}\right] \quad \text{and} \quad \frac{2}{3}\left[\cos\frac{(n-4)\pi}{3} - \cos\frac{n\pi}{3}\right]$$

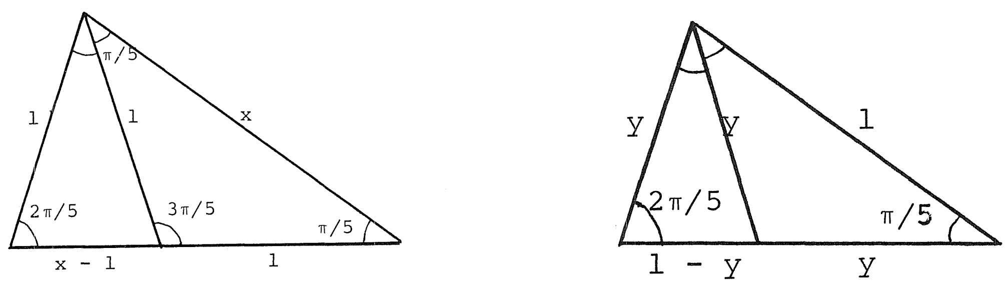
are 0, 1, −1. This problem appeared in the *American Mathematical Monthly* in May, 1938 as Problem E 300 (solution by Emma Lehmer) and again in February, 1956, as Problem E 1172. In slightly altered form it had appeared in the *Monthly* in 1932 as Problem 3497 (solution by Morgan Ward). It appeared as Problem B-6 in the 1974 William Lowell Putnam Contest.

The case of four sums ($q = 4$) yields in each case only two different or three different sums, depending on whether n is odd or even and the differences in the values are successive powers of 2, as the reader can verify. This appeared in *Mathematics Magazine* in November-December, 1952 as Problem 177 (solution by E. P. Starke).

3. THE CASE $k = 2, q = 5$

This case was treated tersely in the solution to a problem posed by E. P. Starke in the March, 1939, issue of the *National Mathematical Magazine*, where the differences were observed to be simple and predictable but the sums themselves were not seen to be reducible to simple form. We shall, therefore, treat this very interesting case at length, along with generalizations.

Consideration of the following two figures yields values of x and y:



(a) (b)

Figure 1

$$\frac{x}{1} = \frac{1}{x-1} \qquad \frac{1}{y} = \frac{y}{1-y} \qquad x = \frac{1+\sqrt{5}}{2} = \alpha \qquad y = -\frac{1-\sqrt{5}}{2} = -\beta,$$

where the signs are chosen so that x,y are positive. We note α is the golden ratio and the α,β are those of the Binet formulas for elements of Fibonacci and Lucas sequences, i.e., if

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n > 2, \quad \text{and} \quad L_1 = 1, \quad L_2 = 3, \quad L_n = L_{n-1} + L_{n-2}, \quad n > 2,$$

then

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively [3].

From Fig. 1a, one sees that

$$\cos\frac{\pi}{5} = \frac{\alpha}{2} \quad \text{and} \quad \cos\frac{2\pi}{5} = \frac{\alpha - 1}{2}$$

and from $\alpha + \beta = 1$ one concludes that

$$\cos\frac{2\pi}{5} = \frac{-\beta}{2}\;.$$

From these one can construct a table of values for $\cos (n\pi)/5$. Then

$$S(n,2,5,r) = \binom{n}{r} + \binom{n}{r+5} + \binom{n}{r+10} + \cdots = \frac{1}{5}\sum_{\ell=0}^{4}\left(2\cos\frac{\ell\pi}{5}\right)^n \cos\frac{\ell(n-2r)\pi}{5}$$

$$= \frac{1}{5}\left[2^n + \alpha^n \cos\frac{(n-2r)\pi}{5} + (-\beta)^n \cos\frac{2(n-2r)\pi}{5} + (-\alpha)^n \cos\frac{4(n-2r)\pi}{5} + \beta^n \cos\frac{3(n-2r)\pi}{5}\right]$$

$$= \frac{1}{5}\left[2^n + 2\alpha^n \cos\frac{(n-2r)\pi}{5} + 2(-\beta)^n \cos\frac{2(n-2r)\pi}{5}\right] \text{ for } r = 0, 1, 2, 3, 4.$$

Let us examine, for example, $S(10m, 2, 5, 0)$:

$$\binom{10m}{0}+\binom{10m}{5}+\binom{10m}{10}+\cdots=\frac{1}{5}[2^{10m}+2\alpha^{10m}+2\beta^{10m}]=\frac{1}{5}[2^{10m}+2L_{10m}],$$

where L_{10m} is a Lucas number. For $n = 10m + 1$,

$$S(10m+1,2,5,0)=\frac{1}{5}[2^{10m+1}+2\alpha^{10m+1}\cdot(\alpha/2)-2\beta^{10m+1}(\beta/2)]=\frac{1}{5}[2^{10m+1}+\alpha^{10m+2}+\beta^{10m+2}]$$

$$=\frac{1}{5}[2^{10m+1}+L_{10m+2}].$$

We can continue to reduce these sums to the form $1/5[2^n + A]$, where A is a Lucas number or twice a Lucas number and can, in fact, form the following table for the values of A:

Table 1

n	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$
$10m$	$2L_{10m}$	L_{10m-1}	$-L_{10m+1}$	$-L_{10m+1}$	L_{10m-1}
$10m+1$	L_{10m+2}	L_{10m+2}	$-L_{10m}$	$-2L_{10m+1}$	$-L_{10m}$
$10m+2$	L_{10m+1}	$2L_{10m+2}$	L_{10m+1}	$-L_{10m+3}$	$-L_{10m+3}$
$10m+3$	$-L_{10m+2}$	L_{10m+4}	L_{10m+4}	$-L_{10m+2}$	$-2L_{10m+3}$
$10m+4$	$-L_{10m+5}$	L_{10m+3}	$2L_{10m+4}$	L_{10m+3}	$-L_{10m+5}$
$10m+5$	$-2L_{10m+5}$	$-L_{10m+4}$	L_{10m+6}	L_{10m+6}	$-L_{10m+4}$
$10m+6$	$-L_{10m+7}$	$-L_{10m+7}$	L_{10m+5}	$2L_{10m+6}$	L_{10m+5}
$10m+7$	$-L_{10m+6}$	$-2L_{10m+7}$	$-L_{10m+6}$	L_{10m+8}	L_{10m+8}
$10m+8$	L_{10m+7}	$-L_{10m+9}$	$-L_{10m+9}$	L_{10m+7}	$2L_{10m+8}$
$10m+9$	L_{10m+10}	$-L_{10m+8}$	$-2L_{10m+9}$	$-L_{10m+8}$	L_{10m+10}

Thus we have formulas for all sums of the form

$$\sum_{t=0}\binom{n}{r+5t},\qquad r=0,1,2,3,4,$$

and since

$$\sum_{i=0}^{n}\binom{n}{i}=2^n,$$

we note that the sum of the five elements on any row of the above table must be zero and, furthermore, it is clear from the method of generating Pascal's Triangle that each element of Table 1 must be the sum of the element above it and to the left of that. The following is the table of high and low values of the elements in Table 1:

Table 2

H	L
$2L_{10m}$	$-L_{10m+1}$
L_{10m+2}	$-2L_{10m+1}$
$2L_{10m+2}$	$-L_{10m+3}$
L_{10m+4}	$-2L_{10m+3}$
$2L_{10m+4}$	$-L_{10m+5}$
L_{10m+6}	$-2L_{10m+5}$
$2L_{10m+6}$	$-L_{10m+7}$
L_{10m+8}	$-2L_{10m+7}$
$2L_{10m+8}$	$-L_{10m+9}$
L_{10m+10}	$-2L_{10m+9}$

The differences between the highest value of the sums for given n and the lowest value is, therefore, always of the form

$$(2L_n + L_{n+1})/5.$$

That

$$(2L_n + L_{n+1})/5 = F_{n+1}$$

is proved easily by induction. We note that for each n there are only three different values for the five sums and that differences between the high and low values are Fibonacci numbers. Furthermore, the differences between the high and middle values, the middle and low values are again Fibonacci numbers. In fact, the three Fibonacci numbers have consecutive subscripts.

4. THE CASE $k = 3$, $q = 5$

In this case we are dealing with five sums of trinomial coefficients, and, for $r = 0, 1, 2, 3, 4$,

$$S(n,3,5,r) = \begin{bmatrix} n \\ r \end{bmatrix}_3 + \begin{bmatrix} n \\ r+5 \end{bmatrix}_3 + \begin{bmatrix} n \\ r+10 \end{bmatrix}_3 + \cdots = \frac{1}{5}\sum_{\ell=0}^{4}\left[2\cos\frac{2\ell\pi}{5} + 1\right]^n \cos\frac{2(n-r)\ell\pi}{5} .$$

But since

$$2\cos\frac{2\pi}{5} + 1 = 2\left(-\frac{\beta}{2}\right) + 1 = -\beta + 1 = \alpha = 2\cos\frac{8\pi}{5} + 1$$

and

$$2\cos\frac{4\pi}{5} + 1 = 2\left(-\frac{\alpha}{2}\right) + 1 = -\alpha + 1 = \beta = 2\cos\frac{6\pi}{5} + 1,$$

$$S(n,3,5,r) = \frac{1}{5}\left[3^n + \alpha^n\cos\frac{2(n-r)\pi}{5} + \beta^n\cos\frac{4(n-r)\pi}{5} + \beta^n\cos\frac{6(n-r)\pi}{5} + \alpha^n\cos\frac{8(n-r)\pi}{5}\right]$$

$$= \frac{1}{5}\left[3^n + 2\alpha^n\cos\frac{2(n-r)\pi}{5} + 2\beta^n\cos\frac{4(n-r)\pi}{5}\right] .$$

These sums reduce in each case to the form $1/5[3^n + B]$, where B is found in Table 3:

Table 3

n	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$
$10m$	$2L_{10m}$	L_{10m-1}	$-L_{10m+1}$	$-L_{10m+1}$	L_{10m-1}
$10m+1$	L_{10m}	$2L_{10m+1}$	L_{10m}	$-L_{10m+2}$	$-L_{10m+2}$
$10m+2$	$-L_{10m+3}$	L_{10m+1}	$2L_{10m+2}$	L_{10m+1}	$-L_{10m+3}$
$10m+3$	$-L_{10m+4}$	$-L_{10m+4}$	L_{10m+2}	$2L_{10m+3}$	L_{10m+2}
$10m+4$	L_{10m+3}	$-L_{10m+5}$	$-L_{10m+5}$	L_{10m+3}	$2L_{10m+4}$
$10m+5$	$2L_{10m+5}$	L_{10m+4}	$-L_{10m+6}$	$-L_{10m+6}$	L_{10m+4}
$10m+6$	L_{10m+5}	$2L_{10m+6}$	L_{10m+5}	$-L_{10m+7}$	$-L_{10m+7}$
$10m+7$	$-L_{10m+8}$	L_{10m+6}	$2L_{10m+7}$	L_{10m+6}	$-L_{10m+8}$
$10m+8$	$-L_{10m+9}$	$-L_{10m+9}$	L_{10m+7}	$2L_{10m+8}$	L_{10m+7}
$10m+9$	L_{10m+8}	$-L_{10m+10}$	$-L_{10m+10}$	L_{10m+8}	$2L_{10m+9}$

Again, differences of the sums are Fibonacci numbers. If one examines cases for larger values of k and uses the fact that, for

$$q = 5, \qquad 1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0,$$

one sees that the sums will be expressible in the form

$$\frac{1}{5}\left[k^n \pm C\right],$$

where C is a Lucas number or twice a Lucas number, and the differences will be consecutive Fibonacci numbers, in the cases where $k \equiv 2,3 \pmod 5$. In other cases, the sums take on a constant value or take on two values which differ by 1.

5. THE CASE OF $k = 2,\ q = 6$

Here

$$S(n,2,6,r) = \frac{1}{6}\sum_{\ell=0}^{5}\left(2\cos\frac{\ell\pi}{6}\right)^n\cos\frac{\ell(n-2r)\pi}{6} = \frac{1}{6}\left[2^n + 2(\sqrt{3})^n\cos\frac{(n-2r)\pi}{6} + 2\cos\frac{2(n-2r)\pi}{6}\right]$$

$r = 0, 1, \cdots, 5$, and the sums take the form $\frac{1}{6}[2^n + D]$, where, for $r = 0$, for example, D can be found in Table 4.

Table 4

n	D
$12m$	$2.3^{6m} + 2$ (this breakd down for $m = 0$)
$12m + 1$	$3^{6m+1} + 1$
$12m + 2$	$3^{6m+1} - 1$
$12m + 3$	-2
$12m + 4$	$-3^{6m+2} - 1$
$12m + 5$	$-3^{6m+3} + 1$
$12m + 6$	$-2.3^{6m+3} + 2$
$12m + 7$	$-3^{6m+4} + 1$
$12m + 8$	$3^{6m+4} - 1$
$12m + 9$	-2
$12m + 10$	$3^{6m+5} - 1$
$12m + 11$	$3^{6m+6} + 1$

The other sums, for $r = 1,2,3,4,5$ can be computed easily and, not surprisingly, the largest and smallest sums differ by a power of 3 or twice a power of 3.

6. THE CASE OF $k = 2,\ q = 8$

The Pell numbers P_n are defined by the following:

$$P_1 = 1, \qquad P_2 = 2, \qquad P_n = 2P_{n-1} + P_{n-2}, \qquad n > 2,$$

and we shall define the Pell-Lucas sequence Q_n as satisfying the same recursion relation but $Q_1 = 2,\ Q_2 = 6.$ The roots of the auxiliary equation $x^2 - 2x - 1 = 0$ are, in this case,

$$\gamma = 1 + \sqrt{2} \qquad \text{and} \qquad \delta = 1 - \sqrt{2}$$

and the Binet-type formulas in this case are, analogously,

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \qquad \text{and} \qquad Q_n = \gamma^n + \delta^n .$$

For $q = 8$, the sums $S(n,2,8,r)$ for $r = 0, 1, 2, \cdots, 7$ can be written

$$S(n,2,8,r) = \binom{n}{r} + \binom{n}{r+8} + \binom{n}{r+16} + \cdots = \frac{1}{8}\sum_{\ell=0}^{7}\left(2\cos\frac{\ell\pi}{8}\right)^n\cos\frac{\ell(n-2r)\pi}{8} = \frac{1}{8}\left[2^n + \left(2\cos\frac{\pi}{8}\right)^n\right.$$

$$\cdot\cos\frac{(n-2r)\pi}{8} + \left(2\cos\frac{2\pi}{8}\right)^n\cos\frac{2(n-2r)\pi}{8} + \left(2\cos\frac{3\pi}{8}\right)^n\cos\frac{3(n-2r)\pi}{8}$$

$$\left. + \left(2\cos\frac{5\pi}{8}\right)^n\cos\frac{5(n-2r)\pi}{8} + \left(2\cos\frac{6\pi}{8}\right)^n\cos\frac{6(n-2r)\pi}{8} + \left(2\cos\frac{7\pi}{8}\right)^n\cos\frac{7(n-2r)\pi}{8}\right]$$

$$= \frac{1}{8}\left[2^n + 2\cdot 2^{n/4}\gamma^{n/2}\cos\frac{(n-2r)\pi}{8} + 2\cdot 2^{n/2}\cos\frac{2(n-2r)\pi}{8} + 2\cdot 2^{n/4}(-\delta)^{n/2}\,\frac{3(n-2r)\pi}{8}\right] .$$

Table 5

	$r = 0$	$r = 1$	$r = 2$	$r = 3$
$16m$	$2^{8m+1} + 2^{4m+1}Q_{8m}$	$2^{4m+2}P_{8m}$	-2^{8m+1}	$-2^{4m+2}P_{8m}$
$16m+1$	$2^{8m+1} + 2^{4m+2}P_{8m+1}$	$-2^{8m+1} + 2^{4m+2}P_{8m+1}$	$-2^{8m+1} + 2^{4m+2}P_{8m}$	$-2^{8m+1} - 2^{4m+2}P_{8m}$
$16m+2$	$2^{4m+1}Q_{8m+1}$	$2^{8m+2} + 2^{4m+3}P_{8m+1}$	$2^{4m+1}Q_{8m+1}$	-2^{8m+2}
$16m+3$	$-2^{8m+2} + 2^{4m+1}Q_{8m+1}$	$2^{8m+2} + 2^{4m+1}Q_{8m+2}$	$2^{8m+2} + 2^{4m+1}Q_{8m+2}$	$-2^{8m+2} + 2^{4m+1}Q_{8m+1}$
$16m+4$	-2^{8m+3}	$2^{4m+3}P_{8m+2}$	$2^{8m+3} + 2^{4m+2}Q_{8m+2}$	$2^{4m+3}P_{8m+2}$
$16m+5$	$-2^{8m+3} - 2^{4m+3}P_{8m+2}$	$-2^{8m+3} + 2^{4m+3}P_{8m+2}$	$2^{8m+3} + 2^{4m+3}P_{8m+3}$	$2^{8m+3} + 2^{4m+3}P_{8m+3}$
$16m+6$	$-2^{4m+2}Q_{8m+3}$	-2^{8m+4}	$2^{4m+2}Q_{8m+3}$	$2^{8m+4} + 2^{4m+4}P_{8m+3}$
$16m+7$	$2^{8m+4} - 2^{4m+2}Q_{8m+4}$	$-2^{8m+4} - 2^{4m+2}Q_{8m+3}$	$-2^{8m+4} + 2^{4m+2}Q_{8m+3}$	$2^{8m+4} + 2^{4m+2}Q_{8m+4}$
$16m+8$	$2^{8m+5} - 2^{4m+3}Q_{8m+4}$	$-2^{4m+4}P_{8m+4}$	-2^{8m+5}	$2^{4m+4}P_{8m+4}$
$16m+9$	$2^{8m+5} - 2^{4m+4}P_{8m+5}$	$2^{8m+5} - 2^{4m+4}P_{8m+5}$	$-2^{8m+5} - 2^{4m+4}P_{8m+4}$	$-2^{8m+5} + 2^{4m+4}P_{8m+4}$
$16m+10$	$-2^{4m+3}Q_{8m+5}$	$2^{8m+6} - 2^{4m+5}P_{8m+5}$	$-2^{4m+3}Q_{8m+5}$	-2^{8m+6}
$16m+11$	$-2^{8m+6} - 2^{4m+3}Q_{8m+5}$	$2^{8m+6} - 2^{4m+3}Q_{8m+6}$	$2^{8m+6} - 2^{4m+3}Q_{8m+6}$	$-2^{8m+6} - 2^{4m+3}Q_{8m+5}$
$16m+12$	-2^{8m+7}	$-2^{4m+5}P_{8m+6}$	$2^{8m+7} - 2^{4m+4}Q_{8m+6}$	$-2^{4m+5}P_{8m+6}$
$16m+13$	$-2^{8m+7} + 2^{4m+5}P_{8m+6}$	$-2^{8m+7} - 2^{4m+5}P_{8m+6}$	$2^{8m+7} - 2^{4m+5}P_{8m+7}$	$2^{8m+7} - 2^{4m+5}P_{8m+7}$
$16m+14$	$2^{4m+4}Q_{8m+7}$	-2^{8m+8}	$-2^{4m+4}Q_{8m+7}$	$2^{8m+8} - 2^{4m+6}P_{8m+7}$
$16m+15$	$2^{8m+8} + 2^{4m+4}Q_{8m+8}$	$-2^{8m+8} + 2^{4m+4}Q_{8m+7}$	$-2^{8m+8} - 2^{4m+4}Q_{8m+7}$	$2^{8m+8} - 2^{4m+4}Q_{8m+8}$

Continued, next page.

Table 5 (Cont'd)

	$r = 4$	$r = 5$	$r = 6$	$r = 7$
$16m$	$2^{8m+1} - 2^{4m+1}Q_{8m}$	$-2^{4m+2}P_{8m}$	-2^{8m+1}	$2^{4m+2}P_{8m}$
$16m+1$	$2^{8m+1} - 2^{4m+2}P_{8m+1}$	$2^{8m+1} - 2^{4m+2}P_{8m+1}$	$-2^{8m+1} - 2^{4m+2}P_{8m}$	$-2^{8m+1} + 2^{4m+2}P_{8m}$
$16m+2$	$-2^{4m+1}Q_{8m+1}$	$2^{8m+2} - 2^{4m+3}P_{8m+1}$	$-2^{4m+1}Q_{8m+1}$	-2^{8m+2}
$16m+3$	$-2^{8m+2} - 2^{4m+1}Q_{8m+1}$	$2^{8m+2} - 2^{4m+1}Q_{8m+2}$	$2^{8m+2} - 2^{4m+1}Q_{8m+2}$	$-2^{8m+2} - 2^{4m+1}Q_{8m+1}$
$16m+4$	-2^{8m+3}	$-2^{4m+3}P_{8m+2}$	$2^{8m+3} - 2^{4m+2}Q_{8m+2}$	$-2^{4m+3}P_{8m+2}$
$16m+5$	$-2^{8m+3} + 2^{4m+3}P_{8m+2}$	$-2^{8m+3} - 2^{4m+3}P_{8m+2}$	$2^{8m+3} - 2^{4m+3}P_{8m+3}$	$2^{8m+3} - 2^{4m+3}P_{8m+3}$
$16m+6$	$2^{4m+2}Q_{8m+3}$	-2^{8m+4}	$-2^{4m+2}Q_{8m+3}$	$2^{8m+4} - 2^{4m+4}P_{8m+3}$
$16m+7$	$2^{8m+4} + 2^{4m+2}Q_{8m+4}$	$-2^{8m+4} + 2^{4m+2}Q_{8m+3}$	$-2^{8m+4} - 2^{4m+2}Q_{8m+3}$	$2^{8m+4} - 2^{4m+2}Q_{8m+4}$
$16m+8$	$2^{8m+5} + 2^{4m+3}Q_{8m+4}$	$2^{4m+4}P_{8m+4}$	-2^{8m+5}	$-2^{4m+4}P_{8m+4}$
$16m+9$	$2^{8m+5} + 2^{4m+4}P_{8m+5}$	$2^{8m+5} + 2^{4m+4}P_{8m+5}$	$-2^{8m+5} + 2^{4m+4}P_{8m+4}$	$-2^{8m+5} - 2^{4m+4}P_{8m+4}$
$16m+10$	$2^{4m+3}Q_{8m+5}$	$2^{8m+6} + 2^{4m+5}P_{8m+5}$	$2^{4m+3}Q_{8m+5}$	-2^{8m+6}
$16m+11$	$-2^{8m+6} + 2^{4m+3}Q_{8m+5}$	$2^{8m+6} + 2^{4m+3}Q_{8m+6}$	$2^{8m+6} + 2^{4m+3}Q_{8m+6}$	$-2^{8m+6} + 2^{4m+3}Q_{8m+5}$
$16m+12$	-2^{8m+7}	$2^{4m+5}P_{8m+6}$	$2^{8m+7} + 2^{4m+4}Q_{8m+6}$	$2^{4m+5}P_{8m+6}$
$16m+13$	$-2^{8m+7} - 2^{4m+5}P_{8m+6}$	$-2^{8m+7} + 2^{4m+5}P_{8m+6}$	$2^{8m+7} + 2^{4m+5}P_{8m+7}$	$2^{8m+7} + 2^{4m+5}P_{8m+7}$
$16m+14$	$-2^{4m+4}Q_{8m+7}$	-2^{8m+8}	$2^{4m+4}Q_{8m+7}$	$2^{8m+8} + 2^{4m+6}P_{8m+7}$
$16m+15$	$2^{8m+8} - 2^{4m+4}Q_{8m+8}$	$-2^{8m+8} - 2^{4m+4}Q_{8m+7}$	$2^{8m+8} - 2^{4m+4}Q_{8m+7}$	$2^{8m+8} + 2^{4m+4}Q_{8m+8}$

One can reduce these sums to the form $\frac{1}{8}[2^n + E]$, where E is found in Table 5. $S(n,6,8,r)$ is similar.

Differences between the largest and smallest sums are, in this case, powers of 2 times Pell or Pell-Lucas numbers.

Further cases yield more differences which satisfy increasingly complicated linear recursion relations or combinations of such relations. Some of these, along with other techniques for handling such problems will appear in a later paper. Some generalizations to multinomial coefficients appear in [1].

REFERENCES

1. L. Carlitz, "Some Sums of Multinomial Coefficients," to appear, *The Fibonacci Quarterly.*
2. A. A. Cournot, *Bull. Sci. Math.,* Feb., 1829.
3. V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers,* Houghton-Mifflin, 1969.
4. E. Netto, *Lehrbuch der Combinetorik,* 2nd ed., Teubner, Berlin, 1927.
5. Chr. Ramus, "Solution générale dun problème d'analyse combinatoire," *J. Reine Angew. Math.,* 11 (1834), pp. 353–355.
6. A. M. and I. M. Yaglom, *Challenging Mathematical Problems with Elementary Solutions,* Vol. I, Holder Day, San Francisco, 1964.

****