nacci sequence does not select pleasing combinations, and one comes to appreciate the problem involved in selecting bright, true colors which harmonize.

This short article certainly will pose more questions than it answers, since mathematicians are not usually accustomed to thinking about color theory as used in painting; Fritz Faiss has devoted fifty years to the study of color theory in art. Fibonacci numbers seem to form a link from art to music; perhaps some creative person will compose a Fibonacci ballet, or harmonize Fibonacci color schemes with Fibonacci music.

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ON THE DENSITY OF THE IMAGE SETS OF CERTAIN ARITHMETIC FUNCTIONS—II

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1. INTRODUCTION

Throughout this article, we will be using the following notation: $n \geq 0$ is an arbitrary nonnegative integer and $n = \sum_{j=0}^{k} d_j b^j$ its representation as an integer in base b, $b \geq 2$ arbitrary. Define

$$(1.1) \qquad T(n) = n + \sum_{j=0}^{k} d_j \qquad [T(0) = 0]$$

$$\mathcal{R} = \{n \mid n = T(x) \text{ for some } x\} \text{ and}$$

$$\mathcal{C} = \{n \mid n \neq T(x) \text{ for any } x\}.$$

It has been shown ([1]) that the set $\mathcal{C}$ is infinite for any base b. More generally, it is true that $\mathcal{C}$ has asymptotic density and that $\mathcal{C}$ is a set of positive density; these results are derived from the following more general theorem and its corollary (proofs of which may be found in [2]).

Theorem: Let

$$n = \sum_{j=0}^{k} d_j b^j, \; b \geq 2 \text{ arbitrary},$$

and define

$$T(n) = n + \sum_{j=0}^{k} f(d_j, j) \text{ and } \mathcal{R} = \{n \mid n = T(x) \text{ for some } x\},$$

where $f(d_j, j)$ satisfies:

a. $f(0, j) = 0$ for all integers $j \geq 0$;
b. $f(d, j) = o(b^j)$ for all j and all digits d such that $1 \leq d \leq b - 1$.

Then the density of $\mathcal{R}$ exists and is equal to L, where L is computable, as follows: let

$$\lambda_{d,k} = \left|\{T(x)\,|\,db^k \leq x \leq (d+1)b^k - 1\} \cap \{T(x)\,|\,(d+1)b^k \leq x \leq (d+2)b^k - 1\}\right|,\ 0 \leq d \leq b - 2$$

$$\varepsilon_k = \sum_{d=0}^{b-2} \lambda_{d,k} / b^{k+1}$$

$$D(b^k - 1) = \left|\{T(x)\,|\,0 \leq x \leq b^k - 1\}\right|$$

$$A_k = D(b^k - 1)/b^k.$$

Then

$$L = A_{k_0} - \sum_{j=k_0}^{\infty} \varepsilon_j = A_k - \sum_{j=k}^{\infty} \varepsilon_j$$

for all $k \geq k_0$, where k_0 is an integer having the property that for all $k \geq k_0$, the sets $\{T(x)\,|\,0 \leq x \leq b^k - 1\}$, $\{T(x)\,|\,b^k \leq x \leq 2b^k - 1\}$, ..., $\{T(x)\,|\,(b-1)b^k \leq x \leq b^{k+1} - 1\}$ are pairwise disjoint, except possibly for adjacent pairs.

Corollary: If $f(d, j) = f(d)$ depends only on the digit d and if $f(0) = 0$ and $f(b - 1) \neq 0$ then $L < 1$.

Now it is easy to see that when $T(n)$ is the function defined by formula (1.1), we have $k_0 = 0$ and that the value of the $\lambda_{d,k}$ does not depend on the digit d. Hence, if we let $\lambda_k = \lambda_{d,k}$ for each digit d, our equation for L becomes

$$L = A_0 - \sum_{j=0}^{\infty} \varepsilon_j = 1 - \sum_{j=1}^{\infty} (b - 1)\lambda_j / b^{j+1}. \tag{1.2}$$

2. COMPUTATION OF THE DENSITY WHEN B IS ODD

Henceforth, let $T(n)$ be the function n + the sum of its digits, the function defined by formula (1.1). It is not difficult to prove that when b is odd, $\mathcal{R}$ is the set of all nonnegative even integers, so that $L = 1/2$ whenever b is odd ([1], [2]). We now give another proof of this fact, independent of the proof in [2], using formula (1.2).

Our principal objective is the proof of the following

Theorem 2.1: $\lambda_k = k(b - 1)/2$ for all odd bases b.

Using this result and equation (1.2), we see that

$$L = 1 - (b - 1)^2/2b\sum_{j=1}^{\infty} j/b^j = 1 - ((b - 1)^2/2b)(b/(b - 1)^2) = 1/2$$

$$= 1/2 \text{ whenever } b \text{ is odd.}$$

The proof of Theorem 2.1 depends on the following two lemmas:

Lemma 2.2: If b is odd then there exists an $x < b^k$ such that $T(x) = T(b^k)$ for all natural numbers k.

Proof: The proof is by induction on k. If $k = 1$ then we have

$$T((b+1)/2) = b + 1 = T(b).$$

Assume that

$$T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T(b^k)$$

and assume that the following claim is true.

Claim: d_0 can be chosen so that $d_0 \geq (b-1)/2$. Then

$$\begin{aligned} &T((b-1)b^k + d_{k-1}b^{k-1} + \cdots + d_1 b + d_0 - (b-1)/2) \\ &= b^{k+1} - b^k + b - 1 + T\left(\sum_{j=0}^{k-1} d_j b^j\right) - (b-1) \\ &= b^{k+1} - b^k + T(b^k) = b^{k+1} - b^k + b^k + 1 = T(b^{k+1}). \end{aligned}$$

So that all remains to be done is to prove the above claim. Observe that

$$\begin{aligned} &T(d_{k-1}b^{k-1} + \cdots + d_2 b^2 + (d_1 + 1)b + d_0' - (b+1)/2) \\ &= T(d_{k-1}b^{k-1} + \cdots + d_2 b^2 + d_1 b + d_0'), \ (b+1)/2 \leq d_0' \leq b - 1. \end{aligned}$$

Therefore the claim is proved if $d_1 \neq 0$. If $d_m = d_{m-1} = \cdots = d_1 = 0$ and if $d_{m+1} \neq 0$, we sill show that there exists

$$y = \sum_{j=0}^{m} d_j' b^j, \ d_0' \geq (b-1)/2$$

such that

$$T(d_{k-1}b^{k-1} + d_{k-2}b^{k-2} + \cdots + (d_{m+1} - 1)b^{m+1} + y) = T\left(\sum_{j=0}^{k-1} d_j b^j\right),$$

i.e.,

$$T(y) = T(b^{m+1} + d_0), \ d_0 \leq (b-3)/2$$

and this will finish the proof of the claim.

Now if $d_0 = 0$ then the existence of such a y is guaranteed by the induction hypothesis. If $d_0' = (b-1)/2$ then we have

$$T\left(\sum_{j=1}^{m} d_j' b^j + (b-1)/2\right) = T(b^{m+1}).$$

Hence

$$\begin{aligned} &T\left(\sum_{j=1}^{m} d_j' b^j + (b+1)/2\right) = T(b^{m+1} + 1) \\ &\vdots \\ &T\left(\sum_{j=1}^{m} d_j' b^j + b - 2\right) = T(b^{m+1} + (b-3)/2) \end{aligned}$$

and we are done if $d_0' = (b-1)/2$. Suppose now that $d_0' \geq (b+1)/2$, so that

$$T\left(\sum_{j=1}^{m} d_j' b^j + d_0'\right) = T(b^{m+1})$$

$$T\left(\sum_{j=1}^{m} d_j'b^j + d_0' + 1\right) = T(b^{m+1} + 1)$$

$$\vdots$$

$$T\left(\sum_{j=1}^{m} d_j'b^j + b - 1\right) = T(b^{m+1} + b - 1 - d_0').$$

We then obtain the following equations:

$$T\left(\sum_{j=2}^{m} d_j'b^j + (d_1' + 1)b + (b - 1)/2\right) = T(b^{m+1} + b - d_0')$$

$$T\left(\sum_{j=2}^{m} d_j'b^j + (d_1' + 1)b + (b + 1)/2\right) = T(b^{m+1} + b - d_0' + 1)$$

$$\vdots$$

$$T\left(\sum_{j=2}^{m} d_j'b^j + (d_1' + 1)b + d_0' - 1\right) = T\left(b^{m+1} + (b - 3)/2\right).$$

Note that by induction we may assume that $d_1' \neq b - 1$. The claim has now been completely proved, so that the proof of the lemma is complete as well.

Remark: Lemma 2.2 is not valid in general if b is even. For example, if $k = 1$, there is no $x < b$ satisfying $T(x) = T(b) = b + 1$, since $x < b$ implies that $T(x) = 2x$ and $b + 1$ is odd.

Lemma 2.3: If b is odd and

$$x = \sum_{j=1}^{m} (b - 1)b^m + r_0,\ r_0 \geq (b + 1)/2,$$

then there exists a $y = b^{m+1} + \sum_{j=0}^{m} d_j b^j$ with $d_0 \leq (b - 1)/2$ such that $T(x) = T(y)$.

Proof: Again, the proof is inductive. If $m = 1$ then $x = (b - 1)b + r_0$, $r_0 \geq (b + 1)/2$. Now

$$T\left((b - 1)b + (b + 1)/2\right) = b^2 - b + b - 1 + b + 1 = b^2 + b$$
$$= T\left(b^2 + (b - 1)/2\right).$$

Hence,

$$T\left((b - 1)b + (b + 3)/2\right) = T\left(b^2 + (b + 1)/2\right) = T(b^2 + b)$$

$$T\left((b - 1)b + (b + 5)/2\right) = T(b^2 + b + 1)$$

$$\vdots$$

$$T\left((b - 1)b + b - 1\right) = T\left(b^2 + b + (b - 5)/2\right)$$

and therefore the statement is true for $m = 1$. Assuming that the statement is true for all natural numbers $\leq m$, consider

$$x = \sum_{j=1}^{m+1} (b - 1)b^j + r_0,\ r_0 \geq (b + 1)/2.$$

We have

$$\begin{aligned}T(x) &= T\left((b-1)b^{m+1}\right) + T\left(\sum_{j=1}^{m}(b-1)b^m + r_0\right)\\ &= b^{m+2} - b^{m+1} + b - 1 + T(y)\\ &= b^{m+2} + b - T(b^{m+1}) + T(y) = b^{m+2} + b + T\left(\sum_{j=0}^{m} d_j b^j\right)\\ &= T\left(b^{m+2} + \sum_{j=0}^{m} d_j b^j + (b-1)/2\right).\end{aligned}$$

If $d_0 = 0$, we are obviously done. If $d_0 \neq 0$, assume by induction that $d_1 \neq b - 1$ (cf. the case $m = 1$). Since

$$\begin{aligned}&T\left(b^{m+2} + \sum_{j=1}^{m} d_j b^j + d_0'\right)\\ &= T\left(b^{m+2} + \sum_{j=2}^{m} d_j b^j + (d_1 + 1)b + d_0' - (b+1)/2\right),\ d_0' \geq (b+1)/2,\end{aligned}$$

the result is proved.

Proof of Theorem 2.1: If m and n are integers with $m \leq n$, define

$$\Omega(m, n) = \{T(x) \mid m \leq x \leq n\}.$$

Then

$$\lambda_k = |\Omega(0, b^k - 1) \cap \Omega(b^k, 2b^k - 1)|.$$

It is easy to see that the theorem is true when $k = 1$, so it suffices to prove that $\lambda_{k+1} - \lambda_k = (b-1)/2$ for all natural numbers k. Observe that if

$$T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T\left(b^k + \sum_{j=0}^{k-1} d_j' b^j\right)$$

and if $d_1' \neq b - 1$, then we can choose $d_0' \leq (b-1)/2$ since

$$T\left(b^k + \sum_{j=0}^{k-1} d_j' b^j\right) = T\left(b^k + \sum_{j=2}^{k-1} d_j' b^j + (d_1' + 1)b + d_0' - (b+1)/2\right)$$

for all $d_0' \geq (b+1)/2$. Also, suppose that we have

$$T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T\left(b^k + \sum_{j=2}^{k-1} d_j' b^j + (b-1)b + d_0'\right).$$

Since $T\left(\sum_{j=0}^{k-1} d_j b^j\right) \leq b^k - 1 + k(b-1)$, it is evident that $d_{k-1}' < b - 1$, so Lemma 2.3 says that we can choose $d_0' \leq (b-1)/2$ in this case as well.

Clearly

$$T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T\left(b^k + \sum_{j=0}^{k-1} d_j' b^j\right),\ d_0' \leq (b-1)/2$$

if and only if

$$T\left((b-1)b^k + \sum_{j=0}^{k-1} d_j b^j\right) = T\left(b^{k+1} + \sum_{j=1}^{k-1} d_j' b^j + d_0' + (b-1)/2\right).$$

By the same type of reasoning used to prove Lemma 2.2, we can see that the only values in $\Omega(b^{k+1}, 2b^{k+1} - 1)$ which we need to consider, besides the values $T(b^{k+1} + d_0'), 0 \leq d_0' \leq (b - 3)/2$, are values of the form

$$T\left(b^{k+1} + \sum_{j=0}^{k} d_j' b^j\right),$$

where $d_0' \geq (b - 1)/2$.

We therefore obtain the following correspondence (corresponding values on the left-hand side belonging to $\Omega(0, b^k - 1)$ if and only if corresponding values on the right-hand side belong to $\Omega(0, b^{k+1} - 1)$:

$$T\left(b^k + \sum_{j=0}^{k-1} d_j' b^j\right) \longleftrightarrow T\left(b^{k+1} + \sum_{j=0}^{k-1} d_j' b^j + (b - 1)/2\right),$$

where $d_0' \leq (b - 1)/2$.

The only other values in $\Omega(b^{k+1}, 2b^{k+1} - 1)$ which are left to consider are the values of the form $T(b^{k+1} + d_0')$, $0 \leq d_0' \leq (b - 3)/2$. By Lemma 2.2, there exists an integer

$$\sum_{j=0}^{k-1} d_j b^j, \; d_0 \geq (b - 1)/2,$$

such that

$$T\left(\sum_{j=0}^{k-1} d_j b^j\right) = T(b^k).$$

Hence

$$T\left((b - 1)b^k + \sum_{j=0}^{k-1} d_j b^j - (b - 1)/2\right) = T(b^{k+1})$$

and therefore

$$T\left((b - 1)b^k + \sum_{j=0}^{k-1} d_j b^j - (b - 1)/2 + 1\right) = T(b^{k+1} + 1)$$

$$\vdots$$

$$T\left((b - 1)b^k + \sum_{j=0}^{k-1} d_j b^j - 1\right) = T(b^{k+1} + (b - 3)/2)$$

i.e., the values $T(b^{k+1} + d_0')$, $0 \leq d_0' \leq (b - 3)/2$ all belong to $\Omega(0, b^{k+1} - 1)$. Since each of these values are different from each other and from all the other values in $\Omega(b^{k+1}, 2b^{k+1} - 1)$, we conclude that $\lambda_{k+1} - \lambda_k = (b - 1)/2$, Q.E.D.

3. AN ESTIMATE OF THE DENSITY WHEN $B = 10$

In contrast to the above result, the λ_k behave somewhat irregularly when b is even, as the following table, constructed for the case $b = 10$, shows.

The values in the table were computed essentially by finding the first integer in $\Omega(0, b^k - 1)$ which also belongs to $\Omega(b^k, 2b^k - 1)$; this appears to be difficult to do in general if b is even.

By using the table below, we obtain the following estimate of the density for base 10.

Theorem 3.1: When $b = 10$, the density of $\mathcal{R}$ is approximately 0.9022222; the error made by using this figure is less than 10^{-7}.

The Values of λ_k and $\lambda_{k+1} - \lambda_k$ for the Case $b = 10$, $1 \le k \le 50$

k	λ_k	$\lambda_{k+1} - \lambda_k$	k	λ_k	$\lambda_{k+1} - \lambda_k$
1	0		26	181	6
2	9	9	27	188	7
3	16	7	28	195	7
4	23	7	29	202	7
5	30	7	30	209	7
6	37	7	31	210	1
7	44	7	32	246	36
8	51	7	33	252	6
9	58	7	34	250	-2
10	65	7	35	249	-1
11	72	7	36	255	6
12	90	18	37	260	5
13	90	0	38	267	7
14	95	5	39	274	7
15	102	7	40	281	7
16	109	7	41	240	-41
17	116	7	42	321	81
18	123	7	43	327	6
19	130	7	44	313	-14
20	137	7	45	320	7
21	142	5	46	329	9
22	169	27	47	335	6
23	188	19	48	339	4
24	169	-19	49	346	7
25	175	6	50	353	7

Proof: Since $\max\{x \mid x \in \Omega(0, b^k - 1)\} = b^k - 1 + k(b - 1)$, it is clear that $\lambda_k < k(b - 1)$ for all k. Formula (1.2) says that

$$L = 1 - \sum_{k=1}^{7}(b \dot{-} 1)\lambda_k / b^{k+1} - \sum_{k=8}^{\infty}(b - 1)\lambda_k / b^{k+1}.$$

Now

$$\sum_{k=j}^{\infty}(b - 1)\lambda_k / b^{k+1} < ((b - 1)^2/b) \cdot \sum_{k=j}^{\infty} k/b^k$$

and

$$\sum_{k=8}^{\infty} k/b^k = \frac{(1 - 1/b)8(1/b)^8 + (1/b)^9}{(1 - 1/b)^2}.$$

Using the table and the above equations, our result is readily verified.

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