LEXICOGRAPHIC ORDERING AND FIBONACCI REPRESENTATIONS

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The Zeckendorf theorem [1], which essentially states that every positive integer can be represented uniquely as a finite sum of distinct Fibonacci numbers 1, 2, 3, 5, ..., 8, where no two consecutive Fibonacci numbers appear, led to so much new work that the entire January 1972 issue of the *Fibonacci Quarterly* was devoted to representations.

Now, through consideration of the ordering of the terms in a representation and the ordering of the integers, we study mappings of one integer into another by increasing the subscripts of the terms in a representation. We are led to number sequences related to the solutions of Wythoff's game [2], [3], and the generalized Wythoff's game [4]. We investigate representations using Fibonacci numbers, Pell numbers, generalized Fibonacci numbers arising from the Fibonacci polynomials, Lucas numbers, and Tribonacci numbers.

1. The Fibonacci Numbers

If we define the Fibonacci numbers in the usual way,

$$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1},\ n \geq 1,$$

then every positive integer N can be written in its Fibonacci-Zeckendorf representation as

$$N = \alpha_2 F_2 + \alpha_3 F_3 + \alpha_4 F_4 + \cdots + \alpha_k F_k, \tag{1.1}$$

where $\alpha_i \in \{0, 1\}$, $\alpha_i \alpha_{i-1} = 0$, or a representation as a sum of distinct Fibonacci numbers where no two consecutive Fibonacci numbers may be used. Such a representation is unique [5] and is also called the first canonical form of N.

If, instead, we write the Fibonacci representation of N in the second canonical form, we replace F_2 with F_1, and

$$N = \alpha_1 F_1 + \alpha_3 F_3 + \alpha_4 F_4 + \cdots + \alpha_k F_k, \tag{1.2}$$

where $\alpha_i \in \{0, 1\}$, $\alpha_2 = 0$, $\alpha_i \alpha_{i-1} = 0$. Such a representation is also unique.

Notice that, if the smallest Fibonacci number used in the representation has an odd subscript, the two forms are the same, but if the smallest Fibonacci number used has an even subscript, it can be written in either form. For example, the Zeckendorf representation of $8 = F_6$ becomes $8 = F_5 + F_3 + F_1$, and $11 = F_6 + F_4 = F_6 + F_3 + F_1$.

We next need some results on the ordering of the terms in a representation. A lexicographic ordering was earlier considered by Silber [7]. We define a lexicographic ordering as follows:

Let positive integers M and N each be represented in terms of a strictly increasing sequence of integers $\{a_n\}$ so that

$$M = \sum_{i=1}^{k} \alpha_i a_i, \qquad N = \sum_{i=1}^{k^*} \beta_i a_i, \tag{1.3}$$

where $\alpha_i, \beta_i \in \{0, 1, \ldots, p\}$. Let $\alpha_i = \beta_i$ for all $i > m$. If $\alpha_m > \beta_m$ only if $M > N$, then we say that the representation is a *lexicographic ordering*.

Theorem 1.1

The Zeckendorf representation of the positive integers in terms of Fibonacci numbers is a lexicographic ordering.

Proof: Let M and N be the two positive integers given in (1.3), where $a_n = F_{n+1}$, $p = 1$, and $\alpha_i \alpha_{i-1} = 0$, $\beta_i \beta_{i-1} = 0$. If $\alpha_i = \beta_i$ for all $i > m$, and if $\alpha_m > \beta_m$, then $\alpha_m = 1$ and $\beta_m = 0$, and we compare the truncated parts of the numbers.

$$M^* = \alpha_2 F_2 + \alpha_3 F_3 + \cdots + \alpha_{m-1} F_{m-1} + F_m \geq F_m$$

$$N^* = \beta_2 F_2 + \beta_3 F_3 + \cdots + \beta_{m-1} F_{m-1} \leq F_{m-1} + F_{m-3} + F_{m-5} + \cdots \leq F_m - 1,$$

so that $M^* > N^*$ and $M > N$, since it is well known that

$$F_{2k} + F_{2k-2} + \cdots + F_2 = F_{2k-1} - 1,$$

$$F_{2k-1} + F_{2k-3} + \cdots + F_3 = F_{2k} - 1.$$

Application: Let f^* be the transformation that advances by one the subscripts on each Fibonacci number used in the Zeckendorf representation of the positive integers M and N. If

$$M \xrightarrow{f^*} M' \quad \text{and} \quad N \xrightarrow{f^*} N',$$

and if $M > N$, then $M' > N'$.

Theorem 1.2

The Fibonacci representation of integers in the second canonical form is a lexicographic ordering.

The proof of Theorem 1.2 is very similar to that of Theorem 1.1. Next, we let f be the transformation that advances by one the subscripts of the Fibonacci numbers used in the representation in the second canonical form of the positive integers M and N. If

$$M \xrightarrow{f} M' \quad \text{and} \quad N \xrightarrow{f} N',$$

and if $M > N$, then $M' > N'$.

Let $A = \{A_n\}$ and $B = \{B_n\}$ be the sets of positive integers for which the smallest Fibonacci number used in the Zeckendorf representation occurred respectively with an even or with an odd subscript. Since the Zeckendorf representation is unique, sets A and B cover the set of positive integers and are disjoint.

Notice that, if the smallest subscript for a Fibonacci number used in the Zeckendorf representation for a number is odd, then the first and second canonical forms are the same. Thus, under f or f^*, every element of B is mapped into an element of A. But every element of A can be written in either canonical form, and under f every element of A is mapped into an element of A. Thus, every positive integer n is mapped into an element of A, or, aided by the lexicographic ordering theorems,

$$A_n \xrightarrow{f} A_{A_n}$$

$$B_n \xrightarrow{f} A_{B_n}$$

$$n \xrightarrow{f} A_n$$

$$A_n \xrightarrow{f^*} B_n$$

so that

$$A_{A_n} + 1 = B_n \tag{1.4}$$

follows, as well as

$$A_n + n = B_n. \tag{1.5}$$

Compare to the numbers a_n and b_n, where (a_n, b_n) is a safe pair for Wythoff's game [3], [4]. If one uses the Zeckendorf representation of positive integers using the Lucas numbers 2, 1, 3, 4, 7, ..., since the Lucas numbers are complete and have a unique Zeckendorf representation, we could make similar mappings. This is essentially developed in [4] but in a different way. For later comparison, we recall [3], [4], that

$$A_n = [n\alpha], \tag{1.6}$$

where $[x]$ is the greatest integer in x and $\alpha = (1 + \sqrt{5})/2$ is the positive root of $y^2 - y - 1 = 0$.

2. The Pell Numbers

Let us go to the Pell sequence $\{P_n\}$, defined by

$$P_1 = 1,\ P_2 = 2,\ P_{n+2} = 2P_{n+1} + P_n,\ n \geq 1.$$

The Pell sequence boasts of a unique Zeckendorf representation [6]. Consider the positive integers and the three sets $A = \{A_n\}$, $B = \{B_n\}$, and $C = \{C_n\}$, where $A_n = B_n - 1$ and $C_n = 2B_n + n$, and A, B, and C contain numbers in their natural order of the form

$$\begin{aligned} A_n &= 1 + \alpha_2 P_2 + \alpha_3 P_3 + \cdots + \alpha_k P_k, \\ B_n &= \alpha_2 P_2 + \alpha_3 P_3 + \cdots + \alpha_k P_k,\ \alpha_2 \neq 0, \\ C_n &= \alpha_3 P_3 + \cdots + \alpha_k P_k,\ \alpha_3 \neq 0, \end{aligned} \tag{2.1}$$

where $\alpha_i \in \{0, 1, 2\}$, and if $\alpha_i = 2$, then $\alpha_{i-1} = 0$.

Since we next wish to map the positive integers into set B, we will need a lexicographic ordering theorem for the Pell numbers.

Theorem 2.1

The Zeckendorf representation of the positive integers, in terms of Pell numbers, is a lexicographic ordering.

Proof: Let M and N be two positive integers given by

$$M = \sum_{i=1}^{k} \alpha_i P_i, \qquad N = \sum_{i=1}^{k} \beta_i P_i,$$

where α_i, $\beta_i \in \{0, 1, 2\}$ except α_1, $\beta_1 \neq 2$; and if $\alpha_i = 2$, then $\alpha_{i-1} = 0$, or if $\beta_i = 2$, then $\beta_{i-1} = 0$. If $\alpha_i = \beta_i$ for all $i > m$, and if $\alpha_m > \beta_m$, then $\alpha_m = 2$ and $\beta_m = 1$, or $\alpha_m = 1$ and $\beta_m = 0$, or $\alpha_m = 1$ and $\beta_m = 0$. We compare the truncated parts of the numbers when $\alpha_m = 2$ and $\beta_m = 1$:

$$M^* = \alpha_1 P_1 + \alpha_2 P_2 + \cdots + 2P_m \geq 2P_m$$

$$N^* = \beta_1 P_1 + \beta_2 P_2 + \cdots + P_m \leq P_m + P_m - 2 < 2P_m,$$

Since, if $\beta_1, \beta_2, \beta_3, \ldots, \beta_{m-1}$ are taken as large as possible, whether m is even or odd,

$$2(P_{2k-1} + \cdots + P_3) + P_1 = P_{2k} - 1 = P_m - 1,$$

$$2(P_{2k} + P_{2k-2} + \cdots + P_2) = P_{2k+1} - 1 = P_m - 1,$$

so that $M^* > N^*$ and $M > N$. If $\alpha_m = 2$ and $\beta_m = 0$, then N^* is even smaller. If $\alpha_m = 1$ and $\beta_m = 0$, then $M^* \geq P_m$, but notice that, if the coefficients β_i are taken as large as possible, we can only reach $N^* = P_m - 1$, and again $M^* > N^*$, making $M > N$. By definition (1.3), we have proved Theorem 2.1.

In an entirely similar manner, we could prove Theorem 2.2, where we write the second canonical form by replacing P_2 by $2P_1$ and $2P_2$ by $P_2 + 2P_1$ in the Zeckendorf representation, where again if $2P_k$ appears, then P_{k-1} is not used in that representation. This second canonical form is again unique [6]. We write:

Theorem 2.2

The Pell number representation of integers in the second canonical form is a lexicographic ordering.

Let f be the transformation that advances by one the subscripts of each Pell number used in the representation in the second canonical form of the positive integers M and N, and let f^* be the transformation that is used for the Zeckendorf form. Then, as before, if

$$M \xrightarrow{f} M' \quad \text{and} \quad N \xrightarrow{f} N',$$

and if $M > N$, then $M' > N'$, and the same for transformation f^*.

Now, we consider A_n, B_n, and C_n of (2.1), and mappings of the integers under f and f^*. We must first put B_n into the second canonical form. In the representation for B_n, replace P_2 by $2P_1$, or replace $2P_2$ by $P_2 + 2P_1$, since the smallest term of B_n is either P_2 or $2P_2$. Now, under f, B_n is mapped into B_{B_n}, while under f^*, B_n goes into C_n, applying the lexicographic theorems for Pell numbers.

$$P_2 \xrightarrow{f^*} P_3, \text{ or } 2 \xrightarrow{f^*} 5;$$

$$2P_1 \xrightarrow{f} 2P_2, \text{ or } 2 \xrightarrow{f} 4;$$

$$2P_2 \xrightarrow{f^*} 2P_3, \text{ or } 4 \xrightarrow{f^*} 10;$$

$$P_2 + 2P_1 \xrightarrow{f} P_3 + 2P_2, \text{ or } 4 \xrightarrow{f} 5 + 2 \cdot 2 = 9.$$

Thus, the image of B_n under f is one less than the image of B_n under f^*, and

$$B_{B_n} + 1 = C_n. \tag{2.2}$$

We know where the A_n's go under f: into B_n, since the A_n's start with a one, while their images start with a P_2. The B_n's (second form) have $2P_1$, so their images start with $2P_2$, clearly a B_n. Now, where do the C_n's go? Each C_n begins with 5 or 10. Replace $2P_3 = 10$ by $5 + 2 \cdot 2 + 1 = P_3 + 2P_2 + P_1$, and replace $1P_3 = 5$ by $2P_2 + P_1 = 2 \cdot 2 + 1$ and under f,

$$10 \to P_4 + 2P_3 + P_2 = 12 + 2 \cdot 5 + 2 = 24,$$

and

$$5 \to 2P_3 + P_2 = 2 \cdot 5 + 2 = 12.$$

Thus A_n, modified B_n, and modified C_n are all carried into B_n by f and

$$B_n \xrightarrow{f^*} C_n.$$

For later comparison, we note that

$$B_n = [n(1 + \sqrt{2})], \tag{2.3}$$

where $[x]$ is the greatest integer in x, and $(1 + \sqrt{2})$ is the positive root of $y^2 - 2y - 1 = 0$.

3. Generalized Fibonacci Numbers (Arising from Fibonacci Polynomials)

Next, consider the sequence of generalized Fibonacci numbers $\{u_n\}$,

$$u_0 = 0,\ u_1 = 1, \text{ and } u_{n+1} = ku_n + u_{n-1},\ n \geq 1.$$

[Note that, if the Fibonacci polynomials are given by $f_0(x) = 0$, $f_1(x) = 1$, and $f_{n+1}(x) = xf_n(x) + f_{n-1}(x)$, $n \geq 1$, then $u_n = f_n(k)$.] Let set B be the set of positive integers whose Zeckendorf representation has the smallest u_n used with an even subscript, and set 0 the set of integers whose Zeckendorf representation has the smallest u_n used with an odd subscript. We know from [6] that N has a unique representation of the form

$$N = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m, \tag{3.1}$$

where

$$\alpha_1 \in \{0, 1, \ldots, k - 1\},$$

$$\alpha_i \in \{0, 1, 2, \ldots, k\},\ i > 1,$$

$$\alpha_i = k \Longrightarrow \alpha_{i-1} = 0,$$

so that sets B and 0 cover the positive integers without overlapping.

We wish to demonstrate a second canonical form for elements of set B. We do this in two parts: Let α_{2k} be the coefficient of the least u_{2k} used; then $\alpha_{2k} = 1, 2, 3, \ldots, k$. Take $1u_{2k}$ and replace it by $ku_{2k-1} + u_{2k-2}$, and continue until you obtain $1u_2$, and replace that by ku_1,

$$u_{2k} = k(u_{2k-1} + u_{2k-3} + \cdots + u_3 + u_1).$$

Thus,

$$B_n = R + ku_{2k-1} + ku_{2k-3} + \cdots + ku_3 + ku_1.$$

If f is again the transformation that increases the subscripts by one for integers written in the second canonical form, and f^* the transformation for the Zeckendorf form, then, if we can again use lexicographic ordering,

$$B_n \xrightarrow{f} R' + ku_{2k} + ku_{2k-2} + \cdots + ku_2$$

$$B_n \xrightarrow{f^*} R' + u_{2k+1},$$

but from [6],

$$u_{2k+1} - 1 = k(u_{2k} + u_{2k-2} + \cdots + u_4 + u_2),$$

so that the images differ by 1. Now, under f, we see that all of the elements of 0 are mapped into set B and set B in second canonical form is also mapped into set B. Thus, provided we have lexicographic ordering, *the positive integers* n map into B_n under f. If we split set 0 into sets A whose elements use $1u_1$ in their representations and $C = \{C_n\}$, where C_n does not use $1u_1$ in its representation, then

$$B_n \xrightarrow{f^*} C_n, \qquad \text{and} \qquad B_n \xrightarrow{f} B_{B_n},$$

and since the images differ by 1,

$$B_{B_n} + 1 = C_n,\ n > 0. \tag{3.2}$$

The general lexicographic theorm should not be difficult.

Theorem 3.1

The Zeckendorf representation of the positive integers in terms of the generalized Fibonacci numbers $\{u_n\}$ is a lexicographic ordering.

Proof: Let M and N be positive integers which have Zeckendorf representations

$$M = \sum_{j=1}^{n} M_j u_j \qquad \text{and} \qquad N = \sum_{j=1}^{n} N_j u_j.$$

Compare the higher-ordered terms from highest to lowest. If $M_j = N_j$ for all $j > m$, and $M_m > N_m$, then we prove that $M > N$. It suffices to let $M = M_m u_m$ and $M_m \geq N_m + 1$.

$$N \leq N^* = ku_{2j-1} + ku_{2j-3} + ku_{2j-5} + \cdots + ku_3 + (k-1)u_1 = u_{2j} - 1$$

or

$$N \leq N^* = ku_{2j} + ku_{2j-2} + \cdots + ku_2 = u_{2j+1} - 1.$$

Thus $M \geq M^* > N^* \geq N$, so that $M > N$, proving Theorem 3.1.

This shows that, if two numbers M and N in Zeckendorf form are compared, then the one with the larger coefficient in the first place that they differ, coming down from the higher side, is larger. Now, what need be said about the second canonical form? If both M and N are in the second canonical form, and they differ in the jth place, whereas their smallest nonzero coefficient occurs in a position smaller than the jth place, then the original test suffices. If they both differ in the smallest position, then again the one with the larger coefficient there is larger, as their second canonical extensions are identical.

Theorem 3.2

The representation of positive integers in the second canonical form using generalized Fibonacci numbers $\{u_n\}$ is a lexicographic ordering.

Under transformation f, using the second canonical form, if $M = N + 1$, then

$$M \xrightarrow{f} M', \quad \text{and} \quad N \xrightarrow{f} N',$$

such that $M' > N' + k - 1$. For example,

$$u_1 = 1 \xrightarrow{f} u_2 = k, \; 2 = 2u_1 \xrightarrow{f} 2u_2 = 2k, \text{ and } 2k > k + k - 1,$$

taking $M = 2$ and $N = 1$.

We now return to sets A and C which made up set 0 and with set B covered the positive integers. Sets A, B, and C can be characterized as the positive integers written, in natural order, in the form

$$A_n = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_m u_m, \; \alpha_1 \neq 0, \; k,$$

(3.3) $$B_n = \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_m u_m, \; \alpha_2 \neq 0,$$

$$C_n = \alpha_3 u_3 + \alpha_4 u_4 + \cdots + \alpha_m u_m, \; \alpha_3 \neq 0, \; \alpha_i \in \{0, 1, 2, 3, \ldots, k\}.$$

For the numbers B_n, we can write:

Theorem 3.3

$$B_{B_n+1} - B_{B_n} = k + 1, \text{ and if } m \neq B_n,$$

$$B_{m+1} - B_m = k.$$

Also, it was proved by Molly Olds [18] that

Theorem 3.4

$$C_n = kB_n + n.$$

4. The Tribonacci Numbers

The Tribonacci numbers $\{T_n\}$ are

$$T_0 = 0, \; T_1 = 1, \; T_2 = 1, \; T_{n+3} = T_{n+2} + T_{n+1} + T_n, \; n \geq 0.$$

The Tribonacci numbers are complete with respect to the positive integers, and the positive integers again have a unique Zeckendorf representation in terms of Tribonacci numbers (see [8]). That is, a positive integer N has a unique representation in the form

(4.1) $$N = \alpha_2 T_2 + \alpha_3 T_3 + \cdots + \alpha_k T_k,$$

where $\alpha_i \in \{0, 1\}$, $\alpha_i \alpha_{i-1} \alpha_{i-2} = 0$.

Now, consider the numbers A_n, B_n, and C_n listed in Table 4.1. Here, because we want completeness in the array, we take A_n as the smallest positive integer not yet used, and we define Δ_n as the number of C_k's less than A_n, and φ_n as the number of C_k's less than B_n. Then, we compute B_n and C_n as

$$B_n = 2A_n - \Delta_n, \tag{4.2}$$

$$C_n = 2B_n - \varphi_n. \tag{4.3}$$

We write the Tribonacci recurrence relation:

$$n + A_n + B_n = C_n. \tag{4.4}$$

TABLE 4.1

n	A_n	B_n	C_n
1	1	2	4
2	3	6	11
3	5	9	17
4	7	13	24
5	8	15	28
6	10	19	35
7	12	22	41
8	14	26	48
9	16	30	55
10	18	33	61

Now, $A = \{A_n\}$ is the set of positive integers whose Zeckendorf representation has smallest term T_k, where $k \equiv 2 \bmod 3$; $B = \{B_n\}$ contains those positive integers using smallest term T_k, where $k \equiv 3 \bmod 3$; and $C = \{C_n\}$ has smallest term T_k, where $k \equiv 1 \bmod 3$, $k > 3$. We have suppressed $T_1 = 1$ in the above; thus, every positive integer belongs to A, B, or C by completeness, where A, B, and C are disjoint.

We write a second canonical form by rewriting each A_n by replacing T_2 by T_1; replacing $T_3 = 2$ in each B_n by $T_2 + T_1$; and leaving the numbers C_n alone.

Note that, instead of saying "A_n has smallest term T_{3m+2}," we could say "A_n has $3m + 1$ leading zeros."

Theorem 4.1

Each A_n has $k \equiv 1 \bmod 3$ leading zeros in the Zeckendorf representation and can be written so that

$$A_n = T_2 + \alpha_3 T_3 + \alpha_4 T_4 + \cdots + \alpha_r T_r, \text{ where } \alpha_i \in \{0, 1\}.$$

Each B_n has $k \equiv 2 \bmod 3$ leading zeros and can be written as

$$B_n = T_3 + \alpha_4 T_4 + \cdots + \alpha_r T_r, \text{ where } \alpha_i \in \{0, 1\}.$$

Each C_n has $k \equiv 0 \bmod 3$ leading zeros, $k \geq 3$, and can also be written

$$C_n = T_4 + \alpha_5 T_5 + \cdots + \alpha_r T_r, \text{ where } \alpha_i \in \{0, 1\}.$$

Proof: Let T_{3m+2} have a nonzero coefficient. Replace T_{3m+2} by

$$T_{3m+1} + T_{3m} + T_{3m-1} = T_{3m+1} + T_{3m} + T_{3(m-1)+2}.$$

Continue until the right member ultimately lands in slot 2. The similar replacement for T_{3m} in B_n and T_{3m+1} in C_n will establish the forms given above.

Theorem 4.2

The Zeckendorf representation of the positive integers in terms of the Tribonacci numbers $\{T_n\}$ is a lexicographic ordering. The representation in the second canonical form is also a lexicographic ordering.

Proof: Write M and N in their Zeckendorf representations,

$$M = \sum_{j=2}^{n} M_j T_j \qquad \text{and} \qquad N = \sum_{j=2}^{n} N_j T_j.$$

If $M_j = N_j$ for all $j > m$ and $M_m > N_m$, then $M_m = 1$ and $N_m = 0$, and we prove that $M > N$. We let M^* and N^* be the truncated parts of the numbers M and N. Then

$$M^* = M_2 T_2 + M_3 T_3 + \cdots + M_m T_m \geq T_m,$$

$$N^* = N_2 T_2 + N_3 T_3 + \cdots + N_{m-1} T_{m-1}.$$

Since $N_i N_{i-1} N_{i-2} = 0$, N^* is as large as possible when both N_{m-1} and N_{m-2} are nonzero. Either $m = 3k$ or $m = 3k + 1$ or $m = 3k - 1$. We use three summation formulas given by Waddill and Sacks [9].

If $m = 3k$, then

$$N^* \leq \sum_{i=1}^{k} (T_{3i-1} + T_{3i-2}) - T_1 = T_{3k} - 1 < T_m \leq M^*.$$

If $m = 3k + 1$,

$$N^* \leq \sum_{i=1}^{k} (T_{3i} + T_{3i-1}) = T_{3k+1} - 1 < T_m \leq M^*.$$

If $m = 3k - 1$,

$$N^* \leq \sum_{i=1}^{k} (T_{3i-2} + T_{3i-3}) - T_1 = T_{3k-1} - 1 < T_m \leq M^*.$$

Thus in all three cases, $M^* > N^*$ so that $M > N$, and the Zeckendorf representation is a lexicographic ordering. The same summation identities would show that the second canonical form is also lexicographic.

Next, let f be the transformation that increases the subscripts by one for integers written in the second canonical form, and f^* the similar transformation for the Zeckendorf form. Now, the numbers in set A are ordered, and since we have lexicographic ordering for the second canonical form,

$$n \xrightarrow{f} A_n, \; A_n \xrightarrow{f} A_{A_n}, \; B_n \xrightarrow{f} A_{B_n}, \; C_n \xrightarrow{f} A_{C_n}.$$

Since we have lexicographic ordering for the Zeckendorf form,

$$A_n \xrightarrow{f^*} B_n, \; B_n \xrightarrow{f^*} C_n, \; C_n \xrightarrow{f^*} A_{C_n}.$$

But each A_{A_n} is one less than B_n, and each A_{B_n} is one less than C_n, so that

$$A_{A_n} + 1 = B_n, \quad \text{and} \quad A_{B_n} + 1 = C_n. \tag{4.5}$$

(4.5) reminds one of $a_{a_n} + 1 = b_n$ from Wythoff's game [3], [4]. Note that $\{C_n\}$ clearly maps into $\{A_n\}$ because they were of the form whose least term had subscript $k \equiv 2 \bmod 3$, so that an upward shift of one yields $k \equiv 3 \bmod 3$ and, hence, A_{C_n}.

Comments: Under f, A_n maps to A_{A_n}, and under f^*, A_n maps to B_n. If A_n is in second canonical form, then $A_n + 1 = A_n + T_2$ is also in second canonical form. Thus, using the Zeckendorf and then the second form for A_n,

$$A_n + T_1 \xrightarrow{f^*} B_n + T_2 = B_n + 1,$$

$$A_n + T_2 \xrightarrow{f} A_{A_n} + T_3 = A_{A_n} + 2,$$

so that $A_{A_n} + 1 = B_n$. Clearly $B_n + 1$ is an A_j since the B_n's have T_3 as the lowest nonzero Tribonacci number, but $B_n + 1$ has T_2. Thus,

$$A_{A_n} + 1 = B_n \quad \text{and} \quad B_n + 1 = A_{A_n+1} \tag{4.6}$$

so that

$$A_{A_n+1} - A_{A_n} = 2.$$

We also have shown that there are A_n of the A_j's less than B_n.

Under f, B_n maps to A_{B_n}, and under f^*, B_n maps to C_n. Therefore,

$$A_{B_n} + 1 = C_n,$$

which shows that there are B_n of the A_j's less than C_n. Also, $C_n + 1$ is an A_j since each C_n can be written with the least summand T_4. Therefore,

$$C_n + 1 = A_{B_n+1},$$

and

$$A_{B_n} + 1 = C_n \quad \text{and} \quad C_n + 1 = A_{B_n+1} \tag{4.7}$$

give us

$$A_{B_n+1} - A_{B_n} = 2.$$

Next, we look at C_n and $C_n + 1$.

$$C_n \xrightarrow{f} A_{C_n} \quad \text{and} \quad C_n + 1 = C_n + T_1 \xrightarrow{f} A_{C_n+1} = A_{C_n} + 1.$$

Since $C_n + 1$ is A_{B_n+1}, the one is T_1 in $C_n + 1$. We conclude that

$$A_{C_n+1} - A_{C_n} = 1.$$

This gives all the recurrent differences for the A sequence.

We now turn to the B sequence.

$$1 = (A_{C_n+1} - A_{C_n}) \xrightarrow{f^*} (B_{C_n+1} - B_{C_n}) = 2,$$

$$2 = (A_{A_n+1} - A_{A_n}) \xrightarrow{f^*} (B_{A_n+1} - B_{A_n}) = 4,$$

$$1 + 1 = 2 = (A_{B_n+1} - A_{B_n}) \xrightarrow{f^*} (B_{B_n+1} - B_{B_n}) = 1 + 2 = 3.$$

We look first at

$$C_n \xrightarrow{f} A_{C_n} \quad \text{and} \quad C_n + 1 \xrightarrow{f} A_{C_n+1}$$

because $C_n + 1$ is an A_j so 1 in it is T_1. Thus

$$A_{C_n+1} = A_{C_n} + T_2 \xrightarrow{f^*} B_{C_n+1} = B_{C_n} + T_3 = B_{C_n} + 2,$$

and

$$B_{C_n+1} - B_{C_n} = 2.$$

Now, in second canonical form, A_n has T_1 but no T_2, but $A_n + 1$ has T_1 and T_2, or, $A_n + 1 = A_n + T_2$.

$$A_n \xrightarrow{f} A_{A_n} \xrightarrow{f^*} B_{A_n}$$

$$A_n + 1 = A_n + T_2 \xrightarrow{f} A_{A_n} + T_3 \xrightarrow{f^*} B_{A_n} + T_4 = B_{A_n} + 4,$$

$$A_n + 1 \xrightarrow{f} A_{A_n+1} \xrightarrow{f^*} B_{A_n+1} = B_{A_n} + 4.$$

Thus,

$$B_{A_n+1} - B_{A_n} = 4.$$

Next, let $B_n = R_n + T_3 = R_n + T_2 + T_1$ be in second canonical form.

$$B_n \xrightarrow{f} A_{B_n} = R_n' + T_3 + T_2 \xrightarrow{f^*} R_n'' + T_4 + T_3,$$

$$B_n + 1 = R_n + T_3 + T_1 \xrightarrow{f} R_n' + T_4 + T_2 \xrightarrow{f^*} R_n'' + T_5 + T_3,$$

$$B_n + 1 \xrightarrow{f} A_{B_n+1} \xrightarrow{f^*} B_{B_n+1} = R_n'' + T_5 + T_3.$$

Therefore,

$$B_{B_n+1} - B_{B_n} = (R_n'' + T_5 + T_3) - (R_n'' + T_4 + T_3) = T_5 - T_4 = 3.$$

Finally, for the third difference of B numbers,

$$C_n \xrightarrow{f} A_{C_n} \xrightarrow{f^*} B_{C_n},$$

$$C_n + 1 = C_n + T_1 \xrightarrow{f} A_{C_n} + T_2 \xrightarrow{f^*} B_{C_n} + T_3,$$

$$C_n + 1 \xrightarrow{f} A_{C_n+1} \xrightarrow{f^*} B_{C_n+1}.$$

Therefore,

$$B_{C_n+1} - B_{C_n} = T_3 = 2.$$

Lastly, the three differences of consecutive C_j's are found by using the above differences of A_j's and B_j's and (4.4).

$$\begin{aligned} C_{A_n+1} - C_{A_n} &= (A_n + 1 + A_{A_n+1} + B_{A_n+1}) - (A_n + A_{A_n} + B_{A_n}) \\ &= (A_n + 1 - A_n) + (A_{A_n+1} - A_{A_n}) + (B_{A_n+1} - B_{A_n}) \\ &= 1 + 2 + 4 = 7. \\ C_{B_n+1} - C_{B_n} &= (B_n + 1 - B_n) + (A_{B_n+1} - A_{B_n}) + (B_{B_n+1} - B_{B_n}) \\ &= 1 + 2 + 3 = 6. \\ C_{C_n+1} - C_{C_n} &= (C_n + 1 - C_n) + (A_{C_n+1} - A_{C_n}) + (B_{C_n+1} - B_{C_n}) \\ &= 1 + 1 + 2 = 4. \end{aligned}$$

We summarize all the possible differences of successive members of the A, B, and C sequences as:

<u>Theorem 4.3</u>

$$A_{A_n+1} - A_{A_n} = 2,\ A_{B_n+1} - A_{B_n} = 2,\ A_{C_n+1} - A_{C_n} = 1;$$

$$B_{A_n+1} - B_{A_n} = 4,\ B_{B_n+1} - B_{B_n} = 3,\ B_{C_n+1} - B_{C_n} = 2;$$

$$C_{A_n+1} - C_{A_n} = 7,\ C_{B_n+1} - C_{B_n} = 6,\ C_{C_n+1} - C_{C_n} = 4.$$

Returning to (4.6), we know that there are A_n of the A_j's less than B_n. Then, B_n is n plus the number of A_j's less than B_n, plus the number of C_k's less than B_n, or,

$$B_n = n + A_n + \varphi_n.$$

Then

$$C_n = 2B_n - \varphi_n = 2B_n - (B_n - n - A_n) = B_n + A_n + n,$$

a consistency proof that the C_n's are properly defined by the array of Table 4.1.

Theorem 4.4

The number of C_j's less than A_n is

$$\Delta_n = 2A_n - B_n.$$

Proof: We show that $2A_n - B_n$ increments by 1 if and only if $n = B_m$, and zero otherwise, applying Theorem 4.3:

$$2(A_{A_n+1} - A_{A_n}) - (B_{A_n+1} - B_{A_n}) = 2(2) - 4 = 0,$$

$$2(A_{B_n+1} - A_{B_n}) - (B_{B_n+1} - B_{B_n}) = 2(2) - 3 = 1,$$

$$2(A_{C_n+1} - A_{C_n}) - (B_{C_n+1} - B_{C_n}) = 2(1) - 2 = 0.$$

Note well that $\{A_n\}$, $\{B_n\}$, and $\{C_n\}$ are sets whose disjoint union is the set of positive integers. From (4.7), we see that

$$A_{B_n} + 1 = C_n = A_{B_n+1} - 1,$$

$$A_{B_n} < C_n < A_{B_n+1}.$$

From $A_{C_n+1} - A_{C_n} = 1$, there are no C_j's between those two A_k's. From (4.6), we see that

$$A_{A_n} + 1 = B_n = A_{A_n+1} - 1,$$

$$A_{A_n} < B_n < A_{A_n+1}.$$

Thus, $2A_n - B_n$ counts the number of C_j's less than A_n.

Theorem 4.4 shows that B_n is properly defined in the array of Table 4.1. We know from earlier work that $(B_n - A_n - n)$ counts the number of C_j's less than B_n and agrees with the definition of C_n in the array. Since each B_n and C_n is followed by some A_k, the choice of A_n as the first positive integer not yet used guarantees that the sets in the array cover the positive integers.

Nota bene: If $(2A_n - B_n)$ counts the number of C_j's less than A_n, it also counts the number of B_j's less than n. Further, $(B_n - A_n - n)$ counts the number of C_j's less than B_n; it also counts the number of B_j's less than A_n, and the number of A_j's less than n. These follow immediately from the lexicographic ordering by moving backward. Summarizing:

Theorem 4.5

(a) $(2n - 1 - A_n)$ counts the number of C_j's less than n;

(b) $(2A_n - B_n)$ counts the number of B_j's less than n;

(c) $(B_n - A_n - n)$ counts the number of A_j's less than n.

Next, we make application of a theorem of Moser and Lamdek [11];

Theorem (Leo Moser and J. Lamdek, 1954)

Let $f(n)$ be a nondecreasing function of nonnegative integers defined on the positive integers,

$$F(n) = f(n) + n, \qquad G(n) = f^*(n) + n, \tag{A}$$

where $f^*(n)$ is the number of positive integers x satisfying $0 \leq f(x) < n$. Then, $F(n)$ and $G(n)$ are complementary sequences. Conversely, every two increasing complementary sequences $F(n)$ and $G(n)$ decompose into form (A), with $f(n)$ nondecreasing.

Let $f^*(n) = B_n - A_n - n$; then

$$G(n) = B_n - A_n \quad \text{and} \quad F(n) = A_n + n = C_n - B_n,$$

since $C_n = B_n + A_n + n$. Thus, $(B_n - A_n)$ and $(C_n - B_n)$ are complementary sequences.

Let $f^*(n) = 2A_n - B_n$; then

$$G(n) = 2A_n - B_n + n = C_n - 2B_n + A_n = (C_n - B_n) - (B_n - A_n)$$

and

$$F(n) = B_n + n = C_n - A_n = (C_n - B_n) + (B_n - A_n).$$

Thus, $G(n) = (C_n - B_n) - (B_n - A_n)$ and $F(n) = (C_n - B_n) + (B_n - A_n)$ are complementary sets.

Let $f^*(n) = 2n - 1 - A_n$; then

$$G(n) = 3n - 1 - A_n \quad \text{and} \quad F(n) = C_n + n.$$

Thus, $F(n)$ and $G(n)$ are complementary sets. We have just proved:

Theorem 4.6

The three sequences $\{A_n\}$, $\{B_n\}$, and $\{C_n\}$ are such that their disjoint union is the set of positive integers. That is, they form a triple of complementary sequences. Further, their differences $(B_n - A_n)$ and $(C_n - B_n)$ form a pair of complementary sequences, *and* the sum and differences of this pair of complementary sequences form another pair of complementary sequences:

$$(C_n - A_n) \quad \text{and} \quad (C_n - 2B_n + A_n = 2A_n - B_n + n).$$

5. The r-nacci Numbers

The r-nacci numbers $\{R_n\}$ are given by [14]

$$R_0 = 0,\ R_1 = 1,\ R_j = 2^{j-2},\ j = 2, 3, \ldots, r + 1,$$

and

$$R_{n+r} = R_{n+r-1} + R_{n+r-2} + \cdots + R_n. \tag{5.1}$$

The Fibonacci numbers $\{F_n\}$ are the case $r = 2$, while the Tribonacci numbers $\{T_n\}$ have $r = 3$, and the Quadranacci numbers $\{Q_n\}$ have $r = 4$.

We have the sequence of identities

$$(5.2)\quad r = 2: \qquad \begin{aligned} F_2 + F_4 + F_6 + \cdots + F_{2n} &= F_{2n+1} - 1, \\ F_3 + F_5 + F_7 + \cdots + F_{2n+1} &= F_{2n+2} - 1. \end{aligned}$$

$$(5.3)\quad r = 3: \qquad \begin{aligned} (T_2 + T_3) + (T_5 + T_6) + \cdots + (T_{3n-1} + T_{3n}) &= T_{3n+1} - 1, \\ (T_3 + T_4) + (T_6 + T_7) + \cdots + (T_{3n} + T_{3n+1}) &= T_{3n+2} - 1, \\ T_2 + (T_4 + T_5) + (T_7 + T_8) + \cdots + (T_{3n+1} + T_{3n+2}) &= T_{3n+3} - 1. \end{aligned}$$

$$(5.4)\quad r = 4: \qquad \begin{aligned} &(Q_2 + Q_3 + Q_4) + (Q_6 + Q_7 + Q_8) + \cdots \\ &\qquad + (Q_{4n-2} + Q_{4n-1} + Q_{4n}) = Q_{4n+1} - 1, \\ &(Q_3 + Q_4 + Q_5) + (Q_7 + Q_8 + Q_9) + \cdots \\ &\qquad + (Q_{4n-1} + Q_{4n} + Q_{4n+1}) = Q_{4n+2} - 1, \\ &Q_2 + (Q_4 + Q_5 + Q_6) + (Q_8 + Q_9 + Q_{10}) + \cdots \\ &\qquad + (Q_{4n} + Q_{4n+1} + Q_{4n+2}) = Q_{4n+3} - 1, \\ &Q_2 + Q_3 + (Q_5 + Q_6 + Q_7) + (Q_9 + Q_{10} + Q_{11}) + \cdots \\ &\qquad + (Q_{4n+1} + Q_{4n+2} + Q_{4n+3}) = Q_{4n+4} - 1. \end{aligned}$$

Note that R_1 is never used on the left. Generalizing to the r-nacci numbers, we make groups of $(r - 1)$ terms, writing r equations:

$$(5.5)\qquad \begin{aligned} &(R_2 + R_3 + \cdots + R_r) + (R_{r+2} + \cdots + R_{2r}) + \cdots \\ &\qquad + (R_{(k-1)r+2} + \cdots + R_{kr}) = R_{kr+1} - 1, \\ &(R_3 + R_4 + \cdots + R_{r+1}) + (R_{r+3} + \cdots + R_{2r+1}) + \cdots \\ &\qquad + (R_{(k-1)r+3} + \cdots + R_{kr+1}) = R_{kr+2} - 1, \\ &R_2 + (R_4 + \cdots + R_{r+2}) + (R_{r+4} + \cdots + R_{2r+2}) + \cdots \\ &\qquad + (R_{(k-1)r+4} + \cdots + R_{kr+2}) = R_{kr+3} - 1, \\ &R_2 + R_3 + (R_5 + \cdots + R_{r+3}) + (R_{r+5} + \cdots + R_{2r+3}) + \cdots \\ &\qquad + (R_{(k-1)r+5} + \cdots + R_{kr+3}) = R_{kr+4} - 1, \\ &\cdots \qquad \cdots \qquad \cdots \qquad \cdots \end{aligned}$$

(5.5)—continued

$$R_2 + R_3 + \cdots + R_{r-1} + (R_{r+1} + \cdots + R_{2r}) + (R_{2r+2} + \cdots + R_{3r+2}) + \cdots + (R_{kr+1} + \cdots + R_{kr+(r-1)}) = R_{kr+r} - 1.$$

Notice that the proof of Eqs. (5.5) is very simple. In any of the equations, add $1 = R_1$ to the left, and observe that

$$R_1 + R_2 + R_3 + \cdots + R_i = R_{i+1} \text{ for } i = 1, 2, \ldots, r - 1,$$

and that R_{i+1} can be added to the next group of $(r - 1)$ consecutive terms to get R_{i+r+1}, which can be added to the next group of $(r - 1)$ consecutive terms. Repeat until reaching R_{kr+i}.

The r-nacci numbers, which are the generalized Fibonacci polynomials of [13] evaluated at $x = k = 1$, again give a unique Zeckendorf representation for each positive integer N,

(5.6) $$N = \alpha_2 R_2 + \alpha_3 R_3 + \cdots + \alpha_k R_k,$$

where $\alpha_i \in \{0, 1\}$, and $\alpha_i \alpha_{i-1} \alpha_{i-2} \ldots \alpha_{i-r+1} = 0$.

Now let $A_i = \{a_{i,n}\}$ be the set of positive integers whose unique Zeckendorf representation has smallest term R_k, $k \geq 2$ (we have suppressed R_1), where $k \equiv i \bmod r$, $i = 2, 3, \ldots, r + 1$. Thus, every positive integer belongs to one of the sets A_i by completeness, where the sets A_i are disjoint.

Theorem 5.1

Each $a_{i,n}$ can be written so that

$$a_{i,n} = R_i + \alpha_{i+1} R_{i+1} + \alpha_{i+2} R_{i+2} + \cdots + \alpha_p R_p,$$

where $\alpha_i \in \{0, 1\}$ and $i = 2, 3, \ldots, r + 1$.

Proof: Let $N = a_{i,n}$ have R_{mr+i} as the smallest term used in its unique Zeckendorf representation. Write R_{mr+i} as

$$R_{mr+i-1} + R_{mr+i-2} + \cdots + R_{mr+i-r}.$$

Then rewrite $R_{(m-1)r+i}$ as

$$R_{(m-1)r+i-1} + R_{(m-1)r+i-2} + \cdots + R_{(m-1)r+i-r},$$

and continue replacing the smallest term used until the smallest term obtained is R_i, which is one of $R_2, R_3, \ldots, R_{r+1}$.

Theorem 5.2

The Zeckendorf representation of the positive integers in terms of the r-nacci numbers $\{R_n\}$ is a lexicographic ordering.

Proof: Write M and N in their Zeckendorf representations,

$$M = \sum_{j=2}^{n} M_j R_j \quad \text{and} \quad N = \sum_{j=2}^{n} N_j R_j,$$

where $M_j, N_j \in \{0, 1\}$. If $M_j = N_j$ for all $j > m$ and $M_m > N_m$, then $M_m = 1$ and $N_m = 0$, and we prove that $M > N$. Let M^* and N^* be the truncated parts of the numbers M and N. Then

$$M^* = M_2R_2 + M_3R_3 + \cdots + M_mR_m \geq R_m,$$

$$N^* = N_2R_2 + N_3R_3 + \cdots + N_{m-1}R_{m-1}.$$

Since $N_i N_{i-1} \ldots N_{i-r+1} = 0$, N^* is as large as possible when $N_{m-1}, N_{m-2}, \ldots, N_{m-r+1}$ are nonzero. Then $m = rk + i$ for some $i = 1, 2, \ldots, r$. But Eqs. (5.5) show that N^* at its largest is $R_m - 1$, so that $N^* < R_m \leq M^*$, and thus $M > N$, so that the Zeckendorf representation is a lexicographic ordering.

6. The Rising Diagonals of Pascal's Triangle

The numbers $u(n; p, 1)$ of Harris and Styles [15] lie on the rising diagonals of Pascal's triangle with characteristic equation

$$x^{p+1} - x^p - 1 = 0.$$

We define $u(n; p, 1) = u_n$, where $n \geq 0$ and $p \geq 0$ are integers, by

$$(6.1) \qquad u_n = u(n; p, 1) = \sum_{i=0}^{[n/(p+1)]} \binom{n - ip}{i}, \; n \geq 1, \; u(0; p, 1) = 1,$$

where $[x]$ is the greatest integer function, and $\binom{n}{k}$ is a binomial coefficient. We note that, if $p = 1$,

$$u(n - 1; 1, 1) = F_n,$$

and if $p = 0$,

$$u(n; 0, 1) = 2^n.$$

Also,

$$u_0 = u_1 = u_2 = \cdots = u_p = 1, \; u_{p+1} = 2.$$

We write Pascal's triangle in left-justified form. Then $u(n; p, 1)$ is the sum of the term in the leftmost column and nth row (the top row is the zeroth row) and the terms obtained by starting at this term and moving p units up and one unit right throughout the array. We also have

(6.2) $$u_n = u_{n-1} + u_{n-p-1}$$

with the useful identity, for any given value of p,

(6.3) $$\sum_{i=0}^{n} u_i = u_{n+p+1} - 1.$$

Now, each positive integer N has a unique Zeckendorf representation in terms of $\{u(n;\ p,\ 1)\}$ for each given p, as developed by Mohanty [16]:

(6.4) $$N = \sum_{i=p}^{s} a_i u(i;\ p,\ 1),$$

with $a_s = 1$ and $a_i = 1$ or 0, $p \leq i < s$. Here, s is the largest integer such that F_s is involved in the sum, and $u_1 = u_2 = \cdots = u_{p-1} = 1$ are not used in any sum. If $a_i a_{i+j} = 0$ for all $i \geq p$ and $j = 1, 2, \ldots, p - 1$, then we have the unique Zeckendorf representation using the least number of terms. If $a_i + a_{i+j} \geq 1$ for all $i \geq p$ and $j = 1, 2, \ldots, p - 1$, then we have a third form, which also is a unique representation.

The results of Mohanty can be restated. Let A_i be the set of positive integers whose unique Zeckendorf representation in terms of $u(n;\ p,\ 1)$ has smallest term u_n, $n \geq p$, where $n \equiv i \bmod (p + 1)$, $i = 0, 1, 2, \ldots, p$. Then every positive integer belongs to one of the sets A_i, where the sets A_i are disjoint. Further, every element in set A_i can be rewritten uniquely so that the smallest term used is u_{p+i}, $i = 0, 1, 2, \ldots, p$, by replacing the smallest term repeatedly, as,

$$\begin{aligned} u_n &= u_{n-1} + u_{n-1-p} = u_{n-1} + u_{n-p-2} + u_{n-2p-2} \\ &= u_{n-1} + u_{n-(p+1)} + u_{n-2(p+1)} + \cdots + u_{p+i}. \end{aligned}$$

We write a second canonical form by replacing $u_p = 1$ by $u_{p-1} = 1$ whenever it occurs, but notice that only set A_p is affected.

We can establish the identity

(6.5) $$\sum_{k=1}^{n} u_{(p+1)k+i} = u_{(p+1)n+i+1} - 1$$

for each integer i, $0 \leq i \leq p$, by mathematical induction. For each value of p, when $n = 1$, we have, by (6.2):

$$u_{(p+1)\cdot 1+i} = u_{(p+1)\cdot 1+i+1} - u_p = u_{(p+1)\cdot 1+i+1} - 1.$$

If (6.5) holds for all integers $n \leq t$, then

$$\sum_{k=1}^{t+1} u_{(p+1)k+i} = \sum_{k=1}^{t} u_{(p+1)k+i} + u_{(p+1)(t+1)+i}$$

$$= (u_{(p+1)t+i+1} - 1) + u_{(p+1)t+p+i+1} - 1$$

$$= u_{(p+1)t+(p+1)+i+1} - 1$$

$$= u_{(p+1)(t+1)+i+1} - 1,$$

the form of (6.5) when $n = t + 1$, so that (6.5) holds for all integers n by mathematical induction.

We are now ready for our main theorem.

Theorem 6.1

The Zeckendorf representation of the positive integers in terms of

$$\{u(n;\ p,\ 1)\}$$

is a lexicographic ordering. The representation in second canonical form is also a lexicographic ordering.

Proof: Write M and N in their Zeckendorf representation using the least number of terms,

$$M = \sum_{i=p}^{n} M_i u_i \quad \text{and} \quad N = \sum_{i=p}^{n} N_i u_i,$$

where M_i, $N_i \in \{0, 1\}$ and $M_i M_{i+j} = 0$ for all $i \geq p$ and $j = 1, 2, \ldots, p - 1$. If $M_i = N_i$ for all $i > m$ and $M_m > N_m$, then $M_m = 1$ and $N_m = 0$, and we prove that $M > N$. Let M^* and N^* be the truncated parts of the numbers M and N. Then

$$M^* = M_p u_p + M_{p+1} u_{p+1} + \cdots + M_m u_m \geq u_m,$$

$$N^* = N_p u_p + N_{p+1} u_{p+1} + \cdots + N_{m-1} u_{m-1}.$$

Since $N_i N_{i+j} = 0$ for $j = 1, 2, \ldots, p - 1$, N^* is as large as possible when N_{m-1} is nonzero, but then $N_{m-2} = N_{m-3} = \cdots = N_{m-p} = 0$. The next largest possible u_i used is u_{m-p-1}, then u_{m-2p-1}, etc. Now, we can represent $(m - 1)$ as

$$m - 1 = (p + 1)k + i,$$

where $0 \leq i \leq p$. By (6.5), for any value of $(m - 1)$, we always have

$$N^* \leq \sum_{k=1}^{[(m-1-i)/(p+1)]} u_{(p+1)k+i} = u_m - 1 < M^*.$$

Thus, $M > N$, and the Zeckendorf representation is a lexicographic ordering.

Note that the same proof can be used in the second canonical form because only the smallest term in the Zeckendorf representation is changed.

7. Applications to the Generalized Fibonacci Numbers $u(n;\, 2,\, 1)$

Let us concentrate now on the sequence $u(n-1;\, 2,\, 1) = u_n$, where we take $p = 2$ in Section 6. We write

(7.1) $$u_1 = 1,\ u_2 = 2,\ u_3 = 3, \text{ and } u_{n+3} = u_{n+2} + u_n.$$

Theorem 7.1

Each positive integer N enjoys a unique Zeckendorf representation in the form

$$N = \sum_{i=1}^{k} \alpha_i u_i,\ \alpha_i \alpha_{i+1} = 0,\ \alpha_i \alpha_{i+2} = 0,$$

where $\alpha_i \in \{0,\, 1\}$.

Each positive integer N can be put into one of three sets A, B, or C according to the smallest u_k used in the unique Zeckendorf representation of N, by whether $k \equiv 1 \bmod 3$ for A, $k \equiv 2 \bmod 3$ for B, or $k \equiv 3 \bmod 3$ for C. Let $A = \{A_n\}$, $B = \{B_n\}$, and $C = \{C_n\}$ be the listing of the elements of A, B, and C in natural order. Note that we can rewrite each unique Zeckendorf representation by changing only the smallest term used to make a *new form* where the smallest term appearing is u_1, u_2, or u_3. If the smallest term appearing is u_k, we replace the smallest term repeatedly:

$$\begin{aligned} u_k = u_{3m} = u_{3m-1} + u_{3m-3} &= u_{3m-1} + u_{3m-4} + u_{3m-6} = \cdots \\ &= u_{3m-1} + u_{3m-4} + \cdots + u_3, \\ u_k = u_{3m+1} = u_{3m} + u_{3m-2} &= u_{3m} + u_{3m-3} + u_{3m-5} = \cdots \\ &= u_{3m} + u_{3m-3} + \cdots + u_1, \\ u_k = u_{3m+2} = u_{3m+1} + u_{3m-1} &= u_{3m+1} + u_{3m-2} + u_{3m-4} \\ &= u_{3m+1} + u_{3m-2} + \cdots + u_2. \end{aligned}$$

We can summarize as

Theorem 7.2

Each member of set A has a representation in the form

$$A_n = 1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_m u_m,\ \alpha_i \in \{0,\, 1\};$$

each member of set B has a representation in the form

$$B_n = 2 + \alpha_3 u_3 + \alpha_4 u_4 + \cdots + \alpha_m u_m,\ \alpha_i \in \{0,\, 1\};$$

and each member of set C has a representation in the form

$$C_n = 3 + \alpha_4 u_4 + \alpha_5 u_5 + \cdots + \alpha_m u_m,\ \alpha_i \in \{0,\, 1\}.$$

There are some instant results:

(7.2) $$B_n - 1 = A_j, \qquad C_n - 1 = B_j.$$

Now, let $H = \{H_n\} = A \cup C$, where the elements of H are listed in natural order. We write the second canonical representation for sets A, B, and C, by replacing $u_1 = 1$ by $u_0 = 1$ in the representation of A_n but leaving B_n and C_n represented as in Theorem 7.2. Let f be the transformation that advances by one the subscripts of each of the summands u_n for each representation that is in *second canonical form*. Let f^* be the transformation that advances the subscripts by one of each summand u_n used in the Zeckendorf representation of N.

Theorem 7.3

$$N \xrightarrow{f} H \xrightarrow{f} A \xrightarrow{f^*} B \xrightarrow{f^*} C$$

Proof: It is clear that $A_n \xrightarrow{f^*} B_n \xrightarrow{f^*} C_n$ by the lexicographic ordering theorem (Theorem 6.1). Consider the sequence 1, 2, 3, ..., H_n; then, since H and B are complementary sets, we have

$$H_n = n + (\text{number of } B_j\text{'s less than } H_n).$$

Thus, by Theorem 6.1,

$$(\text{number of } B_j\text{'s less than } H_n) = (\text{number of } A_j\text{'s less than } n)$$
$$= C_n - B_n - n.$$

Here we have assumed the equivalence of the definitions of A_n, B_n, and C_n and the following (see [17]):

$$A_n = \text{smallest positive integer not yet used,}$$
$$B_n = A_n + n,$$
$$C_n = B_n + H_n.$$

We now consider the sequence 1, 2, 3, ..., B_n; then

$$B_n = n + (\text{number of } H_j\text{'s less than } B_n).$$

From $j = B_n - n = A_n$ and Theorem 6.1, we conclude

(7.3) $$H_{A_n} + 1 = B_n,$$

but we also get that

$$A_n = (\text{number of } A_j\text{'s less than } C_n)$$

from Theorem 6.1. From 1, 2, ..., C_n, then

$$C_n = n + (\text{number of } A_j\text{'s less than } C_n) + (\text{number of } B_j\text{'s less than } C_n)$$

$$= n + A_n + (\text{number of } B_j\text{'s less than } C_n),$$

or

$$(\text{number of } B_j\text{'s less than } C_n) = C_n - (A_n + n) = C_n - B_n = H_n.$$

We therefore conclude from $C_n - 1 = B_j$ that

$$B_{H_n} + 1 = C_n. \tag{7.4}$$

From Theorem 6.1,

$$(\text{number of } B_j\text{'s less than } C_n) = (\text{number of } A_j\text{'s less than } B_n) = H_n.$$

Therefore, since $B_n - 1 = A_j$, we conclude

$$A_{H_n} + 1 = B_n. \tag{7.5}$$

From (7.5) and (7.3), we conclude

$$H_{A_n} = A_{H_n}. \tag{7.6}$$

We would normally have that $A_n \xrightarrow{f^*} B_n$ and $A_n \xrightarrow{f} B_n - 1 = H_{A_n} = A_{H_n}$. Also, $B_n \xrightarrow{f} C_n = H_{B_n}$. But, $C_n \xrightarrow{f} A_j$ for some j, so that set N under f goes into set H. From Theorem 6.1, $A_n \xrightarrow{f} H_{A_n} = B_n - 1 = A_{H_n}$ and $B_n \xrightarrow{f} H_{B_n} = C_n$, and $C_n \xrightarrow{f} H_{C_n}$. Now, $H_{C_n} = A_{B_n}$ as B and H are complementary, and these are the only elements left.

From (7.5), we conclude that

$$A_{A_n} + 1 = B_{H_n} \quad \text{and} \quad A_{C_n} + 1 = B_{B_n}, \tag{7.7}$$

since $H_{B_n} = C_n$. Since $H_{A_n} + 1 = B_n$,

$$H_{A_{H_n}} + 1 = B_{H_n} = A_{H_{H_n}}. \tag{7.8}$$

Note that, if we remove all $H_{B_n} = C_n$ from the ordered sequence H_n, then all we have left are the A_n, and these are $H_{H_n} = A_n$. Thus,

$$A_{A_n} + 1 = B_{H_n}. \tag{7.9}$$

Putting it together, $A_{A_n} + 1 = B_{H_n}$ and $B_{H_n} + 1 = C_n$ imply that $A_{A_n+1} = C_n + 1$, since $C_n + 1 = A_j$ always. Thus,

$$A_{A_n+1} - A_{A_n} = 3. \tag{7.10}$$

From $B_{H_n} + 1 = C_n$, one concludes that, because H and B are complementary, $B_{B_n} + 1 \neq C_j$, and since no two B_j's are consecutive, $B_{B_n} + 1 = A_j$. From

$$A_{C_n} + 1 = B_{B_n} \quad \text{and} \quad B_{B_n} + 1 = A_j = A_{C_n+1},$$

we have

$$A_{C_n+1} - A_{C_n} = 2. \tag{7.11}$$

We consider 1, 2, 3, ..., H_n. Then

$$H_n = n + (\text{number of } B_j\text{'s less than } H_n),$$

and

$$C_n - B_n - n = (\text{number of } B_j\text{'s less than } H_n)$$

$$= (\text{number of } A_j\text{'s less than } n) = (\text{number of } C_j\text{'s less than } A_n).$$

Therefore,

$$C_{B_n} - B_{B_n} - B_n = (\text{number of } C_j\text{'s less than } A_{B_n}).$$

But, $H_{B_n} = C_n$, so $H_{B_n} - B_n = C_n - B_n = H_n$. Therefore, we conclude that

(7.12) $$C_{H_n} + 1 = A_{B_n}.$$

No two C_j's have a difference of 2. Now, can $A_{B_n} + 1 = B_j$? The answer is no, since $A_{A_n} + 1 = B_n$ and H and B are complementary sequences. Then $A_{B_n+1} - A_{B_n} \geq 1$ so that $C_{B_n+1} - C_{B_n} \geq 3$, and (7.10) implies that $C_{A_n+1} - C_{A_n} = 6$, while (7.11) implies that $C_{C_n+1} - C_{C_n} = 4$.

By considering the mappings under f^*, we now conclude that:

Theorem 7.4

$$A_{A_n+1} - A_{A_n} = 3, \quad A_{B_n+1} - A_{B_n} = 1, \quad A_{C_n+1} - A_{C_n} = 2;$$

$$B_{A_n+1} - B_{A_n} = 4, \quad B_{B_n+1} - B_{B_n} = 2, \quad B_{C_n+1} - B_{C_n} = 3;$$

$$C_{A_n+1} - C_{A_n} = 6, \quad C_{B_n+1} - C_{B_n} = 3, \quad C_{C_n+1} - C_{C_n} = 4.$$

Finally, we list the first few members of A, B, C, and H in Table 7.1.

TABLE 7.1

n	A_n	B_n	H_n	C_n
1	1	2	1	3
2	4	6	3	9
3	5	8	4	12
4	7	11	5	16
5	10	15	7	22
6	13	19	9	28

Notice that we may extend the table with the recurrences:

$$C_n + A_n = A_{B_n},$$

$$B_n + H_n = C_n,$$

$$n + A_n = B_n,$$

$$A_n + 2C_n + B_n = C_{B_n},$$
$$A_n + C_n + B_n = B_{B_n}.$$

We have two corollaries to Theorem 7.4:

Corollary 7.4.1

$$(\text{Number of } A_j\text{'s less than } n) = C_n - B_n - n = f(n),$$
$$(\text{Number of } B_j\text{'s less than } n) = C_n - 2A_n - 1 = g(n),$$
$$(\text{Number of } C_j\text{'s less than } n) = 3B_n - 2C_n = h(n).$$

Proof: $f(1) = 0$ and

$$f(A_m + 1) - f(A_m) = 1,$$
$$f(B_m + 1) - f(B_m) = 0,$$
$$f(C_m + 1) - f(C_m) = 0.$$

Thus, $f(n)$ increments by one only when n passes A_m, so that $f(n)$ counts the number of A_j's less than n.

Next, $g(1) = 0$, and

$$g(A_m + 1) - g(A_m) = 0,$$
$$g(B_m + 1) - g(B_m) = 1,$$
$$g(C_m + 1) - g(C_m) = 0.$$

Thus, $g(n)$ increments by one only when n passes B_m, so that $g(n)$ counts the number of B_j's less than n.

Similarly, $h(1) = 0$, and

$$h(A_m + 1) - h(A_m) = 0,$$
$$h(B_m + 1) - h(B_m) = 0,$$
$$h(C_m + 1) - h(C_m) = 1.$$

Thus, $h(n)$ increments by one only when n passes C_m, so that $h(n)$ counts the number of C_j's less than n.

Corollary 7.4.2

Let $u_{m+1} - u_m = \begin{cases} p, & m \in A; \\ q, & m \in B; \\ r, & m \in C. \end{cases}$

Then

$$u_m = (C_m - B_m - m)p + (C_m - 2A_m - 1)q + (3B_m - 2C_m)r + u_1.$$

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