BI-UNITARY AMICABLE AND MULTIPERFECT NUMBERS

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1. INTRODUCTION

In what follows, lower-case letters will be used to denote natural numbers, with p and q always representing primes. As usual, (c, d) will symbolize the greatest common divisor of c and d. If $cd = n$ and $(c, d) = 1$, then d is said to be a unitary divisor of n. If $(c, d)^*$ denotes the greatest common unitary divisor of c and d, then d is said to be a bi-unitary divisor of n if $cd = n$ and $(c, d)^* = 1$. The notion of a bi-unitary divisor was first introduced by Subbarao & Suryanarayana in 1971 (see [6]).

We shall symbolize by $\sigma(n)$, $\sigma^*(n)$, and $\sigma^{**}(n)$, respectively, the sums of the (positive) divisors, unitary divisors, and bi-unitary divisors of n. It is well known that $\sigma(p^a) = (p^{a+1} - 1)/(p - 1)$ and $\sigma^*(p^a) = (p^a + 1)$ *and* that both σ and σ^* are multiplicative functions. It is not difficult to verify that $\sigma^{**}(p^a) = \sigma(p^a)$ if a is odd and $\sigma^{**}(p^a) = \sigma(p^a) - p^{a/2}$ if a is even *and* that σ^{**} is multiplicative. It follows that $\sigma^{**}(n) = \sigma(n)$ if every exponent in the prime-power decomposition of n is odd and that $\sigma^{**}(n) = \sigma^*(n)$ if n is cube-free. It is also immediate that $\sigma^{**}(n)$ is even unless $n = 2^a$ or $n = 1$.

2. BI-UNITARY MULTIPERFECT NUMBERS

A number n is said to be perfect if $\sigma(n) = 2n$ and to be multiperfect if $\sigma(n) = kn$, where $k \geqslant 3$. Perfect and multiperfect numbers have been studied extensively. Subbarao & Warren [7] have defined n to be a unitary perfect number if $\sigma^*(n) = 2n$, and Wall [11] has defined n to be a bi-unitary perfect number if $\sigma^{**}(n) = 2n$. Five unitary perfect numbers have been found to date (see [10]), while Wall [11] has proved that 6, 60, and 90 are the *only* bi-unitary perfect numbers.

If $\sigma^*(n) = kn$, where $k \geqslant 3$, n is said to be a unitary multiperfect number. The properties of such numbers have been studied by Harris & Subbaro [4] and by Hagis [3]. It is known that, if n is a unitary multiperfect number, then $n > 10^{102}$ and n has at least 46 distinct prime factors (including 2). No unitary multiperfect numbers have, as yet, been found.

We shall state that n is a bi-unitary multiperfect number if $\sigma^{**}(n) = kn$, where $k \geqslant 3$. It is easy to show that every such number is even.

Theorem 1: There are no odd bi-unitary multiperfect numbers.

Proof: Suppose that $\sigma^{**}(n) = kn$, where $k \geqslant 3$, and

$$n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}, \text{ with } 3 \leqslant p_1 < p_2 < \cdots < p_s.$$

Suppose, also, that $k = 2^c M$, where $2 \nmid M$ and $c \geqslant 0$. Since

$$\sigma^{**}(n) = \prod_{i=1}^{s} \sigma^{**}(p_i^{a_i})$$

and since $2 | \sigma^{**}(p_i^{a_i})$ for $i = 1, 2, \ldots, s$, we see that $s \leqslant c$. Also,

$$2^c \leqslant 2^c M = k = \sigma^{**}(n)/n \leqslant \sigma(n)/n$$

$$= \prod_{i=1}^{s} \sigma(p_i^{a_i})/p_i^{a_i} < \prod_{i=1}^{s} p_i/(p_i - 1) < 2^s \leqslant 2^c.$$

This contradiction completes the proof.

Using the CDC CYBER 750 at the Temple University Computing Center a search was made for all bi-unitary multiperfect numbers less than 10^7. The search required about 1.5 hours of computer time, and thirteen numbers were found, nine with $k = 3$ and four with $k = 4$. They, along with the three bi-unitary perfect numbers, are listed in Table 1.

TABLE 1

The Bi-Unitary Perfect and Multiperfect Numbers Less than 10**7

1.	6 = 2.3	k = 2
2.	60 = 2**2.3.5	k = 2
3.	90 = 2.3**2.5	k = 2
4.	120 = 2**3.3.5	k = 3
5.	672 = 2**5.3.7	k = 3
6.	2160 = 2**4.3**3.5	k = 3
7.	10080 = 2**5.3**2.5.7	k = 3
8.	22848 = 2**6.3.7.17	k = 3
9.	30240 = 2**5.3**3.5.7	k = 4
10.	342720 = 2**6.3**2.5.7.17	k = 3
11.	523776 = 2**9.3.11.31	k = 3
12.	1028160 = 2**6.3**3.5.7.17	k = 4
13.	1528800 = 2**5.3.5**2.7**2.13	k = 3
14.	6168960 = 2**7.3**4.5.7.17	k = 4
15.	7856640 = 2**9.3**2.5.11.31	k = 3
16.	7983360 = 2**8.3**4.5.7.11	k = 4

3. BI-UNITARY AMICABLE NUMBERS

m and n are said to be amicable numbers if $\sigma(m) = \sigma(n) = m + n$. A history of these numbers may be found in [5]. If $\sigma^*(m) = \sigma^*(n) = m + n$, then m and n are said to be unitary amicable numbers (see [2]). Similarly, we shall say that m and n are bi-unitary amicable numbers if $\sigma^{**}(m) = \sigma^{**}(n) = m + n$.

Theorem 2: If $(m; n)$ is a bi-unitary amicable pair, then m and n have the same parity.

Proof: Assume that $m + n$ is odd. Then $\sigma^{**}(m)$ is odd, and it follows that $m = 2^a$. Similarly, $n = 2^a$, and we have a contradiction.

Theorem 3: Suppose that $(m; n)$ is a bi-unitary amicable pair *and* that $m = 2^a M$ and $n = 2^b N$ where $M \equiv N \equiv 1 \pmod 2$ and $a < b$. If $\omega(M) = s$ and $\omega(M) = t$ [where $\omega(L)$ denotes the number of distinct prime factors of L], then $s \leqslant a$ and $t \leqslant a$.

Proof: If $p^c \| M$, then $2 | \sigma^{**}(p^c)$, and we see that $2^s | \sigma^{**}(m)$. But,

$$\sigma^{**}(m) = m + n = 2^a(M + 2^{b-a}N) = 2^a K \text{ where } K \text{ is odd,}$$

and it follows that $s \leqslant a$. Similarly, $t \leqslant a$.

Corollary 3.1: If $(2M; 2^b N)$, where $b > 1$ *and* M and N are odd, is a bi-unitary amicable pair, then $M = p^c$ and $N = q^d$.

Theorem 4.1: Suppose that $(m; n)$ is a bi-unitary amicable pair such that $m = aM$ and $n = aN$ where $(a, M) = (a, N) = 1$. If b is a natural number such that $\sigma^{**}(b)/b = \sigma^{**}(a)/a$ *and* $(b, M) = (b, N) = 1$, then $(bM; bN)$ is a bi-unitary amicable pair.

Proof: $\sigma^{**}(bM) = \sigma^{**}(b)\sigma^{**}(M) = a^{-1}b\sigma^{**}(a)\sigma^{**}(M) = a^{-1}b\sigma^{**}(aM) = a^{-1}b(aM + aN) = bM + bN$. Similarly, $\sigma^{**}(bN) = bM + bN$.

The proofs of the next two theorems are similar to that of Theorem 4.1 and are, therefore, omitted.

Theorem 4.2: Suppose that $(m; n)$ is a unitary amicable pair such that $m = aM$ and $n = aN$ where $(a, M) = (a, N) = 1$ *and* where M and N are cube-free. If

$$\sigma^{**}(b)/b = \sigma^{*}(a)/a \text{ and } (b, M) = (b, N) = 1,$$

then $(bM; bN)$ is a bi-unitary amicable pair.

Theorem 4.3: Suppose that $(m; n)$ is an amicable pair such that $m = aM$ and $n = aN$ where $(a, M) = (a, N) = 1$ *and* where every exponent in the prime-power decomposition of M and N is odd. If

$$\sigma^{**}(b)/b = \sigma(a)/a \text{ and } (b, M) = (b, N) = 1,$$

then $(bM; bN)$ is a bi-unitary amicable pair.

A computer search among distinct natural numbers a and b such that $2 \leqslant a, b \leqslant 10^4$ yielded 667 cases where $\sigma^{**}(b)/b = \sigma^{**}(a)/a$, 1325 cases where $\sigma^{**}(b)/b = \sigma^{*}(a)/a$, and 673 cases where $\sigma^{**}(b)/b = \sigma(a)/a$.

Example 1: Since $(8 \cdot 17 \cdot 41 \cdot 179; 8 \cdot 23 \cdot 5669)$ is a bi-unitary amicable pair, and since

$$\sigma^{**}(144)/144 = \sigma^{**}(8)/8,$$

it follows from Theorem 4.1 that $(144 \cdot 17 \cdot 41 \cdot 179; 144 \cdot 23 \cdot 5669)$ is also a bi-unitary amicable pair.

Example 2: Since $(135 \cdot 2 \cdot 19 \cdot 47; 135 \cdot 2 \cdot 29 \cdot 31)$ is a unitary amicable pair, and since

$$\sigma^{**}(2925)/2925 = \sigma^{*}(135)/135,$$

it follows from Theorem 4.2 that $(2925 \cdot 2 \cdot 19 \cdot 47; 2925 \cdot 2 \cdot 29 \cdot 31)$ is a bi-unitary amicable pair.

Example 3: Since $(47 \cdot 7 \cdot 19 \cdot 2663;\ 45 \cdot 11 \cdot 73 \cdot 479)$ is an amicable pair, and since

$$\sigma^{**}(450)/450 = \sigma(45)/45,$$

it follows from Theorem 4.3 that $(450 \cdot 7 \cdot 19 \cdot 2663;\ 450 \cdot 11 \cdot 73 \cdot 479)$ is a bi-unitary amicable pair.

A search was made for all bi-unitary amicable pairs $(m;\ n)$ such that $m < n$ and $m \leqslant 10^6$. The search required about five minutes on the CDC CYBER 750 and sixty pairs were found. These are listed in Table 2.

TABLE 2

The Bi-Unitary Amicable Pairs with Smallest Member Less than 10**6

1.	114 = 2.3.19	126 = 2.3**2.7
2.	594 = 2.3**3.11	846 = 2.3**2.47
3.	1140 = 2**2.3.5.19	1260 = 2**2.3**2.5.7
4.	3608 = 2**3.11.41	3952 = 2**4.13.19
5.	4698 = 2.3**4.29	5382 = 2.3**2.13.23
6.	5940 = 2**2.3**3.5.11	8460 = 2**2.3**2.5.47
7.	6232 = 2**3.19.41	6368 = 2**5.199
8.	7704 = 2**3.3**2.107	8496 = 2**4.3**2.59
9.	9520 = 2**4.5.7.17	13808 = 2**4.863
10.	10744 = 2**3.17.79	10856 = 2**3.23.59
11.	12285 = 3**3.5.7.13	14595 = 3.5.7.139
12.	13500 = 2**2.3**3.5**3	17700 = 2**2.3.5**2.59
13.	41360 = 2**4.5.11.47	51952 = 2**4.17.191
14.	44772 = 2**2.3.7.13.41	49308 = 2**2.3.7.587
15.	46980 = 2**2.3**4.5.29	53820 = 2**2.3**2.5.13.23
16.	60858 = 2.3**3.7**2.23	83142 = 2.3**2.31.149
17.	62100 = 2**2.3**3.5**2.23	62700 = 2**2.3.5**2.11.19
18.	67095 = 3**3.5.7.71	71145 = 3**3.5.17.31
19.	67158 = 2.3**2.7.13.41	73962 = 2.3**2.7.587
20.	73360 = 2**4.5.7.131	97712 = 2**4.31.197
21.	79650 = 2.3**3.5**2.59	107550 = 2.3**2.5**2.239
22.	79750 = 2.5**3.11.29	88730 = 2.5.19.467
23.	105976 = 2**3.13.1019	108224 = 2**6.19,89
24.	118500 = 2**2.3.5**3.79	131100 = 2**2.3.5**2.19.23
25.	141664 = 2**5.19.233	153176 = 2**3.41.467
26.	142310 = 2.5.7.19.107	168730 = 2.5.47.359
27.	177750 = 2.3**2.5**3.79	196650 = 2.3**2.5**2.19.23

TABLE 2—continued

28. 185368 = 2**3.17.29.47	203432 = 2**3.59.431
29. 193392 = 2**4.3**2.17.79	195408 = 2**4.3**2.23.59
30. 217840 = 2**4.5.7.389	287600 = 2**4.5**2.719
31. 241024 = 2**7.7.269	309776 = 2**4.19.1019
32. 298188 = 2**2.3**3.11.251	306612 = 2**2.3**3.17.167
33. 308220 = 2**2.3.5.11.467	365700 = 2**2.3.5**2.23.53
34. 308992 = 2**8.17.71	332528 = 2**4.7.2969
35. 356408 = 2**3.13.23.149	399592 = 2**3.199.251
36. 399200 = 2**5.5**2.499	419800 = 2**3.5**2.2099
37. 415264 = 2**5.19.683	446576 = 2**4.13.19.113
38. 415944 = 2**3.3**2.53.109	475056 = 2**4.3**2.3299
39. 462330 = 2.3**2.5.11.467	548550 = 2.3**2.5**2.23.53
40. 545238 = 2.3**3.23.439	721962 = 2.3**2.19.2111
41. 600392 = 2**3.13.23.251	669688 = 2**3.97.863
42. 608580 = 2**2.3**3.5.7**2.23	831420 = 2**2.3**2.5.31.149
43. 609928 = 2**3.11.29.239	686072 = 2**3.191.449
44. 624184 = 2**3.11.41.173	691256 = 2**3.71.1217
45. 627440 = 2**4.5.11.23.31	865552 = 2**4.47.1151
46. 635624 = 2**3.11.31.233	712216 = 2**3.127.701
47. 643336 = 2**3.29.47.59	652664 = 2**3.17.4799
48. 669900 = 2**2.3.5**2.7.11.29	827700 = 2**2.3.5**2.31.89
49. 671580 = 2**2.3**2.5.7.13.41	739620 = 2**2.3**2.5.7.587
50. 699400 = 2**3.5**2.13.269	774800 = 2**4.5**2.13.149
51. 726104 = 2**3.17.19.281	796696 = 2**3.53.1879
52. 785148 = 2**2.3.7.13.719	827652 = 2**2.3.7.59.167
53. 796500 = 2**2.3**3.5**3.59	1075500 = 2**2.3**2.5**3.239
54. 815100 = 2**2.3.5**2.11.13.19	932100 = 2**2.3.5**2.13.239
55. 818432 = 2**8.23.139	844768 = 2**5.26399
56. 839296 = 2**7.79.83	874304 = 2**6.19.719
57. 898216 = 2**3.11.59.173	980984 = 2**3.47.2609
58. 930560 = 2**8.5.727	1231600 = 2**4.5**2.3079
59. 947835 = 3**3.5.7.17.59	1125765 = 3**3.5.31.269
60. 998104 = 2**3.17.41.179	1043096 = 2**3.23.5669

4. BI-UNITARY ALIQUOT SEQUENCES

The function s^{**} is defined by $s^{**}(n) = \sigma^{**}(n) - n$, the sum of the bi-unitary *aliquot* divisors of n. $s^{**}(1) = 0$ and we define $s^{**}(0) = 0$. A t-tuple of distinct natural numbers $(n_0;\ n_1;\ \ldots;\ n_{t-1})$ with $n_i = s^{**}(n_{i-1})$ for $i = 1, 2, \ldots, t - 1$ and $s^{**}(n_{t-1}) = n_0$ is called a *bi-unitary t-cycle*. A bi-unitary 1-cycle is a bi-unitary perfect number; a bi-unitary 2-cycle is a bi-unitary amicable pair. All of the bi-unitary t-cycles with $t > 2$ and smallest member less than 10^5 are listed in Table 3.

TABLE 3

The Bi-Unitary t-Cycles with $t > 2$ and First Member Less than 10**5

t = 4

(162;174;186;198), (1026;1374;1386;1494), (1620;1740;1860;1980),

(10098;15822;19458;15102), (10260;13740;13860;14940),

(41800;51800;66760;83540), (51282;58158;62802;76878)

t = 6

(12420;16380;17220;23100;26820;18180)

t = 13

(6534;8106;10518;10530;17694;11826;13038;14178;16062;16074;12726;

11754;7866)

It is not difficult to modify Theorems 4.1, 4.2, 4.3 so that one can obtain "new" bi-unitary t-cycles from known t-cycles (see [1]), unitary t-cycles (see [8] and [9]), and bi-unitary t-cycles. For example, since

$$\sigma^{**}(20)/20 = \sigma^{**}(2)/2,$$

it follows from Table 3 that

(100980; 158220; 194580; 151020) and (512820; 581580; 628020; 768780)

are bi-unitary 4-cycles.

The *bi-unitary aliquot sequence* $\{n_i\}$ with leader n is defined by

$$n_0 = n,\ n_1 = s^{**}(n_0),\ n_2 = s^{**}(n_1),\ \ldots,\ n_i = s^{**}(n_{i-1}),\ \ldots .$$

Such a sequence is said to be *terminating* if $n_k = 1$ for some index k (so that $n_i = 0$ for $i > k$). This will occur if $n_{k-1} = p$ or p^2. A bi-unitary aliquot sequence is said to be *periodic* if there is an index k such that $(n_k;\ n_{k+1};\ \ldots;\ n_{k+t-1})$ is a bi-unitary t-cycle. A bi-unitary aliquot sequence which is neither terminating nor periodic is (obviously) *unbounded*. Whether or not unbounded bi-unitary aliquot sequences exist is an open question. I would conjecture that such sequences *do* exist.

An investigation was made of all bi-unitary aliquot sequences with leader $n \leqslant 10^5$. About 2.5 hours of computer time was required. 69045 sequences were found to be terminating; 15560 were periodic (6477 ended in 1-cycles, 5556 in

2-cycles and 3527 in t-cycles with $t > 2$); and in 15395 cases an $n_k > 10^{12}$ was encountered and (for practical reasons) the sequence was terminated with its behavior undetermined. The "first" sequence with unknown behavior has leader $n_0 = 2160$. $n_{306} = 1{,}301{,}270{,}618{,}226$ is the first term of this sequence which exceeds 10^{12}.

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