SUMMATION OF CERTAIN RECIPROCAL SERIES RELATED TO FIBONACCI AND LUCAS NUMBERS

Richard André-Jeannin
Ecole Nationale d'Ingénieurs de Sfax, Tunisia
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1. Introduction

Some years ago, R. Backstrom and B. Popov (see [1], [2], [3]) computed sums of the form

$$\sum \frac{1}{F_{an+b} + c} \quad \text{and} \quad \sum \frac{1}{L_{an+b} + c},$$

for certain values of a, b, and c. For instance, Backstrom obtained

$$(1) \qquad \sum_{n=0}^{\infty} \frac{1}{L_{2n} + 3} = \frac{2\sqrt{5} + 1}{10}, \quad \sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 1} = \frac{\sqrt{5}}{2},$$

and he also gave the estimate

$$(2) \qquad \sum_{n=0}^{\infty} \frac{1}{L_{2n} + 2} \approx \frac{1}{8} + \frac{1}{4 \log \alpha} = 0.64452\ldots,$$

where α is the golden ratio. Recently, G. Almkvist [4] has given an exact formula connecting the last sum with Jacobi's theta functions.

The aim of this note is to obtain new results of the same kind. For example, we show that

$$(3) \qquad \sum_{n=0}^{\infty} \frac{1}{L_{2n} + \sqrt{5}} = \frac{1}{\alpha} = 0.618\ldots,$$

which can be compared with (2), and the surprising result

$$(4) \qquad \sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 3/\sqrt{5}} = 1.$$

In the final section, following Almkvist's method, we express the series

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 2/\sqrt{5}},$$

in terms of the theta functions, with the estimate

$$(5) \qquad \sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 2/\sqrt{5}} \approx \frac{\sqrt{5}}{4 \log \alpha} - \frac{\sqrt{5} \cdot \Pi^2}{(\log \alpha)^2 (e^{\Pi^2/\log \alpha} + 2)}.$$

2. Main Result

Theorem: Let s be a positive integer. then

$$(6) \qquad \sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + L_s/\sqrt{5}} = \frac{s}{2F_s}, \quad s \text{ even}, \ s \neq 0,$$

and

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + \sqrt{5}F_s} = \frac{1}{L_s}\left(\frac{s-1}{2} + \frac{1}{1+\alpha^{-s}}\right), \quad s \text{ odd.} \tag{7}$$

3. Preliminaries

As noticed by Almkvist, it is probably better for this study to use direct calculus rather than Fibonaccian identities.

Lemma 1: Let q, σ be real numbers, with $q > 1$, and let s be a positive integer. Then the following equality holds:

$$\sum_{n=0}^{\infty} \frac{1}{q^{n+\sigma} + q^{-n-\sigma} + q^{s/2} + q^{-s/2}} = \frac{1}{q^{s/2} - q^{-s/2}} \sum_{n=0}^{s-1} \frac{1}{1 + q^{n+\sigma-s/2}}. \tag{8}$$

Proof: One can readily verify that

$$\frac{1}{q^{n+\sigma} + q^{-n-\sigma} + q^{s/2} + q^{-s/2}} = \frac{1}{q^{-s/2} - q^{s/2}}\left(\frac{1}{1 + q^{n+\sigma+s/2}} - \frac{1}{1 + q^{n+\sigma-s/2}}\right).$$

Hence, by the telescoping effect, for $N \ge s - 1$,

$$\sum_{n=0}^{N} \frac{1}{q^{n+\sigma} + q^{-n-\sigma} + q^{s/2} + q^{-s/2}} = \frac{1}{q^{-s/2} - q^{s/2}}\left(\sum_{n=N-s+1}^{N} \frac{1}{1 + q^{n+\sigma+s/2}} - \sum_{n=0}^{s-1} \frac{1}{1 + q^{n+\sigma-s/2}}\right).$$

Letting $N \to \infty$, we obtain (8) (since $q > 1$).

Lemma 2: Let a and s be positive integers, let b be any integer, and define $T_s(a, b)$ by

$$T_s(a, b) = \sum_{n=0}^{s-1} \frac{1}{1 + \alpha^{a(2n-s)+b}}.$$

Then

$$T_s(1, 0) = \frac{s-1}{2} + \frac{1}{1 + \alpha^{-s}} \tag{9}$$

and

$$T_s(1, 1) = \frac{s}{2}. \tag{10}$$

Remark: Here, α is the golden ratio or any positive real number.

Proof:

$$T_{2s}(1, 0) = \sum_{n=0}^{2s-1} \frac{1}{1 + \alpha^{2n-2s}} = \frac{1}{1 + \alpha^{-2s}} + \frac{1}{2} + \sum_{k=1}^{s-1}\left(\frac{1}{1 + \alpha^{2k}} + \frac{1}{1 + \alpha^{-2k}}\right)$$

$$= \frac{1}{1 + \alpha^{-2s}} + \frac{1}{2} + s - 1 = \frac{1}{1 + \alpha^{-2s}} + s - \frac{1}{2}.$$

On the other hand,

$$T_{2s+1}(1, 0) = \sum_{n=0}^{2s} \frac{1}{1 + \alpha^{2n-2s-1}}$$

$$= \frac{1}{1 + \alpha^{-2s-1}} + \sum_{k=1}^{s}\left(\frac{1}{1 + \alpha^{2k-1}} + \frac{1}{1 + \alpha^{-2k+1}}\right)$$

$$= \frac{1}{1 + \alpha^{-2s-1}} + s.$$

This concludes the proof of (9). The proof of (10) follows the same pattern.

4. Proof of The Theorem and of Other Identities

As usual, the Fibonacci and Lucas numbers are defined by

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - (-1)^n\alpha^{-n}), \quad L_n = \alpha^n + (-1)^n\alpha^{-n}.$$

Let a, b be integers, with $a \geq 1$. Put $\sigma = b/2a$ and $q = \alpha^{2a}$ in (8) to get

$$(11) \qquad \sum_{n=0}^{\infty} \frac{1}{\alpha^{2an+b} + \alpha^{-2an-b} + \alpha^{as} + \alpha^{-as}} = \frac{1}{\alpha^{as} - \alpha^{-as}} T_s(a, b),$$

where $T_s(a, b)$ is defined above. Let us examine different cases according to the parity of a, b, and s.

First case: b even, s or a even. Since (11) can be written as

$$(12) \qquad \sum_{n=0}^{\infty} \frac{1}{L_{2anb} + L_{as}} = \frac{1}{\sqrt{5}F_{as}} T_s(a, b),$$

letting $s = 1$ in (11), we obtain Backstrom's Theorem V. Namely,

$$\sum_{n=0}^{\infty} \frac{1}{L_{2an+b} + L_a} = \frac{1}{\sqrt{5}F_a} \frac{1}{1 + \alpha^{b-a}}, \quad a \text{ even}, b \text{ even}.$$

Letting $b = 0$, $a = 1$, and applying (9), we obtain Backstrom's Theorem IV, which is

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + L_s} = \frac{1}{\sqrt{5}F_s}\left(\frac{s-1}{2} + \frac{1}{1 + \alpha^{-s}}\right), \quad s \text{ even}.$$

Second case: b even, s and a odd. Formula (11) becomes

$$(13) \qquad \sum_{n=0}^{\infty} \frac{1}{L_{2an+b} + \sqrt{5}F_{as}} = \frac{1}{L_{as}} T_s(a, b).$$

With $s = 1$ in (13), we have

$$\sum_{n=0}^{\infty} \frac{1}{L_{2an+b} + \sqrt{5}F_a} = \frac{1}{L_a} \frac{1}{1 + \alpha^{b-a}}, \quad b \text{ even}, a \text{ odd}.$$

Letting $a = 1$ and $b = 0$ in (13) and applying (9), we get (7) so that

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + \sqrt{5}F_s} = \frac{1}{L_s}\left(\frac{s-1}{2} + \frac{1}{1 + \alpha^{-s}}\right), \quad s \text{ odd}.$$

As special cases, we have

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + \sqrt{5}} = \frac{1}{\alpha} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{L_{2n} + 2\sqrt{5}} = \frac{1}{4} + \frac{\alpha}{8}.$$

Third case: b odd, s or a even. Formula (11) becomes

$$(14) \qquad \sum_{n=0}^{\infty} \frac{1}{F_{2an+b} + L_{as}/\sqrt{5}} = \frac{1}{F_{as}} T_s(a, b).$$

With $s = 1$ in (14), we get

$$\sum_{n=0}^{\infty} \frac{1}{F_{2an+b} + L_a/\sqrt{5}} = \frac{1}{F_a} \frac{1}{1 + \alpha^{b-a}},\ b \text{ odd},\ a \text{ even}.$$

When $a = 2$ and $b = 1$, we have

$$\sum_{n=0}^{\infty} \frac{1}{F_{4n+1} + 3/\sqrt{5}} = \frac{1}{\alpha}.$$

Now, put $a = 1$, $b = 1$ in (14) and use (10) to get (6) so that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + L_s/\sqrt{5}} = \frac{s}{2F_s},\ s \text{ even}.$$

As special cases, we mention (4) and

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 7/\sqrt{5}} = \frac{2}{3}.$$

<u>Last case</u>: b odd, s and a odd. Formula (11) becomes

$$\sum_{n=0}^{\infty} \frac{1}{F_{2an+b} + F_{as}} = \frac{\sqrt{5}}{L_{as}} T_s(a, b). \tag{15}$$

Putting $s = 1$ in (15), we obtain Backstrom's Theorem II, which is

$$\sum_{n=0}^{\infty} \frac{1}{F_{2an+b} + F_a} = \frac{\sqrt{5}}{L_a} \frac{1}{1 + \alpha^{b-a}},\ b \text{ odd},\ a \text{ odd}.$$

With $a = b = 1$ in (15), we obtain Backstrom's Theorem I, which is

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + F_s} = \frac{s\sqrt{5}}{2L_s},\ s \text{ odd}.$$

Remark: Consider the recurrence relation

$$W_n = pW_{n-1} + W_{n-2},\ n \ge 2,\ p > 0,$$

and the solutions

$$U_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{\Delta}}, \quad V_n = \alpha^n + (-1)^n \alpha^{-n},$$

where $\Delta = p^2 + 4$, $\alpha = \dfrac{p + \sqrt{\Delta}}{2} > 1$.

The results above could be generalized with U_n, V_n, $\sqrt{\Delta}$ in place of F_n, L_n, $\sqrt{5}$.

5. A New Tantalizing Problem

Let us return to (6). When putting $s = 0$ in the left-hand side, we obtain the convergent series

$$\sum_{0}^{\infty} \frac{1}{F_{2n+1} + 2/\sqrt{5}} = 1.161685787\ldots, \tag{16}$$

where the number on the right is not one we recognize. Using the limit process introduced by Backstrom ([1], p. 20), we would have

$$\lim_{s \to 0} \frac{s}{2F_s} = \frac{\sqrt{5}}{4 \log \alpha} = 1.16168590\ldots,$$

so we see that $\sqrt{5}/(4 \log \alpha)$ is a good estimate of the sum (16).

Using the method introduced by Almkvist, we can now express (16) in terms of the theta functions. In fact, we have

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 2/\sqrt{5}} = \sqrt{5} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(1 + q^{2n+1})^2},$$

where $q = \alpha^{-1}$. By a classical formula (see, e.g., [5], p. 471), we can write

$$\sum_{n=0}^{\infty} \frac{q^{2n+1}}{(1 + q^{2n+1})^2} = -\frac{1}{8\Pi^2} \frac{\nu_3''}{\nu_3},$$

where (with Almkvist's notations)

$$\nu_3 = \sqrt{-\frac{\Pi}{\log q}} \sum_n e^{\Pi^2 n^2/\log q}$$

and

$$\nu_3'' = \frac{2\Pi^2}{\log q}\sqrt{-\frac{\Pi}{\log q}} \sum_n \left(1 + \frac{2\Pi^2 n^2}{\log q}\right) e^{\Pi^2 n^2/\log q}.$$

(The summation is over all integers n.)

After some calculus, we obtain the final formula

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1} + 2/\sqrt{5}} = \frac{\sqrt{5}}{4 \log \alpha} - \frac{\Pi^2\sqrt{5}}{(\log \alpha)^2} \frac{\sum_{n=1}^{\infty} n^2 e^{-\Pi^2 n^2/\log \alpha}}{1 + 2\sum_{n=1}^{\infty} e^{-\Pi^2 n^2/\log \alpha}},$$

which can be compared with Almkvist's formula for

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n} + 2}.$$

Limiting ourselves to the first term ($n = 1$), we get the estimate (5).

References

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