THE TRIANGLE OF SMALLEST PERIMETER WHICH CIRCUMSCRIBES A SEMICIRCLE

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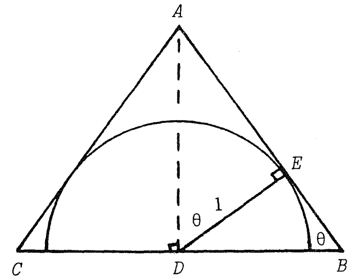
Let ABC be an isosceles triangle which circumscribes a semicircle of radius 1, with the diameter of the semicircle contained in the base BC of the triangle. Such triangles may be parameterized by the base angle θ shown in the figure, where D is the circle's center and E is the point of tangency on side AB. Our objective is to determine the circumscribing isosceles triangle of smallest perimeter.



Since $DE = 1$, we see that the perimeter p of ABC is given by

$$p = 2(AE + EB + BD) = 2(\tan\theta + \cot\theta + \csc\theta).$$

The derivative is

$$p' = 2(\sec^2\theta - \csc^2\theta - \csc\theta\cot\theta),$$

which can be easily rewritten in the form

$$p' = 2(1 - \cos\theta - \cos^2\theta)(1 + \cos\theta)/\cos^2\theta\sin^2\theta.$$

It is now evident that $p' = 0$ has just one solution in $0 < \theta < \pi/2$, namely, at the point where $\cos\theta = 1/G$; here $G = (\sqrt{5} + 1)/2$ denotes the Golden Ratio. Using the relation $G^2 = G + 1$, we then have

$$\sin^2\theta = 1 - \cos^2\theta = 1 - 1/G^2 = 1/G,$$

from which it follows that $\csc\theta = G^{1/2}$ and $\cot\theta = G^{-1/2}$. The perimeter $p_{\min}$ of the optimally circumscribed triangle is then

$$p_{\min} = 2(G^{1/2} + G^{-1/2} + G^{1/2}) = 2G^{1/2}(2 + 1/G).$$

Since $2 + 1/G = 2 + (G - 1) = G + 1 = G^2$, we see that the minimal perimeter is $2G^{5/2}$.

The triangle shown in the figure is in fact the circumscribing triangle of minimal length. The relevant dimensions are:

$$p_{\min} = 2G^{5/2} \doteq 6.66, \qquad \theta = \arccos(1/G) \doteq 51.8^0,$$

$$AE = BD = G^{1/2}, \quad BE = G^{-1/2}, \quad AC = G^{3/2}, \quad AD = G.$$

The unexpected appearance of the Golden Ratio makes this minimization problem of special interest. It would also be of interest to know of other problems which give rise to the "Golden Right Triangle," whose three sides are in the proportion $1 : G^{1/2} : G$.

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