MAXIMAL REPRESENTATIONS OF POSITIVE INTEGERS BY PELL NUMBERS

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1. INTRODUCTION

In [4], the unique *Zeckendorf representations* of positive and negative integers by distinct Pell numbers was *minimal*, i.e., the number of terms in each representational sum was the least possible.

Here we show how to represent positive integers maximally by means of Pell numbers. That is, each positive integer is to be given as a sum in a *maximal* representation by using the greatest number of terms involving distinct Pell numbers (see Table 1).

Short tables for minimal and maximal representations of positive integers in terms of (i) Fibonacci numbers and (ii) Lucas numbers, are given in [3].

Our theory for Pell numbers will be analogous to that used for Fibonacci numbers in [1], where a "Dual-Zeckendorf theorem" is established. Enough variations and complications exist, however, to make this investigation worthwhile *per se*. (Theorems for Lucas numbers corresponding to those for Fibonacci numbers may be found in [2].)

Positive *Pell numbers* are defined by the recurrence

$$P_{n+2} = 2P_{n+1} + P_n,\ n \geq 0, \tag{1.1}$$

with

$$P_0 = 0,\ P_1 = 1. \tag{1.2}$$

Thus, the first few Pell numbers are

$$\begin{cases} n = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ P_n = & 0 & 1 & 2 & 5 & 12 & 29 & 70 & 169 & 408 & 985 & \cdots \end{cases} \tag{1.3}$$

Repeated use of (1.1) leads to

$$P_{k+1} = 2(P_k + P_{k-2} + P_{k-4} + \cdots + P_{k-2t+2}) + P_{k-2t+1}, \tag{1.4}$$

in which

$$\begin{cases} t = 1, 2, \ldots, \frac{k}{2} & k \text{ even}, \\ t = 1, 2, \ldots, \frac{k+1}{2} & k \text{ odd}. \end{cases} \tag{1.5}$$

Consequently,

$$P_{k+1} - 1 = 2(P_k + P_{k-2} + P_{k-4} + \cdots + P_4 + P_2) \quad k \text{ even} \tag{1.6}$$

and

$$P_{k+1} - 1 = 2(P_k + P_{k-2} + P_{k-4} + \cdots + P_1) - 1 \quad k \text{ odd}. \tag{1.7}$$

Also, by repeated use of (1.1),

$$\sum_{i=1}^{k-1} P_i = \begin{cases} \frac{P_k+P_{k-1}-1}{2} & k \geq 2, \\ \frac{P_{k+1}-P_k-1}{2} & k \geq 2. \end{cases} \tag{1.8}$$

2. MAXIMAL REPRESENTATION THEOREM

Theorem: Every positive integer n has a unique representation in the form

$$n = \sum_{i=1}^{k} \beta_i P_i \quad (\beta_i = 0, 1, \text{ or } 2), \tag{2.1}$$

where

$$\beta_i = 0 \Rightarrow \beta_{i-1} = 2 \quad (2 \leq i < k) \tag{2.2}$$

and

$$\beta_k = 1 \text{ or } 2. \tag{2.3}$$

For a given n in (2.1), the unique integer k satisfies

$$P_k + P_{k-1} - 1 < n \leq P_{k+1} + P_k - 1. \tag{2.4}$$

Proof:

(i) *Maximality*. Suppose n satisfies (2.4). Then, equivalently from (1.8),

$$2\sum_{i=1}^{k-1} P_i < n \leq 2\sum_{i=1}^{k} P_i. \tag{2.5}$$

But, by the Zeckendorf theorem [1] for a positive integer, namely, the left-hand side of (2.5), we have

$$2\sum_{i=1}^{k} P_i - n = \sum_{i=1}^{\infty} \alpha_i P_i \quad (\geq 0), \tag{2.6}$$

with

$$\alpha_i = 0, 1, \text{ or } 2 \text{ and } \alpha_i = 2 \Rightarrow \alpha_{i-1} = 0 \quad (i \geq 1). \tag{2.7}$$

Now, from (2.5),

$$2\sum_{i=1}^{k} P_i - n < 2\left(\sum_{i=1}^{k} P_i - \sum_{i=1}^{k-1} P_i\right) = 2P_k. \tag{2.8}$$

This implies $\alpha_i = 0$ in (2.6) for $i > k$, and $\alpha_k = 0$ or 1. Consequently, (2.6) can be rewritten as

$$2\sum_{i=1}^{k} P_i - n = \sum_{i=1}^{k} \alpha_i P_i \quad (\alpha_k = 0 \text{ or } 1) \tag{2.9}$$

which, in turn, may be expressed as

$$n = \sum_{i=1}^{k} (2 - \alpha_i) P_i = \sum_{i=1}^{k} \beta_i P_i \quad (\beta_k = 1 \text{ or } 2), \tag{2.10}$$

where

$$\beta_i = 2 - \alpha_i \quad (i = 1, 2, \ldots, k) \tag{2.11}$$

in accordance with the statements (2.1) and (2.3) of the Theorem.

Lastly, by (2.11), the characteristic Zeckendorf condition (2.7) for Pell numbers becomes $\beta_i = 0 \Rightarrow \beta_{i-1} = 2$, which confirms the requirement (2.2) in the enunciation of our Theorem.

(ii) *Uniqueness.* Assume that the positive integer n has two different representations

$$n = \sum_{i=1}^{m} \beta_i P_i = \sum_{i=1}^{m'} \beta_i' P_i, \tag{2.12}$$

where $\beta_m, \beta'_{m'} = 1$ or 2 and $\beta_i = 0 \Rightarrow \beta_{i-1} = 2$ for $i = 2, 3, \ldots, m-1$ and $\beta_i' = 0 \Rightarrow \beta_{i-1}' = 2$ for $i = 2, 3, \ldots, m'-1$.

Suppose $m > m'$. Now

$$\begin{aligned} \sum_{i=1}^{m} \beta_i P_i &\geq \begin{cases} 2(P_m + P_{m-2} + \cdots + P_2) & m \text{ even} \\ 2(P_m + P_{m-2} + \cdots + P_1) - 1 & m \text{ odd} \end{cases} \\ &\geq P_{m+1} - 1, \end{aligned} \tag{2.13}$$

by (1.6) and (1.7), whereas

$$\begin{aligned} \sum_{i=1}^{m'} \beta_i' P_i \leq 2\sum_{i=1}^{m'} P_i \leq 2\sum_{i=1}^{m-1} P_i &= P_{m+1} - P_m - 1 \quad \text{by (1.8)} \\ &< P_{m+1} - 1. \end{aligned} \tag{2.14}$$

Conclusions (2.13) and (2.14) involve a contradiction. Similarly for $m < m'$.

Hence, $m' = m$.

If $\alpha_i = 2 - \beta_i$, $\alpha_i' = 2 - \beta_i'$ $(i = 1, 2, \ldots, m)$, then (2.12) leads to

$$\sum_{i=1}^{m} (2 - \alpha_i) P_i = \sum_{i=1}^{m} (2 - \alpha_i') P_i, \tag{2.15}$$

where $\alpha_i = 2 \Rightarrow \alpha_{i-1} = 0$, $\alpha_i' = 2 \Rightarrow \alpha_{i-1}' = 0$, whence

$$\sum_{i=1}^{m} \alpha_i P = \sum_{i=1}^{m} \alpha_i' P_i. \tag{2.16}$$

Both sides of (2.16) are Zeckendorf representations of positive integers by Pell numbers, the uniqueness of which [4] yields $\alpha_i' = \alpha_i$.

Thus, $\beta_i' = \beta_i$.

Consequently, the uniqueness of (2.1) with (2.2) and (2.3) is demonstrated.

Remarks:

(a) Implications (2.2) and (2.7), which characterize the representations, are one-way only.

(b) As a numerical illustration of (2.4), take $n = 25$. Then

$$P_4 + P_3 - 1\,(=16) < 25 \leq P_5 + P_4 - 1\,(=40)$$

so that $k = 4$ here. Likewise, when $n = 999$, then $k = 8$.

(c) Integers having identical maximal and minimal representations are worthy of a separate investigation. Please see the Concluding Remarks.

3. CONCLUDING REMARKS

In Table 1, which may be extended indefinitely, the pattern of digits 0 (blank space), 1, and 2 reveals the visible mechanism of the representation. Two successive zeros do not occur in this table. Observe that in the maximal representations (Table 1) we write $2\ (= P_2) = 2P_1\ (= 2P_1 + P_0)$, whereas in the minimal representations (Table 2) we retain $2 = P_2$.

For the *Pell-Lucas numbers* Q_n, defined by the recurrence relation

$$Q_{n+2} = 2Q_{n+1} + Q_n \quad (n \geq 0) \tag{3.1}$$

with

$$Q_0 = 2,\ Q_1 = 2, \tag{3.2}$$

we observe that they are all even. It follows that there can be no representation of integers, maximal or minimal, involving Pell-Lucas numbers, since odd integers would necessarily be excluded.

TABLE 1. Maximal Representations of Positive Integers by Sums of Pell Numbers

n^+	P_1	P_2	P_3	P_4	P_5		P_1	P_2	P_3	P_4	P_5		P_1	P_2	P_3	P_4	P_5
1	1					26	2	1	2	1		51	1	2	1	1	1
2	2					27	1	2	2	1		52	2	2	1	1	1
3	1	1				28	2	2	2	1		53	2		2	1	1
4	2	1				29	1	2		2		54	1	1	2	1	1
5	1	2				30	2	2		2		55	2	1	2	1	1
6	2	2				31	2		1	2		56	1	2	2	1	1
7	2		1			32	1	1	1	2		57	2	2	2	1	1
8	1	1	1			33	2	1	1	2		58	1	2		2	1
9	2	1	1			34	1	2	1	2		59	2	2		2	1
10	1	2	1			35	2	2	1	2		60	2		1	2	1
11	2	2	1			36	2		2	2		61	1	1	1	2	1
12	2		2			37	1	1	2	2		62	2	1	1	2	1
13	1	1	2			38	2	1	2	2		63	1	2	1	2	1
14	2	1	2			39	1	2	2	2		64	2	2	1	2	1
15	1	2	2			40	2	2	2	2		65	2		2	2	1
16	2	2	2			41	2		2		1	66	1	1	2	2	1
17	1	2		1		42	1	1	2		1	67	2	1	2	2	1
18	2	2		1		43	2	1	2		1	68	1	2	2	2	1
19	2		1	1		44	1	2	2		1	69	2	2	2	2	1
20	1	1	1	1		45	2	2	2		1	70	2		2		2
21	2	1	1	1		46	1	2		1	1	71	1	1	2		2
22	1	2	1	1		47	2	2		1	1	72	2	1	2		2
23	2	2	1	1		48	2		1	1	1	73	1	2	2		2
24	2		2	1		49	1	1	1	1	1	74	2	2	2		2
25	1	1	2	1		50	2	1	1	1	1	75	1	2		1	2

TABLE 2. Minimal Representations of Positive Integers by Sums of Pell Numbers

n^+	P_1	P_2	P_3	P_4	P_5	P_6		P_1	P_2	P_3	P_4	P_5	P_6		P_1	P_2	P_3	P_4	P_5	P_6
1	1						26		1		2			51			2	1	1	
2		1					27	1	1		2			52	1		2	1	1	
3	1	1					28		2		2			53				2	1	
4		2					29					1		54	1			2	1	
5			1				30	1				1		55		1		2	1	
6	1		1				31		1			1		56	1	1		2	1	
7		1	1				32	1	1			1		57		2		2	1	
8	1	1	1				33		2			1		58					2	
9		2	1				34			1		1		59	1				2	
10			2				35	1		1		1		60		1			2	
11	1		2				36		1	1		1		61	1	1			2	
12				1			37	1	1	1		1		62		2			2	
13	1			1			38		2	1		1		63			1		2	
14		1		1			39			2		1		64	1		1		2	
15	1	1		1			40	1		2		1		65		1	1		2	
16		2		1			41				1	1		66	1	1	1		2	
17			1	1			42	1			1	1		67		2	1		2	
18	1		1	1			43		1		1	1		68			2		2	
19		1	1	1			44	1	1		1	1		69	1		2		2	
20	1	1	1	1			45		2		1	1		70						1
21		2	1	1			46			1	1	1		71	1					1
22			2	1			47	1		1	1	1		72		1				1
23	1		2	1			48		1	1	1	1		73	1	1				1
24				2			49	1	1	1	1	1		74		2				1
25	1			2			50		2	1	1	1		75			1			1

Further references to Zeckendorf representations may be found in [4].

Finally, a natural question to ask is this: Are there any numbers for which the maximal and minimal representations are the same? Examination of Tables 1 and 2 leads us to the reasonable conviction that this situation arises only when all the coefficients of the Pell numbers in the summations are unity. That is, the required numbers are $\sum_{i=1}^{k-1} P_i$ for $k \geq 2$ [see (1.8)], namely, 1, 3, 8, 20, 49, 119, ... Compare this with the corresponding situation for Fibonacci numbers in [3] and [5].

Properties of the sequence of numbers, 1, 3, 8, 20, 49, 119, ..., are the subject of a further research article.

REFERENCES

1. J. L. Brown, Jr. "A New Characterization of the Fibonacci Numbers." *The Fibonacci Quarterly* **3.1** (1965):1-8.
2. J. L. Brown, Jr. "Unique Representations of Integers as Sums of Distinct Lucas Numbers." *The Fibonacci Quarterly* **7.3** (1969):243-52.
3. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.
4. A. F. Horadam. "Zeckendorf Representations of Positive and Negative Integers by Pell Numbers." In *Applications of Fibonacci Numbers* **5**:305-16. Ed. G. E. Bergum, A. N. Philippou, & A. F. Horadam. Dordrecht: Kluwer, 1993.
5. D. A. Klarner. "Representation of N as a Sum of Distinct Elements from Special Sequences." *The Fibonacci Quarterly* **4.4** (1966):289-306.
6. J. Pihko. "On Fibonacci and Lucas Representations and a Theorem of Lekkerkerker." *The Fibonacci Quarterly* **26.3** (1988):256-61.

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