ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-585 ***Proposed by Herrmann Ernst, Siegburg, Germany***

Let (d_n) denote a sequence of positive integers d_n with $d_1 \geq 3$ and $d_{n+1} - d_n \geq 1$, $n = 1, 2, \ldots$. We introduce the following sets of sequences (d_n):

$$A = \left\{(d_n)\colon \sum_{k=1}^{\infty} \frac{1}{F_{d_k}} \leq 1\right\};$$

$$B = \left\{(d_n)\colon \frac{1}{F_{d_n}} < \sum_{k=n}^{\infty} \frac{1}{F_{d_k}} < \frac{1}{F_{d_n-1}} \text{ for all } n \in N\right\};$$

$$C = \left\{(d_n)\colon 0 \leq \frac{1}{F_{d_n-1}} - \frac{1}{F_{d_n}} - \frac{1}{F_{d_{n+1}-1}} \text{ for all } n \in N\right\}.$$

Show that:

- *(a)* there is a bijection $f\colon]\,0, 1] \to B$, $f(x) = (d_n(x))_{n=1}^{\infty}$;
- *(b)* B is a subset of A with $A \setminus B \neq \emptyset$;
- *(c)* C is a subset of B with $B \setminus C \neq \emptyset$.

H-586 ***Proposed by H.-J. Seiffert, Berlin, Germany***

Define the sequence of Fibonacci and Lucas polynomials by

$$F_0(x) = 0,\ \ F_1(x) = 1,\ \ F_{n+1}(x) = xF_n(x) + F_{n-1}(x),\ n \in N,$$
$$L_0(x) = 2,\ \ L_1(x) = x,\ \ L_{n+1}(x) = xL_n(x) + L_{n-1}(x),\ n \in N,$$

respectively. Show that, for all complex numbers x and all positive integers n,

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k F_{3k}(x) = \frac{xF_{2n+1}(x) - F_{2n}(x) + (-x)^{n+2} F_n(x) + (-x)^{n+1} F_{n-1}(x)}{2x^2 - 1}$$

and

$$\sum_{k=0}^{[n/2]} \binom{n-k}{k} x^k L_{3k}(x) = \frac{xL_{2n+1}(x) - L_{2n}(x) + (-x)^{n+2} L_n(x) + (-x)^{n+1} L_{n-1}(x)}{2x^2 - 1}.$$

<u>H-587</u> ***Proposed by N. Gauthier & J. R. Gosselin, Royal Military College of Canada***

Let x and y be indeterminates and let

$$\alpha \equiv \alpha(x,y)=\frac{1}{2}(x+\sqrt{x^2+4y}),\ \beta \equiv \beta(x,y)=\frac{1}{2}(x-\sqrt{x^2+4y})$$

be the distinct roots of the characteristic equation for the generalized Fibonacci sequence $\{H_n(x,y)\}_{n=0}^{n=\infty}$, where

$$H_{n+2(}(x,y)=xH_{n+1}(x,y)+yH_n(x,y).$$

If the initial conditions are taken as $H_0(x,y)=0$, $H_1(x,y)=1$, then the sequence gives the generalized Fibonacci polynomials $\{F_n(x,y)\}_{n=0}^{n=\infty}$. On the other hand, if $H_0(x,y)=2$, $H_1(x,y)=x$, then the sequence gives the generalized Lucas polynomials $\{L_n(x,y)\}_{n=0}^{n=\infty}$.

Consider the following 2×2 matrices,

$$A=\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix},\ B=\begin{pmatrix}\beta & 1\\ 0 & \beta\end{pmatrix},\ C=\begin{pmatrix}\alpha & 1\\ 0 & \beta\end{pmatrix},\ D=\begin{pmatrix}\beta & 1\\ 0 & \alpha\end{pmatrix},\ I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

and let n and m be nonnegative integers. [By definition, a matrix raised to the power zero is equal to the unit matrix I.]

a. Express $f_{n,m}(x,y)\equiv[(A-B)^{-1}(A^n-B^n)]^m$ in closed form, in terms of the Fibonacci polynomials.

b. Express $g_{n,m}(x,y)\equiv[A^n+B^n]^m$ in closed form, in terms of the Lucas polynomials.

c. Express $h_{n,m}(x,y)\equiv[C^n+D^n]^m$ in closed form, in terms of the Fibonacci and Lucas polynomials.

<u>H-588</u> ***Proposed by José Luiz Díaz-Barrero & Juan José Egozcue, Barcelona, Spain***

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} F_{k+2} \ge \frac{n^{n+1}}{(n+1)^n}\prod_{k=1}^{n}\left\{\frac{L_{k+1}^{\frac{n+1}{n}}-F_{k+1}^{\frac{n+1}{n}}}{L_{k+1}-F_{k+1}}\right\},$$

where F_n and L_n are, respectively, the n^{th} Fibonacci and Lucas numbers.

SOLUTIONS

A Fractional Problem

<u>H-574</u> ***Proposed by J. L. Diaz-Barrero, Barcelona, Spain***
(Vol. 39, no. 4, August 2001)

Let n be a positive integer greater than or equal to 2. Determine

$$\frac{F_n+L_nP_n}{(F_n-L_n)(F_n-P_n)}+\frac{L_n+F_nP_n}{(L_n-F_n)(L_n-P_n)}+\frac{P_n+F_nL_n}{(P_n-F_n)(P_n-L_n)},$$

where F_n, L_n, and P_n are, respectively, the n^{th} Fibonacci, Lucas, and Pell numbers.

Solution by Paul S. Bruckman, Berkeley, CA

We employ certain well-known results from finite difference theory. For any well-defined, complex-valued function $f(x)$ with complex domain D, and for any three distinct values

$x_i \in D$, $i = 1, 2, 3$, define the "second-order" divided difference of f, valued at (x_1, x_2, x_3), as follows:

$$\Delta^2(f(x)) \mid (x_1, x_2, x_3) \equiv \theta_1 f(x_1) + \theta_2 f(x_2) + \theta_3 f(x_3), \tag{1}$$

where $\theta_1 = 1/\{(x_1 - x_2)(x_1 - x_3)\}$, $\theta_2 = 1/\{(x_2 - x_1)(x_2 - x_3)\}$, $\theta_3 = 1/\{(x_3 - x_1)(x_3 - x_2)\}$.

For brevity, we may also denote the left member of (1) as $\Delta^2(f(x))$ when no confusion is likely to arise. If f is a polynomial, the second-order divided difference has the following properties:

$$\Delta^2(f(x)) = 0 \text{ if degree}(f) = 0 \text{ or } 1; \;\; \Delta^2(f(x)) = 1 \text{ if degree}(f) = 2. \tag{2}$$

Given distinct values x_1, x_2, x_3, let $\sigma_1 = x_1 + x_2 + x_3$, $\sigma_2 = x_1x_2 + x_2x_3 + x_3x_1$. Consider the following expression:

$$U(x_1, x_1, x_3) \equiv \sigma_2\Delta^2(1) + (1 - \sigma_1)\Delta^2(x) + \Delta^2(x^2). \tag{3}$$

Using (1), this becomes

$$U(x_1, x_2, x_3) = \sigma_2(\theta_1 + \theta_2 + \theta_3) + (1 - \sigma_1)(x_1\theta_1 + x_2\theta_2 + x_3\theta_3) + (x_1^2\theta_1 + x_2^2\theta_2 + x_3^2\theta_3).$$

After expansion (using the definitions of σ_1 and σ_2), this simplifies to

$$U(x_1, x_1, x_3) = (x_1 + x_2x_3)\theta_1 + (x_2 + x_1x_3)\theta_2 + (x_3 + x_1x_2)\theta_3. \tag{4}$$

On the other hand, since $\Delta^2(1) = \Delta^2(x) = 0$ and $\Delta^2(x^2) = 1$, we see from (3) that $U(x_1, x_1, x_3) = 1$. This yields the following general identity,

$$(x_1 + x_2x_3)\theta_1 + (x_2 + x_1x_3)\theta_2 + (x_3 + x_1x_2)\theta_3 = 1 \tag{5}$$

which is true for any distinct values x_1, x_2, and x_3.

We now need to show that P_n, L_n, and F_n are distinct if $n \geq 2$. Note that $P_1 = L_1 = F_1 = 1$; $P_2 = 2, L_2 = 3, F_2 = 1$; $P_3 = 5, L_3 = 4, F_3 = 2$. Since $P_{n+2} = 2P_{n+1} + P_n$, $L_{n+2} = L_{n+1} + L_n$, and $F_{n+2} = F_{n+1} + F_n$, it follows by an easy inductive proof that, if $n \geq 3$, $P_n > L_n > F_n$, while $L_2 > P_2 > F_2$. Therefore, if $n \geq 2$, we may let $x_1 = F_n$, $x_2 = L_n$, $x_3 = P_n$ in (5), proving that the given expression simplifies to 1.

Also solved by G. Arora, D. Iannucci, H.-J. Seiffert, P. & G. Stănică, and the proposer.

A Remarkable Problem

H-575 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 39, no. 4, August 2001)***

Problem Statement: "Four Remarkable Identities for the Fibonacci-Lucas Polynomials"

For n a nonnegative integer, the following Fibonacci-Lucas identities are known to hold:

$$L_{2n+2} = 5F_{2n+1} - L_{2n}; \;\; F_{2n+3} = L_{2n+2} - F_{2n+1}.$$

The corresponding identities for the Fibonacci $\{F_n(u)\}_{n=0}^{\infty}$ and the Lucas $\{L_n(u)\}_{n=0}^{\infty}$ polynomials, defined by

$$F_0(u) = 0,\; F_1(u) = 1,\; F_{n+2}(u) = uF_{n+1}(u) + F_n(u),$$
$$L_0(u) = 2,\; L_1(u) = u,\; L_{n+2}(u) = uL_{n+1}(u) + L_n(u),$$

respectively, are:

$$L_{2n+2}(u) = (u^2 + 4)F_{2n+1}(u) - L_{2n}(u); \;\; F_{2n+3}(u) = L_{2n+2}(u) = F_{2n+1}(u). \tag{1}$$

For m, n nonnegative integers, with the convention that a discrete sum with a negative upper limit is identically zero, prove the following generalizations of (1).

Case a: $$(2n+2)^{2m}L_{2n+2}(u)=(u^2+4)\left[\sum_{l=0}^{m}\binom{2m}{2l}(2n+1)^{2l}\right]F_{2n+1}(u)$$
$$+u\left[\sum_{l=0}^{m-1}\binom{2m}{2l+1}(2n+1)^{2l+1}\right]L_{2n+1}(u)-[(2n)^{2m}]L_{2n}(u).$$

Case b: $$(2n+3)^{2m}F_{2n+3}(u)=\left[\sum_{l=0}^{m}\binom{2m}{2l}(2n+2)^{2l}\right]L_{2n+2}(u)$$
$$+u\left[\sum_{l=0}^{m-1}\binom{2m}{2l+1}(2n+2)^{2l+1}\right]F_{2n+2}(u)-[(2n+1)^{2m}]F_{2n+1}(u).$$

Case c: $$(2n+2)^{2m+1}F_{2n+2}(u)=u\left[\sum_{l=0}^{m}\binom{2m+1}{2l}(2n+1)^{2l}\right]F_{2n+1}(u)$$
$$+\left[\sum_{l=0}^{m}\binom{2m+1}{2l+1}(2n+1)^{2l+1}\right]L_{2n+1}(u)-[(2n)^{2m+1}]F_{2n}(u).$$

Case d: $$(2n+3)^{2m+1}L_{2n+3}(u)=u\left[\sum_{l=0}^{m}\binom{2m+1}{2l}(2n+2)^{2l}\right]L_{2n+2}(u)$$
$$+(u^2+4)\left[\sum_{l=0}^{m}\binom{2m+1}{2l+1}(2n+2)^{2l+1}\right]F_{2n+2}(u)$$
$$-[(2n+1)^{2m+1}]L_{2n+1}(u).$$

Solution by the proposer

Start from the identity

$$x^{n+2}+x^{-(n+2)}=(x+x^{-1})(x^{n+1}+x^{-(n+1)})-(x^n+x^{-n}),$$

which is valid for any variable x and number n. Next, introduce the differential operator $D\equiv x\frac{d}{dx}$ and note that $D^m x^\lambda=\lambda^m x^\lambda$ for m a nonnegative integer and λ an arbitrary number. Acting on the identity with D^m then gives

$$(n+2)^m(x^{n+2}+(-1)^m x^{-(n+2)})=D^m[(x+x^{-1})(x^{n+1}+x^{-(n+1)})]-n^m(x^n+(-1)^m x^{-n}). \qquad (*)$$

Now let f and g be two arbitrary differentiable functions of x and note that

$$D(fg)=(Df)g+f(Dg);\ D^2(fg)=(D^2f)g+2(Df)(Dg)+f(D^2g);$$
$$D^3(fg)=(D^3f)g+3(D^2f)(Dg)+3(Df)(D^2g)+f(D^3g);\ etc.\ldots$$

The general term is

$$D^m(fg)=\sum_{l=0}^{m}\binom{m}{l}(D^{m-l}f)(D^l g), \qquad (**)$$

as can easily be established by induction on m, so we skip the details.

Insertion of $(**)$ in $(*)$ with $f=(x+x^{-1})$ and $g=(x^{n+1}+x^{-(n+1)})$ gives

$$(n+2)^m(x^{n+2}+(-1)^m x^{-(n+2)})$$
$$=\sum_{l=0}^{m}\binom{m}{l}(x+(-1)^{m-l}x^{-1})(n+1)^l(x^{n+1}+(-1)^l x^{-(n+1)})-n^m(x^n+(-1)^m x^{-n}). \qquad (***)$$

It is well known that the Fibonacci and Lucas polynomials can be represented in Binet form as follows:

$$F_n(u)=\frac{\alpha^n(u)-\beta^n(u)}{\alpha(u)-\beta(u)};\; L_n(u)=\alpha^n(u)+\beta^n(u);$$
$$\alpha(u)=\tfrac{1}{2}(u+\sqrt{u^2+4});\; \beta(u)=\tfrac{1}{2}(u-\sqrt{u^2+4}).$$

We now set $x=\alpha(u)\equiv\alpha$ in $(***)$ and invoke the property $\alpha^{-1}=-\beta$ to get

$$(n+2)^m(\alpha^{n+2}+(-1)^{n+m+2}\beta^{n+2})$$
$$=\sum_{l=0}^{m}\binom{m}{l}(\alpha+(-1)^{m+l+1}\beta)(n+1)^l(\alpha^{n+1}+(-1)^{m+l+1}\beta^{n+1})-n^m(\alpha^n+(-1)^{n+m}\beta^n).$$

Next, separate the sum over all l in the right-hand member of the above into a sum over even values $(2l)$ and one over odd values $(2l+1)$ and make the following substitutions to obtain the four cases given in the problem statement.

Case a: $m\to 2m; n\to 2n$;
Case c: $m\to 2m+1;\; n\to 2n$;
Case b: $m\to 2m;\; n\to 2n+1$;
Case d: $m\to 2m+1;\; n\to 2n+1$.

The algebra is straightforward and we skip the details. This completes the solution.

Also solved by P. S. Bruckman and H.-J. Seiffert.

General IZE

H-576 *Proposed by Paul S. Bruckman, Berkeley, CA*
(Vol. 39, no. 4, August 2001)

Define the following constant, $C_2\equiv\prod_{p>2}\{1-1/(p-1)^2\}$, as an infinite product over all odd primes p.

(A) Show that $C_2=\sum_{n=1}^{\infty}\mu(2n-1)/\{\phi(2n-1)\}^2$, where $\mu(n)$ and $\phi(n)$ are the Möbius and Euler functions, respectively.

(B) Let $\sum_{d\mid n}\mu(n/d)2^d$. Show that $C_2=\prod_{n=2}^{\infty}\{\zeta^*(n)\}^{-R(n)/n}$, where $\zeta(n)=\sum_{k=1}^{\infty}k^{-n}$ is the Riemann Zeta function (with $n>1$) and $\zeta^*(n)=\sum_{k=1}^{\infty}(2k-1)^{-n}=(1-2^{-n})\zeta(n)$.

Note: C_2 is the "twin-primes" constant that enters into Hardy and Littlewood's "extended" conjectures regarding the distribution of twin primes and Goldbach's Conjecture.

Solution by the proposer

Solution to Part (A): C_2 is easily shown to be a well-defined constant in $(0,1)$. We may express the product defining C_2 as a Euler product:

$$C_2=\prod_{p>2}\{1-1/(p-1)^2\}=\prod_{p>2}[1+\mu(p)/\{\phi(p)\}^2]$$
$$=\prod_{p>2}[1+\mu(p)/\{\phi(p)\}^2+\mu(p^2)/\{\phi(p^2)\}^2+\mu(p^3)/\{\phi(p^3)\}^3+\cdots]$$

$$= \sum_{n=1,\, n \text{ odd}}^{\infty} \mu(n) / \{\phi(n)\}^2 = \sum_{n=1}^{\infty} \mu(2n-1) / \{\phi(2n-1)\}^2.$$

Solution to Part (B): From the expression for C_2,

$$-\log C_2 = \sum_{p>2} \{2\log(1-p^{-1}) - \log(1-2p^{-1})\}$$

$$= \sum_{n=1} \sum_{p>2} (2^n - 2) p^{-n} / n = \sum_{n=2} (2^n - 2) g*(n) / n.$$

Taking the logarithm of the Euler product for the "modified" Zeta function, we obtain

$$\log \zeta^*(s) = -\sum_{p>2} \log(1-p^{-s})$$

valid for all s with $\mathrm{Re}(s) > 1$. Then

$$\log \zeta^*(s) = \sum_{m=1}^{\infty} \sum_{p>2} p^{-ms} / m = \sum_{m=1}^{\infty} g*(ms) / m.$$

By a variant form of Möbius inversion, we obtain

$$g*(s) = \sum_{m=1}^{\infty} \mu(m) \log \zeta^*(ms) / m. \tag{$*$}$$

Then

$$-\log C_2 = \sum_{n=2}^{\infty} (2^n - 2) / n \sum_{m=1}^{\infty} \mu(m) \log \zeta^*(mn) / m$$

$$= \sum_{N=2}^{\infty} \zeta^*(N) / N \sum_{d|N} \mu(N/d)(2^d - 2) = \sum_{N=2}^{\infty} \log \zeta^*(N) R(N) / N$$

since

$$\sum_{d|N} \mu(N/d) = 0 \quad (\text{for } N > 1).$$

Now, taking the antilogarithm leads to the expression given in (B).

❖❖❖