A NEW LARGEST SMITH NUMBER

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1. INTRODUCTION

In 1982, Albert Wilansky, a mathematics professor at Lehigh University wrote a short article in the *Two-Year College Mathematics Journal* [6]. In that article he identified a new subset of the composite numbers. He defined a **Smith number** to be a composite number where the sum of the digits in its prime factorization is equal to the digit sum of the number. The set was named in honor of Wilansky's brother-in-law, Dr. Harold Smith, whose telephone number 493-7775 when written as a single number 4,937,775 possessed this interesting characteristic. Adding the digits in the number and the digits of its prime factors 3, 5, 5 and 65,837 resulted in identical sums of 42.

Wilansky provided two other examples of numbers with this characteristic: 9,985 and 6,036. Since that time, many things have been discovered about Smith numbers including the fact that there are infinitely many Smith numbers [4]. The largest Smith numbers were produced by Samuel Yates. Using a large repunit and large palindromic prime, Yates was able to produce Smith numbers having ten million digits and thirteen million digits. Using the same large repunit and a new large palindromic prime, the author is able to find a Smith number with over thirty-two million digits.

2. NOTATIONS AND BASIC FACTS

For any positive integer n, we let $S(n)$ denote the sum of the digits of n. For any positive integer n, we let $S_p(n)$ denote the sum of the digits of the prime factorization of n. For example, $S(27) = 2+7 = 9$ and $S_p(27) = S_p(3\cdot 3\cdot 3) = 3+3+3 = 9$. Hence, 27 is a Smith number.

A *repunit*, denoted R_n, is a number consisting of a string of n one's. For example, $R_4 = 1111$. Currently, the largest known prime repunit is R_{1031}, which was shown to be prime by Hugh Williams and Harvey Dubner in 1985.

The following facts are used in constructions of very large Smith numbers.

Fact 1: If you multiply $9R_n$ by any natural number less than $9R_n$, then the digit sum is $9n$, i.e., $S(M * 9R_n) = 9n = S(9R_n)$ when $M < 9R_n$ (for a proof, see [3]).

Keith Wayland and Sham Oltikar in [5] provided the following.

Fact 2: If $S(u) > S_p(u)$ and $S(u) \equiv S_p(u)$ (mod 7), then $10^k \cdot u$ is a Smith number, where $k = (S(u) - S_p(u))/7$.

3. PRIOR LARGEST SMITH NUMBERS

In 1987, Dubner discovered the large palindromic prime $M = 10^{4594} + 3 * 10^{2297} + 1$. When this prime is raised to a power t, the digit sum will be the sum of the digits of the numbers in front of each power of 10^{2297}. As long as each coefficient of a power of 10^{2297} is less than $9R_{1031}$, when it is multiplied by $9R_{1031}$ that coefficient has a digit sum of $9*1031$. Since the largest coefficient occurs in the middle, it is sufficient to bound it by $9R_{1031}$.

Suppose that $N = 9R_{1031} * M^t$ with each coefficient of a power of 10^{2297} being less than $9R_{1031}$, then each of the $2t+1$ powers of 10^{2297} contributes $9*1031$ to the digit sum. Hence,

$$S(N) = (2t+1)*9*1031.$$

On the other hand, the prime factorization of N is simply $3*3*R_{1031}*M^t$ and so

$$S_p(N) = 3+3+1031+5t$$

because $S_p(M) = 5$. For any positive t, we have $S(N) > S_p(N)$ and

$$\begin{aligned} S(N) - S_p(N) &= 18553t + 8242 \\ &= 3t + 3 \pmod 7 \\ &= 3(t+1) \pmod 7. \end{aligned}$$

This result will be 0 (mod 7) when $t \equiv 6 \pmod 7$. Yates [7] was able to find the optimal t value that is congruent to 6 (mod 7) and has a coefficient of 10^{2297*t} less than $9R_{1031}$ was 1476. In this case, the coefficient of $10^{2297*1476}$ is $7.85*10^{1029}$ and increasing t by 7 causes the middle coefficient to be greater than $9R_{1031}$. Finally, the computation

$$(S(N) - S_p(N))/7 = (18553*1476+8242)/7 = 3913210$$

and Oltikar and Wayland's fact say that

$$9R_{1031}*(10^{4594} + 3*10^{2297} + 1)^{1476} * 10^{3913210}$$

is a Smith number. This number has $1031+1476*4594+1+3913210 = 10{,}694{,}986$ digits.

This number did not remain as the largest Smith number for very long because Dubner found a new larger palindromic prime in 1990. Yates used Dubner's prime, $10^{6572} + 3*10^{3286} + 1$, to produce the Smith number

$$9R_{1031}*(10^{6572} + 3*10^{3286} + 1)^{1476} * 10^{3913210}$$

which has 13,614,514 digits. Yates published his finding in a poem serving as a tribute to Martin Gardner [8].

4. NEW LARGEST SMITH NUMBER

Chris Caldwell is keeping on the World Wide Web a list of the 5000 largest primes [1] which is changing monthly. He also has available for retrieval a list of the largest palindromic primes [2]. In his list, we found a new very large palindromic prime with a small middle. The list credits Daniel Heuer for using the program Primeform in 2001 to discover that $M = 10^{28572} + 8*10^{14286} + 1$ is prime.

Suppose $N = 9R_{1031} * M^t$ with each coefficient of a power of 10^{14286} being less than $9R_{1031}$. Since Heuer's new palindromic prime has a middle digit 8, $S(N) - S_p(N)$ is now $18548t + 8242 \equiv 5t + 3 \pmod 7$.. This will be 0 (mod 7) when $t \equiv 5 \pmod 7$. The optimal t value that is congruent to 5 (mod 7) and has a coefficient of $10^{14286*t}$ less than $9R_{1031}$ turns out to be 1027. Then, the exponent to use on 10 is $(S(N) - S_p(N)/7 = (18548*1027+8242)/7 = 2722434$. Hence the new Smith number is

$$9R_{1031}*(10^{28572}+8*10^{14286}+1)^{1027}*10^{2722434}$$

which has $1031+1027*28572+1+2722434=32{,}066{,}910$ digits.

While there are larger palindomic primes in Caldwell's list, the larger ones have middle terms that are not single-digit numbers. Then you must use a much smaller t value on the palindromic prime so that the middle coefficient in the trinomial expansion is bounded by $9R_{1031}$. This limitation forces the number of digits in the resulting Smith number to be much smaller than 32 million. In fact, using the larger palindromic prime,

$$M=10^{35352}+2049402*10^{17673}+1,$$

the optimal t value is $t=157$ and yields a Smith number having only 5,968,187 digits.

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