COMPLEX FACTORIZATIONS OF THE FIBONACCI AND LUCAS NUMBERS

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1. INTRODUCTION

It is always fascinating to see what results when seemingly different areas of mathematics overlap. This article reveals one such result; number theory and linear algebra (with the help of orthogonal polynomials) are intertwined to yield complex factorizations of the Fibonacci and Lucas numbers. In Sections 2 and 3, respectively, we derive these complex factorizations:

$$F_n = \prod_{k=1}^{n-1}\left(1-2i\cos\frac{\pi k}{n}\right),\ n \geq 2, \tag{1.1}$$

and

$$L_n = \prod_{k=1}^{n}\left(1-2i\cos\frac{\pi(k-\frac{1}{2})}{n}\right),\ n \geq 1. \tag{1.2}$$

Along the way, we also establish the general forms:

$$F_n = i^{n-1}\frac{\sin(n\cos^{-1}(-\frac{i}{2}))}{\sin(\cos^{-1}(-\frac{i}{2}))},\ n \geq 1, \tag{1.3}$$

and

$$L_n = 2i^n\cos(n\cos^{-1}(-\tfrac{i}{2})),\ n \geq 1. \tag{1.4}$$

A simple proof of (1.1) can be obtained by considering the roots of Fibonacci polynomials (see Webb and Parberry [7] and Hoggatt and Long [3]). This paper proves (1.1) by considering how the Fibonacci numbers can be connected to Chebyshev polynomials by determinants of a sequence of matrices, and then illustrates a connection between the Lucas numbers and Chebyshev polynomials (and hence proves (1.2)) by using a slightly different sequence of matrices. (1.3) and (1.4) are not new developments (see Morgado [4] and Rivlin [5]); however, they are of interest here because they fall out of the derivations of (1.1) and (1.2) quite naturally.

In order to simplify the derivations of (1.1) and (1.2), we present the following lemma (the proof is included for completeness).

Lemma 1: Let $\{\mathbf{H}(n), n = 1, 2, \dots\}$ be a sequence of tridiagonal matrices of the form:

$$\mathbf{H}(n) = \begin{pmatrix} h_{1,1} & h_{1,2} & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & & \\ & h_{3,2} & h_{3,3} & \ddots & \\ & & \ddots & \ddots & h_{n-1,n} \\ & & & h_{n,n-1} & h_{n,n} \end{pmatrix}. \tag{1.5}$$

Then the successive determinants of $\mathbf{H}(n)$ are given by the recursive formula:

$$\begin{aligned} &|\mathbf{H}(1)| = h_{1,1}, \\ &|\mathbf{H}(2)| = h_{1,1}h_{2,2} - h_{1,2}h_{2,1}, \\ &|\mathbf{H}(n)| = h_{n,n}|\mathbf{H}(n-1)| - h_{n-1,n}h_{n,n-1}|\mathbf{H}(n-2)|. \end{aligned} \tag{1.6}$$

Proof: We prove Lemma 1 by the second principle of finite induction, computing all determinants by cofactor expansion. For the basis step, we have:

$$|\mathbf{H}(1)| = |(h_{1,1})| = h_{1,1},$$

$$|\mathbf{H}(2)| = \left|\begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix}\right| = h_{1,1}h_{2,2} - h_{1,2}h_{2,1},$$

$$|\mathbf{H}(3)| = \left|\begin{pmatrix} h_{1,1} & h_{1,2} & 0 \\ h_{2,1} & h_{2,2} & h_{2,3} \\ 0 & h_{3,2} & h_{3,3} \end{pmatrix}\right| = h_{3,3}|\mathbf{H}(2)| - h_{2,3}\left|\begin{pmatrix} h_{1,1} & h_{1,2} \\ 0 & h_{3,2} \end{pmatrix}\right| = h_{3,3}|\mathbf{H}(2)| - h_{2,3}h_{3,2}|\mathbf{H}(1)|.$$

For the inductive step, we assume $|\mathbf{H}(k)| = h_{k,k}|\mathbf{H}(k-1)| - h_{k-1,k}h_{k,k-1}|\mathbf{H}(k-2)|$ for $3 \le k \le n$. Then

$$|\mathbf{H}(k+1)| = \left|\begin{pmatrix} h_{1,1} & h_{1,2} & & & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & & & & \\ & h_{3,2} & h_{3,3} & \ddots & & & \\ & & \ddots & \ddots & h_{k-2,k-1} & & \\ & & & h_{k-1,k-2} & h_{k-1,k-1} & h_{k-1,k} & \\ & & & & h_{k,k-1} & h_{k,k} & h_{k,k+1} \\ & & & & & h_{k+1,k} & h_{k+1,k+1} \end{pmatrix}\right|$$

$$= h_{k+1,k+1}|\mathbf{H}(k)| - h_{k,k+1}\left|\begin{pmatrix} h_{1,1} & h_{1,2} & & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & & & \\ & h_{3,2} & h_{3,3} & \ddots & & \\ & & \ddots & \ddots & h_{k-2,k-1} & \\ & & & h_{k-1,k-2} & h_{k-1,k-1} & h_{k-1,k} \\ & & & & 0 & h_{k+1,k} \end{pmatrix}\right|$$

$$= h_{k+1,k+1}|\mathbf{H}(k)| - h_{k,k+1}h_{k+1,k}|\mathbf{H}(k-1)|. \; \square$$

2. COMPLEX FACTORIZATION OF THE FIBONACCI NUMBERS

In order to derive (1.1), we introduce the sequence of matrices $\{\mathbf{M}(n), n = 1, 2, \dots\}$, where $\mathbf{M}(n)$ is the $n \times n$ tridiagonal matrix with entries $m_{k,k} = 1$, $1 \le k \le n$, and $m_{k-1,k} = m_{k,k-1} = i$, $2 \le k \le n$. That is,

$$\mathbf{M}(n) = \begin{pmatrix} 1 & i & & & \\ i & 1 & i & & \\ & i & 1 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & 1 \end{pmatrix}. \tag{2.1}$$

According to Lemma 1, successive determinants of $\mathbf{M}(n)$ are given by the recursive formula:

$$\begin{aligned} &|\mathbf{M}(1)| = 1, \\ &|\mathbf{M}(2)| = 1^2 - i^2 = 2, \\ &|\mathbf{M}(n)| = 1|\mathbf{M}(n-1)| - i^2|\mathbf{M}(n-2)| = |\mathbf{M}(n-1)| + |\mathbf{M}(n-2)|. \end{aligned} \tag{2.2}$$

Clearly, this is also the Fibonacci sequence, starting with F_2. Hence,

$$F_n = |\mathbf{M}(n-1)|, \ n \geq 2. \tag{2.3}$$

There are a variety of ways to compute the matrix determinant (see Golub and Van Loan [2] for more details). In addition to the method of cofactor expansion, the determinant of a matrix can be found by taking the product of its eigenvalues. Therefore, we will compute the spectrum of $\mathbf{M}(n)$ in order to find an alternate formulation for $|\mathbf{M}(n)|$.

We now introduce another sequence of matrices $\{\mathbf{G}(n), n = 1, 2, \dots\}$, where $\mathbf{G}(n)$ is the $n \times n$ tridiagonal matrix with entries $g_{k,k} = 0$, $1 \leq k \leq n$, and $g_{k-1,k} = g_{k,k-1} = 1$, $2 \leq k \leq n$. That is,

$$\mathbf{G}(n) = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}. \tag{2.4}$$

Note that $\mathrm{M}(n) = \mathbf{I} + i\mathbf{G}(n)$. Let λ_k, $k = 1, 2, \dots, n$, be the eigenvalues of $\mathbf{G}(n)$ (with associated eigenvectors $\mathbf{x}_k$). Then, for each j,

$$\mathbf{M}(n)\mathbf{x}_j = [\mathbf{I} + i\mathbf{G}(n)]\mathbf{x}_j = \mathbf{I}\mathbf{x}_j + i\mathbf{G}(n)\mathbf{x}_j = \mathbf{x}_j + i\lambda_j\mathbf{x}_j = (1 + i\lambda_j)\mathbf{x}_j.$$

Therefore, $\mu_k = 1 + i\lambda_k$, $k = 1, 2, \dots, n$, are the eigenvalues of $\mathbf{M}(n)$. Hence,

$$|\mathbf{M}(n)| = \prod_{k=1}^{n}(1 + i\lambda_k), \ n \geq 1. \tag{2.5}$$

In order to determine the λ_k's, we recall that each λ_k is a zero of the characteristic polynomial $p_n(\lambda) = |\mathbf{G}(n) - \lambda\mathbf{I}|$. Since $\mathbf{G}(n) - \lambda\mathbf{I}$ is a tridiagonal matrix, i.e.,

$$\mathbf{G}(n) - \lambda\mathbf{I} = \begin{pmatrix} -\lambda & 1 & & & \\ 1 & -\lambda & 1 & & \\ & 1 & -\lambda & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -\lambda \end{pmatrix}, \tag{2.6}$$

we use Lemma 1 to establish a recursive formula for the characteristic polynomials of $\{\mathbf{G}(n), n = 1, 2, \dots\}$:

$$\begin{aligned} &p_1(\lambda) = -\lambda, \\ &p_2(\lambda) = \lambda^2 - 1, \\ &p_n(\lambda) = -\lambda p_{n-1}(\lambda) - p_{n-2}(\lambda). \end{aligned} \tag{2.7}$$

This family of characteristic polynomials can be transformed into another family $\{U_n(x), n \geq 1\}$ by the transformation $\lambda \equiv -2x$:

$$\begin{aligned} U_1(x) &= 2x, \\ U_2(x) &= 4x^2 - 1, \\ U_n(x) &= 2xU_{n-1}(x) - U_{n-2}(x). \end{aligned} \tag{2.8}$$

The family $\{U_n(x), n \geq 1\}$ is the set of Chebyshev polynomials of the second kind. It is a well-known fact (see Rivlin [5]) that defining $x \equiv \cos\theta$ allows the Chebyshev polynomials of the second kind to be written as:

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}. \tag{2.9}$$

From (2.9), we can see that the roots of $U_n(x) = 0$ are given by $\theta_k = \frac{\pi k}{n+1}$, $k = 1, 2, \ldots, n$, or $x_k = \cos\theta_k = \cos\frac{\pi k}{n+1}$, $k = 1, 2, \ldots, n$. Applying the transformation $\lambda \equiv -2x$, we now have the eigenvalues of $\mathbf{G}(n)$:

$$\lambda_k = -2\cos\frac{\pi k}{n+1}, \quad k = 1, 2, \ldots, n. \tag{2.10}$$

Combining (2.3), (2.5), and (2.10), we have

$$F_{n+1} = |\mathbf{M}(n)| = \prod_{k=1}^{n}\left(1 - 2i\cos\frac{\pi k}{n+1}\right), \quad n \geq 1, \tag{2.11}$$

which is identical to the complex factorization (1.1).

From (2.6), we can think of Chebyshev polynomials of the second kind as being generated by determinants of successive matrices of the form

$$\mathbf{A}(n, x) = \begin{pmatrix} 2x & 1 & & & \\ 1 & 2x & 1 & & \\ & 1 & 2x & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2x \end{pmatrix}, \tag{2.12}$$

where $\mathbf{A}(n, x)$ is $n \times n$. If we note that $\mathbf{M}(n) = i\mathbf{A}(n, -\frac{i}{2})$, then we have:

$$|\mathbf{M}(n)| = i^n\left|\mathbf{A}\left(n, -\frac{i}{2}\right)\right| = i^n U_n\left(-\frac{i}{2}\right). \tag{2.13}$$

Combining (2.3), (2.9), and (2.13) yields

$$F_{n+1} = i^n \frac{\sin((n+1)\cos^{-1}(-\frac{i}{2}))}{\sin(\cos^{-1}(-\frac{i}{2}))}, \quad n \geq 1 \tag{2.14}$$

Since it is also true that

$$F_1 = 1 = i^0 \frac{\sin(\cos^{-1}(-\frac{i}{2}))}{\sin(\cos^{-1}(-\frac{i}{2}))},$$

(1.3) holds.

3. COMPLEX FACTORIZATION OF THE LUCAS NUMBERS

The process by which we derive (1.2) is similar to that of the derivation of (1.1), but it has its own intricacies. Consider the sequence of matrices $\{\mathbf{S}(n), n = 1, 2, \ldots\}$, where $\mathbf{S}(n)$ is the $n \times n$ tri-diagonal matrix with entries $s_{1,1} = \frac{1}{2}$, $s_{k,k} = 1$, $2 \leq k \leq n$, and $s_{k-1,k} = s_{k,k-1} = i$, $2 \leq k \leq n$. That is,

$$\mathbf{S}(n) = \begin{pmatrix} \frac{1}{2} & i & & & \\ i & 1 & i & & \\ & i & 1 & \ddots & \\ & & \ddots & \ddots & i \\ & & & i & 1 \end{pmatrix}. \tag{3.1}$$

According to Lemma 1, successive determinants of $\mathbf{S}(n)$ are given by the recursive formula:

$$\begin{aligned} |\mathbf{S}(1)| &= \tfrac{1}{2}, \\ |\mathbf{S}(2)| &= \tfrac{1}{2} - i^2 = \tfrac{3}{2}, \\ |\mathbf{S}(n)| &= 1|\mathbf{S}(n-1)| - i^2|\mathbf{S}(n-2)| = |\mathbf{S}(n-1)| + |\mathbf{S}(n-2)|. \end{aligned} \tag{3.2}$$

Clearly, each number in this sequence is half of the corresponding Lucas number. We have

$$L_n = 2|\mathbf{S}(n)|, \; n \geq 1. \tag{3.3}$$

Unlike the derivation in the previous section, we will not compute the spectrum of $\mathbf{S}(n)$ directly. Instead, we will first note the following:

$$|\mathbf{S}(n)| = \frac{1}{2}\left|(\mathbf{I} + \mathbf{e}_1\mathbf{e}_1^{\mathrm{T}})\mathbf{S}(n)\right|, \tag{3.4}$$

where $\mathbf{e}_j$ is the j^{th} column of the identity matrix. (This is true because $|\mathbf{I} + \mathbf{e}_1\mathbf{e}_1^{\mathrm{T}}| = 2$.) Furthermore, we can express the right-hand side of (3.4) in the following way:

$$\frac{1}{2}\left|(\mathbf{I} + \mathbf{e}_1\mathbf{e}_1^{\mathrm{T}})\mathbf{S}(n)\right| = \frac{1}{2}\left|\mathbf{I} + i\left(\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}}\right)\right|, \tag{3.5}$$

where $\mathbf{G}(n)$ is the matrix given in (2.4). Let γ_k, $k = 1, 2, \ldots, n$ be the eigenvalues of $\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}}$ (with associated eigenvectors $\mathbf{y}_k$). Then, for each j,

$$(\mathbf{I} + i(\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}}))\mathbf{y}_j = \mathbf{I}\mathbf{y}_j + i(\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}})\mathbf{y}_j = \mathbf{y}_j + i\gamma_j\mathbf{y}_j = (1 + i\gamma_j)\mathbf{y}_j.$$

Therefore,

$$\frac{1}{2}\left|\mathbf{I} + i(\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}})\right| = \frac{1}{2}\prod_{k=1}^{n}(1 + i\gamma_k). \tag{3.6}$$

In order to determine the γ_k's, we recall that each γ_k is a zero of the characteristic polynomial $q_n(\gamma) = |\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}} - \gamma\mathbf{I}|$. Since $\left|(\mathbf{I} - \frac{1}{2}\mathbf{e}_1\mathbf{e}_1^{\mathrm{T}})\right| = \frac{1}{2}$, we can alternately represent the characteristic polynomial as

$$q_n(\gamma) = 2\left|\left(\mathbf{I} - \tfrac{1}{2}\mathbf{e}_1\mathbf{e}_1^{\mathrm{T}}\right)\left(\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}} - \gamma\mathbf{I}\right)\right|. \tag{3.7}$$

Since $q_n(\gamma)$ is twice the determinant of a tridiagonal matrix, i.e.,

$$q_n(\gamma) = 2\left|\left(\mathbf{I} - \tfrac{1}{2}\mathbf{e}_1\mathbf{e}_1^{\mathrm{T}}\right)\left(\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^{\mathrm{T}} - \gamma\mathbf{I}\right)\right| = 2\left|\begin{pmatrix} -\frac{\gamma}{2} & 1 & & & \\ 1 & -\gamma & 1 & & \\ & 1 & -\gamma & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -\gamma \end{pmatrix}\right|, \tag{3.8}$$

we can use Lemma 1 to establish a recursive formula for $q_n(\gamma)$:

$$\begin{aligned} q_1(\gamma) &= -\tfrac{\gamma}{2}, \\ q_2(\gamma) &= \tfrac{\gamma^2}{2} - 1, \\ q_n(\gamma) &= -\gamma q_{n-1}(\gamma) - q_{n-2}(\gamma). \end{aligned} \tag{3.9}$$

This family of polynomials can be transformed into another family $\{T_n(x),\ n \geq 1\}$ by the transformation $\gamma \equiv -2x$:

$$\begin{aligned} T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x). \end{aligned} \tag{3.10}$$

The family $\{T_n(x),\ n \geq 1\}$ is the set of Chebyshev polynomials of the first kind. Rivlin [5] shows that defining $x \equiv \cos\theta$ allows the Chebyshev polynomials of the first kind to be written as

$$T_n(x) = \cos n\theta. \tag{3.11}$$

From (3.11), we can see that the roots of $T_n(x) = 0$ are given by

$$\theta_k = \frac{\pi\left(k - \frac{1}{2}\right)}{n},\ k = 1, 2, \ldots, n, \quad \text{or} \quad x_k = \cos\theta_k = \cos\frac{\pi\left(k - \frac{1}{2}\right)}{n},\ k = 1, 2, \ldots, n.$$

Applying the transformation $\gamma \equiv -2x$ and considering that the roots of (3.7) are also roots of $|\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^T - \gamma\mathbf{I}| = 0$, we now have the eigenvalues of $\mathbf{G}(n) + \mathbf{e}_1\mathbf{e}_2^T$:

$$\gamma_k = -2\cos\frac{\pi\left(k - \frac{1}{2}\right)}{n},\ k = 1, 2, \ldots, n. \tag{3.12}$$

Combining (3.3)-(3.6) and (3.12), we have

$$L_n = \prod_{k=1}^{n}\left(1 - 2i\cos\frac{\pi\left(k - \frac{1}{2}\right)}{n}\right),\ n \geq 1, \tag{3.13}$$

which is identical to the complex factorization (1.2).

From (3.8), we can think of Chebyshev polynomials of the first kind as being generated by determinants of successive matrices of the form

$$\mathbf{B}(n, x) = \begin{pmatrix} x & 1 & & & \\ 1 & 2x & 1 & & \\ & 1 & 2x & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 2x \end{pmatrix}, \tag{3.14}$$

where $\mathbf{B}(n, x)$ is $n \times n$. If we note that $\mathbf{S}(n) = i\mathbf{B}(n, -\frac{i}{2})$, then we have

$$|\mathbf{S}(n)| = i^n \left|\mathbf{B}\left(n, -\frac{1}{2}\right)\right| = i^n T_n\left(-\frac{i}{2}\right). \tag{3.15}$$

Combining (3.3), (3.11), and (3.15) yields

$$L_n = 2i^n \cos\left(n\cos^{-1}\left(-\frac{i}{2}\right)\right),\ n \geq 1, \tag{3.16}$$

which is exactly (1.4).

4. CONCLUSION

This method of exploiting special properties of the Chebyshev polynomials allows us to find other interesting factorizations as well. For instance, the factorizations

$$F_{2n+2} = \prod_{k=1}^{n}\left(3 - 2\cos\frac{\pi k}{n+1}\right), \; n \geq 1, \tag{4.1}$$

$$n = \prod_{k=1}^{n-1}\left(2 - 2\cos\frac{\pi k}{n}\right), \; n \geq 2, \tag{4.2}$$

and

$$2^{1-n} = \prod_{k=1}^{n}\left(1 - \cos\frac{\pi(k-\frac{1}{2})}{n}\right), \; n \geq 1, \tag{4.3}$$

can be derived with judicious choices of entries in tridiagonal matrices (Strang [6] presents a family of tridiagonal matrices that can be used to derive (4.1)). It is also possible to compare these factorizations with the Binet-like general formulas (see Burton [1]) for second-order linear recurrence relations in order to determine which products converge to zero, converge to a non-zero number, or diverge as n approaches infinity.

One final note: an interesting (but fairly straightforward) problem for students of complex variables is to prove that (1.3) is equivalent to Binet's formula for the Fibonacci Numbers.

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