

Correlation Clustering with Sherali-Adams

Vincent Cohen-Addad*

Euiwoong Lee[†]

Alantha Newman[‡]

Abstract

Given a complete graph $G = (V, E)$ where each edge is labeled $+$ or $-$, the CORRELATION CLUSTERING problem asks to partition V into clusters to minimize the number of $+$ -edges between different clusters plus the number of $-$ -edges within the same cluster. CORRELATION CLUSTERING has been used to model a large number of clustering problems in practice, making it one of the most widely studied clustering formulations. The approximability of CORRELATION CLUSTERING has been actively investigated [BBC04, CGW05, ACN08], culminating in a 2.06-approximation algorithm [CMSY15], based on rounding the standard LP relaxation. Since the integrality gap for this formulation is 2, it has remained a major open question to determine if the approximation factor of 2 can be reached, or even breached.

In this paper, we answer this question affirmatively by showing that there exists a $(1.994 + \varepsilon)$ -approximation algorithm based on $O(1/\varepsilon^2)$ rounds of the Sherali-Adams hierarchy. In order to round a solution to the Sherali-Adams relaxation, we adapt the *correlated rounding* originally developed for CSPs [BRS11, GS11, RT12]. With this tool, we reach an approximation ratio of $2 + \varepsilon$ for CORRELATION CLUSTERING. To breach this ratio, we go beyond the traditional triangle-based analysis by employing a *global charging scheme* that amortizes the total cost of the rounding across different triangles.

1 Introduction

Clustering is a central problem in unsupervised machine learning and data mining. Given a dataset and information regarding the similarity of pairs of elements, a “good” clustering is a partition of the elements into groups such that similar elements belong to the same group, while dissimilar elements belong to different groups. Since its introduction by Bansal, Blum, and Chawla [BBC04], CORRELATION CLUSTERING has been one of the most widely studied formulations for clustering. Given a graph $G = (V, E)$ where each edge is either labeled $+$ or $-$, the goal is to find a clustering (partition) (V_1, \dots, V_k) of V that minimizes the number of unsatisfied edges, namely the $+$ -edges between different clusters and the $-$ -edges within the same cluster. Thanks to the simplicity and modularity of the formulation, CORRELATION CLUSTERING has found a spectacular number of applications, e.g., finding clustering ensembles [BGU13], duplicate detection [ARS09], community

*Google Research.

[†]University of Michigan.

[‡]Laboratoire G-SCOP (CNRS, Grenoble-INP). Supported in part by French ANR Project DAGDigDec (ANR-21-CE48-0012).

mining [CSX12], disambiguation tasks [KCMNT08], automated labelling [AHK⁺09, CKP08] and many more.

When G is a general graph, there is an $O(\log n)$ -approximation algorithm [CGW05, DEFI06] via the equivalence to UNDIRECTED MULTICUT, which is hard to approximate within any constant factor assuming the Unique Games Conjecture (UGC) [CKK⁺06]. For the maximization version where the goal is to maximize the number of +edges within the same cluster plus the number of −edges between different clusters, Charikar, Guruswami, and Wirth [CGW05] and Swamy [?] gave 0.766-approximation algorithms based on rounding semidefinite programs.

A lot of effort has focused on understanding the approximability of the original version introduced by [BBC04]: the unweighted case on a complete graph. (For the rest of the paper, CORRELATION CLUSTERING denotes this version.) In this case, [BBC04] gave a PTAS for the maximization version and an $O(1)$ -approximation for the minimization version. Charikar, Guruswami, and Wirth gave a 4-approximation based on rounding the standard linear programming (LP) relaxation and proved APX-hardness [CGW05]. Ailon, Charikar, and Newman gave a combinatorial 3-approximation algorithm based on choosing random pivots and a 2.5-approximation by combining this pivot based approach with the standard LP relaxation [ACN08].

The current best approximation ratio in this classic setting is $2.06 - \varepsilon$ for some fixed $\varepsilon > 0$ by Chawla, Makarychev, Schramm, and Yaroslavtsev [CMSY15], which extended the pivot rounding framework of [ACN08] with advanced functions that convert LP values to rounding probabilities. The standard LP relaxation for CORRELATION CLUSTERING has an integrality gap of 2 [CGW05]. The LP-based approximation algorithms (i.e., the 4-approximation of [CGW05], the 2.5-approximation algorithm of [ACN08], and the 2.06-approximation algorithm of [CMSY15]) each prove upper bounds on the integrality gap of this LP. Furthermore, [CMSY15] shows that their rounding framework cannot yield an approximation ratio better than 2.025.

Thus, currently, even reaching the approximation threshold of 2 is an interesting open problem. In this paper, we overcome the aforementioned barriers and give a $(1.994 + \varepsilon)$ -approximation algorithm for CORRELATION CLUSTERING for any $\varepsilon > 0$ using the Sherali-Adams hierarchy.

Theorem 1.1. *For $\varepsilon > 0$, there exists a $(1.994 + \varepsilon)$ -approximation algorithm for CORRELATION CLUSTERING running in time $n^{O(1/\varepsilon^2)}$. Moreover, the integrality gap of the $O(1/\varepsilon^2)$ -round Sherali-Adams relaxation is at most $(1.994 + \varepsilon)$.*

While we will present our algorithm as a randomized algorithm, it can be derandomized using the standard method of conditional expectation. See Section 8 for details. In order to achieve the result, we introduce the following two techniques for CORRELATION CLUSTERING. Our result also implies a marginally better constant factor approximation for the problem of fitting a tree metric or an ultrametric through the framework of [CDK⁺21, AC11].

- To improve beyond the integrality gap of the standard LP, we naturally use the *Sherali-Adams hierarchy* tailored for CORRELATION CLUSTERING, defined in Section 2.1. Previous algorithms [ACN08, CMSY15] proceed by sampling a random pivot $p \in V$ in each iteration and independently deciding whether $v \in V \setminus \{p\}$ belongs to p 's cluster or not. In order to use the power of Sherali-Adams, we adapt the *correlated rounding* that has been used for CONSTRAINT SATISFACTION PROBLEMS (CSPs) [BRS11, GS11, RT12]. One of the main

advantages is that for $u, v \in V \setminus \{p\}$, one can ensure that $\Pr[u \text{ and } v \text{ belong to } p\text{'s cluster}]$ is approximately equal to the value predicted by the Sherali-Adams solution in an amortized sense. See Section 3 for the description of the algorithm.

- Previous analyses [ACN08, CMSY15] employ the elegant *triangle-based* analysis that bounds the ratio

$$\frac{\text{cost}_u(v, w) + \text{cost}_v(w, u) + \text{cost}_w(u, v)}{\text{lp}_u(v, w) + \text{lp}_v(w, u) + \text{lp}_w(u, v)}, \quad (1)$$

for each triangle (u, v, w) , where $\text{cost}_u(v, w)$ is the probability that (v, w) is violated when u is pivot, and $\text{lp}_u(v, w)$ is the probability that v or w belongs to u 's cluster (removing (v, w) from the instance) times the LP contribution of (v, w) . (See Section 4.1 for this basic setup.) The previous analyses bound the above ratio for every triangle individually. With our rounding algorithm, the ratio (1) is already at most 2 for every triangle, and there is only one type of a *bad triangle* that has a ratio close to 2 (i.e., $++-$ triangles with LP values close to 0.5, 0.5, 1 respectively, which are bad triangles for the previous rounding algorithms as well). We prove that the number of such bad triangles is not large compared to *chargeable triangles* that have significantly smaller ratios but still with large denominators. This allows a global charging scheme where we show that the total ratio (the sum of numerators over all triangles / the sum of denominators over all triangles) is strictly less than 2. Sections 4.2 and 4.4 show how we use this scheme to finish the analysis.

1.1 Further Related Work

The pivot-based algorithm of Ailon et al. [ACN08] has been revisited in terms of derandomization [VZW09], parallelism [CDK14], for classification with asymmetric error costs [JKMM20], and for clustering with categorical rather than binary relationships given between elements [AAEG15, BGU13], to name a few settings in which it has been applied and adapted. A related objective function which maximizes the difference between the satisfied and unsatisfied edges has been studied [CW04, AMMN06]. The CORRELATION CLUSTERING problem has also been studied in an online setting [MSS10] and with respect to local guarantees [PM16, CGS17, KMZ19, JKMM21]. Recent progress has lead to constant factor approximation algorithms for the problem in the massively-parallel computation model [CLM⁺21, BCMT22], in the streaming setting [AW22], online setting [CLMP22], and with differential privacy guarantees [BEK21, CFL⁺22, Liu22].

Besides complete graphs, other special classes of graphs have been considered, including complete k -partite graphs [AALvZ12, CMSY15] and the weighted case where the weights of $-$ edges satisfy the triangle inequality [GMT07]. The result for complete k -bipartite graphs match the integrality gap of the standard LP. When the number of clusters k is bounded, Giotis and Guruswami [GG06] and Karpinski and Schudy [KS09] showed that a PTAS exists.

In terms of using the Sherali-Adams hierarchy to design approximation algorithms, there have been numerous negative results [CMM09, GMT09, KMN11] as well as some applications for designing algorithms [YZ14, ADFH20, OS19, HST20].

2 Preliminaries

An instance of CORRELATION CLUSTERING is a complete graph $G = (V, E)$, where $E = E^+ \cup E^-$ and $E^+ \cap E^- = \emptyset$, and the goal is to compute a partition $\{C_1, \dots, C_k\}$ of V minimizing the number of the $+$ edges (u, v) where $u \in C_i$ and $v \in C_j$, $i \neq j$, plus the number of the $-$ edges (u, v) where $u, v \in C_i$. The following standard LP relaxation has been used by most of the previous work [CGW05, ACN08, CMSY15].

$$\begin{aligned} \min \quad & \sum_{ij \in E^+} x_{ij} + \sum_{ij \in E^-} (1 - x_{ij}) \\ & x_{ij} \leq x_{ik} + x_{jk} \quad \forall i, j, k \in V \\ & x_{ij} \geq 0 \quad \forall i, j \in V. \end{aligned}$$

It has an integrality gap of 2 [CGW05]; consider a graph with vertices $\{0, 1, \dots, k\}$ where the edge $(0, i)$ is $+$ for each $i \in [k]$ and the rest are $-$. Letting $x_{0i} = 1/2$ for each $i \in [k]$ and $x_{ij} = 1$ for each $1 \leq i < j \leq k$ ensures that the LP value is $k/2$, but the optimal integral value is $k - 1$.

2.1 Strengthened LP relaxation

In order to overcome the integrality gap for the standard LP relaxation, we consider the following r -rounds of Sherali-Adams relaxation. For a collection of nonempty disjoint sets $S_1, \dots, S_\ell \subseteq V$ such that $\sum_{i=1}^{\ell} |S_i| \leq r$, we have a variable $y_{S_1|S_2|\dots|S_\ell}$ indicating the probability that the optimal partition induced by $S_1 \cup \dots \cup S_\ell$ is exactly (S_1, \dots, S_ℓ) . Note that the order of S_1, \dots, S_ℓ does not matter. For example, for two vertices u and v , $y_{u|v}$ is supposed to indicate the probability that u and v are in different clusters in the optimal solution and y_{uv} indicates the probability that they are in the same cluster so that $y_{u|v} + y_{uv} = 1$. (Similarly, for three distinct vertices u, v, w , $y_{uvw} + y_{u|vw} + y_{v|uw} + y_{w|uv} + y_{u|v|w} = 1$.) We have the following constraints ensuring the consistency of the variables. We use $S_i \cup S_j$ to indicate disjoint unions. Notice that $x_{uv} = 1 - y_{uv} = y_{u|v}$.

$$\min \quad \sum_{ij \in E^+} (1 - y_{ij}) + \sum_{ij \in E^-} y_{ij} \tag{2}$$

$$\text{s.t.} \quad y_{T_1|\dots|T_k} = \sum_{\substack{S_1, \dots, S_\ell: \\ S = S_1 \cup \dots \cup S_\ell \\ \text{and } T_i = S_i \cap T \quad \forall i \in [k]}} y_{S_1|S_2|\dots|S_\ell} \quad \forall T \subseteq S \subseteq V, |S| \leq r, \text{ and } T = T_1 \cup \dots \cup T_k \tag{3}$$

$$y_\emptyset = 1 \tag{4}$$

$$y \geq 0. \tag{5}$$

Note that the constraint (5) requires the y -variables across all possible subscripts to be nonnegative.

3 Algorithm

Let r be a positive integer denoting the number of rounds, and $\delta := 0.1$ throughout the paper. We consider the solution y obtained from r -round of Sherali-Adams. Let $x_{uv} := y_{u|v} = 1 - y_{uv}$ be the distance between u and v . Call an edge (u, v) *short* if $x_{uv} \leq \delta$, *long* if $x_{uv} \geq 1 - \delta$, and *medium* otherwise.

3.1 Rounding Algorithm

At a high-level, our rounding algorithm follows the general framework of [ACN08] and [CMSY15]. The algorithm proceeds in iterations, and in each iteration with the remaining instance $G = (V, E)$, the algorithm chooses a pivot p uniformly at random from V , samples a random set $S \ni p$, creates S as a new cluster, and proceeds with the remaining instance $G \setminus S$.

The most crucial step of the algorithm is to sample S in each iteration. Given a pivot $p \in V$, for each vertex $v \in V \setminus p$, the LP-KWIKCLUSTER algorithm [ACN08] independently puts v into S with probability $(1 - x_{pv})$. The refined algorithm of [CMSY15] also does independent rounding, but puts v into S with probability $(1 - f^s(x_{pv}))$ where $s \in \{+, -\}$ is the sign of (p, v) . (It sets $f^-(x) = x$ and $f^+(x) = 0$ if $x < 0.19$, $(\frac{x-0.19}{0.5095-0.19})^2$ if $x \in [0.19, 0.5095]$, and 1 if $x \geq 0.5095$.) In order to use the power of Sherali-Adams, given a pivot p , we round medium $+$ -edges in a correlated manner while also employing nontrivial $f^s(\cdot)$ functions for other edges. The full algorithm is described as Algorithm 1.

Before we present the full algorithm, we briefly discuss some intuition behind our rounding. In [ACN08] and [CMSY15], the analysis boiled down to analyzing the ratio defined in (1) on each type of triangle. We call a triangle $+++$ if it has three $+$ -edges, and $++-$, $+--$, $---$ triangles are defined similarly. For the LP rounding algorithm of [ACN08], each triangle has a ratio of at most 2, except the $++-$ triangle, which has a ratio of 2.5. In [CMSY15], there is a trade-off by lowering the ratio for the $++-$ triangles but increasing the ratio for the $+++$ triangles, to the point where each ratio is around 2.06.

A key observation is that we can use the *correlated rounding* (Line 12 of the algorithm) to lower the ratio in the case where the $++-$ triangle had ratio 2.5 (without increasing the ratio on other types of triangles such as $+++$). This is because in the correlated rounding, we obtain a sort of *negative correlation*, which is not present in the independent rounding. Specifically, for a $++-$ triangle with distances $(.5, .5, 1)$ (where the $-$ -edge has distance 1 and vertex p is incident to the two $+$ -edges), if we do the independent rounding with p as a pivot, then there is a $1/4$ probability that both of the other two vertices, call them u and v , will be included in p 's cluster and therefore a $1/4$ probability of the bad event that the $-$ -edge will be an intracluster edge. However, with correlated rounding, if p is the pivot, then the events of including u and v in p 's cluster are negatively correlated and exactly one of them is included. Thus, the $-$ -edge, which contributes 0 to the objective function of the LP, is never contained in a cluster when p is the pivot, which results in a lower ratio for this triangle. (In reality, it happens in an approximate and amortized sense, which slightly complicates the analysis.)

One more comment is that we cannot simply use correlated rounding on both $+$ and $-$ -edges incident to the chosen pivot, because this turns out to have an unbounded ratio on $---$ triangles.

There are additional technical reasons that prevent us from using correlated rounding to short or long $+$ edges (see Remark 4.1), so our algorithm only uses correlated rounding on medium $+$ edges incident to the pivot.

Algorithm 1: Rounding procedure **Round** for Correlation Clustering with parameter δ .

- 1 **Input:** Set of vertices V , with edges E^+ and E^- , a fractional solution y from the r -round Sherali-Adams relaxation;
 - 2 Pick a pivot $p \in V$ uniformly at random;
 - 3 $S \leftarrow \{p\}$;
 - 4 **foreach** vertex $v \in V - \{p\}$ **do**
 - 5 **if** $(v, p) \in E^-$ **then**
 - 6 Add v to S independently with probability $1 - \sqrt{x_{pv}}$;
 - 7 **if** $(v, p) \in E^+$ and (v, p) is short **then**
 - 8 Add v to S independently with probability $1 - x_{pv}^2/\delta$;
 - 9 **if** $(v, p) \in E^+$ and (v, p) is long **then**
 - 10 Add v to S independently with probability $1 - x_{pv}$;
 - 11 Define $I_p \leftarrow \{v \in V \setminus \{p\} : (v, p) \text{ is medium } + \text{ edge}\}$;
 - 12 Sample $S' \subseteq I_p$ as prescribed by Lemma 3.1 (i.e.: such that: (1) For each $v \in I_p$, $\Pr[v \in S'] = y_{pv}$; and (2) $\mathbb{E}_{u, v \in I_p} [|\Pr[u, v \in S'] - y_{puv}|] \leq \varepsilon_r$, where $\varepsilon_r = O(1/\sqrt{r})$);
 - 13 $S \leftarrow S \cup S'$;
 - 14 **Output:** Cluster S and the clusters obtained by calling **Round** on $V - S$ (with the fractional solution y and $+$ and $-$ edges induced by $V - S$);
-

The following lemma shows that the correlated rounding procedure in Line 12 can be implemented using the techniques to round convex hierarchies for CSPs [RT12, GS11, BRS11]. It is proved in Section 7.

Lemma 3.1. *In Line 12, one can sample $S' \subseteq I_p$ in time $n^{O(r)}$ such that*

- For each $v \in I_p$, $\Pr[v \in S'] = y_{pv}$.
- $\mathbb{E}_{u, v \in I_p} [|\Pr[u, v \in S'] - y_{puv}|] \leq \varepsilon_r$, where $\varepsilon_r = O(1/\sqrt{r})$.

4 Analysis

In this section, we show that Algorithm 1 guarantees a $(1.994 + \varepsilon_r)$ -approximation.

4.1 Setup and Ideal Cases

Our high-level setup of the analysis also follows from that of [ACN08] and [CMSY15]. Consider the t -th iteration of Algorithm 1 with the current graph $G_t = (V_t, E_t)$. Let $\text{cost}_p^r(u, v)$ be the probability

that (u, v) is violated in the rounding algorithm when p is the pivot, and $\text{lp}_p^r(u, v)$ be the LP value of (u, v) (i.e., x_{uv} if (u, v) is $+$ and y_{uv} if it is $-$) times the probability that (u, v) disappears (i.e., $\Pr[S \cap \{u, v\} \neq \emptyset]$). The superscript r stands for rounding.

We call a set of three distinct vertices a *triangle*. A set of two vertices is called a *degenerate triangle*. For triangle $\{u, v, w\}$, let $\text{cost}^r(u, v, w) = \text{cost}_u^r(v, w) + \text{cost}_v^r(u, w) + \text{cost}_w^r(u, v)$ and $\text{lp}^r(u, v, w) = \text{lp}_u^r(v, w) + \text{lp}_v^r(u, w) + \text{lp}_w^r(u, v)$. For degenerate triangle $\{u, v\}$, let $\text{cost}^r(u, v) = \text{cost}_u^r(u, v) + \text{cost}_v^r(u, v)$ and $\text{lp}^r(u, v) = \text{lp}_u^r(u, v) + \text{lp}_v^r(u, v)$. Let

$$\text{ALG}_t := \mathbb{E}_{u \in V} \sum_{(v, w) \in \binom{V_t}{2}} \text{cost}_u^r(v, w)$$

be the expected cost incurred by this iteration, and

$$\text{LP}_t := \mathbb{E}_{u \in V} \sum_{(v, w) \in \binom{V_t}{2}} \text{lp}_u^r(v, w)$$

be the expected amount of the LP value removed by this iteration. If we could show that for all t ,

$$\text{ALG}_t \leq \alpha \cdot \text{LP}_t \tag{6}$$

then we will get an upper bound on the total cost \mathbf{ALG} as

$$\mathbb{E}[\mathbf{ALG}] = \mathbb{E}\left[\sum_{t=0}^R \text{ALG}_t\right] \leq \alpha \cdot \mathbb{E}\left[\sum_{t=0}^R \text{LP}_t\right] = \alpha \cdot \mathbf{LP}$$

where \mathbf{LP} denotes the total LP value and R is the number of the iterations.

Therefore, in order to prove Theorem 1.1, it suffices to consider one iteration. For the rest of the paper, let us omit the subscript t denoting the iteration. We prove (6), which is equivalent to upper bounding

$$\frac{\text{ALG}}{\text{LP}} = \frac{\mathbb{E}_{u \in V} \sum_{(v, w) \in E} \text{cost}_u^r(v, w)}{\mathbb{E}_{u \in V} \sum_{(v, w) \in E} \text{lp}_u^r(v, w)} = \frac{\sum_{(u, v, w) \in \binom{V}{3}} \text{cost}^r(u, v, w) + \sum_{(u, v) \in \binom{V}{2}} \text{cost}^r(u, v)}{\sum_{(u, v, w) \in \binom{V}{3}} \text{lp}^r(u, v, w) + \sum_{(u, v) \in \binom{V}{2}} \text{lp}^r(u, v)}.$$

Recall that a triangle is $+++$ if it has three $+$ -edges and $+-$, $+-$, $---$ triangles are defined similarly. For a degenerate triangle $\{u, v\}$, $\text{cost}_u^r(u, v)$ and $\text{lp}_u^r(u, v)$ depend only on x_{uv} and the sign of (u, v) . Even for a triangle $\{u, v, w\}$, the values of $\text{cost}_u^r(v, w)$ and $\text{lp}_u^r(v, w)$ only depend on x_{uv}, x_{uw}, x_{vw} and the signs of the edges unless both (u, v) and (u, w) are medium $+$ -edges; v and w are added to $S \cup S'$ independently with the probabilities depending on x_{uv} and x_{uw} respectively. When both (u, v) and (u, w) are medium $+$ -edges, then they are rounded with correlation and $\Pr[v, w \in S' | u \text{ is pivot}]$ must be, ideally, exactly equal to y_{uvw} , but Lemma 3.1 only gives an approximate guarantee amortized over the vertices in I_u .

To gradually overcome the complication arising from correlated rounding, we define the following two idealized versions of $\text{cost}^r(\cdot)$ and $\text{lp}^r(\cdot)$ and analyze them first.

- $\text{cost}^s(\cdot)$ and $\text{lp}^s(\cdot)$ are defined assuming that the correlated rounding for medium +edges are perfect. Formally, $\text{cost}_u^s(\cdot), \text{cost}^s(\cdot), \text{lp}_u^s(\cdot), \text{lp}^s(\cdot)$ are defined identically to $\text{cost}_u^r(\cdot), \text{cost}^r(\cdot), \text{lp}_u^r(\cdot), \text{lp}^r(\cdot)$ respectively, assuming that in Line 12 of Algorithm 1, the condition (2) is replaced by $\Pr[u, v \in S' | p \text{ is pivot}] = y_{puv}$ for every $p \in V, u, v \in I_p$. With this assumption, note that for every triangle $\{u, v, w\}$ both $\text{cost}^s(u, v, w)$ and $\text{lp}^s(u, v, w)$ depend only on the signs of the edges and the Sherali-Adams solution induced by $\{u, v, w\}$ (i.e., $y_{uvw}, y_{u|vw}, y_{uv|w}, y_{v|uw}, y_{u|v|w}$).
- $\text{cost}^i(\cdot)$ and $\text{lp}^i(\cdot)$ are even more idealized versions of $\text{cost}^s(\cdot)$ and $\text{lp}^s(\cdot)$ in the sense that all +edges (instead of just medium +edges) are rounded with correlation. Formally, $\text{cost}_u^i(\cdot), \text{cost}^i(\cdot), \text{lp}_u^i(\cdot), \text{lp}^i(\cdot)$ are defined identically to $\text{cost}_u^s(\cdot), \text{cost}^s(\cdot), \text{lp}_u^s(\cdot), \text{lp}^s(\cdot)$ respectively, additionally assuming that instead of running Line 7, 8, 9, 10 of Algorithm 1, we let $I_p \leftarrow \{v \in V \setminus \{p\} : (v, p) \text{ is } +\}$ in Line 11.

The superscript i stands for ideal and s stands for special (short and long) edges.

Remark 4.1. *The primary reason that we round short and long +edges separately and differentiate $\text{cost}^s(\cdot), \text{lp}^s(\cdot)$ from $\text{cost}^i(\cdot), \text{lp}^i(\cdot)$ is to handle the rounding error ε_r in Lemma 3.1, because it applies to every pair (u, v) participating the correlated rounding and we want the $\text{cost}^i(\cdot)$ and $\text{lp}^i(\cdot)$ values for these pairs (more precisely, the triangle (p, u, v)) to be large enough to absorb it. For instance, if $\varepsilon_r = 0$ for some r , we could have rounded every +edge with correlation and just used $\text{cost}^i(\cdot), \text{lp}^i(\cdot)$.*

We first analyze $\text{cost}^i(T)/\text{lp}^i(T)$ for all triangles. Let $\eta := 1/12$ and $\gamma := 0.054$. Call $++-$ triangle $\{u, v, w\}$ with $(u, v), (u, w)$ being + bad if $x_{uv}, x_{uw} \in [1/2 - \eta, 1/2 + \eta]$ and $x_{vw} > 1 - \eta$. The proof of the following lemma appears in Section 5.

Lemma 4.2. *For any triangle T , $\text{cost}^i(T)/\text{lp}^i(T)$ is bounded as follows.*

Type of T	Upper bound
+++	1.5
+- -	1.5
- - -	1
++ -: bad	2
++ -: not bad	$2 - \gamma$
degenerate	1

Incorporating short and long +edges yields the following bounds whose proofs appear in Section 6.

Lemma 4.3. *For any triangle T , $\text{cost}^s(T)/\text{lp}^s(T)$ is bounded as follows.*

Type of T	Upper bound
+++	$\max(2 - \delta, 1 + 0.5/(1 - \delta)) \leq 1.9$
+- -	$1.5(1 + \delta) \leq 1.65$
- - -	1
++ -: bad	2
++ -: not bad	$2 - \gamma$
degenerate	1

4.2 Handling Bad Triangles

By Lemma 4.3, the only triangles whose ratio is greater than $2 - \gamma$ are bad triangles; $++-$ triangles with LP distances x, y, z such that $x, y \in [0.5 - \eta, 0.5 + \eta]$ and $z \in (1 - \eta, 1]$. (x, y are $+$ -edges and z is a $-$ -edge.) Each bad triangle has a unique *center*, which is the vertex incident on the two $+$ -edges. Let \mathcal{T} be the set of all non-degenerate triangles, and \mathcal{D} be the set of all degenerate triangles. In this subsection, given a parameter $\tau > 0$, we will define the charging function $h_\tau : \mathcal{T} \cup \mathcal{D} \rightarrow \mathbb{R}$ such that

- $h_\tau(T) = -\tau$ for every bad triangle T .
- $h_\tau(T) \leq +3\tau$ for *chargeable* T which will be defined soon.
- $h_\tau(T) = 0$ for all other triangles.
- $\sum_{T \in \mathcal{T} \cup \mathcal{D}} h_\tau(T) \geq 0$.

For any p , let $V_p = \{u : (u, p) \text{ is } + \text{ and } x_{pu} \in [0.5 - \eta, 0.5 + \eta]\}$. Every bad triangle (p, u, v) centered at p has $u, v \in V_p$. Consider a graph $G_p = (V_p, E_p)$ whose vertex set is V_p and (u, v) is an edge if and only if (p, u, v) is a bad triangle centered at p ; in particular, $x_{uv} > 1 - \eta$. We prove the following claim that if (p, u, v) and (p, v, w) are bad triangles, then (p, u, w) cannot be bad.

Claim 1. *Suppose that $(u, v), (v, w) \in E_p$. Then $x_{uw} \leq 0.5 + 5\eta$.*

Proof. Consider the local distribution on $\{p, u, v, w\}$ and let

- $q_0 = y_{p|uvw} + y_{p|u|vw} + y_{p|uv|w} + y_{p|uw|v} + y_{p|u|v|w}$ (i.e., the probability that p does not belong to the same cluster with any of u, v, w).
- $q_u = y_{pu|vw} + y_{pu|v|w}$ (i.e., the probability that p belongs to the same cluster with only u).
- $q_v = y_{pv|uw} + y_{pv|u|w}$ (i.e., the probability that p belongs to the same cluster with only v).
- $q_w = y_{pw|uv} + y_{pw|u|v}$ (i.e., the probability that p belongs to the same cluster with only w).
- $q_{uv} = y_{puv|w}$.
- $q_{uw} = y_{puw|v}$.
- $q_{vw} = y_{pvw|u}$.
- $q_{uvw} = y_{puvw}$.

Then we have

$$q_0 + q_u + q_v + q_w + q_{uv} + q_{uw} + q_{vw} + q_{uvw} = 1 \quad (7)$$

$$1 - x_{pu} = q_u + q_{uv} + q_{uw} + q_{uvw} \in [1/2 - \eta, 1/2 + \eta] \quad (8)$$

$$1 - x_{pv} = q_v + q_{uv} + q_{vw} + q_{uvw} \in [1/2 - \eta, 1/2 + \eta] \quad (9)$$

$$1 - x_{pw} = q_w + q_{uw} + q_{vw} + q_{uvw} \in [1/2 - \eta, 1/2 + \eta] \quad (10)$$

$$\eta \geq 1 - x_{uv} \geq q_{uv} + q_{uvw} \quad (11)$$

$$\eta \geq 1 - x_{vw} \geq q_{vw} + q_{uvw}. \quad (12)$$

By (8) and (11), $q_u + q_{uw} \geq 1/2 - 2\eta$. By (10) and (12), $q_w + q_{uw} \geq 1/2 - 2\eta$. Adding these two inequalities and (9) implies

$$q_u + q_w + 2q_{uw} + (q_v + q_{uv} + q_{vw} + q_{uvw}) \geq 3/2 - 5\eta.$$

Subtracting (7) from the above implies that $q_{uw} \geq 1/2 - 5\eta$. \square

As an example, note that if $x_{pu} = x_{pv} = x_{pw} = 0.5$ and $x_{uv} = x_{vw} = 1$, then both u and w belong to p 's cluster simultaneously if and only if v does not belong to it, which implies that $y_{uw} \geq y_{puw} = 0.5$.

Call (p, u, v) a *chargeable* non-degenerate triangle centered at p if $u, v \in V_p$ and $x_{uv} \leq 1/2 + 5\eta$. (This is irrespective of the sign of edge (u, v) .) Note that the definition $\eta = 1/12$ ensures $1/2 + 5\eta \leq 1 - \eta$, which means that no triangle can be both chargeable and bad. Also, call any $+$ edge (p, u) with $x_{pu} \in [1/2 - \eta, 1/2 + \eta]$ a chargeable degenerate triangle or chargeable edge. It is centered at both p and u . Using the fact that $(u, v), (v, w) \in E_p$ implies that (p, u, w) is a chargeable triangle, one can prove the following claim.

Claim 2. *For any p , the number of bad triangles centered at p is at most the number of chargeable triangles (non-degenerate and degenerate combined) centered at p .*

Proof. The number of bad triangles centered at p is $|E_p|$, the number of chargeable edges centered at p is $|V_p|$, and the number of chargeable non-degenerate triangles centered at p is the number of pairs $(u, w) \in \binom{V_p}{2}$ such that $u \neq w$ and $(u, v), (v, w) \in E_p$ for some $v \in V_p$ (which implies that $(u, w) \notin E_p$ since $1/2 + 5\eta \leq 1 - \eta$). Let F_p denote the set of such pairs. Note that E_p and F_p are disjoint.

Fix $u \in V_p$ and consider the BFS tree on G_p starting from u . With the root being at the zeroth level, the vertices v such that $(u, v) \in F_p$ are exactly the vertices at the second level of the BFS tree. Since there is no triangle in E_p , the number of vertices in the second level is at least $\max_{w \in N(u)} \deg(w) - 1$, where $N(u)$ denotes the neighbors of u in G_p . So,

$$|F_p| \geq \frac{1}{2} \sum_{u \in V_p} (\max_{w \in N(u)} \deg(w) - 1)$$

and the total number of chargeable triangles is

$$|F_p| + |V_p| \geq \frac{1}{2} \sum_{u \in V_p} \max_{w \in N(u)} \deg(w).$$

We finally prove that

$$\sum_{u \in V_p} \max_{w \in N(u)} \deg(w) \geq \sum_{u \in V_p} \deg(u) = 2|E_p|,$$

which finishes the proof the claim. Let $U = \{u \in V_p : \deg(u) > \max_{w \in N(u)} \deg(w)\}$. Note that U is an independent set in G_p .

We would like to show that there exists a matching between U and $V \setminus U$ saturating U . In order to see it, for any $U' \subseteq U$, let $V' = \cup_{u \in U'} N(u)$ and G' be the bipartite graph with vertex set $U' \cup V'$ and the edge set $E_p \cap (U' \times V')$. Let $\deg'(\cdot)$ denote the degree in G' , and note that for $(u, v) \in E'$, $\deg'(u) = \deg(u) > \deg(v) \geq \deg'(v)$ by construction. Without loss of generality, let $U' = \{u_1, \dots, u_k\}$ and $V' = \{v_1, \dots, v_\ell\}$ with $\deg'(u_1) \leq \dots \leq \deg'(u_k)$ and $\deg'(v_1) \leq \dots \leq \deg'(v_\ell)$. If $\ell < k$, since $|E'| = \sum_{i=1}^k \deg'(u_i) = \sum_{i=1}^\ell \deg'(v_i)$, there exists $t \in [\ell]$ such that $\deg'(u_t) \leq \deg'(v_t)$ and $\deg'(u_i) > \deg'(v_i)$ for $i = 1, \dots, t-1$. However, note that all edges from u_1, \dots, u_t go to v_1, \dots, v_{t-1} while $\sum_{i=1}^t \deg'(u_i) > \sum_{i=1}^{t-1} \deg'(v_i)$, which is contradiction. Therefore, $|V'| \geq |U'|$ for all U' , and by Hall's condition, there exists a matching between U and $V_p \setminus U$ saturating U .

Let $(u_1, v_1), \dots, (u_k, v_k)$ be such a matching where $|U| = k$. Let $V' = \{v_1, \dots, v_k\}$. Note that $\max_{w \in N(v)} \deg(w) \geq \deg(v)$ for every $v \notin U$. Therefore,

$$\sum_{u \in V_p} \max_{w \in N(u)} \deg(w) \geq \sum_{v \in V_p \setminus (U \cup V')} \deg(v) + \sum_{i=1}^k ((\max_{w \in N(u_i)} \deg(w)) + (\max_{w \in N(v_i)} \deg(w))) \geq \sum_{u \in V_p} \deg(u),$$

which finishes the proof. \square

So, around each center p , we can let $h_\tau(T) = -\tau$ for every bad triangle T and increase $h_\tau(T)$ by τ for every chargeable triangle T . Chargeable non-degenerate triangles are increased at most three times, and chargeable degenerate triangles are increased at most twice, so $h_\tau(T) \leq 3\tau$ for every T .

4.3 Incorporating Error from Correlated Rounding

Recall that $\text{cost}^r(\cdot), \text{lp}^r(\cdot)$ are defined with respect to actual rounding, and our goal is to bound the ratio ALG/LP where

$$n \cdot ALG = \sum_{u \in V} \sum_{(v,w) \in \binom{V}{2}} \text{cost}_u^r(v, w) = \sum_{u,v,w \in \binom{V}{3}} \text{cost}^r(w, u, v) + \sum_{u,v \in \binom{V}{2}} \text{cost}^r(u, v),$$

and

$$n \cdot LP = \sum_{u \in V} \sum_{(v,w) \in \binom{V}{2}} \text{lp}_u^r(v, w) = \sum_{u,v,w \in \binom{V}{3}} \text{lp}^r(w, u, v) + \sum_{u,v \in \binom{V}{2}} \text{lp}^r(u, v).$$

Call a (non-degenerate) triangle (a, b, c) *rounded with correlation* when one vertex was the pivot, both of the other vertices are rounded in a correlated manner. Let \mathcal{R} be the set of triangles rounded with correlation. Note that a triangle is in \mathcal{R} if and only if it has at least two medium $+$ -edges. We prove the following claim that relates ALG, LP defined using $\text{cost}^r, \text{lp}^r$ to $\text{cost}^s, \text{lp}^s$.

Claim 3.

$$\frac{ALG}{LP} \leq \frac{\sum_{u,v,w \in \binom{V}{3}} \text{cost}^s(w, u, v) + \sum_{T \in \mathcal{R}} O(\varepsilon_r) + \sum_{u,v \in \binom{V}{2}} \text{cost}^s(u, v)}{\sum_{u,v,w \in \binom{V}{3}} \text{lp}^s(w, u, v) - \sum_{T \in \mathcal{R}} O(\varepsilon_r) + \sum_{u,v \in \binom{V}{2}} \text{lp}^s(u, v)}.$$

Proof. Note that $\text{cost}^s(w, u, v) = \text{cost}^r(w, u, v)$ and $\text{lp}^s(w, u, v) = \text{lp}^r(w, u, v)$ if $(w, u, v) \notin \mathcal{R}$, so we only need to worry about triangles rounded with correlation.

Fix a pivot p and let $S' \subseteq I_p$ be the random set actually sampled by the algorithm. Then for any (p, u, v) with $u, v \in I_p$, changing from $\text{cost}_p^r(u, v)$ to $\text{cost}_p^s(u, v)$ increases the total ALG by at most $3|\Pr[u, v \in S' \mid p \text{ pivot}] - y_{puv}|$. Similarly, changing from $\text{lp}_p^r(u, v)$ to $\text{lp}_p^s(u, v)$ decreases the total LP by at most $3|\Pr[u, v \in S' \mid p \text{ pivot}] - y_{puv}|$.

Lemma 3.1 guarantees that

$$\sum_{u, v \in I_p} |\Pr[u, v \in S'] - y_{puv}| \leq |I_p|^2 \cdot \varepsilon_r,$$

and there are $\Omega(|I_p|^2)$ triangles of the form (p, u, v) with $u, v \in I_p$. Therefore,

$$\sum_{u, v \in I_p} \text{cost}_p^r(u, v) \leq \sum_{u, v \in I_p} (\text{cost}_p^s(u, v) + O(\varepsilon_r))$$

and

$$\sum_{u, v \in I_p} \text{lp}_p^r(u, v) \geq \sum_{u, v \in I_p} (\text{lp}_p^s(u, v) - O(\varepsilon_r)),$$

so converting $\text{cost}_p^r, \text{lp}_p^r$ to $\text{cost}_p^s, \text{lp}_p^s$ for all triangles $\{(p, u, v) : u, v \in I_p\}$ only increases the ALG/LP ratio if we increase $\text{cost}_p^s(u, v)$ and decrease $\text{lp}_p^s(u, v)$ for all these triangles by $O(\varepsilon_r)$. Do such a conversion from $\text{cost}^r, \text{lp}^r$ to $\text{cost}^s, \text{lp}^s$ for every pivot. \square

4.4 Finishing Off

We are finally ready to bound ALG/LP .

Lemma 4.4. $ALG/LP \leq 1.994 + O(\varepsilon_r)$.

Proof. By Claim 3, it suffices to bound

$$\frac{\sum_{u, v, w \in \binom{V}{3}} \text{cost}^s(w, u, v) + \sum_{T \in \mathcal{R}} O(\varepsilon_r) + \sum_{u, v \in \binom{V}{2}} \text{cost}^s(u, v)}{\sum_{u, v, w \in \binom{V}{3}} \text{lp}^s(w, u, v) - \sum_{T \in \mathcal{R}} O(\varepsilon_r) + \sum_{u, v \in \binom{V}{2}} \text{lp}^s(u, v)}.$$

Recall that the only triangles whose $\text{cost}^s/\text{lp}^s$ ratio is greater than $2 - \gamma$ are bad triangles (i.e., $++-$ triangle with LP value x, y, z such that $x, y \in [0.5 - \eta, 0.5 + \eta]$ and $z \in (1 - \eta, 1]$). Let $\tau > 0$ to be determined, and consider $h_\tau : \mathcal{T} \cup \mathcal{D} \rightarrow \mathbb{R}$ constructed in Section 4.2. Since $\sum_T h_\tau(T) \geq 0$, adding $h_\tau(T)$ to the numerator only increases the ratio. Let \mathcal{B} be the set of bad triangles, and \mathcal{C} be the set of chargeable triangles (non-degenerate and degenerate). Recall that $h_\tau(T) = -\tau$ for $T \in \mathcal{B}$, $h_\tau(T) \leq 3\tau$ for $T \in \mathcal{C}$ and 0 for other triangles. Then, the final ratio can be upper bounded by $(\sum_{T \in \mathcal{T} \cup \mathcal{D}} \text{cost}^f(T)) / (\sum_{T \in \mathcal{T} \cup \mathcal{D}} \text{lp}^f(T))$ where \mathbb{I} denotes the indicator function and

- $\text{cost}^f(T) := \text{cost}^s(T) + \mathbb{I}[T \in \mathcal{R}]O(\varepsilon_r) - \mathbb{I}[T \in \mathcal{B}]\tau + \mathbb{I}[T \in \mathcal{C}]3\tau$.
- $\text{lp}^f(T) := \text{lp}^s(T) - \mathbb{I}[T \in \mathcal{R}]O(\varepsilon_r)$.

Now we finish by analyzing each triangle individually. Recall that \mathcal{B} and \mathcal{C} are disjoint (as discussed in Section 4.2). We prove the upper bound and lower bounds for $\text{lp}^s(T)$ when T is bad, chargeable, or rounded with correlation.

Claim 4. *For any $T \in \mathcal{B} \cup \mathcal{C}$, $\text{lp}^s(T) \geq \zeta_l := 2 \cdot (1/2 - \eta)^2$. For any $T \in \mathcal{B}$, $\text{lp}^s(T) \leq \zeta_u := 2(1/2 + 2\eta)(1/2 + \eta) + \eta$. For any $T \in \mathcal{R}$, $\text{lp}^s(T) \geq 2\delta^2$.*

We also show the upper bound and lower bounds for $\text{lp}^s(T)$ when T is bad or chargeable, proving Claim 4.

Proof of Claim 4. For the first claim, consider $T \in \mathcal{B} \cup \mathcal{C}$ and assume T is non-degenerate. It means that $T = \{a, b, c\}$ with two +edges (a, b) and (a, c) with $x_{ab}, x_{ac} \in [1/2 - \eta, 1/2 + \eta]$. When b is the pivot, a belongs to b 's cluster with probability at least $1/2 - \eta$, and in that case, (a, c) , who was contributing at least $1/2 - \eta$ to LP, is removed from the graph. One can apply the same argument when c is the pivot to ensure that $\text{lp}^s(T) \geq 2 \cdot (1/2 - \eta)^2$. If $T = \{a, b\}$ is degenerate, $\text{lp}^s(T) \geq 2\text{lp}(a, b) = 2(1/2 - \eta)$.

For the second claim, let $T = \{a, b, c\}$ be a bad triangle with two +edges (a, b) and (a, c) with $x_{ab}, x_{ac} \in [1/2 - \eta, 1/2 + \eta]$ and $x_{bc} \in (1 - \eta, 1]$. Then when b is the pivot, the edge (a, c) will be removed when a or c belongs to the same cluster with b , which happens with probability at most $y_{ba} + y_{bc} \leq 1/2 + 2\eta$. The case for c is symmetric, so even assuming that (b, c) is always removed when a is the pivot, $\text{lp}^s(T) \leq 2 \cdot (1/2 + 2\eta)(1/2 + \eta) + \eta$.

For the third claim, consider $T \in \mathcal{R}$. It means that $T = \{a, b, c\}$ with two +edges (a, b) and (a, c) with $x_{ab}, x_{ac} \in [\delta, 1 - \delta]$. When b is the pivot, a belongs to b 's cluster with probability at least δ , and in that case, (a, c) , who was contributing at least δ to LP, is removed from the graph. One can apply the same argument when c is the pivot to ensure that $\text{lp}^s(T) \geq 2\delta^2$. \square

Note that $2\delta^2 = 0.02$ and with $\eta := 1/12$, we have $0.8612 \geq \zeta_u \geq \zeta_l \geq 0.3472$. We finally compute the ratio for each type of triangle. Note that whenever additive $O(\varepsilon_r)$ is applied, we make sure that the denominator $\text{lp}^s(T)$ is at least some absolute constant.

- For $T \in \mathcal{B}$: Since $\text{lp}^s(T) \geq \zeta_l$, $\text{cost}^f(T)/\text{lp}^f(T) = (\text{cost}^s(T) + O(\varepsilon_r) - \tau)/(\text{lp}^s(T) - O(\varepsilon_r)) \leq 2 + O(\varepsilon_r) - \tau/\zeta_u$.
- For $T \in \mathcal{C}$: Since $\text{lp}^s(T) \geq \zeta_l$ too, $\text{cost}^f(T)/\text{lp}^f(T) \leq 2 - \gamma + O(\varepsilon_r) + 3\tau/\zeta_l$.
- For $T \in \mathcal{R} \setminus (\mathcal{C} \cup \mathcal{B})$: Since $\text{lp}^s(T) \geq 2\delta^2$, so $\text{cost}^f(T)/\text{lp}^f(T) \leq 2 - \gamma + O(\varepsilon_r)$.
- For all other T : $\text{cost}^f(T)/\text{lp}^f(T) = \text{cost}^s(T)/\text{lp}^s(T) \leq 2 - \gamma$.

The maximum ratio is $\max(2 - \tau/\zeta_u, 2 - \gamma + 3\tau/\zeta_l) + O(\varepsilon_r)$. Setting τ such that

$$2 - \tau/0.8612 = 2 - \gamma + 3\tau/0.3472 \quad \Rightarrow \quad \tau = \frac{\gamma}{1/0.8612 + 3/0.3472} \approx 0.0055,$$

the final approximation ratio is $2 - \tau/0.8612 + O(\varepsilon_r) \leq 1.994 + O(\varepsilon_r)$. \square

5 Bounds for $\text{cost}^i(\cdot)/\text{lp}^i(\cdot)$

In this section, we bound $\text{cost}^i(\cdot)/\text{lp}^i(\cdot)$, proving Lemma 4.2. Throughout this section, we consider a fixed triangle T with vertex set $\{a, b, c\}$ and edge set (ab, ac, bc) . For each type of triangle, we compute the worst case ratio for $\text{cost}^i(T)/\text{lp}^i(T)$. For the sake of brevity, in this section, let $\text{cost}(\cdot) := \text{cost}^i(\cdot)$ and $\text{lp}(\cdot) := \text{lp}^i(\cdot)$. We assume $\frac{0}{0} = 0$.

To compute $\text{cost}(T)/\text{lp}(T)$, we have $\text{cost}(T) = \text{cost}_a(bc) + \text{cost}_b(ac) + \text{cost}_c(ab)$ and $\text{lp}(T) = \text{lp}_a(bc) + \text{lp}_b(ac) + \text{lp}_c(ab)$. We use $\text{cost}_a(bc)$ to denote the probability that edge bc is violated given that a is chosen as a pivot (when triangle abc is still intact). We use $\text{lp}_a(bc)$ to denote the probability that edge bc is decided (i.e., at least one of b or c is chosen to be in the cluster with the pivot a) times the contribution of edge bc to the LP objective function.

5.1 + + + Triangles

Lemma 5.1. *For a + + + triangle T with vertex set $\{a, b, c\}$ and edge set (ab, ac, bc) ,*

$$\frac{\text{cost}(T)}{\text{lp}(T)} = \frac{\text{cost}_a(bc) + \text{cost}_b(ac) + \text{cost}_c(ab)}{\text{lp}_a(bc) + \text{lp}_b(ac) + \text{lp}_c(ab)} \leq \frac{3}{2 + y_{abc} + y_{a|b|c}}.$$

Proof. We have

$$\begin{aligned} \text{cost}_a(bc) &= (y_{ab} - y_{abc}) + (y_{ac} - y_{abc}), \\ \text{lp}_a(bc) &= (1 - y_{bc})((y_{ab} - y_{abc}) + (y_{ac} - y_{abc}) + (y_{abc})), \\ \text{cost}_b(ac) &= (y_{ab} - y_{abc}) + (y_{bc} - y_{abc}), \\ \text{lp}_b(ac) &= (1 - y_{ac})((y_{ab} - y_{abc}) + (y_{bc} - y_{abc}) + (y_{abc})), \\ \text{cost}_c(ab) &= (y_{ac} - y_{abc}) + (y_{bc} - y_{abc}), \\ \text{lp}_c(ab) &= (1 - y_{ab})((y_{ac} - y_{abc}) + (y_{bc} - y_{abc}) + (y_{abc})). \end{aligned}$$

We can write the costs using the following shorthand notation.

$$x = y_{ab|c}, \quad y = y_{ac|b}, \quad z = y_{bc|a}, \quad p = y_{abc} \quad \text{and} \quad q = y_{a|b|c}.$$

We have the following relations:

$$y_{ab} = y_{ab|c} + y_{abc} = x + p, \quad y_{ac} = y_{ac|b} + y_{abc} = y + p, \quad y_{bc} = y_{bc|a} + y_{abc} = z + p.$$

Notice that $x + y + z + p + q = 1$. Then we have the following.

$$\begin{aligned} \text{cost}_a(bc) &= x + y, \\ \text{lp}_a(bc) &= (1 - z - p)(x + y + p) = (1 - z - p)(1 - z - q). \end{aligned}$$

The costs for edges ab and ac are analogous. Thus,

$$\begin{aligned} \frac{\text{cost}(T)}{\text{lp}(T)} &= \frac{2(x + y + z)}{(1 - x - p)(1 - x - q) + (1 - y - p)(1 - y - q) + (1 - z - p)(1 - z - q)} \\ &= \frac{2(1 - p - q)}{(1 + p + q)(1 - p - q) + 3pq + x^2 + y^2 + z^2}. \end{aligned} \tag{13}$$

For fixed p, q , the ratio in (13) is maximized when $x^2 + y^2 + z^2$ is minimized, which occurs when $x = y = z = (1 - p - q)/3$. Therefore, for fixed p and q , the ratio in (13) is at most

$$\begin{aligned}
\frac{2(1-p-q)}{(1+p+q)(1-p-q) + 3pq + \frac{(1-p-q)^2}{3}} &= \frac{2(1-p-q)}{(1+p+q + \frac{1-p-q}{3})(1-p-q) + 3pq} \\
&= \frac{2(1-p-q)}{(\frac{4}{3} + \frac{2p}{3} + \frac{2q}{3})(1-p-q) + 3pq} \\
&= \frac{3(1-p-q)}{(2+p+q)(1-p-q) + \frac{9}{2}pq} \\
&\leq \frac{3}{2+p+q}.
\end{aligned}$$

□

5.2 --- Triangles

Lemma 5.2. For a --- triangle T with vertex set $\{a, b, c\}$ and edge set (ab, ac, bc) ,

$$\frac{\text{cost}(T)}{lp(T)} = \frac{\text{cost}_a(bc) + \text{cost}_b(ac) + \text{cost}_c(ab)}{lp_a(bc) + lp_b(ac) + lp_c(ab)} \leq 1.$$

Proof. All edges are ---edges with costs as follows.

$$\begin{aligned}
\text{cost}_a(bc) &= (1 - \sqrt{x_{ab}})(1 - \sqrt{x_{ac}}), \\
lp_a(bc) &= (1 - x_{bc})(1 - \sqrt{x_{ab}\sqrt{x_{ac}}}), \\
\text{cost}_b(ac) &= (1 - \sqrt{x_{ab}})(1 - \sqrt{x_{bc}}), \\
lp_b(ac) &= (1 - x_{ac})(1 - \sqrt{x_{bc}\sqrt{x_{ab}}}), \\
\text{cost}_c(ab) &= (1 - \sqrt{x_{ac}})(1 - \sqrt{x_{bc}}), \\
lp_c(ab) &= (1 - x_{ab})(1 - \sqrt{x_{bc}\sqrt{x_{ac}}}).
\end{aligned}$$

So we have

$$\frac{\text{cost}(T)}{lp(T)} = \frac{(1 - \sqrt{x_{ab}})(1 - \sqrt{x_{ac}}) + (1 - \sqrt{x_{ab}})(1 - \sqrt{x_{bc}}) + (1 - \sqrt{x_{ac}})(1 - \sqrt{x_{bc}})}{(1 - x_{bc})(1 - \sqrt{x_{ab}\sqrt{x_{ac}}}) + (1 - x_{ac})(1 - \sqrt{x_{bc}\sqrt{x_{ab}}}) + (1 - x_{ab})(1 - \sqrt{x_{bc}\sqrt{x_{ac}}})}.$$

For ease of notation, let $X = x_{ab}, Y = x_{ac}$ and $Z = x_{bc}$. Then we have

$$\frac{\text{cost}(T)}{lp(T)} = \frac{(1 - \sqrt{X})(1 - \sqrt{Y}) + (1 - \sqrt{X})(1 - \sqrt{Z}) + (1 - \sqrt{Y})(1 - \sqrt{Z})}{(1 - Z)(1 - \sqrt{X}\sqrt{Y}) + (1 - Y)(1 - \sqrt{Z}\sqrt{X}) + (1 - X)(1 - \sqrt{Z}\sqrt{Y})}. \quad (14)$$

We will show that the expression in (14) is always at most 1 by showing that the denominator is always at least as large as the numerator for any $X, Y, Z \in [0, 1]$. This is equivalent to the following inequality.

$$X + Y + Z + 2\sqrt{XY} + 2\sqrt{XZ} + 2\sqrt{YZ} \leq 2\sqrt{X} + 2\sqrt{Y} + 2\sqrt{Z} + Z\sqrt{XY} + Y\sqrt{XZ} + X\sqrt{YZ},$$

which is in turn equivalent to the following inequality.

$$(\sqrt{X} + \sqrt{Y} + \sqrt{Z})(\sqrt{X} + \sqrt{Y} + \sqrt{Z}) \leq (\sqrt{X} + \sqrt{Y} + \sqrt{Z})(2 + \sqrt{XYZ}).$$

Now it remains to prove that when $X, Y, Z \in [0, 1]$, the following inequality holds.

$$\sqrt{X} + \sqrt{Y} + \sqrt{Z} \leq 2 + \sqrt{XYZ}.$$

This is true if the following inequality holds for all $A, B, C \in [0, 1]$.

$$A + B + C \leq 2 + ABC.$$

To see that this last inequality is true, set $A = 1 - \alpha, B = 1 - \beta$ and $C = 1 - \gamma$ for $\alpha, \beta, \gamma \in [0, 1]$. Then we have

$$\begin{aligned} A + B + C &= 3 - \alpha - \beta - \gamma \leq 2 + (1 - \alpha)(1 - \beta)(1 - \gamma) \iff \\ 1 - \alpha - \beta - \gamma &\leq (1 - \alpha)(1 - \beta)(1 - \gamma) \iff \\ \alpha\beta\gamma &\leq \alpha\beta + \alpha\gamma + \beta\gamma. \end{aligned}$$

The last inequality is clearly true for $\alpha, \beta, \gamma \in [0, 1]$. □

5.3 + - - Triangles

Lemma 5.3. For a + - - triangle T with vertex set $\{a, b, c\}$ and edge set (ab, ac, bc) ,

$$\frac{\text{cost}(T)}{\text{lp}(T)} = \frac{\text{cost}_a(bc) + \text{cost}_b(ac) + \text{cost}_c(ab)}{\text{lp}_a(bc) + \text{lp}_b(ac) + \text{lp}_c(ab)} \leq 1.5.$$

Proof. Since ab is a +edge, we have

$$\begin{aligned} \text{cost}_c(ab) &= \sqrt{x_{ac}}(1 - \sqrt{x_{bc}}) + \sqrt{x_{bc}}(1 - \sqrt{x_{ac}}), \\ \text{lp}_c(ab) &= x_{ab}(1 - \sqrt{x_{ac}\sqrt{x_{bc}}}). \end{aligned}$$

For -edges ac and bc , we have

$$\begin{aligned} \text{cost}_b(ac) &= (1 - x_{ab})(1 - \sqrt{x_{bc}}), \\ \text{lp}_b(ac) &= (1 - x_{ac})(1 - x_{ab}\sqrt{x_{bc}}), \end{aligned}$$

$$\begin{aligned} \text{cost}_a(bc) &= (1 - x_{ab})(1 - \sqrt{x_{ac}}), \\ \text{lp}_a(bc) &= (1 - x_{bc})(1 - x_{ab}\sqrt{x_{ac}}). \end{aligned}$$

Then

$$\frac{\text{cost}(T)}{\text{lp}(T)} = \frac{\sqrt{x_{ac}}(1 - \sqrt{x_{bc}}) + \sqrt{x_{bc}}(1 - \sqrt{x_{ac}}) + (1 - x_{ab})(2 - \sqrt{x_{bc}} - \sqrt{x_{ac}})}{x_{ab}(1 - \sqrt{x_{ac}\sqrt{x_{bc}}}) + (1 - x_{ac})(1 - x_{ab}\sqrt{x_{bc}}) + (1 - x_{bc})(1 - x_{ab}\sqrt{x_{ac}})}$$

For ease of notation, let $X = x_{ab}, Y = x_{ac}$ and $Z = x_{bc}$. Notice that we have triangle inequality on these values (i.e., $X + Y \geq Z, X + Z \geq Y$ and $Y + Z \geq X$). Without loss of generality, we assume $Y \leq Z$. Then

$$\begin{aligned}
\frac{\text{cost}(T)}{lp(T)} &= \frac{\sqrt{Y}(1 - \sqrt{Z}) + \sqrt{Z}(1 - \sqrt{Y}) + (1 - X)(2 - \sqrt{Z} - \sqrt{Y})}{X(1 - \sqrt{Y}\sqrt{Z}) + (1 - Y)(1 - X\sqrt{Z}) + (1 - Z)(1 - X\sqrt{Y})} \\
&= \frac{2 - 2\sqrt{YZ} - X(2 - \sqrt{Z} - \sqrt{Y})}{2 - Y - Z + X(Y\sqrt{Z} + Z\sqrt{Y} - \sqrt{YZ} - \sqrt{Z} - \sqrt{Y})} \\
&= \frac{2 - 2\sqrt{YZ} + X(\sqrt{Z} + \sqrt{Y} - 2)}{2 - Y - Z + X(\sqrt{Y} + \sqrt{Z} - 1)(\sqrt{YZ} - 1)}. \tag{15}
\end{aligned}$$

First observe that if $Y = Z = 1$, then the ratio is $0/0$. Thus, we assume that Y and Z are not both equal to 1. We consider two cases: i) $\sqrt{Y} + \sqrt{Z} \leq 1$, and ii) $\sqrt{Y} + \sqrt{Z} > 1$. In case i) we will show that ratio is at most 1.

Claim 5. *If $\sqrt{Y} + \sqrt{Z} \leq 1$, then*

$$\frac{2 - 2\sqrt{YZ} + X(\sqrt{Z} + \sqrt{Y} - 2)}{2 - Y - Z + X(\sqrt{Y} + \sqrt{Z} - 1)(\sqrt{YZ} - 1)} \leq 1.$$

Proof. The claim is equivalent to showing the following.

$$2 - 2\sqrt{YZ} + X(\sqrt{Z} + \sqrt{Y} - 2) \leq 2 - Y - Z + X(\sqrt{Y} + \sqrt{Z} - 1)(\sqrt{YZ} - 1).$$

We can rewrite this as

$$Y + Z \leq 2\sqrt{YZ} + X + X(\sqrt{Y} + \sqrt{Z} - 1)(\sqrt{YZ} - 2).$$

Notice that $X(\sqrt{Y} + \sqrt{Z} - 1)(\sqrt{YZ} - 2) \geq 0$, since both of the last two terms are nonpositive. Thus, it suffices to show

$$Y + Z \leq 2\sqrt{YZ} + X.$$

Since we assume that $Z \geq Y$, we have

$$Y + Z = 2Y + Z - Y \leq 2\sqrt{YZ} + Z - Y \leq 2\sqrt{YZ} + X.$$

◇

Now let us now consider case ii) where $\sqrt{Y} + \sqrt{Z} > 1$.

Claim 6. *For all $X, Y, Z \in [0, 1]$ with X, Y, Z obeying triangle inequality, the following ratio*

$$\frac{2 - 2\sqrt{YZ} - X + X(\sqrt{Y} + \sqrt{Z} - 1)}{2 - Y - Z + X(\sqrt{Y} + \sqrt{Z} - 1)(\sqrt{YZ} - 1)}$$

attains its maximum value when $Z = \min\{1, X + Y\}$.

Proof. Consider $X, Y, Z \in [0, 1]$ such that $Z < X + Y \leq 1$. Then we show that we can increase Z and decrease Y without decreasing the ratio. For $X, Y, Z \in [0, 1]$, let $c = \sqrt{Y} + \sqrt{Z}$. Notice that $c \in (1, 2)$. We can rewrite the ratio in the claim as

$$\frac{2 - 2(c\sqrt{Y} - Y) + X(c - 2)}{2 - c^2 + 2(c\sqrt{Y} - Y) + X(c - 1)((c\sqrt{Y} - Y) - 1)}.$$

The numerator is maximized and the denominator is minimized when Y is minimized. Thus, we can decrease Y to Y' and increase $Z = (c - \sqrt{Y})^2$ to $Z' = (c - \sqrt{Y'})^2$ until $Z' = X + Y'$ or $Z' = 1$. \diamond

Claim 7. *Assuming $Z = 1$, the maximum value of the ratio in (15) for $X, Y \in [0, 1]$ and $X + Y \geq 1$ is 1.1184.*

Proof. By Claim 6, we can set $Z = 1$. Then (15) becomes

$$\frac{2 - X}{\sqrt{Y}(1 - X) + 1}.$$

Since both numerator and denominator are always nonnegative for $X, Y \in [0, 1]$, the ratio is maximized when Y is minimized, which occurs when $Y = 1 - X$ (since $X + Y \geq Z = 1$).

Now if $Y = 1 - X$, then we have:

$$f(X) = \frac{2 - X}{\sqrt{1 - X}(1 - X) + 1}. \quad (16)$$

Taking the derivative of this, we obtain

$$f'(X) = 0 \iff \sqrt{1 - X}(4 - X) - 2 = 0.$$

This last equation is satisfied when $X = .64470$ and the value of (16) for this value of X is at most 1.1184. \diamond

So if $X + Y \geq 1$, then the lemma holds. It remains to consider the case in which $Z = X + Y < 1$. Recall that $\sqrt{Y} + \sqrt{Z} = \sqrt{Y} + \sqrt{X + Y} > 1$ also holds. In this case, observe that the ratio in (15) is at most

$$\frac{2 - 2\sqrt{Y(X + Y)} + X(\sqrt{Y} - 1)}{2 - 2Y - X + XY\sqrt{X + Y} - X\sqrt{Y}}. \quad (17)$$

Claim 8. *For $X + Y < 1$, $\sqrt{Y} + \sqrt{X + Y} > 1$ and $X, Y \in (0, 1]$, the maximum value of the ratio (17) is 1.5.*

Proof. For each $X \in (0, 1]$, we define the following functions.

$$\begin{aligned} U_X(Y) &:= 2 - 2\sqrt{Y(X + Y)} + X(\sqrt{Y} - 1), \\ V_X(Y) &:= 2 - 2Y - X + XY\sqrt{X + Y} - X\sqrt{Y}. \end{aligned}$$

To show that $U_X(Y)/V_X(Y) \leq 3/2$, we will show that the function $F_X(Y) := (3/2)V_X(Y) - U_X(Y)$ is decreasing on the relevant domain of Y . Then we can evaluate $F_X(Y)$ for $Y = .1$; it is sufficient to show that $F_X(.1) \geq 0$.

To show that $F_X(Y)$ is a decreasing function on the relevant interval, we argue that $F'_X(Y) := (3/2)V'_X(Y) - U'_X(Y) < 0$. We have

$$U'_X(Y) := \frac{\partial U_X}{\partial Y} = \frac{X}{2\sqrt{Y}} - \frac{X+2Y}{\sqrt{Y(X+Y)}},$$

$$V'_X(Y) := \frac{\partial V_X}{\partial Y} = X\sqrt{X+Y} - \frac{X}{2\sqrt{Y}} + \frac{XY}{2\sqrt{X+Y}} - 2,$$

Thus, we have

$$F'_X(Y) = \frac{3}{2} \left(X\sqrt{X+Y} - \frac{X}{2\sqrt{Y}} + \frac{XY}{2\sqrt{X+Y}} - 2 \right) - \frac{X}{2\sqrt{Y}} + \frac{X+2Y}{\sqrt{Y(X+Y)}},$$

and we want to show $F_X(Y) < 0$ for $Y \in (0, .1]$. This is equivalent to showing

$$X \left(\frac{3}{2}\sqrt{X+Y} - \frac{5}{4}\frac{1}{\sqrt{Y}} + \frac{3}{4}\frac{Y}{\sqrt{X+Y}} \right) + \frac{X+2Y}{\sqrt{Y(X+Y)}} < 3,$$

which is equivalent to showing

$$X \left(\frac{6(X+Y)\sqrt{Y}}{4\sqrt{Y(X+Y)}} - \frac{5\sqrt{X+Y}}{4\sqrt{Y(X+Y)}} + \frac{3Y^{3/2}}{4\sqrt{Y(X+Y)}} + \frac{4}{4\sqrt{Y(X+Y)}} \right) + \frac{2Y}{\sqrt{Y(X+Y)}} < 3.$$

Using the facts that $X+Y < 1$ and $-\sqrt{X+Y} < \sqrt{Y} - 1$, we have

$$\frac{X(6(X+Y)\sqrt{Y} - 5\sqrt{X+Y} + 3Y^{3/2} + 4) + 8Y}{4\sqrt{Y(X+Y)}} < \frac{X(6\sqrt{Y} + 5(\sqrt{Y} - 1) + 3Y^{3/2} + 4) + 8Y}{4\sqrt{Y(X+Y)}}.$$

Now, it suffices to show

$$\frac{X(6\sqrt{Y} + 5(\sqrt{Y} - 1) + 3Y^{3/2} + 4) + 8Y}{4\sqrt{Y(X+Y)}} < 3,$$

which is equivalent to showing

$$X(11\sqrt{Y} - 1 + 3Y^{3/2}) + 8Y < 12\sqrt{Y(X+Y)}.$$

Equivalently, we want to show for all $X, Y \in [0, 1]$,

$$H(X, Y) := X(11 + 3Y) - \frac{X}{\sqrt{Y}} + 8\sqrt{Y} - 12\sqrt{X+Y} < 0.$$

In fact, we will show that for each fixed $Y \in [0, 1]$, the function $H_Y(X) := H(X, Y)$ is convex for $X \in [0, 1]$. Thus, we need to check if $H(X, Y) < 0$ only for the extreme values of X , which are $X = 0$ and $X = 1 - Y$. In these cases, we have $H(0, Y) = 8\sqrt{Y} - 12\sqrt{Y} < 0$, and

$$H(1 - Y, Y) = (1 - Y)(11 + 3Y) - \frac{1 - Y}{\sqrt{Y}} + 8\sqrt{Y} - 12 = -8Y - 3Y^2 - \frac{1}{\sqrt{Y}} + 9\sqrt{Y} - 1.$$

It can be verified that this quantity is always negative for $Y \in [0, 1]$.

Now we show that for each fixed $Y \in [0, 1]$, the function $H_Y(X) := H(Y, X)$ is convex. We take the derivative with respect to X , which is

$$\frac{\partial}{\partial X} \left(X(11 + 3Y) - \frac{X}{\sqrt{Y}} + 8\sqrt{Y} - 12\sqrt{X+Y} \right) = -\frac{6}{\sqrt{X+Y}} + 3Y - \frac{1}{\sqrt{Y}} + 11$$

and the second derivative which is

$$\frac{\partial^2}{\partial^2 X} \left(X(11 + 3Y) - \frac{X}{\sqrt{Y}} + 8\sqrt{Y} - 12\sqrt{X+Y} \right) = \frac{3}{(X+Y)^{3/2}}.$$

Thus, since the second derivative is positive for all $Y, X \in [0, 1]$, the function is thus convex with respect to X . \diamond

\square

5.4 $++-$ Triangles

Lemma 5.4. *For a $++-$ triangle T with vertex set $\{a, b, c\}$ and edge set (ab, ac, bc) ,*

$$\frac{\text{cost}(T)}{\text{lp}(T)} = \frac{\text{cost}_a(bc) + \text{cost}_b(ac) + \text{cost}_c(ab)}{\text{lp}_a(bc) + \text{lp}_b(ac) + \text{lp}_c(ab)} \leq 2.$$

Proof. Edge bc is the $-$ edge, so we have

$$\begin{aligned} \text{cost}_a(bc) &= y_{abc}, \\ \text{lp}_a(bc) &= y_{bc}(y_{ab} - y_{abc} + y_{ac}). \end{aligned}$$

Since ab and ac are both $+$ edges, we have

$$\begin{aligned} \text{cost}_b(ac) &= (1 - x_{ab})\sqrt{x_{bc}} + (1 - \sqrt{x_{bc}})x_{ab}, \\ \text{lp}_b(ac) &= x_{ac}(1 - x_{ab}\sqrt{x_{bc}}), \end{aligned}$$

$$\begin{aligned} \text{cost}_c(ab) &= (1 - x_{ac})\sqrt{x_{bc}} + (1 - \sqrt{x_{bc}})x_{ac}, \\ \text{lp}_c(ab) &= x_{ab}(1 - x_{ac}\sqrt{x_{bc}}). \end{aligned}$$

We use the following for ease of notation.

$$X = y_{ab} = x + p, \quad Y = y_{ac} = y + p, \quad Z = y_{bc} = z + p, \quad A = 1 - X, \quad B = 1 - Y, \quad C = 1 - Z.$$

$$\begin{aligned}
\frac{\text{cost}(T)}{lp(T)} &= \frac{y_{abc} + (1 - x_{ab})\sqrt{x_{bc}} + (1 - \sqrt{x_{bc}})x_{ab} + (1 - x_{ac})\sqrt{x_{bc}} + (1 - \sqrt{x_{bc}})x_{ac}}{y_{bc}(y_{ab} - y_{abc} + y_{ac}) + x_{ac}(1 - x_{ab}\sqrt{x_{bc}}) + x_{ab}(1 - x_{ac}\sqrt{x_{bc}})} \\
&= \frac{y_{abc} + y_{ab}\sqrt{1 - y_{bc}} + (1 - \sqrt{1 - y_{bc}})(1 - y_{ab}) + y_{ac}\sqrt{1 - y_{bc}} + (1 - \sqrt{1 - y_{bc}})(1 - y_{ac})}{y_{bc}(y_{ab} - y_{abc} + y_{ac}) + (1 - y_{ac})(1 - (1 - y_{ab})\sqrt{1 - y_{bc}}) + (1 - y_{ab})(1 - (1 - y_{ac})\sqrt{1 - y_{bc}})} \\
&= \frac{p + X\sqrt{1 - Z} + (1 - \sqrt{1 - Z})(1 - X) + Y\sqrt{1 - Z} + (1 - \sqrt{1 - Z})(1 - Y)}{Z(X - p + Y) + (1 - Y)(1 - (1 - X)\sqrt{1 - Z}) + (1 - X)(1 - (1 - Y)\sqrt{1 - Z})} \\
&= \frac{p + X\sqrt{C} + (1 - \sqrt{C})A + Y\sqrt{C} + (1 - \sqrt{C})B}{Z(X - p + Y) + B(1 - A\sqrt{C}) + A(1 - B\sqrt{C})} \\
&= \frac{p + \sqrt{C}(X + Y - (1 - X) - (1 - Y)) + A + B}{Z(X - p + Y) + A + B - 2AB\sqrt{C}} \\
&= \frac{p + \sqrt{C}(2X + 2Y - 2) + A + B}{Z(X - p + Y) + A + B - 2AB\sqrt{C}}. \tag{18}
\end{aligned}$$

Claim 9. *The ratio in (18) is maximized when $A = B$ (which implies $X = Y$ and $x = y$).*

Proof. Fix $W = X + Y$. Then $A + B = 2 - W$. Then the ratio in (18) is equal to

$$\frac{p + \sqrt{1 - z - p}(2W - 2) + 2 - W}{(z + p)(W - p) + 2 - W - 2AB\sqrt{1 - z - p}}.$$

This ratio is maximized when the denominator is minimized, which occurs when the term $2AB\sqrt{1 - z - p}$ is maximized. For fixed $A + B = 2 - W$, this occurs when $A = B = \frac{2 - W}{2}$. \diamond

Then we have

$$\begin{aligned}
\frac{\text{cost}(T)}{lp(T)} &\leq \frac{p + \sqrt{1 - z - p}(2W - 2) + 2 - W}{(p + z)(W - p) + 2 - W - \frac{(2 - W)(2 - W)\sqrt{1 - z - p}}{2}} \\
&\leq \frac{p + \sqrt{1 - z - p}(2W - 2) + 2 - W}{(p + z)(W - p) + \sqrt{1 - z - p}(2W - 2 - \frac{W^2}{2}) + 2 - W}. \tag{19}
\end{aligned}$$

Notice that $W = x + y + 2p$, $Z = z + p$ and $x + y + z + p \leq 1$. Let $w = x + y$. So $w + z + p \leq 1$.

Claim 10. (19) *is maximized when $z = 0$. In other words, we have*

$$\frac{p + \sqrt{1 - z - p}(2W - 2) + 2 - W}{(z + p)(W - p) + 2 - W - \frac{(2 - W)(2 - W)\sqrt{1 - z - p}}{2}} \leq \frac{p + \sqrt{1 - p}(2W - 2) + 2 - W}{p(W - p) + 2 - W - \frac{(2 - W)(2 - W)\sqrt{1 - p}}{2}} \tag{20}$$

Proof. When $W = 0$, then we have $w = p = 0$. In this case, both the numerator and the denominator are zero. So we can assume that $W > 0$.

Fix p, w . Then $z \in [0, 1 - p - 2w]$. First consider the case in which $W \geq 1$. Then as z increases, the numerator decreases and the denominator increases, so the ratio is maximized when $z = 0$.

Next consider the case in which $0 < W < 1$. We want to show that

$$\frac{2 - W + p + \sqrt{1 - z - p}(2W - 2)}{2 - W + p(W - p) + z(W - p) + \sqrt{1 - z - p}(2W - 2 - \frac{W^2}{2})} \leq \frac{2 - W + p + \sqrt{1 - p}(2W - 2)}{2 - W + p(W - p) + \sqrt{1 - p}(2W - 2 - \frac{W^2}{2})}.$$

Let $F = 2 - W + p$ and let $G = 2 - W + p(W - p)$. Notice that $G < F$. Also, let $H = 2 - 2W$ and $I = 2 - 2W + W^2/2$. Notice that $I > H > 0$ $F > G > 0$.

$$\frac{F + \sqrt{1 - z - p}(-H)}{G + z(W - p) + \sqrt{1 - z - p}(-I)} \leq \frac{F + \sqrt{1 - p}(-H)}{G + \sqrt{1 - p}(-I)}.$$

So this inequality holds iff

$$FG + F\sqrt{1 - p}(-I) + G\sqrt{1 - z - p}(-H) + \sqrt{1 - z - p}\sqrt{1 - p}(-H)(-I) \leq FG + G\sqrt{1 - p}(-H) + F\sqrt{1 - z - p}(-I) + \sqrt{1 - z - p}\sqrt{1 - p}(-H)(-I) + z(W - p)(F + \sqrt{1 - p}(-H)),$$

which holds iff

$$F\sqrt{1 - p}(-I) + G\sqrt{1 - z - p}(-H) \leq G\sqrt{1 - p}(-H) + F\sqrt{1 - z - p}(-I) + z(W - p)(F + \sqrt{1 - p}(-H)).$$

We prove this in two steps. The second will be to show that $z(W - p)(F + \sqrt{1 - p}(-H)) > 0$. The first will be to show that

$$F\sqrt{1 - p}(-I) + G\sqrt{1 - z - p}(-H) \leq G\sqrt{1 - p}(-H) + F\sqrt{1 - z - p}(-I).$$

This holds iff

$$\begin{aligned} G\sqrt{1 - z - p}(-H) - G\sqrt{1 - p}(-H) &\leq F\sqrt{1 - z - p}(-I) - F\sqrt{1 - p}(-I) \iff \\ G\sqrt{1 - p}(H) - G\sqrt{1 - z - p}(H) &\leq F\sqrt{1 - p}(I) - F\sqrt{1 - z - p}(I) \iff \\ GH(\sqrt{1 - p} - \sqrt{1 - z - p}) &\leq FI(\sqrt{1 - p} - \sqrt{1 - z - p}), \end{aligned}$$

which holds because $G < F$ and $H < I$. Now we need to show

$$z(W - p)(F + \sqrt{1 - p}(-H)) > 0,$$

which holds iff

$$2 - W + p + \sqrt{1 - p}(2W - 2) \geq 0.$$

Since $2W - 2 < 0$, we have

$$2 - W + p + \sqrt{1 - p}(2W - 2) \geq 2 - W + p + (2W - 2) = W + p \geq 0.$$

Thus, we conclude that we can set $z = 0$ to maximize the ratio. \diamond

Claim 11. For each $p \in [0, 1]$, the righthandside of (20) is maximized either when $w = 1 - p$ or when $w = 0$.

Proof. For fixed p , we have the following function of w .

$$\begin{aligned} f_p(w) &= \frac{\sqrt{1-p}(2w+4p-2)+2-w-p}{p(w+p)+2-w-2p-\frac{(2-w-2p)(2-w-2p)\sqrt{1-p}}{2}} \\ &= \frac{\sqrt{1-p}(4p-2)+2-p+w(2\sqrt{1-p}-1)}{2-2p+p^2-\sqrt{1-p}(2-4p+2p^2)-\frac{\sqrt{1-p}}{2}(-4w+w^2+4pw)+w(p-1)}. \end{aligned}$$

We want to show that $f_p(w) \leq \alpha$ for $p, w \geq 0$ and $p+w \leq 1$. Let $f_p = g_p/h_p$. Then we want to show that $g_p \leq \alpha \cdot h_p$ for $p \in [0, 1]$ and $w \in [0, 1-p]$. Thus, we want to evaluate if the function

$$F_p(w) := \alpha \cdot h_p(w) - g_p(w) \geq 0.$$

Notice that

$$F'_p(w) = \alpha \left(-\sqrt{1-p}(-2+w+2p) + p - 1 \right) - 2\sqrt{1-p} - 1,$$

and

$$F''_p(w) = \alpha \left(-\sqrt{1-p} \right).$$

We conclude that F_p is concave and therefore to find the minimum values of $F_p(w)$ for $w \in [0, 1-p]$, we need to evaluate the endpoints on the interval $w \in [0, 1-p]$. \diamond

Claim 12. When $w = 0$

$$\frac{p + \sqrt{1-p}(2W-2) + 2 - W}{p(W-p) + 2 - W - \frac{(2-W)(2-W)\sqrt{1-p}}{2}} \leq 1.76.$$

Proof. When $w = 0$, then $W = w + 2p = 2p$. So we have

$$\frac{p + \sqrt{1-p}(2W-2) + 2 - W}{p(W-p) + 2 - W - \frac{(2-W)(2-W)\sqrt{1-p}}{2}} = \frac{\sqrt{1-p}(4p-2) + 2 - p}{p^2 + 2 - 2p - \frac{(2-2p)(2-2p)\sqrt{1-p}}{2}}.$$

This function of p is maximized when $p = .71415$ and the ratio is at most 1.7538. \diamond

Claim 13. When $w = 1 - p$, we have

$$\frac{p + \sqrt{1-p}(2W-2) + 2 - W}{p(W-p) + 2 - W - \frac{(2-W)(2-W)\sqrt{1-p}}{2}} \leq \frac{2p\sqrt{1-p} + 1}{1 - \frac{(1-p)(1-p)\sqrt{1-p}}{2}} \leq 2.$$

Proof. We want to show that for $p \in [0, 1]$,

$$f(p) = \frac{2p\sqrt{1-p} + 1}{1 - \frac{(1-p)(1-p)\sqrt{1-p}}{2}} \leq 2. \quad (21)$$

Let

$$F(p) = 2 - (1-p)(1-p)\sqrt{1-p} - (2p\sqrt{1-p} + 1) = 1 - (1-p)^{2.5} - 2p\sqrt{1-p}.$$

Then we want to show that $F(p) \geq 0$ for $p \in [0, 1]$.

$$F'(p) = \frac{2.5(1-p)^2 + 3p - 2}{\sqrt{1-p}}.$$

It can be seen that $F'(p) \geq 0$ for $p \in [0, 1]$. Thus, we can conclude that F is an increasing function and we only need to check that $F(0) \geq 0$. Indeed, we have $F(0) = 0$. \diamond

\square

5.4.1 Ratio for $++-$ triangles that are not bad

Recall that a $++-$ triangle is *bad* if the $+$ -edges have distances in $[1/2 - \eta, 1/2 + \eta]$ and the $-$ -edge has distance in $[1 - \eta, 1]$. Thus, there are two cases in which a $++-$ is *not bad*. Either i) at least one $+$ -edge, say ab , has $x_{ab} \in [0, 1/2 - \eta]$ or $x_{ab} \in [1/2 + \eta, 1]$, or ii) the $-$ -edge, say bc , has $x_{bc} \in [0, 1 - \eta]$.

Lemma 5.5. *For a $++-$ triangle T that is not bad with vertex set $\{a, b, c\}$ and edge set (ab, ac, bc) ,*

$$\frac{\text{cost}(T)}{lp(T)} = \frac{\text{cost}_a(bc) + \text{cost}_b(ac) + \text{cost}_c(ab)}{lp_a(bc) + lp_b(ac) + lp_c(ab)} \leq 1.946.$$

Proof. We first consider the case in which $x_{ab} \in [0, 1/2 - \eta]$ or $x_{ab} \in [1/2 + \eta, 1]$. In other words, $y_{ab} \leq 1/2 - \eta$ or $y_{ab} \geq 1/2 + \eta$. We have the same ratio as in (19) with one modification. Let us assume that $y_{ab} \leq 1/2 - \eta$. Then the maximum value of AB is

$$AB \leq \left(\frac{2-W}{2} - \eta \right) \left(\frac{2-W}{2} + \eta \right).$$

Then we have

$$\frac{\text{cost}(T)}{lp(T)} \leq \frac{p + \sqrt{1-p}(2W-2) + 2-W}{p(W-p) + 2-W - \frac{(2-W-2\eta)(2-W+2\eta)\sqrt{1-p}}{2}}.$$

This does not change much from the earlier analysis: We want to show that this ratio is at most 1.946, so we only need to check the case in which $w = 1 - p$. In this case, $W = 1 + p$. The ratio is at most

$$\frac{\sqrt{1-p}(2p) + 1}{1 - \frac{(1-p-2\eta)(1-p+2\eta)\sqrt{1-p}}{2}}.$$

To show that this ratio is at most 1.946, we can show, as before, that it suffices to check the condition when $p = 0$. When $p = 0$, we have

$$\frac{2}{2 - (1 - 2\eta)(1 + 2\eta)} = \frac{2}{1 + 4\eta^2}.$$

When $\eta = 1/12$, this ratio is at most $1.9459 \leq 1.946$.

The second case is when the $-$ edge bc has $x_{bc} \in [0, \eta]$. In this case, $x_{bc} \leq 1 - \eta$. Therefore, $1 - x_{bc} = y_{bc} = z + p \geq \eta$. Because, as we have seen in Claim 10, the ratio is maximized when $z = 0$, we just need to compute the ratio (21) for $p = \eta$. Recall, the ratio is at most

$$f(p) = \frac{2p\sqrt{1-p} + 1}{1 - \frac{(1-p)(1-p)\sqrt{1-p}}{2}}.$$

So $f(1/12) = 1.9399 \leq 1.946$. □

5.5 Degenerate Triangles

Let $\{u, v\}$ be a degenerate triangle.

Lemma 5.6. $cost(u, v)/lp(u, v) \leq 1$.

Proof. When (u, v) is $+$, $lp(u, v) = cost(u, v) = 2x_{uv}$, so the ratio is 1. When (u, v) is $-$, $lp(u, v) = 2y_{uv}$ always. $cost(u, v) = 2(1 - \sqrt{x_{uv}}) \leq 2(1 - x_{uv}) = 2y_{uv}$, so the ratio is at most 1. □

6 Bounds for $cost^s(\cdot)/lp^s(\cdot)$

In this section we bound $cost^s(\cdot)/lp^s(\cdot)$, proving Lemma 4.3. We do the case analyses for different types of triangles.

6.1 $+++$ Triangles.

Suppose that the vertices are (a, b, c) and the LP values are (x, y, z) , where $x = x_{bc}$, $y = x_{ac}$, and $z = x_{ab}$. For the sake of brevity, for the rest of the proof, we let $cost := cost^s(a, b, c)$ and $lp := lp^s(a, b, c)$. We do the further case analyses depending on how many edges are short.

6.1.1 3 short edges

Note that all edges are rounded independently for every pivot.

$$cost = \frac{y^2}{\delta}(1 - \frac{z^2}{\delta}) + \frac{z^2}{\delta}(1 - \frac{y^2}{\delta}) + \frac{x^2}{\delta}(1 - \frac{z^2}{\delta}) + \frac{z^2}{\delta}(1 - \frac{x^2}{\delta}) + \frac{x^2}{\delta}(1 - \frac{y^2}{\delta}) + \frac{y^2}{\delta}(1 - \frac{x^2}{\delta}).$$

$$lp = x(1 - \frac{y^2 z^2}{\delta \delta}) + y(1 - \frac{x^2 z^2}{\delta \delta}) + z(1 - \frac{x^2 y^2}{\delta \delta}).$$

Without loss of generality, assume $0 \leq x \leq y \leq z \leq \delta$. The triangle inequalities impose additional constraints $z \leq x + y \leq 2y$. We want to show that $T(x, y, z) := cost - (2 - \delta)lp \leq 0$. Note that for fixed y and z , we have that T is convex in x , since the coefficient of x^2 is

$$\frac{2 - 2z^2/\delta - 2y^2/\delta}{\delta} + (2 - \delta)(zy^2/\delta^2 + yz^2/\delta^2) > 0.$$

Therefore, given y and z , $T(x, y, z)$ is maximized when x is smallest or largest possible. So $T(x, y, z) \leq \max(T(z - y, y, z), T(y, y, z))$.

- First consider the case $x = y$. Let

$$\begin{aligned} T'(y, z) &:= T(y, y, z) = \frac{y^2}{\delta} \left(4 - \frac{4z^2}{\delta} \right) - \frac{2y^4}{\delta^2} + \frac{2z^2}{\delta} - (2 - \delta) \left(2y \left(1 - \frac{y^2 z^2}{\delta^2} \right) + z \left(1 - \frac{y^4}{\delta^2} \right) \right) \\ &= -2(2 - \delta)y + \frac{4(1 - z^2/\delta)}{\delta} y^2 + \frac{2(2 - \delta)z^2}{\delta^2} y^3 + y^4 (-2 + (2 - \delta)z/\delta^2) - (2 - \delta)z + 2z^2/\delta. \end{aligned}$$

Then

$$\frac{\partial T'(y, z)}{\partial y} := -2(2 - \delta) + \frac{8(1 - z^2/\delta)}{\delta} y + \frac{6(2 - \delta)z^2}{\delta^2} y^2 + y^3 \cdot 4(-2 + (2 - \delta)z/\delta^2),$$

and

$$\frac{\partial^2 T'(y, z)}{\partial y^2} := \frac{8(1 - z^2/\delta)}{\delta} + \frac{12(2 - \delta)z^2}{\delta^2} y + y^2 \cdot 12(-2 + (2 - \delta)z/\delta^2) > 0,$$

which implies that for fixed $z \leq \delta$, the function $T'(y, z)$ is convex for all $y \in [z/2, z]$, which means that $T'(y, z) \leq \max(T'(z/2, z), T'(z, z))$. For

$$T'(z, z) = 6z^2/\delta - 6z^4/\delta^2 - 3(2 - \delta)(z - z^5/\delta^2),$$

we have

$$\frac{\partial T'(z, z)}{\partial z^2} = 12/\delta - 72z^2/\delta^2 + 60(2 - \delta)z^3/\delta^2,$$

which is nonnegative for every $0 \leq z \leq \delta \leq 0.1$, so $T'(z, z)$ is convex in z and we have $T'(0, 0) = 0$ and $T'(\delta, \delta) = 6\delta - 6\delta^2 - 3(2 - \delta)(\delta - \delta^3) = -3\delta^2 + 6\delta^3 - 3\delta^4 < 0$.

Also for

$$T'(z/2, z) = \frac{3z^2}{\delta} - \frac{9z^4}{8\delta^2} - (2 - \delta) \left(2z - 3z^5/(16\delta^2) \right),$$

one can similarly prove that it is convex for $z \in [0, \delta]$ and check $T'(0, 0) = 0$ and $T'(\delta/2, \delta) = -2(2 - \delta)\delta + 3\delta - (9/8)\delta^2 + (2 - \delta)3\delta^3/16 < 0$.

- We now consider the case when $x + y = z$, so that

$$\text{cost} = \left(\frac{x^2}{\delta} + \frac{y^2}{\delta} \right) \left(2 - 2\frac{z^2}{\delta} \right) - 2\frac{x^2 y^2}{\delta \delta} + 2\frac{z^2}{\delta}.$$

$$lp = x + y - \frac{xy(x + y)z^2}{\delta^2} - z\frac{x^2 y^2}{\delta \delta} + z = 2z - \frac{xyz^3}{\delta^2} - z\frac{x^2 y^2}{\delta \delta}$$

So,

$$T(z - y, y, z) = \left(\frac{x^2}{\delta} + \frac{y^2}{\delta} \right) \left(2 - 2\frac{z^2}{\delta} \right) - 2\frac{x^2 y^2}{\delta \delta} + 2\frac{z^2}{\delta} - (2 - \delta) \left(2z - \frac{xyz^3}{\delta^2} - z\frac{x^2 y^2}{\delta \delta} \right)$$

For fixed $x + y = z$ with $z \leq \delta \leq 0.1$,

$$\left(\frac{x^2}{\delta} + \frac{y^2}{\delta}\right)\left(2 - \frac{2z^2}{\delta}\right) + \frac{(2-\delta)xyz^3}{\delta^2} = (x+y)^2\left(\frac{2-2z^2/\delta}{\delta}\right) + xy\left(\frac{(2-\delta)z^3}{\delta^2} - \frac{2(2-2z^2/\delta)}{\delta}\right)$$

is maximized when $x = 0, y = z$, since the coefficient of xy in the second expression is strictly negative. Therefore, it suffices to check

$$T(0, z, z) = \frac{z^2}{\delta}\left(2 - 2\frac{z^2}{\delta}\right) + 2\frac{z^2}{\delta} - (2-\delta)2z = -2(2-\delta)z + 4\frac{z^2}{\delta} - 2\frac{z^4}{\delta^2}.$$

Again, this function is convex in the interval $[0, \delta]$ and $T(0, 0, 0) = T(0, \delta, \delta) = 0$.

6.1.2 2 short/1 medium

Assume that z is medium. Note that all edges are rounded independently for every pivot. Without loss of generality, assume $0 \leq x \leq y \leq \delta \leq z$.

$$cost = \frac{y^2}{\delta}(1-z) + z\left(1 - \frac{y^2}{\delta}\right) + \frac{x^2}{\delta}(1-z) + z\left(1 - \frac{x^2}{\delta}\right) + \frac{x^2}{\delta}\left(1 - \frac{y^2}{\delta}\right) + \frac{y^2}{\delta}\left(1 - \frac{x^2}{\delta}\right).$$

$$lp = x\left(1 - \frac{y^2}{\delta}z\right) + y\left(1 - \frac{x^2}{\delta}z\right) + z\left(1 - \frac{x^2y^2}{\delta^2}\right).$$

The triangle inequalities impose additional constraints $z \leq x + y \leq 2y \leq 2\delta$. We want to show that $T(x, y, z) := cost - (2-\delta)lp \leq 0$. Note that for fixed y and z , we have that T is convex in x , since the coefficient of x^2 is

$$\frac{2-2z-2y^2/\delta}{\delta} + (2-\delta)(zy^2/\delta^2 + yz/\delta) \geq \frac{2-4\delta-2\delta}{\delta} > 0,$$

for $\delta \leq 0.1$. Therefore, given y and z , $T(x, y, z)$ is maximized when x is smallest or largest possible. So $T(x, y, z) \leq \max(T(z-y, y, z), T(y, y, z))$.

- Let us first consider

$$\begin{aligned} T'(y, z) &:= T(y, y, z) = \frac{y^2}{\delta}(4-4z) - \frac{2y^4}{\delta^2} + 2z - (2-\delta)\left(2y\left(1 - \frac{y^2z}{\delta}\right) + z\left(1 - \frac{y^4}{\delta^2}\right)\right) \\ &= -2(2-\delta)y + \frac{4(1-z)}{\delta}y^2 + \frac{2(2-\delta)z}{\delta}y^3 + y^4(-2/\delta^2 + (2-\delta)z/\delta^2) + \delta z. \end{aligned}$$

Then

$$\frac{\partial T'(y, z)}{\partial y} := -2(2-\delta) + \frac{8(1-z)}{\delta}y + \frac{6(2-\delta)z}{\delta}y^2 + 4y^3(-2/\delta^2 + (2-\delta)z/\delta^2),$$

and

$$\frac{\partial^2 T'(y, z)}{\partial y^2} := \frac{8(1-z)}{\delta} + \frac{12(2-\delta)z}{\delta}y + 12y^2(-2/\delta^2 + (2-\delta)z/\delta^2) > 0,$$

which implies that for fixed $z \leq \delta$, the function $T'(y, z)$ is convex for all $y \in [z/2, \delta]$, which means that $T'(y, z) \leq \max(T'(z/2, z), T'(\delta, z))$. For

$$\begin{aligned} T'(\delta, z) &= 2\delta(1-z) + 2\delta(1-\delta) + 2z(1-\delta) - (2-\delta) \left(2\delta(1-\delta z) + z(1-\delta^2) \right) \\ &= z(2-4\delta - (2-\delta)(1-3\delta^2)) + 4\delta - 2\delta^2 - 2(2-\delta)\delta \\ &= z(2-4\delta - 2 + 6\delta^2 + \delta - 3\delta^3) = z(-3\delta + 6\delta^2 - 3\delta^3) \end{aligned}$$

it is maximized when $z = \delta$, and $T'(\delta, \delta) > 0$.

Also for

$$\begin{aligned} T'(z/2, z) &= 2z + \frac{z^2}{\delta} - \frac{z^3}{\delta} - \frac{z^4}{8\delta^2} - (2-\delta) \left(2z - \frac{z^4}{4\delta} - \frac{z^5}{16\delta^2} \right) \\ &= z(-2 + 2\delta) + z^2/\delta - z^3/\delta + z^4(-1/8\delta^2) + (2-\delta)/4\delta + z^5(2-\delta)/16\delta^2, \end{aligned}$$

one can similarly prove that it is convex for $z \in [0, 2\delta]$ and check $T'(0, 0) = 0$ and

$$T'(\delta, 2\delta) = -4\delta + 4\delta^2 + 4\delta - 8\delta^2 - 2\delta^2 + 4(2-\delta)\delta^3 + 2(2-\delta)\delta^3 < 0.$$

- We now consider the case when $x + y = z$, so that

$$\begin{aligned} cost &= \left(\frac{x^2}{\delta} + \frac{y^2}{\delta} \right) (2-2z) - 2 \frac{x^2 y^2}{\delta} + 2z. \\ lp &= x + y - \frac{xy(x+y)z}{\delta} - z \frac{x^2 y^2}{\delta} + z \end{aligned}$$

So,

$$T(z-y, y, z) = \left(\frac{x^2}{\delta} + \frac{y^2}{\delta} \right) (2-2z) - 2 \frac{x^2 y^2}{\delta} + 2z - (2-\delta) \left(2z - \frac{xyz^2}{\delta} - z \frac{x^2 y^2}{\delta} \right)$$

For fixed $x + y = z$ with $z \leq 2\delta \leq 0.3$,

$$\left(\frac{x^2}{\delta} + \frac{y^2}{\delta} \right) (2-2z) + \frac{(2-\delta)xyz^2}{\delta} = (x+y)^2 \left(\frac{2-2z}{\delta} \right) + xy \left(\frac{(2-\delta)z^2}{\delta} - \frac{2(2-2z)}{\delta} \right)$$

is maximized when $x = z - \delta, y = \delta$, since the coefficient of xy in the second expression is strictly negative. Therefore, it suffices to check

$$T(z-\delta, \delta, z) = ((z-\delta)^2/\delta + \delta)(2-2z) - 2(z-\delta)^2 + 2z - (2-\delta) \left(2z - z^2(z-\delta) - z(z-\delta)^2 \right).$$

It is convex in the interval $[\delta, 2\delta]$ with $T'(0, \delta, \delta) = 0$ and $T'(\delta, \delta, 2\delta) < 0$.

6.1.3 1 short/2 medium

Let us say y, z are medium. When a is the pivot, the edges (a, b) and (a, c) are rounded with correlation (recall that $y = x_{ac}$ and $z = x_{ab}$). Note that $x = x_{bc} = y_{ab|c} + y_{ac|b} + y_{a|b|c}$.

$$\begin{aligned} cost &= y_{ab|c} + y_{ac|b} + \frac{x^2}{\delta}(1-z) + z(1 - \frac{x^2}{\delta}) + \frac{x^2}{\delta}(1-y) + y(1 - \frac{x^2}{\delta}) \\ &\leq x + \frac{x^2}{\delta}(1-z) + z(1 - \frac{x^2}{\delta}) + \frac{x^2}{\delta}(1-y) + y(1 - \frac{x^2}{\delta}). \end{aligned}$$

$$lp = x(1 - y_{a|bc}) + y(1 - \frac{x^2}{\delta}z) + z(1 - \frac{x^2}{\delta}y).$$

Then we compose $cost$ and lp into three parts each and bound their ratios.

- If we consider $x/2 + z(1 - \frac{x^2}{\delta})$ from $cost$ and $z(1 - \frac{x^2}{\delta}y)$ from lp , then $z(1 - \frac{x^2}{\delta}) \leq z(1 - \frac{x^2}{\delta}y)$ and $x/2 \leq z/2 = \frac{z(1-\delta)}{2(1-\delta)} \leq \frac{z(1-\frac{x^2}{\delta}y)}{2(1-\delta)}$. Therefore,

$$x/2 + z(1 - \frac{x^2}{\delta}) \leq (1 + \frac{1}{2(1-\delta)})z(1 - \frac{x^2}{\delta}y).$$

- Similarly,

$$x/2 + y(1 - \frac{x^2}{\delta}) \leq (1 + \frac{1}{2(1-\delta)})y(1 - \frac{x^2}{\delta}z).$$

- Finally, $\frac{x^2}{\delta}(1-z)$ from $cost$ is at most $x(1 - y_{a|bc})$ from lp since $x^2/\delta \leq x$ and $z = y_{a|bc} + y_{a|b|c} + y_{ac|b}$.

Therefore, $cost \leq (1 + \frac{1}{2(1-\delta)})lp$.

6.1.4 2 long/1 medium, or 3 long

Assume $z \geq 0.9$. We have

$$cost = 2x + 2y + 2z - 2xy - 2yz - 2zx$$

and

$$lp = x + y + z - 3xyz$$

Let $w = x + y$. Notice that $w \geq z$ (since $x + y \geq z$). Consider

$$\begin{aligned} &2x + 2y + 2z - 2xy - 2yz - 2zx - 1.5(x + y + z - 3xyz) \\ &= 2z + 2w - 2zw - 2xy - 1.5(z + w - 3zx) = 0.5z + 0.5w - 2zw + xy(-2 + 4.5z). \end{aligned}$$

For fixed w and $z \geq .9$, it is maximized when xy is maximized, which occurs when x and y are equal, so we can assume that $x = y = w/2$. This yields the expression

$$0.5z + 0.5w - 2zw + (-2 + 4.5z)w^2/4$$

This is linear in z , so maximized when $z = 0.9$ or $z = \min(w, 1)$. When $z = 0.9$,

$$0.45 + 0.5w - 1.8w + (-2 + 4.05)w^2/4$$

is negative for all $w \in [0.9, 2]$. When $z = 1$,

$$0.5 + 0.5w - 2w + (-2 + 4.5)w^2/4$$

is negative for all $w \in [1, 2]$. When $z = w$,

$$w - 2w^2 + (-2 + 4.5w)w^2/4$$

is negative for all $w \in [0.9, 1]$.

6.1.5 1 short/2 long

Compared to the 1 medium/2 long case, lp increases and $cost$ increases by a factor of at most $1/(1 - \delta)$, so the ratio is at most $1.5/(1 - \delta) \leq 1.6667$.

6.1.6 1 short/1 medium/1 long

Similarly, compared to the 2 medium/1 long case, lp increases and $cost$ increases by a factor of at most $1/(1 - \delta)$, so the ratio is at most $1.5/(1 - \delta) \leq 1.6667$.

6.1.7 3 medium

This case is checked in Lemma 4.2 and the ratio is at most 1.5.

6.1.8 2 medium/1 long

Let us say y, z are medium. When a is the pivot, the edges (a, b) and (a, c) are rounded with correlation (recall that $y = x_{ac}$ and $z = x_{ab}$). Note that $x = x_{bc} = y_{ab|c} + y_{ac|b} + y_{a|b|c}$.

$$cost = y_{ab|c} + y_{ac|b} + x(1 - z) + z(1 - x) + x(1 - y) + y(1 - x)$$

$$lp = x(1 - y_{a|bc}) + y(1 - xz) + z(1 - xy).$$

Focus on the last four terms of $cost$ and the last two terms of lp and consider

$$\begin{aligned} & x(1 - z) + z(1 - x) + x(1 - y) + y(1 - x) - 1.9[y(1 - xz) + z(1 - xy)] \\ &= 2x + (z + y) - 2x(y + z) - 1.9[y + z - 2xyz]. \end{aligned}$$

For fixed x and $t := y + z$, it is maximized when $y = z = t/2$, yielding

$$2x + t - 2xt - 1.9[t - xt^2/2] = x(2 - 2t + 0.95t^2) - 0.9t.$$

The coefficient of x is strictly positive, so it is maximized when $x = 1$, so that the expression is at most

$$2 - 2.9t + 0.95t^2.$$

It is negative when $t \geq 1.1$ and at most 0.17 when $t = 0.9$. (Note that $t \geq x \geq 0.9$.)

For the remaining two terms $y_{ab|c} + y_{ac|b}$ of $cost$ and the first term $x(1 - y_{a|bc})$ of lp ,

- $y_{ab|c} + y_{ac|b} \leq \frac{1}{x} \cdot x(1 - y_{a|bc}) \leq 1.9x(1 - y_{a|bc})$. So if $t \geq 1.1$, then the overall ratio is at most 1.9.
- If $t \leq 1.1$, then $y_{a|bc} \leq 0.55$ since it contributes to both y and z . Therefore, the above inequality $\frac{1}{x}x(1 - y_{a|bc}) \leq 1.9x(1 - y_{a|bc})$ has an additive slack of at least $0.45x(1.9 - 1/x) \geq 0.3$, which covers the 0.17 excess. Therefore, the overall ratio is at most 1.9 in every case.

6.2 + + - Triangles.

Suppose that the vertices are (a, b, c) , edge (a, b) is $-$, and the LP values are (x, y, z) , where $x = x_{bc}$, $y = x_{ac}$, and $z = x_{ab}$. When both $+$ edges are medium, it is handled in Lemma 4.2. Therefore, we only need to handle when either x or y is short or long. Note that in this case, all edges are rounded independently.

6.2.1 When a +edge is short

Assume that $x \leq y$ and $x \leq \delta$. Let $\alpha = 1.9$ be the targeted ratio.

- We first handle the case $x, y \leq \delta$.

$$\begin{aligned} cost &= \frac{y^2}{\delta}(1 - \sqrt{z}) + \sqrt{z}(1 - \frac{y^2}{\delta}) + \frac{x^2}{\delta}(1 - \sqrt{z}) + \sqrt{z}(1 - \frac{x^2}{\delta}) + (1 - \frac{x^2}{\delta})(1 - \frac{y^2}{\delta}) \\ &\leq 2\sqrt{z} + \frac{y^2}{\delta}(-2\sqrt{z}) + \frac{x^2}{\delta}(-2\sqrt{z}) + 1 + \delta^2 =: cost' \end{aligned}$$

and

$$\begin{aligned} lp &= x(1 - \frac{y^2}{\delta}\sqrt{z}) + y(1 - \frac{x^2}{\delta}\sqrt{z}) + (1 - z)(1 - \frac{x^2}{\delta}\frac{y^2}{\delta}) \\ &\geq x(1 - \delta\sqrt{z}) + y(1 - \delta\sqrt{z}) + (1 - z)(1 - \delta^2) =: lp' \end{aligned}$$

Let $T(x, y, z) = cost' - \alpha lp'$. Since the coefficients of both z and \sqrt{z} are positive, it is maximized when $z = x + y$. Given $z = x + y$, $cost$ is maximized when $x^2 + y^2$ is minimized which is the case when $x = y = z/2$. Therefore,

$$T(x, y, z) \leq 2\sqrt{z} - \frac{z^{2.5}}{\delta} + 1 + \delta^2 - \alpha(z(1 - \delta\sqrt{z}) + (1 - z)(1 - \delta^2)),$$

which is strictly negative for $z \in [0, 2\delta]$.

- Assume $x \leq \delta < y$.

$$cost = y(1 - \sqrt{z}) + \sqrt{z}(1 - y) + \frac{x^2}{\delta}(1 - \sqrt{z}) + \sqrt{z}(1 - \frac{x^2}{\delta}) + (1 - \frac{x^2}{\delta})(1 - y)$$

and

$$lp = x(1 - y\sqrt{z}) + y(1 - \frac{x^2}{\delta}\sqrt{z}) + (1 - z)(1 - \frac{x^2}{\delta}y).$$

Let $T(x, y, z) = cost - \alpha lp$. The coefficient of y is $1 - \sqrt{z} - \sqrt{z} - (1 - \frac{x^2}{\delta}) - \alpha(-x\sqrt{z} + 1 - \frac{x^2}{\delta}\sqrt{z} - \frac{x^2}{\delta}(1 - z)) < 0$, which means that y should be minimized. While decreasing y , if it becomes $y = \delta$, then the above case proves the claimed ratio. (We used the fact that $\frac{y^2}{\delta} = y$ when $y = \delta$.)

Then the only other case where we cannot increase y further is when $z = x + y$. For fixed y , consider $T(x, y, x + y)$ as a function of x .

$$\begin{aligned} T(x, y, x + y) = & 1 + \frac{x^2}{\delta}y + 2\sqrt{x+y}(1 - y - \frac{x^2}{\delta}) \\ & - \alpha \left(1 - \frac{x^2}{\delta}y + (x + y)\frac{x^2}{\delta}y - xy\sqrt{x+y} - y\frac{x^2}{\delta}\sqrt{x+y} \right) \end{aligned}$$

It is an increasing function in x , so the maximum is attained at $x = \delta$. To show this, we can take the derivative. Let $F_y(x) := T(x, y, x + y)$. Recall we have $x + y < 1$.

$$F_y(x) := 1 + \frac{x^2}{\delta}y + 2\sqrt{x+y}(1 - y - \frac{x^2}{\delta}) + \alpha \left(-1 + (1 + \sqrt{x+y} - x - y)\frac{x^2}{\delta}y + xy\sqrt{x+y} \right). \quad (22)$$

$$F'_y(x) := 2\frac{x}{\delta}y - 4\sqrt{x+y}\frac{x}{\delta} + \frac{1}{\sqrt{x+y}}(1 - y - \frac{x^2}{\delta}) \quad (23)$$

$$+ \alpha \left(\left(\frac{1}{2\sqrt{x+y}} - 1 \right) \frac{x^2}{\delta}y + (1 + \sqrt{x+y} - x - y)\frac{2x}{\delta}y + y\sqrt{x+y} + xy\frac{1}{2\sqrt{x+y}} \right). \quad (24)$$

For each $y \in [\delta, 1]$ and all $x \in [0, \delta]$, we can show that $F'_y(x) > 0$ ($F_y(x)$ attains its minimum value of .66 for $x = .1$ and $y = .18$), which shows that $F_y(x)$ is increasing. So we can assume that $x = \delta$, then we have

$$H(y) := 1 + \delta y + 2\sqrt{\delta+y}(1 - y - \delta) + \alpha \left(-1 + \delta y - \delta^2 y - \delta y^2 + 2\delta y\sqrt{\delta+y} \right). \quad (25)$$

$$H'(y) := \delta - 2\sqrt{\delta+y} + \frac{1}{\sqrt{\delta+y}}(1 - y - \delta) + \alpha \left(\delta - \delta^2 - 2\delta y + 2\delta\sqrt{\delta+y} + \delta y\frac{1}{\sqrt{\delta+y}} \right). \quad (26)$$

$$H''(y) := -\frac{2}{\sqrt{\delta+y}} - \frac{1}{2(\delta+y)^{3/2}}(1 - y - \delta) + \alpha \left(-2\delta + \frac{2\delta}{\sqrt{\delta+y}} - \delta y\frac{1}{2(\delta+y)^{3/2}} \right). \quad (27)$$

$H''(y) < 0$ for all y , because $-2 + 2\delta\alpha < 0$. So now we set $H'(y) = 0$ and solve for y .

$$\delta - 2\sqrt{\delta + y} + \frac{1}{\sqrt{\delta + y}}(1 - y - \delta) + \alpha \left(\delta - \delta^2 - 2\delta y + 2\delta\sqrt{\delta + y} + \delta y \frac{1}{\sqrt{\delta + y}} \right) = 0.$$

This function has one root for $y \in [\delta, 1]$ at $y \approx .342$. We can verify that for $x = \delta, y = .342$ and $z = y + x$, the function $T(x, y, z) \leq 0$.

6.2.2 When a +edge is long

Now we assume that one +edge is long and the other +edges is either long or medium. Recall that

$$cost = (1 - x)(1 - y) + y(1 - \sqrt{z}) + \sqrt{z}(1 - y) + x(1 - \sqrt{z}) + \sqrt{z}(1 - x)$$

and

$$lp = (1 - z)(1 - xy) + x(1 - y\sqrt{z}) + y(1 - x\sqrt{z}),$$

so that the ratio is

$$\frac{1 + xy + 2\sqrt{z} - 2y\sqrt{z} - 2x\sqrt{z}}{(1 - z)(1 - xy) + x(1 - y\sqrt{z}) + y(1 - x\sqrt{z})}. \quad (28)$$

So we want to prove the following inequality, assuming triangle inequality on x, y, z and $x \geq .9$.

$$\frac{1 + xy + 2\sqrt{z} - 2y\sqrt{z} - 2x\sqrt{z}}{1 - z - xy + xyz + x + y - 2xy\sqrt{z}} = \frac{1 + xy + 2\sqrt{z}(1 - y - x)}{1 - z - xy + xyz + x + y - 2xy\sqrt{z}} \leq \frac{3}{2}. \quad (29)$$

$$1 + xy + 2\sqrt{z}(1 - y - x) \leq \frac{3}{2}(1 - z - xy + xyz + x + y - 2xy\sqrt{z}). \quad (30)$$

$$-\frac{1}{2} + xy + 2\sqrt{z}(1 - y - x + \frac{3}{2}xy) + \frac{3}{2}z(1 - xy) + \frac{3}{2}xy - \frac{3}{2}(x + y) \leq 0. \quad (31)$$

For $x, y \in [0, 1]$, the coefficients of z and \sqrt{z} are always nonnegative. Thus, we can assume that $z = \min\{x + y, 1\}$.

We consider two cases: $z = 1$ and $z = x + y < 1$. Let us first assume that $z = 1$. We rewrite the ratio as

$$\frac{1 + xy + 2(1 - y - x)}{-2xy + x + y} = \frac{3 + xy - 2x - 2y}{x + y - 2xy}. \quad (32)$$

To upper bound the ratio by $\frac{3}{2}$, it is thus enough to show that

$$3 + xy - 2x - 2y - \frac{3}{2}(x + y - 2xy) \leq 0. \quad (33)$$

Which we can rearrange as

$$3 + xy - 2x - 2y - \frac{3}{2}x - \frac{3}{2}y + 3xy \leq 0. \quad (34)$$

$$3 + 4xy - \frac{7}{2}x - \frac{7}{2}y \leq 0 \quad (35)$$

The LHS is linear in y and so maximized for $y = 0$ or $y = 1$. It is thus enough to show:

$$3 + 4x - \frac{7}{2}x - \frac{7}{2} \leq 0, \quad (36)$$

which holds for $x \leq 1$; and

$$3 - \frac{7}{2}x \leq 0 \quad (37)$$

which holds for $x \geq 6/7$, so it also holds when $x \geq .9$.

Next, we prove the desired upper bound on the ratio for the case $z = x + y < 1$. We have

$$\frac{1 + xy + 2\sqrt{x+y}(1-y-x)}{1 - xy + xy(x+y) - 2xy\sqrt{x+y}} \leq \frac{1 + xy + 2(1-y-x)}{1 - xy + xy(x+y) - 2xy}. \quad (38)$$

We want to show that this ratio is at most $\frac{3}{2}$.

$$1 + xy + 2(1-y-x) \leq \frac{3}{2}(1 - xy + xy(x+y) - 2xy) \quad (39)$$

$$3 + xy - 2y - 2x \leq \frac{3}{2} - \frac{3}{2}xy + \frac{3}{2}xy(x+y) - 3xy \quad (40)$$

$$\frac{3}{2} + \frac{11}{2}xy \leq 2y + 2x + \frac{3}{2}xy(x+y). \quad (41)$$

So we have

$$F_x(y) := \frac{3}{2} + \frac{11}{2}xy - 2y - 2x - \frac{3}{2}xy(x+y), \quad (42)$$

and we want to show that this function is at most 0 when $x \in [.9, 1]$ and $y \in [0, 1]$. Since $F_x''(y) = -3x$, the function is always concave in y for any x . If

$$F_x'(y) = \frac{11}{2}x - 2 - \frac{3}{2}x^2 - 3xy = 0, \quad (43)$$

then $y = y^* = \frac{11}{6} - \frac{2}{3x} - \frac{x}{2}$. However, this value of y is much larger than .1, which is the maximum value of y allowed (i.e., the maximum is outside the interval $[0, .1]$). Thus, it suffices to check the extreme values of $y = 0$ and $y = 1 - x$. When $y = 0$, we have

$$F_x(y) := \frac{3}{2} - 2x \leq 0, \quad (44)$$

when $x \geq 3/4$. When $y = 1 - x$, we have

$$F_x(y) : = \frac{3}{2} + \frac{11}{2}x(1-x) - 2(1-x) - 2x - \frac{3}{2}x(1-x) \quad (45)$$

$$= -\frac{1}{2} - \frac{8}{2}x^2 + \frac{8}{2}x. \quad (46)$$

This is at most 0 when

$$-1 - 8x^2 + 8x \leq 0, \quad (47)$$

and it can be verified that this is the case when $x \in [.9, 1]$.

6.3 + - - Triangles

Suppose that the vertices are (a, b, c) , edge (b, c) is +, and the LP values are (x, y, z) , where $x = x_{bc}$, $y = x_{ac}$, and $z = x_{ab}$.

- x is medium or long: This case is checked in Lemma 4.2 and the ratio is at most 1.5.
- x is short: Note that all triangles are rounded independently.

$$cost = \sqrt{z}(1 - \sqrt{y}) + \sqrt{y}(1 - \sqrt{z}) + (1 - \frac{x^2}{\delta})(1 - \sqrt{z}) + (1 - \frac{x^2}{\delta})(1 - \sqrt{y}),$$

and

$$lp = x(1 - \sqrt{y}\sqrt{z}) + (1 - y)(1 - \frac{x^2}{\delta}\sqrt{z}) + (1 - z)(1 - \frac{x^2}{\delta}\sqrt{y}).$$

Note that the expressions for $cost$ and lp when x is long are identical to the above, except that $\frac{x^2}{\delta}$ is replaced by x . Since $\frac{x^2}{\delta} \leq x \leq \delta$ and $(1 - \frac{x^2}{\delta}) \in [1 - x, (1 + \delta)(1 - x)]$, $cost$ for short x is at most $(1 + \delta)$ times $cost$ for long x , and lp for short x is at least lp for long x . Since the ratio $cost/lp$ for medium/long x is at most 1.5, the ratio for short x is at most $1.5(1 + \delta)$.

6.4 - - - Triangles

This case is already checked in Lemma 4.2, and the ratio is at most 1.

6.5 Degenerate triangles

Let $\{u, v\}$ be a degenerate triangle. Compared to degenerate triangles in Lemma 4.2, the only change happens when (u, v) is a short +edge, which makes $cost(u, v) = 2x_{uv}^2/\delta \leq 2x_{uv}$. Therefore, the ratio is still at most 1.

7 Details of Correlated Rounding

In this section, we prove Lemma 3.1. Recall that given a correlation clustering instance $G = (V, E)$ and a solution y to the r -rounds of Sherali-Adams, we chose a pivot $p \in V$ and let I_p be the set of vertices that have a medium +edge to p . We would like to sample a set $S' \subseteq I_p$ such that (1) for each $v \in I_p$, $\Pr[v \in S] = y_{pv}$ and (2) $\mathbb{E}_{u,v \in I_p} [|\Pr[u, v \in S] - y_{puv}|] \leq \varepsilon_r$, where $\varepsilon_r = O(1/\sqrt{r})$.

Note that sampling $S' \subseteq I_p$ is equivalent to making a binary decision for each $v \in I_p$; whether to put v into S' or not. In this interpretation, we can almost directly import the tools for CSPs (with binary alphabets). For sake of completeness, we show how the framework of Raghavendra and Tan [RT12] used for Max-CSPs with cardinality constraints can be used for our purpose. Similar techniques also have been used for non-constrained CSPs and graph partitioning problems [GS11, BRS11].

Imagine we are interested in a CSP that has n variables $W = \{v_1, \dots, v_n\}$ where each v_i can have a value in $\{0, 1\}$. (Predicates and objective functions are not important here.) The r -rounds of Sherali-Adams for the CSP have variables $x_{S,\alpha}$ for any $S \subseteq W$, $|S| \leq t$ and $\alpha \in \{0, 1\}^S$ (also interpreted as a function $\alpha : S \rightarrow \{0, 1\}$) where $x_{S,\alpha}$ denotes the probability that the variables in S are assigned α . The following constraints ensure that these *local distributions* are consistent. Given $\alpha \in \{0, 1\}^S$ and $T \subseteq S$, let $\alpha|_T \in \{0, 1\}^T$ be the restriction of α to T .

$$x_\emptyset = 1. \tag{48}$$

$$x_{T,\beta} = \sum_{\alpha \in \{0,1\}^S : \alpha|_T = \beta} x_{S,\alpha} \quad T \subseteq S \subseteq W, |S| \leq r, \beta \in \{0, 1\}^T. \tag{49}$$

$$x \geq 0. \tag{50}$$

In our setting where $W = I_p$ and $v_i = 1$ indicates that v_i is put into S' , it is natural to associate $x_{v_i,1} = y_{pv_i}$. The following claim shows that such association can be formally defined for higher-level variables as well.

Claim 14. *Let y be a solution to the r -rounds of Sherali-Adams for CORRELATION CLUSTERING, $p \in V$, $W \subseteq V \setminus \{p\}$ and define $\{x_{S,\alpha}\}_{S \subseteq W, |S| \leq r-1, \alpha \in \{0,1\}^S}$ as*

$$x_{S,\alpha} = \sum_{\substack{S_1, \dots, S_\ell \\ S = S_1 \cup \dots \cup S_\ell \\ \text{and } S_1 = \alpha^{-1}(1)}} y_{S_1 \cup \{p\} | S_2 | \dots | S_\ell}.$$

Then x satisfies the Sherali-Adams constraints for CSP (48), (49), and (50) with $(r-1)$ rounds.

Proof. The only nontrivial constraint is (49). Fix $T \subseteq S \subseteq W$ with $|S| \leq r-1$ and $\beta \in \{0, 1\}^T$.

Let $T_1 = \beta^{-1}(1)$ and $T_0 = \beta^{-1}(0)$.

$$\begin{aligned}
x_{T,\beta} &= \sum_{\substack{T_2, \dots, T_\ell: \\ T_0 = T_2 \cup \dots \cup T_\ell}} y_{T_1 \cup \{p\} | T_2 \dots | T_\ell} \\
&= \sum_{\substack{T_2, \dots, T_\ell: \\ T_0 = T_2 \cup \dots \cup T_\ell}} \sum_{\substack{S_1, \dots, S_q: \\ S = S_1 \cup \dots \cup S_q \text{ and} \\ T_i = S_i \cap T \ \forall i \in [\ell]}} y_{S_1 \cup \{p\} | S_2 \dots | S_q} \\
&= \sum_{S_1: T_1 = S_1 \cap T} \left(\sum_{\substack{T_2, \dots, T_\ell: \\ T_0 = T_2 \cup \dots \cup T_\ell}} \sum_{\substack{S_2, \dots, S_q: \\ S = S_1 \cup \dots \cup S_q \text{ and} \\ T_i = S_i \cap T \ \forall i \in \{2, \dots, \ell\}}} y_{S_1 \cup \{p\} | S_2 \dots | S_q} \right) \\
&= \sum_{S_1: T_1 = S_1 \cap T} \left(\sum_{\substack{S_2, \dots, S_q: \\ S = S_1 \cup \dots \cup S_q}} y_{S_1 \cup \{p\} | S_2 \dots | S_q} \right) \\
&= \sum_{S_1: T_1 = S_1 \cap T} \sum_{\alpha \in \{0,1\}^S: \alpha^{-1}(1) = S_1} x_{S,\alpha} = \sum_{\alpha \in \{0,1\}^S: \alpha|_T = \beta} x_{S,\alpha}.
\end{aligned}$$

□

Therefore, $\{x_{S,\alpha}\}$ is a valid solution to the $(r-1)$ rounds of the Sherali-Adams hierarchy (for CSPs). Then in order to finish Lemma 3.1 it suffices to give a randomized rounding algorithm that outputs 0-1 random variables $\{X_v\}_{v \in W}$ such that $\mathbb{E}[X_v] = x_v = y_{pv}$ for each $v \in W$ and $\mathbb{E}_{u,v \in W} \mathbb{E}[X_u X_v] = \mathbb{E}_{u,v} [x_{uv}] \pm \varepsilon_r = \mathbb{E}_{u,v} [y_{puv}] \pm \varepsilon_r$. At this point, Theorem 4.6 of Raghavendra and Tan [RT12] shows that such rounding exists. For sake of completeness, we reproduce their proof here.

Their rounding is to (1) carefully choose the seed set $S \subseteq W$ with $|S| \leq r-2$, (2) round $\{X_v\}_{v \in S}$ according to the joint distribution $\{x_{S,\alpha}\}_{\alpha \in \{0,1\}^S}$, and (3) for each $u \in W \setminus S$, independently round X_u from the conditional distribution given the rounded values for $\{X_v\}_{v \in S}$. (Since x is a solution for $r-1$ rounds and $|S| \leq r-2$, the conditional rounding is possible.)

[RT12] showed how to find a good seed and analyzed the performance of the rounding using entropy. Recall that for 0-1 random variables X and Y , their entropy, mutual entropy, and conditional entropy are defined as

$$\begin{aligned}
H(X) &:= - \sum_{i \in \{0,1\}} \Pr[X = i] \log \Pr[X = i], \\
I(X; Y) &:= \sum_{i,j \in \{0,1\}} \Pr[X = i, Y = j] \log \frac{\Pr[X = i, Y = j]}{\Pr[X = i] \Pr[Y = j]}, \\
H(X|Y) &:= \sum_{i \in \{0,1\}} \Pr[Y = i] H(X|Y = i).
\end{aligned}$$

The mutual information and the pairwise correlation can be related as follows.

Claim 15 (Fact 4.3 of [RT12]). For any $i, j \in \{0, 1\}$,

$$|\Pr[X = i, Y = j] - \Pr[X = i]\Pr[Y = j]| \leq \sqrt{2I(X; Y)}.$$

When $i = j = 1$, this implies that

$$|\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]| \leq \sqrt{2I(X; Y)}.$$

For a seed S , let $X_S = \{x_v\}_{v \in S}$. We want to find a good seed S with $|S| \leq r - 2$ such that $\mathbb{E}_{u, v \in W} \sqrt{2I(X_u; X_v | X_S)}$ is small. The following lemma guarantees that there exists a good seed.

Lemma 7.1. *There exists $t \leq r - 3$ such that*

$$\mathbb{E}_{w_1, \dots, w_t \in W} \mathbb{E}_{u, v \in W} [I(X_u; X_v | X_{w_1}, \dots, X_{w_t})] \leq 1/(r - 2).$$

Proof. By linearity of expectation, we have that for any $t \leq r - 2$,

$$\begin{aligned} & \mathbb{E}_{u, w_1, \dots, w_t} [H(X_u | X_{w_1}, \dots, X_{w_t})] \\ &= \mathbb{E}_{u, w_1, \dots, w_t} [H(X_u | X_{w_1}, \dots, X_{w_{t-1}})] - \mathbb{E}_{w_1, \dots, w_{t-1}} \mathbb{E}_{u, w_t} [I(X_u; X_{w_t} | X_{w_1}, \dots, X_{w_{t-1}})] \end{aligned}$$

adding the equalities from $t = 1$ to $t = r - 2$, the lemma follows since

$$1 \geq \mathbb{E}_{u \in W} [H(X_u)] - \mathbb{E}_{u, w_1, \dots, w_{r-2} \in W} [H(X_u | X_{w_1}, \dots, X_{w_{r-2}})] = \sum_{1 \leq t \leq r-2} \mathbb{E}_{u, v, w_1, \dots, w_{t-1}} [I(X_u; X_v | X_{w_1}, \dots, X_{w_{t-1}})].$$

□

Therefore, there exists $S \subseteq W$ with $|S| \leq r - 3$ such that $\mathbb{E}_{u, v} [I(X_u; X_v | X_S)] \leq 1/(r - 2)$. Find such an S by exhaustive search, sample $\alpha_S \in \{0, 1\}^S$ from the local distribution (i.e., according to the convex combination of solutions on S), let $X_S = \alpha_S$, and for each $v \in W \setminus S$, round X_v independently conditioned on $X_S = \alpha_S$. (One can also interpret that X_v for $v \in S$ is also independently rounded again conditioned on $X_S = \alpha_S$, though this rounding does not change anything.) Note that for any $v \in W$, the marginal is exactly preserved; i.e., $\mathbb{E}[X_v] = \mathbb{E}_{\alpha_S} \mathbb{E}[X_v | X_S = \alpha_S] = x_v$. Finally,

$$\begin{aligned} & \mathbb{E}_{u, v \in W} |\mathbb{E}[X_u X_v] - x_u x_v| \\ &= \mathbb{E}_{u, v \in W} |\mathbb{E}_{\alpha_S} (\mathbb{E}[X_u | X_S = \alpha_S] \mathbb{E}[X_v | X_S = \alpha_S] - \mathbb{E}[X_u X_v | X_S = \alpha_S])| \\ &\leq \mathbb{E}_{u, v \in W} \mathbb{E}_{\alpha_S} |\mathbb{E}[X_u | X_S = \alpha_S] \mathbb{E}[X_v | X_S = \alpha_S] - \mathbb{E}[X_u X_v | X_S = \alpha_S]| \\ &\leq \mathbb{E}_{u, v \in W} \mathbb{E}_{\alpha_S} \sqrt{2I(X_u; X_v | X_S = \alpha_S)} \\ &\leq \mathbb{E}_{u, v \in W} \sqrt{2I(X_u; X_v | X_S)} \\ &\leq \sqrt{\mathbb{E}_{u, v \in W} 2I(X_u; X_v | X_S)} \leq O(1/\sqrt{r}). \end{aligned}$$

8 Derandomization

In this section, we show that our algorithm can be derandomized. Fix one iteration with $G = (V, E)$. For any $p \in V$, Section 7 shows how to deterministically find of a good seed set $T_p \subseteq I_p$ with $|T_p| \leq r - 2$. Then the algorithm for one iteration can be abstractly described as follows.

1. Sample $p \in V$. Recall $I_p = \{u : (p, u) \text{ is medium } +\}$. Let $S \leftarrow \emptyset$.
2. For each $u \in V \setminus (I_p \cup \{p\})$, independently decide $S \leftarrow S \cup \{u\}$ or not with the probability depending on x_{pv} .
3. Sample $T', S' \subseteq I_p$ as follows.
 - Sample $T' \subseteq T_p$ according to the local distribution of the Sherali-Adams solution induced by $T_p \cup \{p\}$.
 - For each $u \in I_p \setminus T_p$, independently decide $S' \leftarrow S' \cup \{u\}$ or not with the probability according the local distribution of the Sherali-Adams solution induced by $T_p \cup \{p, u\}$ (conditioned on T').
4. Make $\{p\} \cup S \cup T' \cup S'$ as a new cluster.

From the description, it is clear that $\text{cost}_p(u, v)$ and $\text{lp}_p(u, v)$ can be deterministically computed in polynomial time; once T_p is given, one can go over each possible $T' \subseteq T_p$ (there are at most $2^{|T_p|} \leq 2^r$ choices), and the rest of the rounding is independent for each vertex.

Since Lemma 4.4 prove that $ALG/LP \leq 1.994 + \varepsilon$, there exists $p \in V$ such that

$$\frac{\sum_{(u,v) \in E} \text{cost}_p^r(u, v)}{\sum_{(u,v) \in E} \text{lp}_p^r(u, v)}$$

is at most $1.994 + \varepsilon$, and one can deterministically compute such p since $\text{cost}_p(u, v)$ and $\text{lp}_p(u, v)$ are already computed.

Once p is chosen, for each possible $T' \subseteq T_p$, one can compute the expected value of $\sum_{(u,v) \in E} \text{cost}_p^r(u, v)$ and $\sum_{(u,v) \in E} \text{lp}_p^r(u, v)$ conditioned on T' . Find a T' that makes the ratio still $1.994 + \varepsilon$. Conditioned on T' , the rest of the rounding is independent for every vertex $V \setminus (T_p \cup \{p\})$, and one can continue to apply the method of conditional expectations for each vertex to decide whether it belongs to p 's cluster or not. At the end, we deterministically compute a cluster including p whose removal incurs the cost of α and decreases the remaining LP value by β , where $\alpha \leq (1.994 + \varepsilon)\beta$. Iterating this method for every iteration until the end ensures that the total cost is at most $(1.994 + \varepsilon)$ times the original LP value.

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