Julia sets of rational maps with rotational symmetries

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Abstract

By a symmetry of the Julia set of a polynomial, also referred as polynomial Julia set, we mean an Euclidean isometry preserving the Julia set. Each such symmetry is in fact a rotation about the centroid of the polynomial. In this article, a survey of the symmetries of polynomial Julia sets is made. Then the Euclidean isometries preserving the Julia set of rational maps are considered. A rotation preserving the Julia set of a rational map is called a rotational symmetry of its Julia set. A sufficient condition is provided for a rational map to have rotational symmetries whenever the rational map has an exceptional point. Two classes of rational maps are provided whose Julia sets have rotational symmetries of finite orders. Using this, it is proved that $z \mapsto \mu z$ where $\mu^{m+n} = 1$ is a rotational symmetry of the McMullen map $z^m + \frac{\lambda}{z^n}$ for all m, n with $m \geq 2$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Assuming that a normalized polynomial has a simple root at the origin, it is shown that the groups of the rotational symmetries of the polynmial coincide with that of its Newton's method and Chebyshev's method.

Keywords: Complex dynamical system; Rational maps; Fatou and Julia sets; Symmetry group of Julia sets.

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1 Introduction

Complex Dynamics deals with the iteration of analytic functions on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$ For a non-constant rational function R, the extended complex plane $\hat{\mathbb{C}}$ is partitioned into two disjoint sets namely, the Fatou set and the Julia set. The Fatou set of R, denoted by $\mathcal{F}(R)$, is defined as the maximal open subset of $\widehat{\mathbb{C}}$ where $\{R^n\}_{n>0}$ is equicontinuous. The complement of $\mathcal{F}(R)$ in $\widehat{\mathbb{C}}$ is called the Julia set of R and is denoted by $\mathcal{J}(R)$. By definition the Fatou set is open, whereas the Julia set is closed. Also note that $\mathcal{F}(R) = \mathcal{F}(R^k)$ for all $k \geq 1$. Further details can be found in [\[3,](#page-27-0) [15\]](#page-27-1).

A point $z_0 \in \widehat{\mathbb{C}}$ is said to be a fixed point of R if its image under R is itself. A fixed point can be classified according to its multiplier λ . If $z_0 \in \mathbb{C}$, then λ is defined as $\lambda = R'(z_0)$ and whenever ∞ is a fixed point of R, its multiplier is defined as $g'(0)$ where $g(z) = \frac{1}{R(\frac{1}{z})}$. Now, z_0 is said to be an attracting fixed point if λ lies in the unit disk (i.e., $|\lambda| < 1$). In a particular case, z_0 is superattracting whenever $\lambda = 0$. If λ lies in the exterior of the closed unit disk then z_0 is repelling and whenever λ is on the unit circle, z_0 is said to be indifferent. An indifferent fixed point is said to be rationally indifferent or parabolic if and only if λ is a root of unity (i.e., for some $n \in \mathbb{N}$, $\lambda^n = 1$), else it is called irrationally indifferent. A point z^* is called a periodic point of R with period p, in short a p-periodic point if it is a fixed point of R^p but $R^q(z^*) \neq z^*$ for any $q < p$. The classification of z^* can be done in same way considering it to be a fixed point of the rational map R^p . The set $\{z^*, R(z^*), \ldots, R^{p-1}(z^*)\}$ is called a p-periodic cycle. The Fatou set is open but not always connected. A maximal open connected subset of the Fatou set is called a Fatou component. A Fatou component U of of a rational map R is called p-periodic if $R^p(U) \subseteq U$. A Fatou component U is called pre-periodic if it is not periodic but there exist a natural number k such that $R^k(U)$ is periodic. D. Sullivan proved that every Fatou component of a rational map is either periodic or pre-periodic. In fact, there are four types of Fatou components for a rational map R . Let U be a p-periodic Fatou component. Then

- U is said to be an attracting component if $z_0 \in U$, where z_0 is an attracting p-periodic point.
- U is a parabolic component whenever ∂U contains a parabolic p-periodic point.
- U is a Herman ring or a Siegel disk if $R^p : U \mapsto U$ is conformally conjugate to an irrational rotation of some annulus or to the unit disk respectively onto itself.

We say U is a rotation domain if it is either a Herman ring or a Siegel disk. The Julia set is completely invariant under the function and is usually fractal with complicated topology. Though the iterative behavior of the function on its Julia set is chaotic, it often possesses some pattern. More precisely, there may exist a Möbius map σ such that $\sigma(\mathcal{J}(R)) = \mathcal{J}(R)$. The collection of all such maps, denoted by $\mathcal{M}(R)$, is closed under composition of functions and forms a group. This, or sometimes an appropriate subgroup of it, gives an idea, at least approximately about the structure of the Julia set without in fact finding it.

Let X_i be a metric space with the metric d_i for $i = 1, 2$. A map $h: X_1 \to X_2$ is called an isometry if $d_2(h(z), h(w)) = d_1(z, w)$ for all $z, w \in X_1$. An isometry is necessarily one-one. Every analytic Euclidean isometry of $\mathbb C$ is of the form $z \mapsto az+b$ with $|a|=1$. Such an isometry is either a translation (if $a=1$) or a rotation about the point $\frac{b}{1-a}$ (if $a \neq 1$). Similarly, a chordal isometry of $\widehat{\mathbb{C}}$ is a Möbius map of the form $z \mapsto \frac{az-\bar{b}}{bz+\bar{a}}$ where $|a|^2 + |b|^2 = 1$. Here the chordal distance $\rho(z, w)$ between two points in z, w in $\mathbb C$ is given by $\frac{2|z-w|}{\sqrt{z-1}$ $\frac{2|z-w|}{1+|z|^2\sqrt{1+|w|^2}}$ and $\rho(z,\infty) = \frac{2}{\sqrt{1+1}}$ $\frac{2}{1+|z|^2}$. Though no translation is a chordal isometry, all rotations about the origin are isometries with respect to the chordal metric.

Let $\mathcal{M}(R)$ be the set of all Möbius maps preserving the Julia set of R. We consider two subgroups of $\mathcal{M}(R)$, namely

$$
\mathcal{I}(R) = \{s(z) = \frac{az - \bar{b}}{bz + \bar{a}} : |a|^2 + |b|^2 = 1 \text{ and } s(\mathcal{J}(R)) = \mathcal{J}(R)\}\
$$

and

$$
\Sigma R = \{ \sigma(z) = az + b : |a| = 1, a \neq 1 \text{ and } \sigma(\mathcal{J}(R)) = \mathcal{J}(R) \}.
$$

A (non-trivial) translation, i.e., map of the form $z \mapsto z + a$ for some $a \neq 0$ cannot be in $\mathcal{I}(R)$. If ΣR contains rotations with respect to two distinct points then their composition is a translation (see the proof of Theorem [2.4\)](#page-5-0) and is in ΣR . In other words, if ΣR does not contain any translation then each of its elements is a rotation with respect to a point, that depends on R but not on ΣR . Further, if the point is the origin in this case then $\Sigma R \subseteq \mathcal{I}(R)$. We call ΣR is non-trivial if it contains at least one non-identity element.

The study of ΣR , referred as the symmetry group of R, when R is a polynomial is done by Julia, Baker, Eremenko and later by Beardon [\[1,](#page-27-2) [2,](#page-27-3) [4,](#page-27-4) [9\]](#page-27-5). It is known that, for every polynomial p, there is a point $\xi(p)$ such that each element of Σp is a rotation about $\xi(p)$ (see Lemma [2.4\)](#page-5-0). The study of symmetry in rational maps remains relatively underexplored. While some literature exists, such as references [\[10,](#page-27-6) [11,](#page-27-7) [23\]](#page-28-0), discussing rational maps having identical Julia sets, the subject still lacks extensive study. Recently, Ferreira made a systematic study of $\mathcal{I}(R)$ for all rational maps R (see [\[8\]](#page-27-8)). Each element of $\mathcal{I}(R)$ is a rotation of the sphere with respect to some axis passing through the origin. But a rotation in the plane with respect to a non-zero point is not in $\mathcal{I}(R)$. Also, this set $\mathcal{I}(R)$ does not contain any translation.

Thus, Ferreira's work does not accommodate Julia sets that are preserved under two geometrically simple classes of maps, namely translations and rotations of the plane with respect to a non-zero point. The later maps are called rotational symmetries. This article presents a survey of results on rotational symmetries of polynomial Julia sets and the related issue of identical Julia sets for two different polynomials.

Let R be a rational map analytic at $z_0 \in \mathbb{C}$. If its Taylor series about z_0 is given by $a_k(z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + \cdots$ for some $k > 0$ where $a_k \neq 0$ then we say the local degree of R at z_0 , denoted by $deg(R, z_0)$ is k. The map R is like $z \mapsto z^k$ near z_0 . The local degree of R at ∞ or at a pole is defined by a change of coordinate using $z \mapsto \frac{1}{z}$. More precisely, if $R(\infty) \in \mathbb{C}$ then $\deg(R, \infty)$ is defined as the degree of $R(\frac{1}{z})$ $\frac{1}{z}$ at 0. If $R(\infty) = \infty$ then $\deg(R, \infty)$ is defined as $\deg(\frac{1}{R(\frac{1}{z})}, 0)$. A point w is exceptional for a rational map R if $deg(R, w)$ is equal to the degree of R. This is equivalent to the statement that $\{z: R^n(z) = w \text{ for some } n \geq 0\}$ is finite. There can be at most two exceptional points for any rational map (Theorem 4.1.2., [\[3\]](#page-27-0)). This article shows, under some condition that a rational map with an exceptional point has rotational symmetry. Two classes of rational maps are presented whose Julia sets have rotational symmetries.

In Section 2, a systematic discussion of the symmetry group of polynomial Julia sets is made. Results relating the rotational symmetries of polynomial Julia sets with two polynomials with identical Julia sets are dealt with in Section [3.](#page-7-0) Section 4 deals with the rotational symmetries of rational Julia sets. If a rational map R has an exceptional point then existence of rotational symmetries of the Julia sets of R is proved under some condition (see Theorem [4.2\)](#page-11-0). We also introduce two forms of rational maps whose Julia sets have rotational symmteries of finite order (see Theorem [4.7,](#page-15-0) [4.8\)](#page-18-0).

All the polynomials and rational maps are assumed to be of degree at least two, unless stated otherwise. By a translation or a rotation, we mean a non-identity translation or rotaion respectively, unless stated otherwise.

2 Symmetries of polynomial Julia sets

Let

$$
p(z) = a_d z^d + a_{d-1} z^{d-1} + \dots + a_0 \tag{1}
$$

where $a_d \neq 0$ and $d \geq 2$. The centroid of p, denoted by ξ is defined as $\xi =$ $-\frac{a_{d-1}}{a}$ $\frac{d_{d-1}}{da_d}$. For $c \in \mathbb{C}$, the equation $p(z) = c$ has d number of roots counting with multiplicities. A root z^* is counted m-times here if it is with multiplicity m i.e., $p(z) - c = (z - z^*)^m h(z)$ for some analytic function h in a neighborhood of z^* such that $h(z^*) \neq 0$. If the roots of $p(z) = c$ are z_1, z_2, \ldots, z_d , repeated according to their

multiplicities, then p can be expressed as $p(z) = c + a_d \prod_{i=1}^d (z - z_i)$. Comparing the coefficients of z^{d-1} on both the sides, we get $\sum_{i=1}^{d} z_i = -\frac{a_{d-1}}{a_d}$ $\frac{d-1}{a_d}$. Therefore, ξ is the average of the roots of $p(z) = c$ i.e., $\frac{1}{d} \sum_{i=1}^{d} z_i = -\frac{a_{d-1}}{da_d}$ $\frac{d_{d-1}}{da_d}$. It is important to note that ξ is independent of c. Observe that $\xi = 0$ if and only if $a_{d-1} = 0$. A polynomial whose centroid is at the origin, is called centered.

For any polynomial p as given in [\(1\)](#page-3-0), consider the affine map $\psi(z) = Az + \xi$, where ξ is the centroid of p and A is such that $A^{d-1} = \frac{1}{a}$ $\frac{1}{a_d}$. Then the polynomial $g = \psi^{-1} \circ p \circ \psi$ is monic and centered. Such a polynomial is called normalized. The fact that every polynomial is conjugate to a normalized polynomial is crucial for investigating the symmetries of polynomial Julia sets. In fact, we have the following for all rational maps (see Theorem 3.1.4., [\[3\]](#page-27-0)).

Lemma 2.1. If R_1 and R_2 are two rational maps such that $R_1 = \psi^{-1} \circ R_2 \circ \psi$ for some Möbius map ψ then $\mathcal{J}(R_2) = \psi(\mathcal{J}(R_1)).$

For every $\sigma \in \Sigma_p$, $\psi^{-1} \circ \sigma \circ \psi$ is an Euclidean isometry preserving the Julia set of g. Conversely, if $\gamma \in \Sigma$ g then $\psi \circ \gamma \circ \psi^{-1}$ is an Euclidean isometry preserving the Julia set of p. Thus, we have the following.

Lemma 2.2. For every polynomial p, there is an affine map T such that $g =$ $\psi^{-1} \circ p \circ \psi$ is normalized and $\Sigma p = \psi \circ (\Sigma g) \circ \psi^{-1}$.

Note that the affine conjugacy ψ maps 0, the centroid of g to that of p. Now onwards, we discuss the symmetry group of normalized polynomials and without loss of generality assume that p is normalized. If p does not have any constant term in its expression, i.e., if $p(0) = 0$ then take z^{α} common from the expression of p where α is the multiplicity of 0 as a root of p.

If p has a non-zero constant term, we take $\alpha = 0$. Hence it is always possible to express p as $p(z) = z^{\alpha} p_1(z)$, where p_1 is a normalized polynomial, $p_1(0) \neq 0$, and $\alpha \in \mathbb{N} \cup \{0\}$ is maximal for this form. Let $\beta_1, \beta_2, \ldots, \beta_k$ be the (non-zero) powers of z in the expression of p_1 and $\beta = \gcd(\beta_1, \beta_2, \ldots, \beta_k)$. Then p_1 can be expressed as $p_1(z) = z^{m_1\beta} + a_2 z^{m_2\beta} + \cdots + a_k z^{m_k\beta} + a_{k+1}$, where $a_i \neq 0$ for $i = 2, 3, \ldots, k+1$, $m_j \in \mathbb{N}$ for $j = 1, 2, ..., k$, and $gcd(m_1, m_2, ..., m_k) = 1$. Hence,

$$
p(z) = z^{\alpha} p_0(z^{\beta}) \tag{2}
$$

where

$$
p_0(z) = z^{m_1} + a_2 z^{m_2} + \dots + a_k z^{m_k} + a_{k+1}
$$
 (3)

is a monic polynomial. Note that p_0 is not necessarily centered whereas p_1 is always so. Further note that, α and β are maximal for the expression [\(2\)](#page-4-0) and they determine p_0 completely.

If $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the distinct roots of p_0 with multiplicities b_1, b_2, \ldots, b_r respectively, then we can write p as

$$
p(z) = z^{\alpha} \prod_{s=1}^{r} (z^{\beta} - \lambda_s)^{b_s}.
$$

Therefore, all the non-zero roots of p can be partitioned into r number of sets $A_s = \{z : z^{\beta} = \lambda_s\}$ where $s = 1, 2, \ldots, r$. Each element of A_s lies on a circle of radius $|\lambda_s|^{1/\beta}$ around the origin, and each of these differs from its nearest one by an argument of $\frac{2\pi}{\beta}$. Hence each rotation about the origin of order β preserves every A_s . In other words, each such rotation takes a root of p to another root with the same modulus and with the same multiplicity. These rotations are going to be the elements of Σp .

Since ∞ is a superattracting fixed point of each polynomial, it has a neighborhood contained in the Fatou set. Thus we have the following.

Lemma 2.3. [\[3\]](#page-27-0) For every polynomial p, $\mathcal{J}(p)$ is bounded.

An Euclidean isometry $\sigma(z) = az+b$, $|a|=1$ is either a translation (if $a=1$) or a rotation about the point $\frac{b}{1-a}$ (if $a \neq 1$). Indeed, $\phi \sigma \phi^{-1}(z) = az$ where $\phi(z) = z - \frac{b}{1-a}$ $\frac{b}{1-a}$. Now we identify possible elements of Σp .

Lemma 2.4. For each polynomial p, there is a point $\xi(p)$ such that every element of Σp is a rotation about $\xi(p)$.

Proof. If there is a translation T in Σp then $T^n \in \Sigma p$ for all $n \in \mathbb{N}$. For $z \in \mathcal{J}(p)$, $T^n(z) \in \mathcal{J}(p)$ whereas $T^n(z) \to \infty$ as $n \to \infty$. This gives that $\mathcal{J}(p)$ is unbounded, contradicting Lemma [2.3.](#page-5-1) Therefore Σp does not contain any non-trivial (i.e., nonidentity) translation.

Suppose that $\sigma, \gamma \in \Sigma p$ are two non-trivial (i.e., non-identity) rotations about two distinct points α and β respectively. Then $\sigma(z) = ze^{i\theta} + \alpha(1 - e^{i\theta})$ and $\gamma(z) =$ $ze^{it} + \beta(1 - e^{it})$, for some $\theta, t \in (0, 2\pi)$. Note that $\sigma^{-1}(z) = ze^{-i\theta} + \alpha(1 - e^{-i\theta})$ and $\gamma^{-1}(z) = ze^{-it} + \beta(1 - e^{-it})$. As Σp is a group under composition of functions, $\gamma \circ \sigma \circ \gamma^{-1} \circ \sigma^{-1} \in \Sigma p$. But $\gamma \circ \sigma \circ \gamma^{-1} \circ \sigma^{-1}(z) = z + (\alpha - \beta)(e^{it} - 1)(1 - e^{i\theta}),$ which is a non-trivial translation. Since this is already known to be impossible, $\alpha = \beta$. Therefore, there is a point $\xi(p)$ such that every element of Σp is a rotation about $\xi(p)$. \Box

Theorem [2.4](#page-5-0) states that all the elements of Σp are rotations with respect to a single point, which possibly depends on p . What can that point be? To answer this question, recall that the average of all solutions of the equation $p(z) = c$ is ξ , the centroid of p, irrespective of the value of c. For every root w of $p(z) = c$, the

average of all solutions of $p(z) = w$ is also ξ . In general, the average of all solutions of $p^{-n}(z) = c$ is ξ , for every $n \in \mathbb{N}$ and $c \in \mathbb{C}$. Now consider a point $z_0 \in \mathcal{J}(p)$. By the backward invariance of the Julia set, every open set containing the Julia set contains the set $\{z : p^n(z) = z_0\}$ for all n. In fact, the set $\{z : p^n(z) = z_0\}$ is in a sense uniformly distributed in the Julia set. To see it, let $\epsilon > 0$ and the Julia set of p be covered by finitely many balls B_i , $i = 1, \dots k$, each with radius $\frac{\epsilon}{2}$. This is possible as $\mathcal{J}(p)$ is compact. Since z_0 is not exceptional (because all exceptional points belong to the Fatou set), there is an n_i such that $z_0 \in p^n(B_i)$ for all $n > n_i$ (by Theorem 6.9.4, [\[3\]](#page-27-0)). If $N = \max_{1 \leq i \leq k} \{n_i\}$ then $z_0 \in p^n(B_i)$ for all $n > N$ and for all *i*. Let $z_i \in B_i$ such that $p^n(z_i) = z_0$. Here z_i depends on *n*. Now the union of balls with radius ϵ and with center at the points of $\{z : p^n(z) = z_0\}$ contains $\mathcal{J}(p)$ for all $n > N$. This is a reason why the centroid is expected to be the point stated in Theorem [2.4.](#page-5-0)

If a normalized polynomial p is affine conjugate to a monomial then its Julia set is a circle whose center is 0, the centroid of p. In this case, Σp contains all the rotations about 0. Therefore, Σp is an infinite set. Beardon proves that the converse of this statement is also true (see Lemma 4, [\[2\]](#page-27-3)).

Consider p which is not conjugate to any monomial. There is a conformal map ϕ in a neighborhood of ∞ , called the Böttcher coordinate into the unit disk such that $\phi \circ p \circ \phi^{-1}(z) = z^d$ where d is the degree of p. The function $|\phi|$ extends continuously to the whole basin of attraction, $\mathcal A$ of ∞ . Using this ϕ , the Green's function $\log |\phi(z)|$ is defined in A with the pole at ∞ and further analysis gives that every element of Σp is rotation about the origin. In fact, Beardon proved the following.

Theorem 2.5 ([\[3\]](#page-27-0)). Let p be a normalized polynomial of degree $d \geq 2$. A rotation σ of finite order about the origin is in Σ g if and only if $p \circ \sigma = \sigma^d \circ p$.

The symmetry group of Julia set of a normalized polynomial is now described.

Theorem 2.6. ([\[3\]](#page-27-0)) If p is a normalized polynomial of the form [\(2\)](#page-4-0) then $\Sigma p = {\sigma : \sigma$ $\sigma(z) = \lambda z, \lambda^{\beta} = 1$.

Remark 2.1. It is easy to observe that the above theorem is true even if p is not monic but only centered.

In view of Lemma [2.2,](#page-4-1) the symmetry group of the Julia set of an arbitrary polynomial q with centroid at ξ that is conjugate to a normalized polynomial p , i.e., $p = \psi^{-1} \circ q \circ \psi$ for $\psi(z) = Az + \xi$ for a suitable A (see Lemma [2.2\)](#page-4-1) is given by

$$
\Sigma q = \{ \sigma : \sigma(z) = \lambda(z - \xi) + \xi, \lambda^{\beta} = 1 \},
$$

where p is as given in Theorem [2.6.](#page-6-0)

We now discuss few examples.

- **Example 2.1.** 1. The polynomial $q(z) = z^3 + 3z^2 + 3z \frac{1}{3}$ $\frac{1}{3}$ is not normalized and its centroid is $\xi = -1$. For the affine map $\psi(z) = z - 1$, the polynomial $\psi^{-1} \circ q \circ \psi(z) = z^3 - \frac{1}{3}$ $\frac{1}{3}$ is clearly normalized. Writing it in the form [\(2\)](#page-4-0), it is observed that $\alpha = 0$ and $\beta = 3$. Therefore, $\Sigma(\psi^{-1} \circ q \circ \psi) = \{z \mapsto \lambda z : \lambda^3 = 1\}.$ Hence, $\Sigma q = \psi \circ (\Sigma p) \circ \psi^{-1} = \{z \mapsto \lambda (z+1) - 1 : \lambda^3 = 1\}$ (see Fig. [1\(](#page-8-0)a)).
	- 2. The polynomial $p(z) = z^3 1.2iz$ is normalized and is in the prescribed form [\(2\)](#page-4-0) with $\alpha = 1$ and $\beta = 2$. Therefore, $\Sigma p = \{z \mapsto \lambda z : \lambda^2 = 1\}$ (see Fig. [1\(](#page-8-0)b)).
	- 3. The symmetry group of the normalized polynomial $p(z) = z^3 z 0.5i$ is trivial as it is in the form [\(2\)](#page-4-0) where $\alpha = 0$ and $\beta = 1$. However, there is a non-Möbius homeomorphism preserving its Julia set. In fact, $p(-\bar{z}) = (-\bar{z})^3 - (-\bar{z}) - 0.5i =$ $-(\overline{z^3}-\overline{z}-\overline{0.5i})=-\overline{p(z)}$. Therefore, $\mathcal{J}(p)$ is preserved under the reflection about the imaginary axis (see Fig. [1\(](#page-8-0)c)). That $z \mapsto -\overline{z}$ is a chordal isometry is used here.

3 Polynomials with the same Julia set

When two polynomials have the same Julia set? This question is closely related to the symmetries of the Julia set. We start with the following result which is proved by Julia in 1922.

Theorem 3.1. [\[9\]](#page-27-5) If two polynomials p and q commute (i.e., $p \circ q = q \circ p$) then $\mathcal{J}(p) = \mathcal{J}(q).$

The above result is also true for all rational maps and a proof can be found in Theorem 4.2.9., [\[3\]](#page-27-0). A kind of converse is obtained by Baker and Eremenko.

Theorem 3.2. [\[1\]](#page-27-2) If two polynomials p and q have the same Julia set $\mathcal J$ then either the polynomials commute or there exists a non-identity Euclidean isometry σ such that $\sigma(\mathcal{J}) = \mathcal{J}$.

After Ritt [\[24,](#page-28-1) [25\]](#page-28-2) initiated the study on commuting rational maps, numerous subsequent studies have been conducted. Examples include references [\[14,](#page-27-9) [19,](#page-28-3) [20\]](#page-28-4). If Σp consists of the identity only then $\mathcal{J}(p) = \mathcal{J}(q)$ guarantees that the polynomials commute. But if the symmetry group of p contains at least one non-identity element, the converse of Theorem [3.1](#page-7-1) may not be true. To see it, consider $p(z) = z^2 - 1$ and $q(z) = -z^2 + 1$. Then $q(z) = -p(-z)$ and this gives that $\mathcal{J}(q) = \sigma(\mathcal{J}(p))$ where $\sigma(z) = -z$. Also, $\sigma \in \Sigma p$ by Theorem [2.6.](#page-6-0) Therefore, $\mathcal{J}(q) = \sigma(\mathcal{J}(p)) = \mathcal{J}(p)$ (see Fig. [2\)](#page-9-0). However $p(q(z)) = z^4 - 2z^2 = -q(p(z))$. More generally, we obtain the following result.

(a) Julia set of $z^3 - \frac{1}{3}$ 3 (b) Julia set of $z^3 - 1.2iz$

(c) Julia set of $z^3 - z - 0.5i$

Figure 1: Symmetries of the Julia sets

Theorem 3.3. Let p be a normalized polynomial of the form [\(2\)](#page-4-0) such that $\alpha \neq 1$ and $\beta \geq 2$. If β does not divide $(\alpha - 1)^2$ then for every non-identity $\sigma \in \Sigma_p$, $q := \sigma \circ p \circ \sigma^{-1}$ has the same Julia set as that of p but $q \circ p \neq p \circ q$.

Proof. For each $\sigma \in \Sigma p$, $\mathcal{J}(p) = \sigma(\mathcal{J}(p))$. Also, as $q = \sigma \circ p \circ \sigma^{-1}$, $\mathcal{J}(q) = \sigma(\mathcal{J}(p))$ (by Theorem 3.1.4., [\[3\]](#page-27-0)). Hence we get $\mathcal{J}(p) = \mathcal{J}(q)$.

Any non-identity $\sigma \in \Sigma p$ is of the form $\sigma(z) = \lambda z$, where $\lambda^{\beta} = 1$ (by Theo-

Figure 2: The Julia set of $z^2 - 1$ and $-z^2 + 1$

rem [2.6\)](#page-6-0). Thus,

$$
q(z) = \sigma p\left(\frac{1}{\lambda}z\right) = \sigma \left[\frac{1}{\lambda^{\alpha}}z^{\alpha}p_0(z^{\beta})\right]
$$

$$
= \frac{1}{\lambda^{\alpha-1}}z^{\alpha}p_0(z^{\beta}).
$$

Therefore,

$$
p(q(z)) = p\left(\frac{1}{\lambda^{\alpha-1}}z^{\alpha}p_0(z^{\beta})\right) = \left[\frac{1}{\lambda^{\alpha-1}}z^{\alpha}p_0(z^{\beta})\right]^{\alpha}p_0\left(\left[\frac{1}{\lambda^{\alpha-1}}z^{\alpha}p_0(z^{\beta})\right]^{\beta}\right)
$$

$$
= \frac{1}{\lambda^{\alpha(\alpha-1)}}z^{\alpha^2}[p_0(z^{\beta})]^{\alpha}p_0\left(z^{\alpha\beta}(p_0(z^{\beta}))^{\beta}\right).
$$

The last equation is due to the fact that $\left(\frac{1}{\lambda^{\alpha}}\right)$ $\frac{1}{\lambda^{\alpha-1}}\big)^\beta=\big(\frac{1}{\lambda^\beta}$ $\frac{1}{\lambda^{\beta}}$ $)^{\alpha-1}$ = 1. However,

$$
q(p(z)) = q(z^{\alpha}p_0(z^{\beta})) = \frac{1}{\lambda^{\alpha-1}}z^{\alpha^2}[p_0(z^{\beta})]^{\alpha}p_0(z^{\alpha\beta}(p_0(z^{\beta}))^{\beta}).
$$

This implies that the polynomials p and q commute if and only if $\frac{1}{\lambda^{\alpha(\alpha-1)}} = \frac{1}{\lambda^{\alpha}}$ $\frac{1}{\lambda^{\alpha-1}},$ which gives $\lambda^{(\alpha-1)^2} = 1$. As $\lambda = e^{\frac{2\pi i}{\beta}}, \alpha \neq 1$ and $\beta > 0, k > 0$ and consequently, β divides $(\alpha - 1)^2$. Therefore, if β does not divide $(\alpha - 1)^2$ then for every non-identity $\sigma \in \Sigma p$, $\mathcal{J}(\sigma \circ p \circ \sigma^{-1}) = \mathcal{J}(p)$ but $p \circ q \neq q \circ p$. Since $\beta \geq 2$, there is a non-identity element in Σp , and we are done. \Box

In 1990, Beardon established a necessary and sufficient condition for polynomials with the same Julia set. In a way, this is a complete description of the relation between the polynomials with identical Julia sets and the rotational symmetry of the Julia set.

Theorem 3.4. ([\[2\]](#page-27-3)) The polynomials p and q share the same Julia set if and only if there is some $\sigma \in \Sigma p$ such that $p \circ q = \sigma q \circ p$.

The previous four theorems are about the uniqueness of the Julia set of different polynomials. Now the uniqueness of polynomials is considered assuming some relation between their Julia sets. In this line, there is a result by Fernandez.

Theorem 3.5. ([\[7\]](#page-27-10)) Let p and q be polynomials of the same degree and with the same leading coefficient. If the Julia set of p is disjoint from the unbounded Fatou component (i.e., the basin of ∞) of q then $p = q$.

As a consequence, it is obtained that if two polynomials with the same degree and the same leading coefficient have the same Julia set then they are the same.

Beardon revealed a beautiful connection between the polynomials with the same degree having identical Julia set.

Theorem 3.6. ([\[4\]](#page-27-4)) Let p be a polynomial of degree d. Any polynomial q of the same degree d has the same Julia set as that of p if and only if $q = \sigma p$ for some $\sigma \in \Sigma p$.

If Σp is trivial then the above theorem gives that if q is a polynomial with the same degree and with the same Julia set as that of p then $q = p$.

We conclude with a result with the same spirit by Schmidt and Steinmetz [\[21\]](#page-28-5).

Theorem 3.7. Let \mathcal{J} be a Julia set of a polynomial which is neither a circle nor a line segment. Then there exists a polynomial p such that any polynomial q with the Julia set $\mathcal J$ can be written in the form $q(z) = \sigma p^{n}(z)$, where σ is a rotation (including identity) with $\sigma(\mathcal{J}) = \mathcal{J}$, and n is a natural number.

4 Symmetries of rational Julia sets

This section deals with symmetries of rational maps that are not polynomials. Recall that $\mathcal{M}(R) = \{\phi : \phi \text{ is a Möbius map such that } \phi(\mathcal{J}(R)) = \mathcal{J}(R)\}\.$ The set $\mathcal{M}(R)$ is respected by conformal conjugacy in the same way as the affine conjugacies respect the set of Euclidean isometries of polynomial Julia sets (see Lemma [2.2\)](#page-4-1).

Lemma 4.1. If two rational maps R and S are conjugate with the Möbius map ϕ , such that $S = \phi \circ R \circ \phi^{-1}$, then $\mathcal{M}(S) = \phi \circ (\mathcal{M}(R)) \circ \phi^{-1}$.

4.1 Rational maps with an exceptional point

Polynomials are rational maps with at least one exceptional point. It is well-known that a rational map has at most two exceptional points. If it has exactly two exceptional points then it is conjugate to z^d for some non-zero integer d (Theorem 4.1.2, [\[3\]](#page-27-0)). To see it, Let ζ_1 and ζ_2 be the two exceptional points of R. Then,

there exists a complex number $a \in \mathbb{C}$ such that for the Möbius map $\phi(z) = a \frac{z-\zeta_1}{z-\zeta_0}$ $\frac{z-\zeta_1}{z-\zeta_2},$ the function $p(z) = \phi \circ R \circ \phi^{-1}(z)$ is z^d for some non-zero integer d. In both the cases, $\mathcal{J}(p)$ is the unit circle. Further, $\mathcal{J}(R) = \phi^{-1}(\mathcal{J}(p))$ by Lemma [2.1.](#page-4-2) Here $\phi^{-1}(z) = \frac{\zeta_2 z - a\zeta_1}{z-a}$ and has its pole at a. Hence the Julia set of R is either a circle (if $|a| \neq 1$ or a straight line $(|a| = 1)$. This is so because the image of the unit circle under any Möbius map is either a circle or a straight line, and it is a straight line if and only if the pole of the map is on the unit circle.

If a rational map R has exactly one exceptional point, say w then for $h(z) = z-w$, $h \circ R \circ h^{-1}$ is a rational map whose only exceptional point is 0. A rotation about w preserves $\mathcal{J}(R)$ if and only if a rotaion by the same angle about the origin preserves $\mathcal{J}(h \circ R \circ h^{-1})$. Since the rotational symmetries are the main concern of this article, we assume without loss of generality that the exceptional point of R is 0. Since 0 is the only exceptional point of $R, R^{-1}(0) = \{0\}$ and therefore, R is of the form

$$
R(z) = \frac{z^d}{a_0 z^d + a_1 z^{d-1} + \dots + a_{d-1} z + a_d},\tag{4}
$$

where $a_d \neq 0$. As R has exactly one exceptional point it is conjugate to a (nonmonomial) polynomial and hence it is conjugate to a normalized polynomial. Two different polynomials that are conjugate to R are conjugate to the same normalized polynomial. Let β be the order of the symmetry group of this normalized polynomial. This β is a property of R, and we call it the order of rotational symmetries of R. The following theorem finds all Möbius maps, arising out of this normalized polynomial and preserving the Julia sets of R.

Theorem 4.2. If R is rational map with only one exceptional point, that is 0 and is of the form [\(4\)](#page-11-1) then $\mathcal{M}(R)$ contains the set $\{z \mapsto \frac{z}{\zeta(1-\lambda)z+\lambda} : \lambda^{\beta} = 1\}$, where $\zeta = -\frac{a_{d-1}}{da}$ $\frac{d_{d-1}}{da_d}$ and β is the order of rotational symmetries of R.

Proof. For any Möbius map φ , if $\varphi \circ R \circ \varphi^{-1}$ is a polynomial, then $\varphi(0) = \infty$ and $\varphi(z) = \frac{Az+B}{z}$ for some $A, B \in \mathbb{C}$ and $B \neq 0$. Let $p_1(z) = \varphi \circ R \circ \varphi^{-1}(z)$. Then

$$
p_1(z) = \varphi \circ R\left(\frac{B}{z - A}\right)
$$

= $\varphi \left(\frac{B^d}{a_d(z - A)^d + Ba_{d-1}(z - A)^{d-1} + \dots + a_0 B^d}\right)$
= $\frac{(a_d(z - A)^d + Ba_{d-1}(z - A)^{d-1} + \dots + a_0 B^d) + AB^{d-1}}{B^{d-1}}.$

Here the centroid of p_1 is $\xi = -\frac{-da_dA + Ba_{d-1}}{da_d}$ $\frac{d_4 + Ba_{d-1}}{da_d} = A + B\zeta$ where ζ is as given in the statement of this theorem. Now consider the affine map $\psi(z) = \alpha z + \xi$, where $\alpha^{d-1} =$ B^{d-1} $\frac{a^{d-1}}{a_d}$. Then $p(z) = \psi^{-1} \circ p_1 \circ \psi(z)$ is a normalized polynomial. Let $p(z) = z^{\alpha} p_0(z^{\beta})$

Figure 3: The conjugacy

where α and β are maximal for this expression. Then $p(z) = T \circ R \circ T^{-1}(z)$, where $T(z) = \psi^{-1} \circ \varphi(z) = \psi^{-1} \left(\frac{Az+B}{z} \right)$ $\left(\frac{x+B}{z}\right) = \frac{(A-\xi)z+B}{\alpha z}$ $\frac{\xi}{\alpha z}$. If $\sigma \in \Sigma p$, where $\sigma(z) = \lambda z$ then $\lambda^{\beta} = 1, T^{-1} \circ \sigma \circ T \in \mathcal{M}(R)$. Now,

$$
T^{-1} \circ \sigma \circ T(z) = T^{-1} \circ \sigma \left(\frac{(A - \xi)z + B}{\alpha z} \right) = T^{-1} \left(\lambda \frac{(A - \xi)z + B}{\alpha z} \right)
$$

$$
= \frac{B}{\alpha \lambda \left(\frac{(A - \xi)z + B}{\alpha z} \right) - A + \xi} = \frac{Bz}{(1 - \lambda)(\xi - A)z + \lambda B}
$$

$$
= \frac{z}{(1 - \lambda)\zeta z + \lambda}.
$$

This gives that $\{z \mapsto \frac{z}{\zeta(1-\lambda)z+\lambda} : \lambda^{\beta} = 1\} \subseteq \mathcal{M}(R)$.

Here are few remarks.

Remark 4.1. 1. Observe that ζ is the centroid of the polynomial $\frac{1}{R(\frac{1}{z})}$.

2. The set $\{z \mapsto \frac{z}{\zeta(1-\lambda)z+\lambda} : \lambda^{\beta} = 1\}$ is independent of the choice of the Möbius maps that conjugate R to a normalized polynomial.

 \Box

This theorem guarantees that, if R is conjugate to a polynomial whose Julia set is invariant under a rotation, then $\mathcal{M}(R)$ contains a non-identity Möbius map. In a particular, we get the following.

Corollary [4](#page-11-1).1. If $a_{d-1} = 0$ in the Equation 4 then $\{z \mapsto \lambda z : \lambda^{\beta} = 1\} \subseteq \mathcal{M}(R)$.

Further, if we consider ΣR then applying few more restrictions we get the equality in the relation between Σp and ΣR .

Corollary 4.2. If $a_{d-1} = 0$ in the Equation [4,](#page-11-1) $\beta \geq 2$ and $\mathcal{J}(R)$ is not translation *invariant then* $\{z \mapsto \lambda z : \lambda^{\beta} = 1\} = \Sigma R$.

Proof. Let $p(z) = \frac{1}{R(\frac{1}{z})}$. The relation $\Sigma p \subseteq \Sigma R$ follows from Corollary [4.1.](#page-12-0) As $\mathcal{J}(R)$ is not translation invariant, ΣR contains rotations about origin. Let there is a rotation $\sigma(z) = \lambda z$, $|\lambda| = 1$ of order $\beta_1 > \beta$. Now $\phi \circ \sigma \circ \phi^{-1}(z) = \frac{z}{\lambda}$, where $\phi(z) = \frac{1}{z}$, is a rotation of order β_1 and as $\mathcal{J}(p) = \phi(\mathcal{J}(R))$, we get $\phi \circ \sigma \circ \phi^{-1} \in \Sigma p$. But it contradicts the maximality of β in the expression of p. Hence, $\{z \mapsto \lambda z :$ $\lambda^{\beta} = 1$ } = ΣR . \Box

Here are two examples of rational maps $\frac{3z^3}{3-z}$ $rac{3z^3}{3-z^3}$ and $rac{z^3}{1-2}$ $\frac{z^3}{1-2z^2}$ that are conjugate to $z^3 - \frac{1}{3}$ $\frac{1}{3}$ and $z^3 - 2z$ respectively via the Möbius map $z \mapsto \frac{1}{z}$. For both the cases $\beta = 3$, giving that the Julia sets of the mentioned rational maps are not translation invariant. Their Julia sets are provided in Fig. [4](#page-13-0) as the boundary of different colors (of blue and yellow in (a) and of red and green in (b)).

 $rac{3z^3}{3-z^3} \sim z^3 - \frac{1}{3}$ 3 $\frac{z^3}{1-2z^2} \sim z^3 - 2z$ Figure 4: Julia sets of $\frac{3z^3}{3-z}$ $rac{3z^3}{3-z^3}$ and $rac{z^3}{1-2}$ $1-2z^2$

4.2 Two new classes of rational maps

A rational map R is said to be exceptional if $\mathcal{J}(R)$ is the whole extended complex plane $\widehat{\mathbb{C}}$, a circle or a line segment in $\widehat{\mathbb{C}}$. This definition is upto conformal conjugacy. In other words, each rational map which is conformally conjugate to an exceptional rational map is also exceptional and it can have an arc of a circle or a straight line as its Julia set. Examples of exceptional rational maps are well-known, e.g., the Julia set of $\frac{z^2-2}{z^2}$ $rac{2-2}{z^2}$ and $rac{z^2-2}{2z}$ $\frac{2-2}{2z}$ are $\widehat{\mathbb{C}}$ and $\mathbb{R} \cup \{\infty\}$ respectively (see Remark [4.6\)](#page-20-0). The Julia set of every Chebyshev polynomial is $[-1, 1]$ (page 11, [\[3\]](#page-27-0)).

A Möbius map ϕ is said to be of order β if ϕ^{β} , β times composition of ϕ with itself, is the identity map. An irrational rotation (i.e., rotation by an angle of an irrational multiple of 2π) and a translation are with infinite order whereas a rational rotation (i.e., rotation by an angle of a rational multiple of 2π) is of finite order. Note that $\mathbb C$ is preserved by every Möbius map and straight lines are preserved by translations whereas circles are preserved by irrational rotations. These facts probably motivate a conjecture mentioned in $\left[5\right]$ which states that if there is a Möbius map of infinite order preserving the Julia set of R then R is exceptional. In 2000, Boyd settles it partially by proving the following.

Theorem 4.3 ([\[5\]](#page-27-11)). Let R be a rational map of degree at least two such that $\mathcal{J}(R) = T(\mathcal{J}(R))$, where $T(z) = z + 1$ and the point at infinity is either periodic or preperiodic. Then $\mathcal{J}(R)$ is either the whole extended complex plane or a horizontal line.

The translation by 1 in the above theorem is not any loss of generality. It is not known, at least to the present authors whether there is a non-exceptional rational map whose Julia set is not a line but is invariant under some translation.

The fact that a translation fixes ∞ and ∞ is a periodic or pre-periodic point of R are important in the above theorem. Levin generalized this result in 2001 (see [\[12\]](#page-27-12)).

Theorem 4.4. Let R be a rational map and ϕ be a Möbius map such that $\phi(\mathcal{J}(R)) =$ $\mathcal{J}(R)$. If a fixed point of ϕ is a periodic or pre-periodic point of R and R is not exceptional then ϕ is of finite order.

The above theorem ensures the existence of rotational symmetries for nonexceptional rational maps satisfying some conditions.

Ferreira [\[8\]](#page-27-8) considers all chordal isometries $z \mapsto \frac{\alpha z-\bar{\beta}}{\beta z+\bar{\alpha}}$, $|\alpha|^2 + |\beta|^2 = 1$, that preserve rational Julia sets. Though this seems to be a natural way to generalize, two significant classes of maps, namely rotations about any non-zero point and translations, being non chordal isometries are left out of the consideration. However, all rotations about the origin are chordal isometries and some results obtained by Ferreira remains useful in our consideration of rotational symmetries. Recall that $\mathcal{I}(R) = \{s(z) = \frac{az-\bar{b}}{bz+\bar{a}} : |a|^2 + |b|^2 = 1 \text{ and } s(\mathcal{J}(R)) = \mathcal{J}(R)\}.$

Lemma 4.5. ([\[8\]](#page-27-8)) Let R be non-exceptional. For a natural number n and a chordal isometry σ , if $R \circ \sigma = \sigma^n \circ R$ then $\sigma \in \mathcal{I}(R)$.

Since every rotation about the origin is a chordal isometry as well as an Euclidean isometry, we have the following useful remark.

Remark 4.2. If σ is a rotation about the origin and $R \circ \sigma = \sigma^n \circ R$ for some natural number n then $\sigma \in \Sigma R$.

The next result offers a necessary condition for a chordal isometry to be in $\mathcal{I}(R)$.

Lemma 4.6. ([\[8\]](#page-27-8)) Let R be a non-exceptional rational map without any parabolic or rotation domain. If $\sigma \in \mathcal{I}(R)$ fixes a superattracting fixed point z_0 of R then $R \circ \sigma = \sigma^m \circ R$ where $m \geq 2$ is the local degree of R at z_0 .

Note that, if the point at infinity is not a point in the Julia set of R, then $\mathcal{J}(R)$ is not translation invariant. As the composition of rotations about two different points is a translation, ΣR contains rotations about a single point whenever $\mathcal{J}(R)$ is not invariant under translation. The following result provides a class of non-exceptional rational maps R such that ΣR contains rotations about the origin.

Theorem 4.7. Let P and Q be two non-monomial and centered polynomials without any common factor except possibly 0 such that $R(z) = \frac{P(z)}{Q(z)}$ is non-exceptional rational map without any parabolic domain or a rotation domain, and its Julia set is not invariant under any (non-trivial) translation. Further, let $P(z) = a_1 z^{\alpha_1} P_0(z^{\beta_1})$ and $Q(z) = a_2 z^{\alpha_2} Q_0(z^{\beta_2})$, where α_i, β_i are maximal for the respective expressions (refer Equation [\(2\)](#page-4-0)) and $a_1, a_2 \in \mathbb{C} \setminus \{0\}$ are the leading coefficients of P and Q respectively. If $\alpha_1 > \alpha_2 + 1$ and $\beta = \gcd(\beta_1, \beta_2) > 1$ then $\Sigma R = \{z \mapsto \lambda z : \lambda^{\beta} = 1\}.$

Proof. It follows from the assumption that

$$
R(z) = az^m \frac{P_0(z^{\beta_1})}{Q_0(z^{\beta_2})},\tag{5}
$$

where $a = \frac{a_1}{a_2}$ $\frac{a_1}{a_2}$ and $m = \alpha_1 - \alpha_2$. Recall from [\(3\)](#page-4-3) that $P_0(z^{\beta_1})$ and $Q_0(z^{\beta_2})$ 2^{β}) are normalized polynomials with non-zero constant terms. Also, $\beta = \gcd(\beta_1, \beta_2) > 1$.

For $\sigma(z) = \lambda z$ with $\lambda^{\beta} = 1$, $R \circ \sigma(z) = a \lambda^m z^m \frac{P_0(z^{\beta_1})}{Q_0(z^{\beta_2})}$ $\frac{P_0(z^{\beta_1})}{Q_0(z^{\beta_2})} = \sigma^m \circ R(z)$. This gives that $\sigma \in \Sigma R$ by the Remark [4.2.](#page-14-0) Therefore, $\{z \mapsto \lambda z : \lambda^{\beta} = 1\} \subseteq \Sigma R$.

In order to prove $\Sigma R \subseteq \{z \mapsto \lambda z : \lambda^{\beta} = 1\}$, let $P_0(z) = z^{m_1} + a_2 z^{m_2} + \cdots + a_k z^{m_k} +$ a_{k+1} and $Q_0(z) = z^{n_1} + b_2 z^{n_2} + \cdots + b_r z^{n_r} + b_{r+1}$ where $a_i \neq 0$ for $i = 2, 3, \ldots, k+1$, $\gcd\{m_1, m_2, \ldots, m_k\} = 1$, and $b_i \neq 0$ for $i = 2, 3, \ldots, r+1$ and $\gcd\{n_1, n_2, \ldots, n_r\} =$ 1. In particular, P_0 and Q_0 have non-zero constant terms. All these follow from the analysis preceeding Equation [\(2\)](#page-4-0). This along with the assumption $m \geq 2$ give that 0 is a superattracting fixed point of R and $\deg(R, 0) = m$.

If there is a rotation about a non-zero point in ΣR then its composition with $z \mapsto \lambda z, \lambda^{\beta} = 1$ would be a non-trivial translation and it has to be in ΣR . But the Julia set of R is not invariant under any translation by the assumption. Therefore, every element of ΣR is a rotaion about a point depending only on R and that point must be the origin because $z \mapsto \lambda z \in \Sigma R$ and $\lambda^{\beta} = 1$.

Let $\sigma \in \Sigma R$ and $\sigma(z) = \mu z$ for some $|\mu| = 1$. As 0 is a superattracting fixed point of R and $\deg(R, 0) = m$, it follows from Lemma [4.6](#page-15-1) that $R \circ \sigma = \sigma^m \circ R$. This implies that

$$
\frac{P_0(\mu^{\beta_1} z^{\beta_1})}{Q_0(\mu^{\beta_2} z^{\beta_2})} = \frac{P_0(z^{\beta_1})}{Q_0(z^{\beta_2})}.
$$

Let $R_1(z) = \frac{P_0(\mu^{\beta_1} z^{\beta_1})}{Q_0(\mu^{\beta_2} z^{\beta_2})}$ $\frac{P_0(\mu^{\beta_1} z^{\beta_1})}{Q_0(\mu^{\beta_2} z^{\beta_2})}$ and $R_2(z) = \frac{P_0(z^{\beta_1})}{Q_0(z^{\beta_2})}$ $\frac{P_0(z^{p_1})}{Q_0(z^{p_2})}$. Then R_1 and R_2 share the same sets of roots and poles.

Since P is not a monomial, P_0 is non-constant. Let $\xi_1, \xi_2, \ldots, \xi_s$ be the distinct roots of $P_0(z)$. Then the roots of R_1 are the solutions of $z^{\beta_1} = \frac{\xi_i}{\mu^{\beta}}$ $\frac{\xi_i}{\mu^{\beta_1}}$, for $i=1,2,\ldots,s$ whereas the roots of R_2 are the solutions of $z^{\beta_1} = \xi_i$, for $i = 1, 2, \ldots, s$. If there is an *i* such that $\xi_i = \frac{\xi_i}{\mu^{\beta}}$ $\frac{\xi_i}{\mu^{\beta_1}}$, then $\mu^{\beta_1} = 1$, and hence $Q_0(\mu^{\beta_2} z^{\beta_2}) = Q_0(z^{\beta_2})$. This gives that

$$
\mu^{n_1 \beta_2} z^{n_1 \beta_2} + b_2 \mu^{n_2 \beta_2} z^{n_2 \beta_2} + \dots + b_r \mu^{n_r \beta_2} z^{n_r \beta_2} + b_{r+1}
$$

= $z^{n_1 \beta_2} + b_2 z^{n_2 \beta_2} + \dots + b_r z^{n_r \beta_2} + b_{r+1}.$

Comparing the coefficients, we get $\mu^{n_i\beta_2} = 1$ for all $i = 1, 2, ..., r$. This gives that $(\mu^{\beta_2})^{\gcd(n_1,n_2,...,n_r)}$ = 1. As $gcd(n_1, n_2, \ldots, n_r) = 1$, we get $\mu^{\beta_2} = 1$. Therefore, $\mu^{\beta} = 1$ where $\beta = \gcd(\beta_1, \beta_2)$.

Now consider the remaining cases, i.e., when there is no i for which $\xi_i = \frac{\xi_i}{\mu^{\beta}}$ $\frac{\xi_i}{\mu^{\beta_1}}.$ Then $\mu^{\beta_1} \neq 1$.

After renaming the roots of R_1 and R_2 , if required, we can write

$$
\xi_1 = \frac{\xi_2}{\mu^{\beta_1}}, \ \xi_2 = \frac{\xi_3}{\mu^{\beta_1}}, \ \ldots, \ \xi_{s-1} = \frac{\xi_s}{\mu^{\beta_1}}, \ \xi_s = \frac{\xi_1}{\mu^{\beta_1}}.\tag{6}
$$

These give that all the roots of P_0 are of the same modulus and any two nearest such pair of roots differ by an argument of $\frac{2\pi}{s}$. Let a_i be the multiplicity of ξ_i as a root of P_0 for $i = 1, 2, 3, \dots, s$. Since

$$
P_0(\mu^{\beta_1} z^{\beta_1}) = \mu^{\beta_1(a_1 + a_2 + \dots + a_s)} (z^{\beta_1} - \frac{\xi_1}{\mu^{\beta_1}})^{a_1} (z^{\beta_1} - \frac{\xi_2}{\mu^{\beta_1}})^{a_2} (z^{\beta_1} - \frac{\xi_3}{\mu^{\beta_1}})^{a_3} \cdots (z^{\beta_1} - \frac{\xi_k}{\mu^{\beta_1}})^{a_s},
$$

$$
P_0(z^{\beta_1}) = (z^{\beta_1} - \xi_1)^{a_1} (z^{\beta_1} - \xi_2)^{a_2} (z^{\beta_1} - \xi_3)^{a_3} \cdots (z^{\beta_1} - \xi_s)^{a_s},
$$

and $P_0(z^{\beta_1}) = P_0(\mu^{\beta_1}z^{\beta_1})$, it follows from Equation[\(6\)](#page-16-0) that $a_1 = a_2 = \cdots = a_s$. In other words, the multiplicity of each ξ_i is the same. Let it be r. Since ξ_i^s is the same for each *i*, let it be denoted by ξ . Hence $P_0(z) = (z^s - \xi)^r$. As β_1 is maximal for the expression of $P(z) = z^{\alpha_1} P_0(z^{\beta_1})$, the gcd of all the powers of z in the expression of $P_0(z)$ is 1. However, the powers of z in the expression $P_0(z) = (z^s - \xi)^r$ have gcd equal to s. Therefore $s = 1$. It follows from Equation [6](#page-16-0) that $\mu^{s\beta_1} = 1$. This leads to $\mu^{\beta_1} = 1$ and this is a contradiction. \Box

Fig. [5](#page-17-0) illustrates Theorem [4.7](#page-15-0) for $\frac{z^2(z^2-2)}{z^2+1}$ $\frac{(z^2-2)}{z^2+1}$ and $\frac{z^3(z^3+1)}{z^6+1}$ $rac{(z^3+1)}{z^6+1}$.

The assumption in the above theorem that $\mathcal{J}(R)$ is not invariant under any translation can be ensured by putting mild restrictions on the degrees of P and Q.

Figure 5: Julia set of rational maps of the form $a \frac{P(z)}{Q(z)}$ $Q(z)$

Remark 4.3. If $\deg(P) > \deg(Q) + 1$ then R has a superattracting fixed point at ∞ . On the other hand, if $deg(P) < deg(Q)$, $R(\infty) = 0$ and since 0 is a superattracting fixed point of R, ∞ is in the Fatou set of R. In both the cases, the Julia set is bounded and therefore, is not invariant under any translation.

The assumption in Theorem [4.7](#page-15-0) that both P and Q are not monomials can be relaxed.

Remark 4.4. If both P and Q are monomials or only Q but not P is a monomial then $R = \frac{F}{C}$ $\frac{P}{Q}$ is polynomial itself and the issue of rotational symmetry is already discussed in Section 2.

If only P but not Q is a monomial then $R = \frac{P}{Q}$ $\frac{P}{Q}$ takes the form of $\frac{az^m}{Q_0(z^{\beta_2})}$ for some $a \neq 0, m > 1, \beta_2 \geq 2$ where $Q_0(z^{\beta_2})$ is a normalized polynomial in z with non-zero constant term. Note that 0 is a superattracting fixed point of R and its Julia set cannot be $\hat{\mathbb{C}}$. If the Julia set of R is not invariant under any translation then $\Sigma R = \{z \mapsto \lambda z : \lambda^{\beta_2} = 1\}$. The proof of $\{z \mapsto \lambda z : \lambda^{\beta_2} = 1\} \subseteq \Sigma R$ is the same as the first part of the proof of Theorem [4.7.](#page-15-0) In order to prove that $\Sigma R \subseteq \{z \mapsto \lambda z : \lambda^{\beta_2} = 1\},$ first note that every element of ΣR is a rotation about the origin (this follows from the arguments used in the proof of Theorem 4.7). Let $Q_0(z) = z^{n_1} + b_2 z^{n_2} + \cdots + b_r z^{n_r} + b_{r+1}$ where $b_i \neq 0$ for $i = 2, 3, \ldots, r+1$ and $gcd\{n_1, n_2, \ldots, n_r\} = 1$. If $z \mapsto \mu z \in \Sigma R$ then $R(\mu z) = \mu^m R(z)$ by Lemma [4.6.](#page-15-1)

Consequently, $Q_0(\mu^{\beta_2} z^{\beta_2}) = Q_0(z^{\beta_2})$. In other words,

$$
\mu^{n_1 \beta_2} z^{n_1 \beta_2} + b_2 \mu^{n_2 \beta_2} z^{n_2 \beta_2} + \dots + b_r \mu^{n_r \beta_2} z^{n_r \beta_2} + b_{r+1}
$$

= $z^{n_1 \beta_2} + b_2 z^{n_2 \beta_2} + \dots + b_r z^{n_r \beta_2} + b_{r+1}.$

Comparing the coefficients, we get $\mu^{n_i\beta_2} = 1$ for all $i = 1, 2, ..., r$. This gives that $(\mu^{\beta_2})^{\gcd(n_1,n_2,...,n_r)} = 1$. Since $\gcd(n_1,n_2,...,n_r) = 1$, $\mu^{\beta_2} = 1$.

Using Theorem [4.7,](#page-15-0) a partial generalization of Theorem [2.6](#page-6-0) is possible.

Theorem 4.8. Let P be a normalized polynomial of degree d and $P(z) = z^{\alpha} P_0(z^{\beta})$ where α, β are maximal for this expression (refer Equation [\(2\)](#page-4-0)). Also, let $R(z) =$ $az^{\nu}P(z)$ where $a \in \mathbb{C} \setminus \{0\}$ and $\nu \in \mathbb{Z}$ such that it has no parabolic domain or any rotation domain. If $\beta \geq 2$ then $\Sigma P = \Sigma R$ for all ν except the possible values of $-d$ and $-d+1$. For $\nu = -d$ or $-d+1$, we have $\Sigma P \subseteq \Sigma R$.

Proof. Since $\beta \geq 2$, it follows from Theorem [2.6](#page-6-0) that $\Sigma P = \{z \mapsto \lambda z : \lambda^{\beta} = 1\}$ contains a non-identity map. Note that

$$
R(z) = \frac{aP_0(z^{\beta})}{z^{-(\nu+\alpha)}}.
$$

For $\nu \geq -\alpha$, R is itself a centered polynomial and we have $\Sigma P = \Sigma R$ by the remark following Theorem [2.6.](#page-6-0)

Let $\nu < -\alpha$. Choose a positive integer m such that $m - (\alpha + \nu)$ is a positive multiple of β . Then for every $\sigma(z) = \lambda z$ with $\lambda^{\beta} = 1$, $R(\sigma(z)) = \lambda^{m} R(z)$. Hence, by Lemma [4.5,](#page-14-1) $\Sigma P \subseteq \Sigma R$. Note that if $\mathcal{J}(R)$ is not $\widehat{\mathbb{C}}$ and is not invariant under any translation then each element of ΣR is a rotation about the origin. This is to be used later.

Now we look into the possibility of $\Sigma R \subseteq \Sigma P$ and hence the equality of ΣP and ΣR when $\nu < -\alpha$.

Let $-d+2 \leq \nu < -\alpha$.

If $-d+2=-\alpha$ (this happens when $\deg(P_0)=1$ and $\beta=2$) then $\nu=-\alpha$. It is already found that $\Sigma P \subseteq \Sigma R$ in this case. We proceed with the other situation, i.e., $-d+2 < -\alpha$. Since $-d+2 \leq \nu$, $\alpha-d+2 \leq \alpha+\nu$. Let $\deg(P_0) = m_1$. Then $m_1\beta = d - \alpha \geq 2 - (\alpha + \nu)$. This gives that ∞ is a superattracting fixed point of R with local degree $d + \nu$. Then, for any $\sigma \in \Sigma R$, where $\sigma(z) = \lambda z$, $R(\sigma(z)) = \sigma^{d+\nu}(R(z))$ by Lemma [4.6.](#page-15-1) This gives that $P_0(\lambda^{\beta} z^{\beta}) = \lambda^{m_1 \beta} P_0(z^{\beta})$. Let $P_0(z) = z^{m_1} + a_2 z^{m_2} + \cdots + a_k z^{m_k} + a_{k+1}$ where each $a_i \neq 0$ and $gcd(m_1, m_2, \cdots m_k) =$ 1. Comparing the coefficients except the leading term, we get $\lambda^{m_i\beta} = 1$ for all $i = 1, 2, 3, \cdots, k$. Since $\gcd\{m_1, m_2, m_3, \cdots, m_k\} = 1$ we have $\lambda^{\beta} = 1$. Therefore, $\Sigma R \subseteq \{z \mapsto \lambda z : \lambda^{\beta} = 1\}.$ It is important to note here that there is no translation in ΣR as ∞ is in the Fatou set of R.

Now consider $\nu < -d$.

Let $\zeta = -(\nu + \alpha)$. Then $R(z) = a \frac{P_0(z^{\beta})}{z^{\zeta}}$ $\frac{d(z^{\mu})}{dz^{\zeta}}$. As $\zeta > d - \alpha = m_1 \beta$, $R(\infty) = 0$ and $R(0) = \infty$. Now, deg(R, 0) = $\zeta \geq 2$ gives that $\{0, \infty\}$ is a 2-cycle of superattracting periodic points of R. For $F(z) = R(R(z))$, each of 0 and ∞ is a superattracting fixed point of F .

Let $\xi_1, \xi_2, \ldots, \xi_s$ be the distinct roots of $P_0(z)$, i.e., $P_0(z) = (z - \xi_1)^{a_1}(z - \xi_1)^{b_2}$ $(\xi_2)^{a_2} \dots (\xi - \xi_s)^{a_s}$, where $\sum_{i=1}^s a_i = m_1$. Then

$$
F(z) = R\left(a\frac{P_0(z^{\beta})}{z^{\zeta}}\right) = a\frac{P_0\left(\left(a\frac{P_0(z^{\beta})}{z^{\zeta}}\right)^{\beta}\right)}{\left(a\frac{P_0(z^{\beta})}{z^{\zeta}}\right)^{\zeta}}
$$

$$
= \frac{z^{\zeta^2}P_0\left(\frac{a^{\beta}(P_0(z^{\beta}))^{\beta}}{z^{\beta\zeta}}\right)}{a^{\zeta-1}(P_0(z^{\beta}))^{\zeta}}.
$$

Now,

$$
P_0 \left(\frac{a^{\beta} (P_0(z^{\beta}))^{\beta}}{z^{\beta \zeta}} \right)
$$

= $\left(\frac{a^{\beta}}{z^{\beta \zeta}} (P_0(z^{\beta}))^{\beta} - \xi_1 \right)^{a_1} \left(\frac{a^{\beta}}{z^{\beta \zeta}} (P_0(z^{\beta}))^{\beta} - \xi_2 \right)^{a_2} \dots \left(\frac{a^{\beta}}{z^{\beta \zeta}} (P_0(z^{\beta}))^{\beta} - \xi_s \right)^{a_s}$
= $\frac{1}{z^{m_1 \beta \zeta}} \left(a^{\beta} (P_0(z^{\beta}))^{\beta} - \xi_1 z^{\beta \zeta} \right)^{a_1} \left(a^{\beta} (P_0(z^{\beta}))^{\beta} - \xi_2 z^{\beta \zeta} \right)^{a_2} \dots \left(a^{\beta} (P_0(z^{\beta}))^{\beta} - \xi_s z^{\beta \zeta} \right)^{a_s} .$

The numerator of this expression is a polynomial, each of whose non-constant terms is of the form $z^{l\beta}$ for some $l > 0$. Since $\beta \geq 2$, the difference of two consecutive powers is at least 2. In particular, this polynomial is centered. Let $AP_1(z^{\beta_1})$ be this polynomial where $P_1(z^{\beta_1})$ is a normalized polynomial in z and such that β_1 is maximal for this expression. Clearly β divides β_1 . By the same argument,

$$
(P_0(z^{\beta}))^{\zeta} = (z^{\beta} - \xi_1)^{\zeta a_1} (z^{\beta} - \xi_2)^{\zeta a_2} \dots (z^{\beta} - \xi_s)^{\zeta a_s}
$$

is a normalized polynomial. This expression can be written as $P_2(z^{\beta})$ for some polynomial P_2 where β is maximal for this expression as it is so for $P_0(z^{\beta})$. Hence, we get

$$
F(z) = B z^{\zeta^2 - m_1 \beta \zeta} \frac{P_1(z^{\beta_1})}{P_2(z^{\beta})}.
$$

where $B = \frac{A}{a^2}$ $\frac{A}{a^{\zeta-1}}$. Note that $\zeta^2 - m_1\beta\zeta = \zeta(\zeta - m_1\beta) > 2$ and $\gcd(\beta_1, \beta) = \beta \ge 2$. As 0 and ∞ are superattracting fixed points of F, these are in $\mathcal{F}(F)$. Hence $\mathcal{J}(F)$ is not invariant under any translation. Also, there is no common root of $P_1(z^{\beta_1})$ and $P_2(z^{\beta})$. It follows from Theorem [4.7](#page-15-0) that $\Sigma F = \{z \mapsto \lambda z : \lambda^{\beta} = 1\}$. As $\mathcal{J}(R) = \mathcal{J}(F)$, we conclude that $\Sigma R = \{z \mapsto \lambda z : \lambda^{\beta} = 1\}.$

Remark 4.5. The Julia set of the rational map R as given in the above theorem cannot be \overline{C} or a line. For $\nu = -d$ or $-d+1$, it is possible.

Remark 4.6. Let $\nu = -d$. The Julia set of $R_1(z) = \frac{z^2-2}{z^2}$ $\frac{2-2}{z^2}$ is $\widehat{\mathbb{C}}$ and ΣR_1 consists of all the Euclidean isometries. Here $\nu = -2$, $d = 2$ and therefore $\nu = -d$. Writing $R_1(z)$ as $\frac{P(z)}{z^2}$ $\frac{P(z)}{z^2}$, we see that $\Sigma P = {\lambda z : \lambda^2 = 1}$ and $\Sigma P \subsetneq \Sigma R_1$.

Consider $\tilde{R}(z) = \frac{z^2-1}{z^2}$ $\frac{z-1}{z^2}$. As per the notations of Theorem [4.8,](#page-18-0) $\nu = -d = -2$ and $P(z) = z^2 - 1$. Note that $\deg(\tilde{R}) = 2$, so \tilde{R} has two critical points, namely 0 and ∞ . Also, $\{0,\infty,1\}$ is a cycle of 3-periodic points of R containing both the critical points. Therefore this is a superattracting cycle and hence, $0, \infty \in \mathcal{F}(R)$. This leads to the conclusion that Julia set of \tilde{R} is not invariant under any translation and \tilde{R} does not have any parabolic or rotation domain. Let $R = \tilde{R} \circ \tilde{R} \circ \tilde{R}$. Then $R(z) = -z^4 \frac{z^4 - 4z^2 + 2}{(2z^2 - 1)^2}$ $\frac{\zeta^4-4z^2+2}{(2z^2-1)^2}$. Since $\mathcal{J}(\tilde{R}) = \mathcal{J}(R)$, the latter is not invariant under any translation. Also all other assumptions of Theorem [4.7](#page-15-0) are satisfied and it follows that $\Sigma R = \{z \mapsto \lambda z : \lambda^2 = 1\}$. Hence $\Sigma \tilde{R} = \{z \mapsto \lambda z : \lambda^2 = 1\}$ which is nothing but ΣP . The Julia set of \tilde{R} is given in red in Fig. [\(6\)](#page-20-1).

Figure 6: Julia set of $\frac{z^2-1}{z^2}$ $\overline{z^2}$

Remark 4.7. To discuss the situation $\nu = -d + 1$, consider the Newton map N₁ applied to $z^2 + 1$. Then $N_1(z) = \frac{z^2-1}{2z}$ $\frac{2-1}{2z}$, and $\mathcal{J}(N_1) = \mathbb{R} \cup {\infty}$. Thus ΣN_1 contains translations by every real number and the rotation about the origin by an angle of π . However, N_1 can be written as $N_1(z) = \frac{P(z)}{2z}$ where $P(z) = z^2 - 1$. In this case $\nu = 1$ and $d = 2$, therefore, $\nu = -d + 1$. Note that $\Sigma P = {\lambda z : \lambda^2 = 1}$. Thus $\Sigma P \subseteq \Sigma N_1$. Here N_1 is exceptional.

Now consider the Newton map N_2 applied to the polynomial $z^3 - 1$. Then $N_2(z) = \frac{2z^3+1}{3z^2}$ $\frac{z^3+1}{3z^2}$. The Fatou set $\mathcal{F}(N_2)$ consists of the basins of the superattracting fixed points of N_2 corresponding to the roots of $z^3 - 1 = 0$, and the Julia set $\mathcal{J}(N_2)$ is connected. Therefore, N_2 is non-exceptional rational map without any parabolic domain or a rotation domain, and its Julia set is not invariant under any (non-trivial) translation. Also note that, $\mathcal{F}(N_2)$ contains exactly three unbounded components, namely the immediate basins of the superattracting fixed points of N_2 .

Express N_2 as $N_2(z) = \frac{2P(z)}{3z^2}$ where $P(z) = z^3 + \frac{1}{2}$ $\frac{1}{2}$, $d = 3$ and $\nu = -2$. As $\{z \mapsto \lambda z : \lambda^3 = 1\} \subseteq \Sigma N_2$ (from the first part of the Theorem [4.8\)](#page-18-0), the elements of ΣN_2 are rotations about the origin and are of finite order. If there is a rotation σ of order bigger than three preserving $\mathcal{J}(N_2)$, then an unbounded component of $\mathcal{F}(N_2)$ will be mapped to a bounded component of $\mathcal{F}(N_2)$ by σ , which is not possible. Since $\Sigma P = \{ z \mapsto \lambda z : \lambda^3 = 1 \}, \ \Sigma P = \Sigma N_2.$

Figure 7: Julia set of $N_2(z) = \frac{2z^3+1}{3z^2}$ $3z^2$

We conclude with a corollary, which is already known and can be found in $[8]$. It is presented here with a simplified proof using Theorem [4.8.](#page-18-0)

Corollary 4.3. For the McMullen map $R_{\lambda}(z) = z^m + \frac{\lambda}{z^n}$, where $m, n \in \mathbb{N}$ with $m \geq 2$ and $\lambda \in \mathbb{C} \setminus \{0\}$, if $R_{\lambda}(z)$ has no parabolic or rotation domain then $\Sigma R_{\lambda} =$ $\{\sigma : \sigma(z) = \mu z, \mu^{m+n} = 1\}.$

Proof. Here $R_{\lambda}(z) = z^m + \frac{\lambda}{z^n} = \frac{z^{m+n} + \lambda}{z^n} = z^{\nu} P(z)$, where $\nu = -n$ and $P(z) =$ $z^{m+n} + \lambda$, which is a normalized polynomial with $\Sigma P = \{z \mapsto \mu z : \mu^{m+n} = 1\}$. Also, note that $-(m+n)+2 \leq \nu < 0$. By Theorem [4.8,](#page-18-0) $\Sigma R_{\lambda} = \Sigma P$. \Box

5 Newton's and Chebyshev's methods

A root-finding method applied to a non-constant polynomial p is a rational map, for which every root of p is an attracting (superattracting if the root is simple) fixed point. A root-finding method can be expected to inherit, at least partially, some dynamical aspects of the polynomial. This section deals with the possible relation between the rotational symmetries of the Julia set of a polynomial and that of some root-finding methods applied to it.

The Newton's method is a classical root-finding method. The Newton's method N_p applied to a polynomial p is defined as

$$
N_p(z) = z - \frac{p(z)}{p'(z)}.
$$

The dynamics of the Newton's method applied to a polynomial has already been determined. It is proved that the Julia set of N_p is connected. It is due to a result of Shishiura (Theorem [\[22\]](#page-28-6)) who proved that the Julia set of a rational map of degree at least two is disconnected if the rational map has at least two weakly repelling fixed point (a fixed point which is either repelling or a multiple fixed point). Other than the roots of p (which are indeed attracting fixed points of N_p), ∞ is only a fixed point of N_p and it is repelling. The following is proved by Yang [\[26\]](#page-28-7).

Theorem 5.1. If p is a normalized polynomial then $\Sigma p \subseteq \Sigma N_p$.

In the same article ([\[26\]](#page-28-7)) it is proved that the Julia set of N_p is a line whenever p has exactly two roots with same multiplicity. In fact, the Julia set is the perpendicular bisector of the line segment joining two roots of p. In this case ΣN_p contains translation, hence can not be equal to Σp . If ΣN_p does not contain any translation and Σp is non-trivial, then Theorem [5.1](#page-22-0) asserts that ΣN_p contains rotations about the origin. In this scenario one can expect equality in Σp and ΣN_p . We apply certain conditions on p and use Theorem [4.7](#page-15-0) to prove the desire equality. As $\mathcal{J}(N_p)$ is connected, N_p does not contain a Herman ring of any period. However, the possibility of existence of a Siegel disk can not be discarded. Note that, a polynomial is called generic if all its roots are simple.

Theorem 5.2. Let p be a normalized, generic polynomial p with a root at the origin and Σp be non-trivial. If N_p does not contain a parabolic domain or a Siegel disk then $\Sigma p = \Sigma N_p$.

Proof. Recall from [\(2\)](#page-4-0) and [\(3\)](#page-4-3) that $p(z) = z^{\alpha}p_0(z^{\beta})$, where α, β are maximal for this expression and p_0 is a monic polynomial. Since 0 is a simple root of p, $\alpha = 1$ and as Σp contains more than one element, $\beta \geq 2$. The latter gives that every root of p_0 gives rise to two distinct roots of p. Therefore, p has at least three distinct roots and hence the Fatou set of N_p contains at least three (in fact, infinitely many) Fatou components. The Julia set cannot be a line. Since ∞ is a repelling fixed point of N_p , it follows from Theorem [4.3](#page-14-2) that $\mathcal{J}(N_p)$ is not invariant under any translation. Further, N_p is not exceptional.

Now, p can be writtes as

$$
p(z) = z \sum_{i=1}^{k} a_i z^{m_i} + a_{k+1} z
$$

where $p_0(z) = z^{m_1} + a_2 z^{m_2} + \cdots + a_k z^{m_k} + a_{k+1}, a_1 = 1$ and $a_i \neq 0$ for $i = 2, 3, \cdots k+1$. Then $p'(z) = \sum_{i=1}^{k} b_i z^{m_i \beta} + a_{k+1}$ where $b_i = a_i (m_i \beta + 1)$, and

$$
N_p(z) = z - \frac{p(z)}{p'(z)} = z - \frac{z(\sum_{i=1}^k a_i z^{m_i \beta}) + a_{k+1} z}{(\sum_{i=1}^k b_i z^{m_i \beta}) + a_{k+1}}
$$

=
$$
\frac{z(m_1 \beta z^{m_1 \beta} + a_2 m_2 \beta z^{m_2 \beta} + \dots + a_k m_k \beta z^{m_k \beta})}{(\sum_{i=1}^k b_i z^{m_i \beta}) + a_{k+1}}
$$

.

Let

$$
P(z) = \frac{1}{m_1 \beta} z (m_1 \beta z^{m_1 \beta} + a_2 m_2 \beta z^{m_2 \beta} + \dots a_{k-1} m_{k-1} \beta z^{m_{k-1} \beta} + a_k m_k \beta z^{m_k \beta})
$$

and

$$
Q(z) = \frac{1}{b_1} \big[\big(\sum_{i=1}^k b_i z^{m_i \beta} \big) + a_{k+1} \big].
$$

Then P and Q are normalized polynomials without any common root (as p is generic) and $N_p(z) = \frac{m_1 \beta}{b_1}$ $P(z)$ $\frac{P(z)}{Q(z)}$. Let $P_0(z) = z^{m_1 - m_k} + \frac{a_2 m_2}{m_1}$ $\frac{2m_2}{m_1}z^{m_2-m_k} + \cdots + \frac{a_{k-1}m_{k-1}}{m_1}$ $\frac{1^{m_{k-1}}}{m_1}z^{m_{k-1}-m_k} +$ $a_k m_k$ $\sum_{m_1}^{km_k}$. Then $P(z) = z^{m_k\beta+1}P_0(z^{\beta_1})$ where $m_k\beta+1$ and β_1 are maximal for this expression. Here $\beta_1 = \beta \gcd(m_1 - m_k, m_2 - m_k, \cdots, m_{k-1} - m_k)$ is a multiple of β . Similarly $Q(z) = Q_0(z^{\beta})$ where β is maximal for this expression. Now,

$$
N_p(z) = \frac{m_1 \beta}{b_1} \frac{z^{m_k \beta + 1} P_0(z^{\beta_1})}{Q_0(z^{\beta})}.
$$

As $gcd(\beta_1, \beta) = \beta \geq 2$ and $m_k \beta + 1 > 1$, it follows from Theorem [\[4.7\]](#page-15-0) that $\Sigma N_p = \{ z \mapsto \lambda z : \lambda^{\beta} = 1 \},$ which is nothing but Σp . \Box

The polynomial $p(z) = z(z^3 - 1)$ is normalized, generic and is with a simple root at the origin. Also $\Sigma p = \{z \mapsto \lambda z : \lambda^3 = 1\}$, non-trivial. Here $N_p(z) = \frac{3z^4}{4z^3}$ $\frac{3z^4}{4z^3-1}$. Note that the critical points of N_p are the roots of p, which are also superattracting fixed points of N_p . Hence N_p can not have any parabolic domain or Siegel disk. Thus, it follows from Theorem [5.2](#page-22-1) that $\Sigma p = \Sigma N_p$ (see Fig. [8\(](#page-24-0)a)).

The preceding example can be generalized with the same argument. Consider the polynomial $p(z) = z(z^{n} - 1)$, $n \ge 2$. Then $N_p(z) = \frac{nz^{n+1}}{(n+1)z^{n-1}}$. The critical points of N_p are the roots of p, and the Fatou set of N_p consists of the basin of attractions corresponding to the roots of p. Therefore, N_p does not have any parabolic or rotation domain. Hence by Theorem [5.2,](#page-22-1) $\Sigma p = \Sigma N_p = \{z \mapsto \lambda z : \lambda^n = 1\}.$

Remark 5.1. The König's methods is a family of root-finding methods $\{K_{p,n}, n =$ $2, 3, \ldots$ } defined by

$$
K_{p,n}(z) = z + (n-1) \frac{\left(\frac{1}{p}\right)^{[n-2]}(z)}{\left(\frac{1}{p}\right)^{[n-1]}(z)},
$$

(a) Julia set of $N_p(z)$ (b) Julia set of $C_p(z)$

Figure 8: The Newton's method and the Chebyshev's method applied to $p(z)$ = $z(z^3-1)$

where $\left(\frac{1}{n}\right)$ $\frac{1}{p} \Big)^{[i]}$ denotes the i-th derivative of $\frac{1}{p}$. For $n=2, K_{p,2}$ is the Newton's method. Some useful properties of $K_{p,n}$ can be found in [\[6\]](#page-27-13). Liu and Gao proved the following results in [\[13\]](#page-27-14).

Theorem 5.3. If p is a normalized polynomial then $\Sigma p \subseteq \Sigma K_{p,n}$.

Theorem 5.4. The Julia set $\mathcal{J}(K_{p,n})$ of $K_{p,n}$ is a straight line if and only if p has exactly two distinct roots with the same multiplicity.

Since ∞ is a repelling fixed point of $K_{p,n}$ and $\mathcal{J}(K_{p,n}) \neq \widehat{\mathbb{C}}$, it follows from Theorem [4.3](#page-14-2) that $\mathcal{J}(K_{p,n})$ is not invariant under any translation whenever p has exactly two distinct roots with different multiplicities or has at least three distinct roots. In this case, there are rotational symmetries of $\mathcal{J}(K_{p,n})$ whenever Σp is nontrivial. This motivates the following conjecture.

Conjecture 5.5. For $n \geq 2$ and a normalized polynomial p, $\sum K_{p,n} = \sum p$.

It is already proved in Theorem [5.2](#page-22-1) that this is true for $n = 2$ under certain conditions.

We now prove Theorem [5.3](#page-24-1) for another root-finding method, namely the Chebyshev's method. The Chebyshev's method of p , denoted by C_p is defined as

$$
C_p(z) = z - \left(1 + \frac{1}{2}L_p(z)\right) \frac{p(z)}{p'(z)}
$$

where $L_p(z) = \frac{p(z)p''(z)}{(p'(z))^2}$ $\frac{\partial^{(2)}(z)p^{(2)}(z)}{(\rho'(z))^2}$. Note that if p is a monomial or a linear polynomial then C_p is a linear polynomial, and this is not of interest here. Now onwards, we assume that p is not a monomial and $\deg(p) \geq 2$. The Chebyshev's method is a third-order convergent method. In other words, the local degree of C_p at each simple root of p is at least three.

Theorem 5.6. For every normalized polynomial p , $\Sigma p \subseteq \Sigma C_p$.

Proof. Recall that $p(z) = z^{\alpha} p_0(z^{\beta})$ where α, β are maximal for this expression. If $\beta = 1$, then Σp contains the identity map only and hence the theorem is trivial.

Let $\beta \geq 2$. Then p has at least two non-zero roots. This is because $p_0(z^{\beta})$ has a non-zero root and $\beta \geq 2$. For every $\sigma \in \Sigma p, \sigma(z) = \lambda z$ where $\lambda^{\beta} = 1$. Note that $p(\lambda z) = \lambda^{\alpha} z^{\alpha} p_0(\lambda^{\beta} z) = \lambda^{\alpha} p(z)$. Differentiating it once and twice, it is found that $p'(\lambda z) = \lambda^{\alpha-1} p'(z)$ and $p''(\lambda z) = \lambda^{\alpha-2} p''(z)$. Thus, $L_p(\sigma(z)) = \frac{\lambda^{2\alpha-2} p(z) p''(z)}{(\lambda^{\alpha-1} p'(z))^2}$ $\frac{z\alpha^{-2}p(z)p''(z)}{(\lambda^{\alpha-1}p'(z))^2}=L_p(z)$ and hence

$$
C_p(\sigma(z)) = \sigma(z) - \left(1 + \frac{1}{2}L_p(\sigma(z))\right) \frac{p(\sigma(z))}{p'(\sigma(z))}
$$

= $\lambda z - \left(1 + \frac{1}{2}L_p(z)\right) \frac{\lambda p(z)}{p'(z)} = \sigma(C_p(z)).$

 \Box

Therefore $\sigma(\mathcal{J}(C_p) = \mathcal{J}(C_p)$ and $\sigma \in \Sigma C_p$.

A generalized version of Theorems [5.1,](#page-22-0) [5.3](#page-24-1) and [5.6](#page-25-0) is established in [\[17\]](#page-28-8). It is shown that if a root-finding method F satisfies a special property, named as the Scaling Theorem, then for any normalized polynomial $p, \Sigma p \subseteq \Sigma F_p$ (Theorem 1.1., [\[17\]](#page-28-8)). Note that the König's methods and the Chebyshev's method satisfy the Scaling Theorem (see [\[6\]](#page-27-13) and [\[16\]](#page-28-9)). An attempt to find equality in Σp and ΣC_p is made in [\[17\]](#page-28-8) and [\[18\]](#page-28-10). The work is done for polynomials of degree up to four. In a general case, the same is done for unicritical polynomial and polynomials with exactly two roots. Considering a polynomial of any degree, the following result proves equality in Σp and ΣC_p under certain conditions.

Theorem 5.7. Let p be a normalized and generic polynomial with a simple root at the origin such that p' is also generic. If Σp is non-trivial and C_p does not contain a parabolic or rotation domain then $\Sigma p = \Sigma C_p$.

Proof. The Chebyshev's method has a repelling fixed point at ∞ . Using the argu-ments used in the proof of the Theorem [5.2,](#page-22-1) it is found that $\mathcal{J}(C_p) \neq \widehat{\mathbb{C}}$ and ΣC_p does not contain any translation. For $p(z) = z \sum_{i=1}^{k} a_i z^{m_i \beta} + a_{k+1} z$ where $a_1 = 1$, observe that $p'(z) = \sum_{i=1}^{k} b_i z^{m_i \beta} + a_{k+1}$, where $b_i = a_i(m_i \beta + 1)$, and $p''(z) = \sum_{i=1}^{k} c_i z^{m_i \beta - 1}$, where $c_i = b_i m_i \beta$, for $i = 1, 2, ..., k$ (see [\(3\)](#page-4-3)). Note that Σp is non-trivial amounts to $\beta \geq 2$. We have

$$
C_p(z) = z - \left(1 + \frac{1}{2}L_p(z)\right) \frac{p(z)}{p'(z)} = z \left(\frac{2(p'(z))^3 - \{2(p'(z))^2 - zp''(z)p_0(z^{\beta})\}p_0(z^{\beta})}{2(p'(z))^3}\right).
$$

Since p, p' are generic, there is no common root of $F_1(z) - F_2(z)$ and $p'(z)$ ³. Let $F_1(z) = 2(p'(z))^3$ and $F_2(z) = \{2(p'(z))^2 - zp''(z)p_0(z^{\beta})\}p_0(z^{\beta}).$ Then

.

.

$$
C_p(z) = z \left(\frac{F_1(z) - F_2(z)}{2(p'(z))^3} \right)
$$

Since $F_1(z) = 2 \left(\sum_{i=1}^k b_i z^{m_i \beta} + a_{k+1} \right)^3$ and,

$$
F_2(z) = \left[2\left(\sum_{i=1}^k b_i z^{m_i \beta} + a_{k+1}\right)^2 - \left(\sum_{i=1}^k c_i z^{m_i \beta}\right) \left(\sum_{i=1}^k a_i z^{m_i \beta} + a_{k+1}\right) \right] \left(\sum_{i=1}^k a_i z^{m_i \beta} + a_{k+1}\right)
$$

The constant terms of $F_1(z)$ and $F_2(z)$ are the same and that is $2a_{k+1}^3$. In other words, there is no constant term in $F_1(z) - F_2(z)$. Further, each of its terms is a constant multiple of some positive power of z^{β} .

Hence, there exist natural numbers ζ and β_1 such that $F_1(z) - F_2(z) = Az^{\zeta} F(z^{\beta_1}),$ where β divides ζ as well as β_1 and A is the coefficient of the leading term of $F_1(z) - F_2(z)$. Choose ζ and β_1 to be maximal for this expression. Note that ζ is the multiplicity of 0 as a root of $F_1(z) - F_2(z)$. By the choice of A, $F(z^{\beta_1})$ is monic. Further, since $\beta_1 \geq 2$, $F(z^{\beta_1})$ is centered. Thus $F(z_1^{\beta_1})$ $\binom{1}{1}$ is a normalized polynomial and is with a non-zero constant term because of the choice of ζ . Note that $(p'(z))^3$ is centered as $\beta \geq 2$ and taking its leading coefficient, say A' common we get the expression of C_p as

$$
C_p(z) = Bz^{\zeta+1} \frac{F(z^{\beta_1})}{Q(z^{\beta})},
$$

where $B = \frac{A}{2A'}$ and $(p'(z))^3 = A'Q(z^{\beta})$. For this latter expression, β is maximal as it is so for p'. Since $gcd(\beta_1, \beta) = 1$, it follows from Theorem [4.7](#page-15-0) that $\Sigma C_p = \{z \mapsto$ $\lambda z : \lambda^{\beta} = 1$. Therefore $\Sigma C_p = \Sigma p$. \Box

Consider the polynomial $p(z) = z(z^3 - 1)$. Then $p'(z) = 4z^3 - 1$ is generic. In [\[17\]](#page-28-8) $\Sigma p = \Sigma C_p$ is proved by analyzing the number of unbounded Fatou components of C_p . It is also proved that the $\mathcal{J}(C_p)$ is connected and F is the union of basins of attraction of the superattracting fixed points of C_p corresponding to the roots of p. Thus $\mathcal{F}(C_p)$ does not consist of any parabolic or rotation domain. The Fig. $(8(b),$ $(8(b),$ illustrating the Fatou and the Julia set of C_p , supports Theorem [5.7.](#page-25-1)

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