Some Nordhaus-Gaddum-type Results

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Abstract

A Nordhaus-Gaddum-type result is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement. In this paper some variations are considered. First, the sums and products of $\psi(G_1)$ and $\psi(G_2)$ are examined where $G_1 \oplus G_2 = K(s,s)$, and ψ is the independence, domination, or independent domination number, inter alia. In particular, it is shown that the maximum value of the product of the domination numbers of G_1 and G_2 is $\lfloor (s/2+2)^2 \rfloor$ for $s \geq 3$. Thereafter it is shown that for $H_1 \oplus H_2 \oplus H_3 = K_p$, the maximum product of the domination numbers of H_1, H_2 and H_3 is $p^3/27 + \Theta(p^2)$.

1 Introduction

In 1956 the original paper [6] by Nordhaus and Gaddum appeared. In it they gave sharp bounds on the sum and product of the chromatic numbers of a graph and its complement. Since then such results have been given for several parameters; see, for example, [2]. They include the following on the domination number, $\gamma(G)$, due to Jaeger & Payan and Payan & Xuong:

Proposition 1 [5, 8] *If G is a graph of order p then $\gamma(G) + \gamma(\bar{G}) \leq p + 1$ and $\gamma(G)\gamma(\bar{G}) \leq p$ and these are sharp. Equality in the product bound requires $\{\gamma(G), \gamma(\bar{G})\} = \{1, p\}$, $\{2, p/2\}$ or $\{3, 3\}$.*

Another direction was pursued by Plesník [9] who extended Nordhaus and Gaddum's results to the case where the complete graph is factored into several factors.

In this paper we look at two variations on the above results. In the second section we extend the concept by considering $G_1 \oplus G_2 = K(s, s)$ rather than $G_1 \oplus G_2 = K_p$. (If G and H are graphs on the same vertex set but with disjoint edge sets, then $G \oplus H$ denotes the graph whose edge set is the union of their edge sets.) In that and the following section we look at parameters including the independence and domination numbers. In the final section we consider the domination number and $G_1 \oplus G_2 \oplus G_3 = K_p$.

In this paper we shall use the terminology of [1]. Specifically, $p(G)$ denotes the number of vertices (order) of a graph G with vertex set $V(G)$ and edge set $E(G)$. Also, $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of G respectively. Further, $N(x)$ denotes the neighborhood of a vertex x and, for $X \subseteq V(G)$, $N(X) = \bigcup_{x \in X} N(x)$, while $\langle X \rangle_G$ is the subgraph of G induced by X. For two disjoint graphs G and H, $G \cup H$ and $G + H$ denote the union and join of G and H respectively. For a real number x, $\lfloor x \rfloor$ denotes the largest integer not more than x, and $\lceil x \rceil$ the smallest integer not less than x.

2 The Relative Complement

Recently, Cockayne [3] suggested the idea of a relative complement of a graph. In this section and the next we look at this concept.

If G is a subgraph of H then the graph $H - E(G)$ is the *complement of G relative to H*. Cockayne [3] posed the question of finding the graphs H with respect to which complements are always unique in the following sense: if G_1 and G_2 are isomorphic subgraphs of H then their complements $H - E(G_1)$ and $H - E(G_2)$ are isomorphic.

We address this question first. We shall use the following easy lemma:

Lemma 1 *Let H be a regular graph with respect to which complements are always unique. Let F_1 and F_2 be two graphs without isolated vertices such that F_1 is a proper spanning subgraph of F_2. Then at most one of F_1 and F_2 is (isomorphic to) an induced subgraph of H.*

PROOF. Suppose F_1 and F_2 are isomorphic to induced subgraphs of H with vertex sets V_1 and V_2 respectively. Then let H_1 (H_2) be the complement of F_1 relative to H formed by removing the edges of a copy of F_1 from the subgraph induced by V_1 (V_2). By the hypothesis $H_1 \cong H_2$. Since H is regular and F_1, F_2 have no isolated vertices, the isomorphism must map V_1 to V_2 and thus $\langle V_1 \rangle_{H_1} \cong \langle V_2 \rangle_{H_2}$. But this is a contradiction since $\langle V_1 \rangle_{H_1}$ is empty while $\langle V_2 \rangle_{H_2}$ is not. QED

Theorem 2 *Let H be a graph without isolated vertices with respect to which complements are always unique. Then H is one of the following:*
(a) $rK(1, s)$,
(b) rK_3,
(c) K_s,
(d) C_5, *or*
(e) $K(s, s)$,
for some integers r and/or s.

PROOF. Consider first the case when H is not regular. Then as the complement of K_2 is unique, every edge links a vertex of minimum degree and one of maximum degree. But as the complement of P_3 is unique, it follows that H has minimum degree 1. Thus H is the union of stars and is given by case (a).

So assume now that H is regular of degree at least 2. If H is disconnected then $2K_2$ is an induced subgraph of H, so that (by the above lemma) each component of H must have order less than four. This yields case (b).

So assume further that H is connected but not complete. Then P_3 is an induced subgraph of H so that K_3 is not. By the previous assumption, H has a cycle; consider the shortest such cycle. This has length at most five, else P_4 and $2K_2$ would both be induced subgraphs of H, a contradiction.

Assume first that H has girth five, and suppose $H \neq C_5$. Then there exists a 5-cycle C and a vertex w such that w is adjacent to exactly one vertex of C. Then P_5 and C_5 are induced subgraphs of H, a contradiction. So this case yields only C_5.

Finally, assume that H has girth four. Then C_4 is an induced subgraph of H, so that P_4 is not. This implies that H has diameter two. Further, H does not contain an odd cycle; for the shortest such cycle is not K_3 (see above) and would thus contain an induced P_4. Hence H is bipartite, has diameter two and is regular, which yields case (e). QED

The above results suggest that the complete bipartite graph $K(s, s)$ is a suitable graph to look at for results on relative complements. Indeed $K(s, s)$ is an obvious replacement for K_p in Nordhaus-Gaddum results. In the next two sections we look at this, and at five possible parameters of a graph G viz:

(i) The independence number $\beta(G)$.

(ii) The domination number $\gamma(G)$.

(iii) The independent domination number $i(G)$ being the minimum cardinality of a maximal independent set.

(iv) The (upper) irredundance number $\mathrm{IR}(G)$ being the maximum cardinality of an irredundant set.

(v) The vertex cover number $\alpha(G)$.

However, it is well-known that $\alpha(G) + \beta(G) = p(G)$, while Cockayne et al. [4]

showed that for bipartite graphs G it holds that $\mathrm{IR}(G) = \beta(G)$. We thus only explicitly state the results for the first three parameters. Note that

$$\gamma(G) \leq i(G) \leq \beta(G).$$

This leads to:

Theorem 3 *Let $s \geq 2$ be an integer and let $G_1 \oplus G_2 = K(s,s)$. Then the following table represents some sharp bounds on the sum and product of $\psi(G_1)$ and $\psi(G_2)$ for certain parameters ψ:*

	SUM				*PRODUCT*			
ψ	*Lower*		*Upper*		*Lower*		*Upper*	
γ	5^*	a	$2s+2$	c	6^*	a	—	
i	5^*	a	$3s$	d	6^*	a	$2s^2$	e
β	$2s$	b	$3s$	d	s^2	b	$\lfloor 9s^2/4 \rfloor$	d

(For entries marked with an asterisk, consider $s \geq 3$.)

PROOF. The proof is in five parts as indicated above.

a) Observe that $\gamma(G_j) \geq 2$ always. Observe further that if $\gamma(G_1) = 2$ and $s \geq 3$, then there are vertices of each partite set of degree at least $s-1$ in G_1, and thus $\gamma(G_2) \geq 3$. Equality is attained by taking for G_1 say the (disjoint) union of two stars $K(1, s-1)$ and joining an end-vertex from each star.

b) Trivially $\beta(G_j) \geq s$. Equality in the sum and product is attained for $G_1 = sK_2$ say.

c) Observe first that if $\gamma(G_1) > s$ then $\gamma(G_2) = 2$. This follows as neither partite set dominates G_1 so there exist vertices x and y from the two partite sets which are isolated in G_1, and thus $\{x, y\}$ dominates G_2. Equality in the bound is attained if G_1 say is empty.

d) Let S be a maximum independent set for G_1 and let A and B be the intersections of S with the two partite sets. Let T be a maximum independent set of G_2; then $T \subseteq V - A$ or $T \subseteq V - B$. It follows that $|S| + |T| \leq 2s + \max\{|A|, |B|\}$, whence the upper bound on the sum. The upper bound on the product follows from that on the sum. Equality is attained if G_1 say is $(\bar{K}_s + \bar{K}_{\lfloor s/2 \rfloor}) \cup \bar{K}_{\lceil s/2 \rceil}$.

e) To prove this bound, observe that it cannot happen that $i(G_1)$ and $i(G_2)$ are both greater than s. For, if $i(G_j) > s$ then there must be isolated vertices in both partite sets in G_j, and this cannot happen in both G_1 and G_2. Equality in the bound is attained if G_1 say is empty. QED

3 Domination Number and Relative Complement

In this section we look at $G_1 \oplus G_2 = K(s,s)$ and the upper bound on $\gamma(G_1)\gamma(G_2)$. While the upper bound on the sum is $2s+2$, we show that the maximum product is asymptotically $s^2/4$. This is in contrast with Proposition 1.

We shall establish:

Theorem 4 *Let $G_1 \oplus G_2 = K(s,s)$. Then:*

$$\gamma(G_1)\gamma(G_2) \leq \lfloor (s/2+2)^2 \rfloor,$$

and this is sharp for $s \geq 3$.

There are at least two constructions which realize this bound for $s \geq 4$. (For $s = 3$ take G_1 (say) empty.) Let $K(s,s)$ have partite sets A and B say. Partition A (B) into two sets, one of size $\lfloor s/2 \rfloor$, say A_1 (B_1), and one of size $\lceil s/2 \rceil$, say A_2 (B_2). Then for the first construction, let the edges of G_1 (say) be given by all edges between A and B_1. Any dominating set of G_1 contains B_2. But the complete bipartite graph $G_1 - B_2$ has domination number 2; so $\gamma(G_1) = \lceil s/2 \rceil + 2$. Similarly, $\gamma(G_2) = \lfloor s/2 \rfloor + 2$.

For the second construction, form the edge set of G_1 from the set of all edges between A and B_1 by removing (the edges of) a matching between A_1 and B_1 and inserting a matching between A_2 and B_2. This yields the same domination numbers—consider for example G_1. Then a vertex $v \in B_2$ has degree one. So in a minimum dominating set we should take v's neighbor, and hence all of A_2. This leaves A_1 to be dominated. But no single vertex dominates A_1, though any two in B_1 do.

3.1 Proof of the Upper Bound

We start the proof of the upper bound by introducing what we call left and right domination numbers. For the rest of the section we shall assume that $K(s, s)$ has partite sets $\mathcal{L}$ and $\mathcal{R}$ (standing for "left" and "right"), and that $G_1 \oplus G_2 = K(s, s)$.

Let $G \subset K(s, s)$. Then the left (right) domination number $l(G)$ ($r(G)$) of G is the minimum cardinality of a set which dominates $\mathcal{L}$ ($\mathcal{R}$). (Recall that a set S dominates a set T if T is contained in $S \cup N(S)$.) Now, it is trivial to observe that

$$1 \le l(G), r(G) \le s,$$

and that

$$\gamma(G) \le l(G) + r(G).$$

We establish some bounds on these new domination numbers, but first we introduce some notation. For $G \subset K(s, s)$ let $\delta^L(G)$ denote the minimum degree of a vertex of $\mathcal{L}$. Define $\delta^R(G)$, $\Delta^L(G)$ and $\Delta^R(G)$ similarly. Note that

$$\delta^L(G_1) + \Delta^L(G_2) = s,$$

and, by counting edges in G_1 and G_2, that

$$\delta^L(G_1) + \delta^R(G_2) \le s.$$

Further, we shall abbreviate parameters by writing ψ_i for $\psi(G_i)$ for a parameter ψ, and $N_i(v)$ for $N_{G_i}(v)$, for $i \in \{1, 2\}$.

This leads to the following generalizations of results for the domination number.

Lemma 5 *Let $s \ge 2$. Then*
a) $l_1 \le 1 + \delta_2^R$;
b) $(l_1 - 2)(r_2 - 1) \le \delta_2^R - 1$;
c) if $\Delta_1^R < s$ then $\gamma_2 \le \Delta_1^R + 2$;
d) if $\gamma_1 > s$ then $l_2 = r_2 = 1$ so that $\gamma_2 = 2$;
and analogous results ($l \leftrightarrow r$ and/or $1 \leftrightarrow 2$).

Proof. **a)** Let $v \in \mathcal{R}$ be a vertex of degree Δ_1^R in G_1. Then $N_2(v) \cup \{v\}$ dominates $\mathcal{L}$ in G_1 so that (as $\delta_2^R + \Delta_1^R = s$) the result follows.
b) If $r_2 = 1$ then $\Delta_2^L = s$ so that $\delta_2^R > 0$. So assume that $r_2 \geq 2$. Let $v \in \mathcal{R}$ have degree Δ_1^R in G_1 and let $X = \mathcal{L} - N_1(v)$. Then partition X into subsets $X_1, \ldots, X_k$ of size at most $r_2 - 1$ such that k is as small as possible. For each i, X_i does not dominate $\mathcal{R}$ in G_2, so there exists a $y_i \in \mathcal{R} - N_2(X_i)$ and thus $X_i \subseteq N_1(y_i)$. Hence

$$l_1 \leq |\{v, y_1, \ldots, y_k\}| = 1 + \left\lceil \frac{s - \Delta_1^R}{r_2 - 1} \right\rceil \leq 2 + \frac{\delta_2^R - 1}{r_2 - 1},$$

whence the result.
c) Let $v \in \mathcal{R}$ with degree Δ_1^R in G_1. Let $M = N_1(v)$, and let $x \in \mathcal{L} - M$. Then let $Y = N_1(x)$ and $Z = \mathcal{R} - Y - \{v\}$. Now, every $y \in Y$ has degree in G_1 at most that of v. But x is a neighbor of y and not of v. Hence $M \not\subseteq N_1(y)$ and therefore $y \in N_2(M)$. Thus, in G_2, M dominates Y, x dominates Z, and v dominates $\mathcal{L} - M$. Hence the set $M \cup \{v, x\}$ is a dominating set for G_2, and the result follows.
d) Neither $\mathcal{L}$ nor $\mathcal{R}$ dominates G_1 and hence there exist $x \in \mathcal{L}$ and $y \in \mathcal{R}$ isolated in G_1. QED

Some comments are in order. Part (b) is an improvement on (a) (except when $r_2 \leq 2$) and is based on a lemma of [8]. One can in fact obtain Nordhaus-Gaddum results for these parameters. Assume for the time being that both G_1 and G_2 are non-empty. Let G be formed from G_1 by adding all edges between vertices of $\mathcal{L}$; thus $\bar{G}$ is G_2 with all edges between vertices of $\mathcal{R}$ added. Further, $\gamma(G) = r(G_1)$ and $\gamma(\bar{G}) = l(G_2)$. Thus by Proposition 1 it holds that $r(G_1)l(G_2) = \gamma(G)\gamma(\bar{G}) \leq p(G) = 2s$. Thus for all G_1 we get

$$r_1 l_2 \leq 2s \quad \text{and} \quad r_1 + l_2 \leq s + 2.$$

(The bound for the sum is a consequence of the one for the product.) This is sharp; consider for example $G_1 = sK_2$.

These parameters may be of interest for further study in themselves, but we now use them to prove Theorem 4. By Lemma 5d, if $\gamma_1 > s$ then $\gamma_2 = 2$ so that $\gamma_1\gamma_2 \leq 4s$. Hence we may assume that $\gamma_1, \gamma_2 \leq s$. By the standard bound on the product given the sum, it is sufficient to prove:

Theorem 6 *If $\gamma_1, \gamma_2 \leq s$ then*

$$\gamma_1 + \gamma_2 \leq s + 4.$$

Proof. Note that the hypothesis implies that the bound of Lemma 5c holds without the restriction on Δ_1^R. Assume $r_1 = \min\{l_1, r_1, l_2, r_2\}$. There are two cases to consider.

Case 1: $r_1 = \min\{l_1, r_1, l_2, r_2\} \leq 2$

Addition of inequalities (c) and (a) of Lemma 5 yields that $l_1 + \gamma_2 \leq s + 3$. Since $\gamma_1 \leq l_1 + r_1$, it follows that $\gamma_1 + \gamma_2 \leq s + 5$. Suppose $\gamma_1 + \gamma_2 = s + 5$. Then $r_1 = 2$. Further, we need equality in the three bounds we used. Thus, (i) $\gamma_2 = \Delta_1^R + 2$, (ii) $l_1 = \delta_2^R + 1$, and (iii) $\gamma_1 = l_1 + r_1 = \delta_2^R + 3$.

Condition (i) implies that $\Delta_1^R \leq s - 2$ and thus $\delta_2^R \geq 2$. But then, by Lemma 5b, condition (ii) requires that $r_2 = 2$. By Lemma 5a, this means that $\gamma_1 + \gamma_2 \leq (l_1 + r_1) + (l_2 + r_2) \leq (3 + \delta_2^R) + (3 + \delta_1^R)$. Hence $\delta_1^R + \delta_2^R \geq s - 1$ and thus (iv) $\delta_1^R \geq s - \delta_2^R - 1 = \Delta_1^R - 1$.

Now, let $v \in \mathcal{R}$ of degree Δ_1^R in G_1. Let $M = N_1(v)$, $X = \mathcal{L} - M$, $Y = N_1(X)$ and $Z = \mathcal{R} - Y - \{v\}$. Condition (ii) implies that dominating X in G_1 requires $|X|$ vertices. Thus for all $y \in Y$, y is adjacent to at most one $x \in X$ in G_1, while we know that no $x \in X$ is isolated in G_1 (by the value of r_2). Together these observations imply that $|Y| \geq |X|$.

Further, condition (iii) implies that for all $m \in M$ it holds that $X \cup \{v, m\}$ does not dominate G_1. Thus for all such m there is a vertex $z_m \in Z$ such that z_m is not adjacent to m in G_1. By (iv) $\deg_{G_1} z_m \geq \deg_{G_1} v - 1$ and $N_1(z_m) \subseteq M$, so that z_m is adjacent to all of $M - \{m\}$. Thus the z_m are distinct, and $|Z| \geq |M|$. Hence $|\mathcal{R}| > |\mathcal{L}|$, a contradiction.

Case 2: $r_1 = \min\{l_1, r_1, l_2, r_2\} \geq 3$

By Lemma 5b, $l_1 \leq 2 + (\delta_2^R - 1)/(r_2 - 1) \leq 2 + (\delta_2^R - 1)/2$ and similarly $r_1 \leq 2 + (\delta_2^L - 1)/2$. Thus $\gamma_1 \leq l_1 + r_1 \leq (\delta_2^L + \delta_2^R)/2 + 3$. By Lemma 5c, $\gamma_2 \leq$

$\min\{\Delta_1^R+2, \Delta_1^L+2\} \leq (\Delta_1^R+\Delta_1^L)/2+2$. Adding these two inequalities yields that $\gamma_1+\gamma_2 \leq s+5$.

Suppose equality occurs. This requires equality in the first inequality (i.e. Lemma 5b). Thus $r_2 = 3$, for it follows from Lemma 5a and $l_1 \geq 3$ that $\delta_2^R \geq 2$. By symmetry, $l_1 = l_2 = r_1 = 3$. Further, equality requires $\gamma_i = l_i + r_i = 6$, and thus $s = 7$. However, a simple calculation which we omit shows that equality is still not possible. This completes the proof of Theorems 4 and 6. QED

4 The Triple Product

In this section we again consider the domination number, but now we look at the complete graph factored into several edge-disjoint graphs. In particular we investigate the upper bound on $\gamma(G_1)\gamma(G_2)\gamma(G_3)$ where $G_1 \oplus G_2 \oplus G_3 = K_p$.

We observe that $\gamma(G_2)+\gamma(G_3) \leq \gamma(G_2 \oplus G_3)+p$. For, let D be a dominating set of $G_2 \oplus G_3$. The vertices D misses in G_2 are disjoint from those it misses in G_3. Thus, in extending D to dominating sets of G_2 and G_3, we need take every other vertex at most once. By Proposition 1 we thus obtain that $\gamma(G_1)+\gamma(G_2)+\gamma(G_3) \leq \gamma(G_1)+\gamma(G_2 \oplus G_3)+p \leq p+1+p = 2p+1$.

But we shall prove:

Theorem 7 *Let $G_1 \oplus G_2 \oplus G_3 = K_p$. Then the maximum value of the product $\gamma(G_1)\gamma(G_2)\gamma(G_3)$ is $p^3/27+\Theta(p^2)$.*

That is, there exist constants c_1 and c_2 such that the maximum triple product always lies between $p^3/27+c_1p^2$ and $p^3/27+c_2p^2$.

4.1 Values for Small p and a General Construction

We look first at the maximum value of the triple product for small p. Using Proposition 1 and $\gamma(G_2)+\gamma(G_3) \leq \gamma(G_2 \oplus G_3)+p$, we get that the maximum triple product is at most the value of:

p	max. prod.	realization
1	1	trivial
2	4	trivial
3	9	G_1 complete
4	18	C_4, and $K_2 \cup 2K_1$ twice
5	27	$C_4 \cup K_1$, and $K_3 \cup 2K_1$ twice
6	40	$K(2,2,2)$, $K_2 \cup 4K_1$ and $2K_2 \cup 2K_1$
7	64	$((K_1 \cup K_2) + \bar{K}_2) \cup \bar{K}_2$ thrice
8	80	*see discussion*

Table 1: Optimal Values of the Triple Product for small p

$$\begin{array}{ll} \max & xyz \\ \text{s.t.} & 1 \leq x \leq y \leq z, \qquad \text{and} \\ & y + z \leq p + p/x \\ & \text{with strict inequality unless } x \in \{1, 2, p/2\} \text{ or } x = 3 \text{ and } p = 9. \end{array}$$

For real optimization this gives an upper bound of approximately $p^3/8$. But for integer optimization, we get the actual maxima for $p \leq 8$ (at least). These are summarized in Table 1.

In some cases these realizations may be obtained via a general construction which we now describe. Let (A, B, C) denote a weak partition of the vertex set of K_p. Then A, B and C will be the sets of vertices isolated in G_1, G_2 and G_3 respectively. Thus G_1 has all the edges between B and C, and some of the edges of the (complete) graphs induced by B and C. Now we observe that:

$$\gamma(G_1) = \begin{cases} |A| + 2 & \text{if } \gamma(\langle B \rangle_{G_1}), \gamma(\langle C \rangle_{G_1}) > 1, \text{ and} \\ |A| + 1 & \text{otherwise.} \end{cases}$$

Thus what matters is whether or not $\langle B \rangle_{G_1}$ or $\langle C \rangle_{G_1}$ has domination number 1 or equivalently, whether or not $\langle B \rangle_{G_3}$ or $\langle C \rangle_{G_2}$ has an isolated vertex.

For $p \geq 12$ it is possible and desirable to choose $|A|$, $|B|$ and $|C| \geq 4$, and to ensure that $\langle B \rangle_{G_1}$, $\langle C \rangle_{G_1}$, $\langle A \rangle_{G_2}$, etc. have no isolated vertices. Thus this construction yields, as a lower bound, the maximum product of three positive integers summing to $p+6$. This shows that the maximum product is at least $p^3/27+2p^2/3$.

For smaller p the best choice of parameters is not so straight-forward. For example, the maximum product for $p=6$ is achieved by taking $|A|=0$, $|B|=2$ and $|C|=4$, and $\langle B\rangle$ empty in G_1. For $p=8$ the maximum product can be achieved by taking $|A|=|B|=2$ and $C=4$ and letting $\langle A\rangle$ and $\langle B\rangle$ be complete in G_3. In both cases the edges of $\langle C\rangle$ are distributed between G_1 and G_2 to ensure that neither $\langle C\rangle_{G_1}$ nor $\langle C\rangle_{G_2}$ has an isolated vertex.

4.2 Proof of the Upper Bound

We shall use the following lemma:

Lemma 8 *Let v_1, v_2, v_3 be not necessarily distinct vertices having degrees d_1, d_2, d_3 in G_1, G_2, G_3 respectively. Then $\gamma_1+\gamma_2+\gamma_3 \le p+6+d_1+d_2+d_3$.*

PROOF. Let $W=\{v_1, v_2, v_3\}$. For $j=1,2,3$, let A_j denote the set of vertices adjacent to all vertices of W in G_j; note that $|A_j| \le d_j$. Then let $Y=A_1\cup A_2\cup A_3$ and $X=V-Y-W$.

By the definition of Y, if $x\in X$ then x is not adjacent to every vertex of W in any G_i ($i=1,2,3$). Thus in at least two of the G_i, W dominates x. We take W as a basis of a dominating set in all three G_i, and then each x need only be counted once. On the other hand, every vertex in Y is dominated by W in one of G_1, G_2 or G_3; thus

$$\gamma_1+\gamma_2+\gamma_3 \le 3|W|+|X|+2|Y| = p+2|W|+|Y|,$$

which concludes the proof. QED

Though we do not need it, one may also show that, under the above hypothesis, $\gamma_1+\gamma_2+\gamma_3 \le p+12+\max\{d_1, d_2, d_3\}$. We omit the proof.

An immediate corollary of Lemma 8 is that:

Theorem 9 *If G_1, G_2 and G_3 have isolated vertices then $\gamma_1+\gamma_2+\gamma_3 \le p+6$.*

This is sharp by the previous discussion.

For the remainder of the proof of the upper bound we need the following result of Payan:

Proposition 2 [7] *If G has order p and minimum degree δ, then*

$$\gamma(G) \leq \frac{p}{\delta+1} \sum_{j=1}^{\delta+1} 1/j \sim p \log \delta / \delta.$$

So assume $G_1 \oplus G_2 \oplus G_3 = K_p$ and the product of the domination numbers of G_1, G_2 and G_3 is at least $p^3/27$. Then each domination number is at least $p/27$. By Proposition 2, this means that there is a constant d such that each minimum degree is at most d. Then by Lemma 8, the sum of the domination numbers is at most $p + 6 + 3d$, and thus Theorem 7 is proved.

Nevertheless, we believe:

Conjecture. *Let $G_1 \oplus G_2 \oplus G_3 = K_p$ where $p = 3\ell + m$ and $0 \leq m \leq 2$. Then the maximum value of the product $\gamma(G_1)\gamma(G_2)\gamma(G_3)$ is $(\ell+2)^{3-m}(\ell+3)^m$.*

These values were established as lower bounds for $p \geq 12$ earlier.

A natural extension of the above is to consider factoring the complete graph into more factors. One can easily get asymptotically $p^r/27$ for r factors, but what is the best value in general?

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